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EXAMPLES OF NONCOMMUTATIVE GROUPS WITH NONTRIVIAL EXIT-BOUNDARY

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A number of new counterexamples are given, disproving certain assumptions about the mutual relations of the exit-boundary (Poisson boundary) of a random walk on a group and the amenability and growth of the group. Random walks are constructed with nontrivial exit-boundary on the affine group of the dyadic-rational line and on the infinite symmetric group.

The present paper is devoted to the investigation of a number of examples, illustrating various situations which arise in the theory of boundaries of random walks on nonabelian groups. The paper is closely connected with [17] (cf. also [4]), where the basic facts of this theory which have been obtained recently are given.

We recall some definitions. Let μ be a countable discrete group, G be a nondegenerate probability measure on G (i.e., $\operatorname{supp}\mu$ generates G as a semigroup). A homogeneous Markov process $\{y_n\}_{n=0}^{\infty}$ with state space G, initial distribution δ_e, (e is the identity of G), and transition probabilities $p(g|h)=\mu(h^{-1}g)$ is called a (right) random walk on G, given by the measure μ. By $(G^{\infty}, \mathbb{P}^{\mu})$ we denote the space of trajectories $\{y_n\}_{n=0}^{\infty}$ of the walk (G,μ) with the usual probability measure P^{μ}.

By the exit-boundary of (G,μ) is meant the quotient space (Γ,ν) of the space of trajectories of the walk, corresponding to its tail σ-algebra. The boundary (Γ,ν) is canonically provided with the structure of a measurable G-space. We stress that for Abelian groups Γ is always a single point. There exist a whole lot of other definitions (stationary boundary, Poisson boundary, etc.), which lead to the same space (Γ,ν), which we shall simply call the boundary of the walk (G,μ) [4, 17]. With the help of these definitions one can get various tests for trivality (= single-pointedness) of the boundary [3, 4, 7, 17]. Until recently the number of groups investigated was not large. In the present paper we investigate several new types of examples, which disprove certain old assumptions about the relations of the boundary, amenability, and growth of a group.

In Sec. 1 we consider the group $G_K = \mathbb{Z}^K \leftthreetimes \sum_{\mathbb{Z}^K} \mathbb{Z}_2$ which is the additive group of configurations on $\mathbb{Z}^K$ (with addition $\operatorname{mod} 2$), extending the natural action of $\mathbb{Z}^K$. We establish effective tests for the trivality of the boundary for finite measures on the

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groups G_K and we show that on G_1 there exists a nonfinite measure for which the boundary is nontrivial, despite the triviality of the boundary for the inverse measure.

In Sec. 2 we construct examples of measures with nontrivial boundary on the affine group of the dyadic-rational line.

In Secs. 3 and 4 we give examples of random walks with nontrivial boundary on the infinite symmetric group $\mathfrak{S}_\infty$ and on a solvable locally finite group of uniformly polynomial growth.

We stress that in this paper we are not pursuing the goal of a complete description of the boundary, but restrict ourselves to merely establishing its triviality or nontriviality.

The idea of using the groups G_K and $\mathfrak{S}_\infty$ for constructing nontrivial examples in the theory of random walks and other situations (cf. [1, 2, 6, 9]) was suggested by A. M. Vershik in connection with the general program of investigations of measures on groups. The author also thanks him for constant support and interest.

1. Extended Configuration Groups (the Groups G_K)

1. Random Walks on the Groups G_K. Let $\mathbb{Z}^K = \sum_{i=1}^{K} \mathbb{Z}$ be the K-dimensional integral lattice, $\mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2) = \sum_{\mathbb{Z}^K} \mathbb{Z}_2$ be the direct sum of isomorphic copies of the group $\mathbb{Z}_2 = \{0,1\}$, indexed by the elements of $\mathbb{Z}^K$ (i.e., the group of finite $\mathbb{Z}_2$-valued functions on $\mathbb{Z}^K$). It is also convenient to speak of $\mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2)$ as the additive group of finite configurations on $\mathbb{Z}^K$ with the operation of pointwise addition $\bmod 2$. We shall denote by $f(x)$ the value of the configuration $f \in \mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2)$ on the element $x \in \mathbb{Z}^K$, and by $\mathrm{supp}\, f$ the support of the configuration f:

$$\mathrm{supp}\, f = \{x \in \mathbb{Z}^K : f(x) \neq 0\} \tag{1}$$

By G_K we denote the group $G_K = \mathbb{Z}^K \leftthreetimes \mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2)$, which is the semidirect product of the group $\mathbb{Z}$ by the group $\mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2)$, in which $\mathbb{Z}^K$ acts by translations. The elements of G_K will be written as ordered pairs $g = (x, f)$, where $x \in \mathbb{Z}^K$, $f \in \mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2)$. Then the group operation in G_K can be written as follows:

$$(x_1, f_1)(x_2, f_2) = (x_1 + x_2, f_1 + x_1 f_2), \tag{2}$$

where xf is the result of the action of x on f:

$$(xf)(y) = f(y - x) \quad x, y \in \mathbb{Z}^K;\ f \in \mathrm{fun}(\mathbb{Z}^K, \mathbb{Z}_2) \tag{3}$$

All the groups G_K are solvable of stage 2, finitely generated and have exponential growth.

