BOOLEAN-VALUED ANALYSIS AND JB-ALGEBRAS

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Introduction

In the last decade Boolean-valued models of set theory find interesting applications to operator algebras and C^*-algebras, see [1–7]. In particular, for type 1 AW^*-algebras a negative solution was given to the Kaplansky problem of uniqueness for decomposition into the direct sum of homogeneous algebras [4] and some presentation was obtained in the form of a direct sum of algebras of strongly continuous operator-functions [7]. It was observed in [8] that Boolean-valued methods can be of similar use in the theory of JB-algebras; however, this direction is not duly developed yet. The purpose of the present article is to fill this gap somehow.

JB-algebras are nonassociative real analogs of C^*-algebras and von Neumann operator algebras. The theory of such algebras stems from the article [9] by Jordan, von Neumann, and Wigner, and it exists as a branch of functional analysis since the middle of the 1960s. Stages of its development are reflected in [10]. The theory of JB-algebras undergoes intensive study, the range of its applications widens. Among the main directions of research are: the structure and classification of JB-algebras, nonassociative integration and quantum probability theory, the geometry of states of JB-algebras, etc. (see [11–13] and references therein).

In §1 of the article we present some definitions, notation, and terminology. In §2 we consider Banach spaces with complete Boolean algebras of projections and establish some results on their Boolean-valued representation. These results provide a uniform approach to AW^*-algebras and JB-algebras (cf. [7]). In §3 we prove a theorem on representation of JB-algebras with a distinguished complete Boolean algebra of central projections in a Boolean-valued model. Some applications presenting Boolean-valued interpretation of the well-known results of Shultz [14] are given in §4. Here we also introduce a new class of $\mathbb{B}$-JBW-algebras which is broader than the class of JBW-algebras. The principal difference between the two classes is in the fact that in general a $\mathbb{B}$-JBW-algebra has a faithful representation in the algebra of selfadjoint operators on