Now let μ be some probability measure on G_K; we shall denote by $\{(x_i, f_i)\}_{i=1}^{\infty}$ the collection of increments of the random walk defined by the measure μ (i.e., all (x_i, f_i) are independent and have distribution μ); then

$$(y_n, \varphi_n) = (x_1, f_1) \cdot \ldots \cdot (x_n, f_n) \tag{4}$$

is the n-th coordinate of the trajectory of the random walk defined by the measure μ. The following relations, which express the next coordinate of the trajectory of the random walk in terms of the preceding one and the corresponding increment, follow from the definition (2) of the group operation in G_K:

$$\begin{cases} y_{n+1} = y_n + x_{n+1} \\ \varphi_{n+1} = \varphi_n + y_n f_{n+1} \end{cases} \tag{5}$$

2. Nontriviality of the Boundary for Finite Measures. We consider an arbitrary finite nondegenerate measure μ on the group G $(\kappa \geqslant 3)$. Since any nondegenerate random walk on $\mathbb{Z}^\kappa$ for $\kappa \geqslant 3$ is nonrecurrent [10], y_n leaves any finite subset $\mathbb{Z}^\kappa$ for almost all trajectories $\{(y_n, \varphi_n)\}_{n=0}^\infty$ of the random walk (G_κ, μ). The measure μ is finite, so for sufficiently large n the support of the configuration $y_n f_{n+1}$ does not intersect any previously given subset of the lattice $\mathbb{Z}^\kappa$. Thus, the values of the configuration φ_n at any fixed point $z \in \mathbb{Z}^\kappa$ almost surely stabilize in n. In other words, the configuration φ_∞ converges pointwise a.s. to some (now nonfinite) configuration. Thus for any $z \in \mathbb{Z}^\kappa$ the corresponding set of trajectories of the random walk

$$A = \left\{ \{(y_n, \varphi_n)\}_{n=0}^\infty : \lim_n \varphi_n(z) = 0 \right\} \tag{6}$$

is a tail. The nontriviality of A obviously follows from the nondegeneracy of the measure μ. Thus, $\Gamma(G_\kappa, \mu)$ is nontrivial. We have proved

THEOREM 1.1. Let μ be a finite nondegenerate probability measure on the group G_κ $(\kappa \geqslant 3)$; then the boundary $\Gamma(G_\kappa, \mu)$ of the random walk defined by the measure μ is nontrivial.

Remark. In the proof of the theorem we have used essentially the nonrecurrence of the walk $\{y_n\}_{n=0}^\infty$ on $\mathbb{Z}^\kappa$ and the finiteness of the set $\bigcup \operatorname{supp} f$, where the union is taken over all elements $(x, f) \in \operatorname{supp} \mu$.

3. Test for Triviality of the Boundary. Now we give a simple sufficient condition for the triviality of the boundary on the groups G_κ. The following useful lemma is due to Furstenberg [13]:

LEMMA 1.1. Let the subgroup $G^\circ \subset G$ be a set of recurrence for the random walk on G, defined by the measure μ. We define on G° a probability measure μ° as follows: $\mu^\circ(g)$ is the probability that after the walk leaves the identity e the first return to G° occurs at the point $g \in G^\circ$. Then the boundaries $\Gamma(G, \mu)$ and $\Gamma(G^\circ, \mu^\circ)$ are canonically isomorphic as spaces with measure. In particular, the triviality of $\Gamma(G, \mu)$ is equivalent with the triviality of $\Gamma(G^\circ, \mu^\circ)$.

THEOREM 1.2. If the measure μ on the group G_κ is such that the walk induced by it on $\mathbb{Z}^\kappa$ is recurrent, then the boundary $\Gamma(G_\kappa, \mu)$ is trivial.

Proof. The hypothesis of the theorem means that the subgroup $G_\kappa^\circ = \{(x, f) \in G_\kappa : x = 0\}$ is recurrent for the walk (G_κ, μ). But the subgroup G_κ° is Abelian, i.e., the boundary of any walk on it is trivial by the Choquet–Deny theorem [12]. On the basis of the lemma we get that $\Gamma(G_\kappa, \mu)$ is trivial.

The theorem just proved combined with Theorem 1.1 allows us to get necessary and sufficient conditions for the triviality of the boundary for finite measures on the groups G_κ.

THEOREM 1.3. The random walk defined by the finite measure μ on the group G_κ has trivial boundary if and only if the projection of the walk onto $\mathbb{Z}^\kappa$ is recurrent. In particular, for symmetric finite measures μ the boundary $\Gamma(G_K,\mu)$ is trivial for $K=1,2$, and nontrivial for $K\geqslant 3$.

4. Boundary for Nonfinite Measures. It follows from Theorem 1.3 that for finite measures μ on the group G_K one has the following alternative: either 1) the boundary $\Gamma(G_K,\mu)$ is trivial, or 2) for almost all trajectories $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ of the random walk (G_K,μ) the configurations φ_n converge pointwise and $\Gamma(G_K,\mu)$ is nontrivial. Thus, for finite measures the triviality or nontriviality of the boundary is completely determined by the presence or absence of stabilization of the sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ ($z\in \mathbb{Z}^K$). For nonfinite measures the situation is more complicated, and examples arise here of different tail behavior.

Proposition 1.1. On the group G_1 there exist a (nonfinite) probability measure μ such that the boundary $\Gamma(G_1,\mu)$ is nontrivial, but nevertheless for a.a. trajectories $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ and all $z\in\mathbb{Z}$ the sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ does not stabilize.

Proof. We consider on the group G_1 the following measure:

$$\begin{aligned}
&\mu(1,0)=\tfrac{1}{8}\\
&\mu(-1,0)=\tfrac{3}{8}\\
&\mu(0,\delta_0)=\tfrac{1}{2}\cdot\tfrac{1}{1\cdot 2}\\
&\mu(0,\delta_0+\delta_1)=\tfrac{1}{2}\cdot\tfrac{1}{2\cdot 3}\\
&\dots\dots\dots\dots\\
&\mu(0,\delta_0+\delta_1+\dots+\delta_n)=\tfrac{1}{2}\cdot\tfrac{1}{(n+1)(n+2)}\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\tag{7}$$

Obviously the projection $\{y_n\}_{n=0}^{\infty}$ of the random walk (G_1,μ) on $\mathbb{Z}$ is nonrecurrent and $y_n\xrightarrow[n]{}-\infty$. Here, since the jumps on $\mathbb{Z}$ are by not more than one, going to $-\infty$, the sequence $\{y_n\}_{n=0}^{\infty}$ goes through all the points 0, −1, −2, ... Moreover, by definition of the measure μ

$$\mu\{(x,f):f(K)=1\}=\frac{1}{2}\left(\frac{1}{(K+1)(K+2)}+\frac{1}{2(K+2)(K+3)}+\dots\right)=\frac{1}{2(K+1)}\tag{8}$$

Since, as is evident from (5), $\varphi_{n+1}(z)=\varphi_n(z)+f_{n+1}(z-y_n)$, by the Borel–Cantelli lemma we get that the equation $f_{n+1}(z-y_n)=1$ a.s. holds an infinite number of times, i.e., the sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ does not stabilize.

On the other hand, the difference $\varphi_n(1)-\varphi_n(0)$ now stabilizes a.s. In fact, from the definition of the measure μ we get that $\varphi_n(1)-\varphi_n(0)\neq\varphi_{n+1}(1)-\varphi_{n+1}(0)$ only when $f_{n+1}=\delta_0+\dots+\delta_{-y_n}$, but the probability of this event is $1/(2(-y_n+1)(-y_n+2))(y_n<0)$. It is easy to show that for almost all trajectories $\sum\limits_{y_n<0}\frac{1}{|y_n|^2}<\infty$, and hence the difference $\varphi_n(1)-\varphi_n(0)$ with probability 1 changes its value a finite number of times, i.e., stabilizes. Thus $\Gamma(G_1,\mu)$ is nontrivial.

Remark. In connection with the problem of the complete description of the boundary for the group G_K, the following question arises. Let $\{A_i\}$ be an increasing sequence of finite subsets, exhausting $\mathbb{Z}^K$. By $\varphi|_A$ we shall denote the restriction of the configuration φ to the finite subset $A\subset\mathbb{Z}^K$. It is easy to see that the stochastic process $\{(y_n,\varphi_n|_A)\}_{n=0}^{\infty}$, where $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ is the original random walk, is Markov. Is it true that the tail sets defined by the final behavior of the trajectories $\{(y_n,\varphi_n|_{A_i})\}_{n=0}^{\infty}$ form a basis of the entire tail σ-algebra of the random walk (G_K,μ)?

5. Boundary for the Inverse Measures. The entropy test for the trivality of the boundary asserts [4, 17], that for a measure μ with finite entropy $H(\mu)$ on a countable group G, the boundary $\Gamma(G,\mu)$ is trivial if and only if the entropy $h(G,\mu)$ of the group with measure vanishes. It is easy to see that if the measure $\check{\mu}$ in inverse to the measure μ (i.e., $\check{\mu}(g)=\mu(g^{-1})$ $\forall g\in G$), then $h(G,\mu)=h(G,\check{\mu})$, so the boundaries $\Gamma(G,\mu)$ and $\Gamma(G,\check{\mu})$ are trivial or nontrivial simultaneously.

On the other hand, the triviality of $\Gamma(G,\mu)$ is equivalent with the convergence of the convolutions of the measure μ to a left-invariant mean on G, and the triviality of $\Gamma(G,\check{\mu})$ is equivalent with the convergence of the convolutions of the measure $\check{\mu}$ to a left-invariant mean, or what is the same, the convergence of the convolutions of the measure μ to a right-invariant mean [4, 17]. Thus, for measures μ with finite entropy, convergence of the sequence of convolutions μ_n of the measure μ to a left-invariant mean is equivalent with convergence to a right-invariant mean.

The following example (partly evoked by an example from M. Rosenblatt [18], which was shown to the author by B. A. Rubshtein) shows that if one waives the finiteness of entropy $H(\mu)$ condition, then this equivalence is lost.

THEOREM 1.4. There exist a solvable group G and a nondegenerate probability measure μ on it such that the boundary $\Gamma(G,\mu)$ is nontrivial, but the boundary of $\Gamma(G,\check{\mu})$ is trivial, i.e., the sequence of convolutions of the measure μ converges to a right-invariant mean on G which is not left-invariant.

Proof. We consider on the group $G=G_1$ the measure μ, defined as follows:

$$\begin{aligned}
&\mu(1,0)=\tfrac{3}{8}\\
&\mu(-1,0)=\tfrac{1}{8}\\
&\mu(0,\delta_0)=\varepsilon_0\\
&\mu(0,\delta_1)=\mu(0,\delta_0+\delta_1)=\tfrac{\varepsilon_1}{2}\\
&\mu(0,\delta_2)=\mu(0,\delta_1+\delta_2)=\mu(0,\delta_0+\delta_2)=\mu(0,\delta_0+\delta_1+\delta_2)=\tfrac{\varepsilon_2}{4}\\
&\cdots\cdots\cdots\cdots\cdots\\
&\mu(0,\delta_n)=\ldots=\mu(0,\delta_0+\ldots+\delta_n)=\tfrac{\varepsilon_n}{2^n}\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}\tag{9}$$

where the positive numbers ε_n are so chosen that $\sum_{n=0}^{\infty}\varepsilon_n=\frac{1}{2}$, $\sum_{n=0}^{\infty}n\varepsilon_n=\infty$. Obviously the measure μ is nondegenerate, and $H(\mu)=\infty$.

We shall show that $\Gamma(G,\mu)$ is nontrivial. By the choice of μ, for almost all trajectories $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ of the random walk (G,μ) we have $y_n\longrightarrow\infty$. Since the measure μ is concentrated on configurations which only burden the positive half-axis of $\mathbb{Z}$, the configurations φ_n converge pointwise a.s., and hence $\Gamma(G,\mu)$ is nontrivial.

We proceed now to the proof of the triviality of $\Gamma(G,\check{\mu})$. We note first of all that by the definition of the group operation in G_1

$$(x,f)^{-1}=(-x,(-x)f)\tag{10}$$

and hence

$$\check{\mu}(1,0)=\tfrac{1}{8},\qquad \check{\mu}(-1,0)=\tfrac{3}{8},\qquad \check{\mu}(0,f)=\mu(0,f)\quad\forall f\in \mathrm{fun}(\mathbb{Z},\mathbb{Z}_2)\tag{11}$$

We shall prove first that for almost all trajectories $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ of the random walk $(G,\check{\mu})$ and all configurations $f\in \operatorname{fun}(\mathbb{Z},\mathbb{Z}_2)$ one has

$$\lim_n \frac{\check{\mu}_n(y_n,f+\varphi_n)}{\check{\mu}_n(y_n,\varphi_n)}=1 \tag{12}$$

(here $\check{\mu}_n$ is the n-th convolution of the measure $\check{\mu}$). We fix $f\in \operatorname{fun}(\mathbb{Z},\mathbb{Z}_2)$. Since $\operatorname{supp} f$ is finite, one can find an $m>0$ such that $f(x)=0$ for all $|x|\geq m$ (i.e., $\operatorname{supp} f \subset [-m+1,m-1]$). We consider the trajectory $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ of the random walk $(G,\check{\mu})$. Obviously

$$(y_n,\varphi_n)=(x_1,f_1)\cdots(x_n,f_n)=(y_n,f_1+y_1f_2+\ldots+y_{n-1}f_n) \tag{13}$$

where (x_i,f_i) are the increments of the random walk. The probability p_0 that $f_n|_{[m,\infty)}\neq 0$ is $p_0=\sum_{i=m}^{\infty}\varepsilon_i$ by definition of the measure μ; analogously, the probability p_k that $(-r)f_n|_{[m,\infty)}=0$ is $p_r=\sum_{i=m+r}^{\infty}\varepsilon_i$. The random walk $\{y_n\}_{n=0}^{\infty}$ on $\mathbb{Z}$, going to $-\infty$, passes through all points 0, −1, −2, ..., but

$$\sum_{r=0}^{\infty}p_r=\sum_{r=0}^{\infty}\sum_{i=m+k}^{\infty}\varepsilon_i=\sum_{k=m}^{\infty}(k-m+1)\varepsilon_k=\infty \tag{14}$$

by the choice of $\{\varepsilon_r\}$. Thus, by the Borel–Cantelli lemma, for almost all trajectories $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ one can find an infinite set of k such that $y_{k-1}f_k|_{[m,\infty)}=0$ (i.e., $(y_k-y_{k-1})|_{[m,\infty)}\neq 0$).

Now we consider the following transformation in the space of trajectories of the random walk. We choose the smallest k such that $y_{k-1}\leq -m$ and $y_{k-1}f_k|_{[m,\infty)}\neq 0$ and we change the increment f_k to $f_k'=f_k+(-y_{k-1})f$, i.e., $y_{k-1}f_k'=y_{k-1}f_k+f$, leaving all other increments unchanged. We denote the trajectory so obtained by $\{(y_n',\varphi_n')\}_{n=0}^{\infty}$. The transformation $\{(y_n,\varphi_n)\}_{n=0}^{\infty}\mapsto\{(y_n',\varphi_n')\}_{n=0}^{\infty}$ is defined a.e., is one-one, and by the definitions of μ and $\check{\mu}$ it preserves the measure in the space of trajectories. Moreover, obviously

$$(y_n',\varphi_n')=(y_n,f+\varphi_n)=(0,f)(y_n,\varphi_n) \tag{15}$$

for all $n\geq k$. (12) now follows directly from this.

Now let F be some bounded $\check{\mu}$-harmonic function on G. We denote by g the element $(0,f)\in G$. Then

$$F(e)=\sum_h F(h)\check{\mu}_n(h) \tag{16}$$

and

$$F(g)=\sum_h F(gh)\check{\mu}_n(h)=\sum_h F(h)\check{\mu}_n(g^{-1}h) \tag{17}$$

Subtracting (17) from (16), we get

$$F(e)-F(g)=\sum_h F(h)(\check{\mu}_n(h)-\check{\mu}_n(g^{-1}h)) \tag{18}$$

By the boundedness of F and (12), the right side of (18) tends to zero with n, i.e., $F(e)=F(g)$. One can prove analogously that $F(x,f)=F(x,0)$ for any element $(x,f)\in G$, i.e., the function $F'(x)\equiv F(x,f)$ is harmonic on the Abelian group $\mathbb{Z}$. Thus it follows from the Choquet–Deny theorem that F is constant. Thus on G there are no nontrivial bounded $\check{\mu}$-harmonic functions, i.e., $\Gamma(G,\check{\mu})$ is a single point. The theorem is proved.

2. Affine Group

Basing ourselves on the results given above, we now consider the group $G = \mathrm{aff}(\mathbb{Z}[\frac{1}{2}])$ of matrices of the form $\begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}$, where $p = 2^k$, $q = \frac{m}{2^n}$ (k, m, n are integers), with the operation of matrix multiplication – the affine group of the dyadic-rational line $\mathbb{Z}[\frac{1}{2}]$. The group G is isomorphic with the semidirect product of the group $\mathbb{Z} \cong \left\{\begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix}\right\}$ by the group $\mathbb{Z}[\frac{1}{2}] \cong \left\{\begin{pmatrix} 1 & \frac{m}{2^n} \\ 0 & 1 \end{pmatrix}\right\}$, is solvable of stage 2, has exponential growth, and can be defined by the generators $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the relation $b^2 a = ab$. It is essential for us that the group G is the homomorphic image of the group $\tilde{G} = \mathbb{Z} \rightthreetimes \mathrm{fun}(\mathbb{Z},\mathbb{Z})$ under the canonical homomorphism $\pi: \tilde{G} \to G$

$$\pi(x,f) = \begin{pmatrix} 2^x & \sum_k 2^k f(k) \\ 0 & 1 \end{pmatrix} \tag{1}$$

Thus, the study of the boundaries of random walks on the group G can be reduced to the study of boundaries of the group $\tilde{G}$ and their behavior under the homomorphism π. More precisely, if $\tilde{\mu}$ is some preimage of the measure μ on $\tilde{G}$, then the boundary of $\Gamma(G,\mu)$ is the quotient-space of the boundary $\Gamma(\tilde{G},\tilde{\mu})$ by the partition into ergodic components with respect to the action of the kernel

$$\mathrm{Ker}\,\pi = \left\{(x,f) \in \tilde{G} : x = 0,\ \sum_k f(k)2^k = 0\right\} \tag{2}$$

of the homomorphism π [7]. Now the theory of random walks on the group $\tilde{G}$ is close to the theory of random walks on the group $G_1 = \mathbb{Z} \rightthreetimes \mathrm{fun}(\mathbb{Z},\mathbb{Z}_2)$.

THEOREM 2.1. For any symmetric finite measure μ on the group $G = \mathrm{aff}(\mathbb{Z}[\frac{1}{2}])$ the boundary $\Gamma(G,\mu)$ is trivial, but there exist nonfinite symmetric and finite nonsymmetric measures on G with nontrivial boundaries.

Before proving the theorem we give the following lemma:

LEMMA 2.1. Let the random walk $(\mathbb{Z},\mu)$ on $\mathbb{Z}$, defined by some nondegenerate probability measure μ, be nonrecurrent; then for almost all trajectories $\{y_n\}_{n=0}^{\infty}$ of the walk $(\mathbb{Z},\mu)$ the sum $\sum_{n=0}^{\infty} 2^{-|y_n|}$ is finite.

Proof. We shall show that actually the integral $\int \sum_{n=0}^{\infty} 2^{-|y_n|}\, d\mathbb{P}^{\mu}(y)$, is finite, where $\mathbb{P}^{\mu}$ is the canonical measure in the space of trajectories $y = \{y_n\}_{n=0}^{\infty}$ of the walk $(\mathbb{Z},\mu)$. In fact,

$$\begin{aligned} \int \sum_n 2^{-|y_n|}\, d\mathbb{P}^{\mu}(y) &= \sum_n \int 2^{-|y_n|}\, d\mathbb{P}^{\mu}(y) = \\ &= \sum_n \sum_{k=-\infty}^{\infty} 2^{-|k|} \mu_n(k) = \sum_{k=-\infty}^{\infty} 2^{-|k|} \sum_n \mu_n(k) = \sum_{k=-\infty}^{\infty} 2^{-|k|} \theta(k) \end{aligned} \tag{3}$$

where $\theta(k) = \sum_{n=0}^{\infty} \mu_n(k)$ is the kernel of the Green's function of the random walk $(\mathbb{Z},\mu)$. Since due to the nonrecurrence the function θ is bounded [10], the sum $\sum_k 2^{-|k|}\theta(k)$ is finite. The lemma is proved.

Proof of Theorem 2.1. 1) Let μ be a symmetric finite measure on the group G; then the projection of the random walk (G,μ) to the subgroup $\left\{\begin{pmatrix} 2^k & 0 \\ 0 & 1\end{pmatrix}\right\}$, isomorphic with $\mathbb{Z}$, is recurrent. Since the group $\left\{\begin{pmatrix} 1 & \frac{m}{2^n} \\ 0 & 1\end{pmatrix}\right\}$ is Abelian, by Lemma 1.1 (cf. the proof of Theorem 1.2) the boundary $\Gamma(G,\mu)$ is trivial.

2) Now we proceed to the construction of a measure μ on G with nontrivial boundary. We set

$$\begin{aligned} &\mu\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\right\} = \mu\left\{\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\right\} = \frac{1}{4} \\ &\mu\left\{\begin{pmatrix} 2^k & 0 \\ 0 & 1\end{pmatrix}\right\} = \frac{1}{2}\mu'(k) \end{aligned} \tag{4}$$

where μ' is some probability measure on $\mathbb{Z}$. Obviously the symmetry (or finiteness) of μ is equivalent with the symmetry (or finiteness) of μ'. Now we choose μ' to be symmetric nonfinite or finite nonsymmetric, defined so that its walk on $\mathbb{Z}$ is nonrecurrent. Let $\tilde{\mu}$ be the preimage of the measure μ on $\tilde{G}$:

$$\begin{aligned} &\tilde{\mu}(0,\delta_0) = \tilde{\mu}(0,-\delta_0) = \frac{1}{4} \\ &\tilde{\mu}(k,0) = \mu\begin{pmatrix} 2^k & 0 \\ 0 & 1\end{pmatrix} = \frac{1}{2}\mu'(k) \end{aligned} \tag{5}$$

Due to the nonrecurrence of the random walk $(\mathbb{Z},\mu')$ we get that for almost all trajectories $\{(y_n,\varphi_n)\}_{n=0}^{\infty}$ of the random walk $(\tilde{G},\tilde{\mu})$ the functions φ_n converge pointwise to some (nonfinite) function φ_∞ (and consequently $\Gamma(\tilde{G},\tilde{\mu})$ is nontrivial). By Lemma 2.1, the sum $\sum_{k<0}\varphi_\infty(k)2^k$ is almost surely finite. Since the action of $\operatorname{Ker}\pi$ on $\Gamma(\tilde{G},\tilde{\mu})$ does not change the quantities

$$\left[\sum_{k<0}\varphi_\infty(k)2^k\right] \tag{6}$$

($[x]$ denotes the greatest integer in the number x), we get that on $\Gamma(\tilde{G},\tilde{\mu})$ there exists a nontrivial measurable $\operatorname{Ker}\pi$-invariant function. Thus, $\Gamma(G,\mu)$ is nontrivial.

COROLLARY. Let μ be a finite measure on $G=\operatorname{aff}(\mathbb{Z}[\frac{1}{2}])$; then the boundary $\Gamma(G,\mu)$ is trivial if and only if the projection of the random walk (G,μ) onto the subgroup $\left\{\begin{pmatrix} 2^k & 0 \\ 0 & 1\end{pmatrix}\right\}$ is recurrent.

3. Infinite Symmetric Group

We consider the symmetric group $\mathfrak{S}_\infty$ of finite permutations of a countable set. The group $\mathfrak{S}_\infty$ is obviously countable and locally finite. It is clear that due to the local finiteness of the group $\mathfrak{S}_\infty$ any finite measure on it is contained in some finite subgroup, and hence has trivial boundary. Nevertheless, there exist on $\mathfrak{S}_\infty$ nonfinite measures with nontrivial boundary.

THEOREM 3.1. On the group $\mathfrak{S}_\infty$ there exists a symmetric probability measure μ for which the boundary of the random walk $\Gamma(\mathfrak{S}_\infty,\mu)$ is nontrivial.

Proof. We shall assume that the group $\mathfrak{S}_\infty$ is realized as the group of finite permutations $g: V\to V$ of some countable set V. On V there is defined a natural right action of $\mathfrak{S}_\infty$:

$$\upsilon . g = g^{-1}(\upsilon) \qquad \upsilon \in V,\ g \in \mathfrak{S}_\infty \tag{1}$$

The basic idea of the proof is to construct a measure μ on $\mathfrak{S}_\infty$, for which the homogeneous Markov process on V with transition probabilities

$$p(x_1|x_0) = \mu\{g : x_0 g = x_1\} = \mu\{g : g^{-1}(x_0) = x_1\}, \tag{2}$$

induced by the action (1) of the group $\mathfrak{S}_\infty$ on V, has nontrivial exit-boundary. The preimage of any tail event of the induced process on V will obviously be a tail event for the random walk, and hence from the nontriviality of the exit-boundary of the induced process, the nontriviality of the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ of the original random walk on $\mathfrak{S}_\infty$ follows.

In what follows it will be convenient to provide the set V with the additional structure of a binary tree and to consider it as the set of sequences $\upsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of finite length $0 \leqslant |\upsilon| = n < \infty$, consisting of zeros and ones (the vertex of the binary tree, the empty sequence $\emptyset$, has length $|\emptyset| = 0$). We denote by V_n the set of vertices of the V_n-th level

$$V_n = \{\upsilon \in V : |\upsilon| = n\},\ \operatorname{card} V_n = 2^n \quad (n \geqslant 0) \tag{3}$$

We define two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ of elements of $\mathfrak{S}_\infty$ as follows:

$$a_n(\varepsilon_1, \ldots, \varepsilon_k) = \begin{cases} (\varepsilon_1, \ldots, \varepsilon_k, 0), & k = n \\ (\varepsilon_1, \ldots, \varepsilon_{k-1}), & k = n+1 \\ (\varepsilon_1, \ldots, \varepsilon_k) \end{cases}$$

$$b_n(\varepsilon_1, \ldots, \varepsilon_k) = \begin{cases} (\varepsilon_1, \ldots, \varepsilon_k, 1), & k = n \\ (\varepsilon_1, \ldots, \varepsilon_{k-1}), & k = n+1 \\ (\varepsilon_1, \ldots, \varepsilon_k) \end{cases} \tag{4}$$

In other words, a_n transposes the elements V_n and $V'_{n+1} = \{(\varepsilon_1, \ldots, \varepsilon_{n+1}) : \varepsilon_{n+1} = 0\}$, and b_n does the same to the elements V_n and $V''_{n+1} = \{(\varepsilon_1, \ldots, \varepsilon_{n+1}) : \varepsilon_{n+1} = 1\}$ (a_n and b_n are not automorphisms of the binary tree!).

Now we define the probability measure μ on $\mathfrak{S}_\infty$ as follows:

$$\mu(a_n) = \mu(b_n) = \frac{d_n}{2} \qquad (n \geqslant 0) \tag{5}$$

where $\sum_{n=0}^{\infty} d_n = 1$, $d_n > 0$. Then for the induced Markov process on V we get the following values of the transition probabilities (2):

$$\begin{cases} p((\varepsilon_1, \ldots, \varepsilon_{n-1}) | (\varepsilon_1, \ldots, \varepsilon_n)) = \frac{d_{n-1}}{2} \\ p((\varepsilon_1, \ldots, \varepsilon_n, 0) | (\varepsilon_1, \ldots, \varepsilon_n)) = \frac{d_n}{2} \\ p((\varepsilon_1, \ldots, \varepsilon_n, 1) | (\varepsilon_1, \ldots, \varepsilon_n)) = \frac{d_n}{2} \\ p((\varepsilon_1, \ldots, \varepsilon_n) | (\varepsilon_1, \ldots, \varepsilon_n)) = 1 - d_n - \frac{d_{n-1}}{2}. \end{cases} \tag{6}$$

Thus, the induced Markov process $\{x_n\}_{n=0}^{\infty}$ on V has nontrivial exit-boundary (consisting of the ends of the binary tree), if the Markov process on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with transition probabilities

$$
\begin{aligned}
p(n-1|n) &= \frac{\alpha_{n-1}}{2} \\
p(n+1|n) &= \alpha_n \\
p(n|n) &= 1-\alpha_n-\frac{\alpha_{n-1}}{2}
\end{aligned} \tag{7}
$$

is nonrecurrent.

Processes of this type on $\mathbb{Z}_+$ have been thoroughly studied (cf., e.g., [8, Chap. 3]). A necessary and sufficient condition for nonrecurrence applicable to our case is the finiteness of the sum

$$\sum_{n=0}^{\infty} \frac{1}{2^n \alpha_n} < \infty \tag{8}$$

Now choosing a sequence $\{\alpha_n\}$ such that (8) holds, we get that the measure μ on $\mathfrak{S}_\infty$, defined by (5), has nontrivial boundary $\Gamma(\mathfrak{S}_\infty, \mu)$. The theorem is proved.

Remarks. 1. All elements of the support of the measure μ constructed are elements of the second order ($a_n^2 = b_n^2 = e$) and the measure μ is thus symmetric.

2. The support $\operatorname{supp}\mu$ of the measure μ constructed generally does not generate the entire group $\mathfrak{S}_\infty$, but only some subgroup of it, but by modifying somewhat the construction given above, it is easy to give an example of a nondegenerate measure $\tilde{\mu}$ on $\mathfrak{S}_\infty$ with nontrivial boundary. In fact, let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be two sequences of positive numbers such that $\sum_{n=0}^{\infty}(\alpha_n + 2^n\gamma_n) = 1$. Now as before we set $\tilde{\mu}(a_n) = \tilde{\mu}(b_n) = \frac{\alpha_n}{2}$ and in addition $\tilde{\mu}(g) = \frac{\gamma_n}{2}$ for all transpositions g of pairs of elements $(\varepsilon_1, \dots, \varepsilon_n) \in V_n$ and $(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}) \in V_{n+1}$ (i.e., transposition of pairs of elements lying on one edge of the binary tree). The measure $\tilde{\mu}$ so constructed is obviously nondegenerate on $\mathfrak{S}_\infty$. Choosing $\{\alpha_n\}$ and $\{\gamma_n\}$ so that $\sum_{n=0}^{\infty} \frac{1}{2^n(\alpha_n+\gamma_n)} < \infty$ (i.e., so that the induced process on V has nontrivial exit-boundary), we get a nondegenerate measure on $\mathfrak{S}_\infty$ with nontrivial boundary.

3. The measure μ with nontrivial boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ can be chosen to have finite entropy $H(\mu)$, as is evident from its construction. On the other hand, for any finite measure μ' on $\mathfrak{S}_\infty$ the entropy $h(\mathfrak{S}_\infty, \mu')$ is zero due to the local finiteness of $\mathfrak{S}_\infty$. This shows that generally an arbitrary probability measure μ on a countable group G cannot be approximated (in any sense) by finite measures $\mu^{(k)}$ such that the entropies $h(G, \mu^{(k)})$ converge to the entropy $h(G, \mu)$.

4. Let the group $\mathfrak{S}_\infty$ be realized as the group of finite permutations of some countable set V (as in the proof of the theorem). It is unknown whether the boundary $\Gamma(\mathfrak{S}_\infty, \mu)$ admits a complete description in terms of the exit-boundary of the induced Markov process on V, and in particular, whether from the triviality of the exit-boundary of the induced process the triviality of $\Gamma(\mathfrak{S}_\infty, \mu)$ follows.

4. Solvable Locally Finite Group

Let $D = \operatorname{fun}(\mathbb{N}, \mathbb{Z}_2)$ be the countable direct sum of groups $\mathbb{Z}_2 = \{0,1\}$, indexed by the natural numbers 1, 2, 3, ... It is convenient to assume that D is the group of functions $f: \mathbb{N} \to \mathbb{Z}_2$ with finite supports $\operatorname{supp} f = \{n \in \mathbb{N} : f(n) \neq 0\}$ (or: D is the group of finite configurations on $\mathbb{N}$ with the operation of pointwise addition $\operatorname{mod} 2$).

We denote by $\mathrm{fun}(D,\mathbb{Z}_2)$ the group of finite functions $F: D \to \mathbb{Z}_2$ with the operation of pointwise addition. In other words, $\mathrm{fun}(D,\mathbb{Z}_2)$ is the countable direct sum D of copies of the group $\mathbb{Z}_2$. The group D acts canonically on $\mathrm{fun}(D,\mathbb{Z}_2)$:

$$fF(h) = F(f+h) \qquad F \in \mathrm{fun}(D,\mathbb{Z}_2);\ f,h \in D \tag{1}$$

By $\mathfrak{D}$ we denote the semidirect product $\mathfrak{D} = D \ltimes \mathrm{fun}(D,\mathbb{Z}_2)$ defined by the action (1). The group $\mathfrak{D}$ as a set consists of pairs of elements (f,F), where $f \in D$, $F \in \mathrm{fun}(D,\mathbb{Z}_2)$, with the group operation

$$(f_1,F_1)(f_2,F_2) = (f_1+f_2, F_1+f_1F_2) \tag{2}$$

Let $\emptyset$ be the identity of the group D:

$$\emptyset(n) = 0 \qquad \forall n \in \mathbb{N} \tag{3}$$

We shall denote by $\delta_n\ (n \in \mathbb{N})$ the generator of the group D:

$$\delta_n(m) = \begin{cases} 1, & n = m \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We denote by φ the identity of the group $\mathrm{fun}(D,\mathbb{Z}_2)$:

$$\varphi(f) = 0 \qquad \forall f \in D \tag{5}$$

We single out another element $\omega \in \mathrm{fun}(D,\mathbb{Z}_2)$:

$$\begin{aligned} &\omega(\emptyset) = 1 \\ &\omega(f) = 0, \quad f \in D,\ f \neq \emptyset \end{aligned} \tag{6}$$

We shall denote the identity $(\emptyset, \varphi)$ of the group $\mathfrak{D}$ by e.

The group $\mathfrak{D}$ is a locally finite solvable group of stage 2. We shall establish some of its more special properties.

LEMMA 4.1. The set $\{\delta_n\}_{n=1}^{\infty} \cup \{\omega\}$ generates the group $\mathfrak{D}$.

Proof. Since $\{\delta_n\}_{n=1}^{\infty}$ generates the group D, it suffices to show that the D-orbit of the element ω generates the entire group $\mathrm{fun}(D,\mathbb{Z}_2)$. Let $F \in \mathrm{fun}(D,\mathbb{Z}_2)$, $\mathrm{supp}\,F = \{f_i\}_{i=1}^{k}$; then obviously $F = f_1\omega + f_2\omega + \ldots + f_k\omega$. The lemma is proved.

LEMMA 4.2. The orders of all elements of $\mathfrak{D}$ do not exceed four.

Proof. Let $g = (f,F) \in \mathfrak{D}$; then since $f+f=\emptyset$ and $F+F=\varphi$ for all $f \in D$, $F \in \mathrm{fun}(D,\mathbb{Z}_2)$, one has $g^2 = (f+f, F+fF) = (\emptyset, F+fF)$, from which $g^4 = (\emptyset, F+fF)^2 = (\emptyset, F+F+fF+fF) = (\emptyset,\varphi) = e$.

LEMMA 4.3. The orders of the finitely generated subgroups of $\mathfrak{D}$ with no more than k generators are bounded for all k.

Proof. We fix some set T of k elements of the group $\mathfrak{D}$. Without loss of generality one can assume the set T is symmetric and contains the identity e, so the group generated by T is $gr(T) = \bigcup_{n=0}^{\infty} T^n$, while the sets T^n do not decrease, i.e., $\{e\} = T^0 \subset T^1 \subset \ldots \subset T^n \subset T^{n+1} \subset \ldots$. We estimate the cardinality of the set T^n. Let $\{(f_i,F_i)\}_{i=1}^{n}$ be some collection of elements of T; then

$$(f_1,F_1)\cdot\ldots\cdot(f_n,F_n) = (f_1+\ldots+f_n, F_1+f_1F_2+\ldots+(f_1+\ldots+f_{n-1})F_n). \tag{7}$$

The subgroup of D consisting of all sums $f_1+\ldots+f_n$ has no more than k generators of order 2, and hence consists of no more than 2^k elements. Thus, the subgroup $\operatorname{fun}(D,\mathbb{Z}_2)$ generated by elements of the form $(f_1+\ldots+f_{n-1})F_n$ has no more than $k\cdot 2^k$ generators of order 2, and consists of no more than 2^{k2^k} elements. Finally we get $|T^n|\leqslant 2^k\cdot 2^{k2^k}$ for all n. The lemma is proved.

We recall that the group G is called a group of uniformly polynomial growth [11, 17], if there exists a collection of polynomials p_k such that

$$|T^n|\leqslant p_k(n) \tag{8}$$

for all subsets $T\subset G$ consisting of no more than k elements. Thus, $\mathfrak{D}$ is a group of uniformly polynomial growth. (We note that the symmetric group $\mathfrak{S}_\infty$ has weakly exponential growth; cf. [17].)

THEOREM 4.1. On the group $\mathfrak{D}$ there exists a nondegenerate symmetric probability measure μ with finite entropy $H(\mu)$, for which the boundary $\Gamma(\mathfrak{D},\mu)$ is nontrivial.

The proof goes by the same scheme as in Theorem 1.1. We consider a probability measure μ on $\mathfrak{D}$, given as follows:

$$\mu(\sigma_n)=p_n,\ \mu(\omega)=q,\ \mu(e)=r \tag{9}$$

where $p_n, q, r>0$, $\sum_{n=1}^{\infty} p_n+q+r=1$. The measure μ is obviously symmetric, and on the basis of Lemma 4.1 is nondegenerate. Let $\{(h_n,H_n)\}_{n=0}^{\infty}$ be a random walk defined by the measure μ, i.e.,

$$(h_n,H_n)=(f_1,F_1)\cdot\ldots\cdot(f_n,F_n) \tag{10}$$

where (f_i,F_i) are independent $\mathfrak{D}$-valued random walks with distribution μ (increments of the random walk). Since the supports of all F_i are either empty or consist of the unique point ω, for the nontriviality of the boundary $\Gamma(\mathfrak{D},\mu)$ it is sufficient that the random walk $\{h_n\}_{n=0}^{\infty}$ on D is nonrecurrent (then the functions H_n will be a.s. pointwise stabilized; cf. the Remark on Theorem 1.1). It is known that if, for example, $\lim_n \frac{p_{n+1}}{p_n}=1$ then the walk $\{h_n\}_{n=0}^{\infty}$ on D is nonrecurrent [10], and consequently $\Gamma(\mathfrak{D},\mu)$ is nontrivial. The theorem is proved.

Remarks. 1. We recall that for finitely generated groups of polynomial growth Gromov's theorem [14] combined with the triviality of the boundary for nilpotent groups [5] shows that the boundary is trivial for any measure. The theorem proved gives an example of the fact that for groups with an infinite number of generators the situation is very different from the case of finitely generated groups.

2. The somewhat more "complicated" group $D\leftthreetimes\operatorname{fun}(D;D)$ was considered by Hulanicki [15] (cf. also [16]), who proved the nonsymmetry of its group algebra. The results of this section carry over almost word for word to this group.

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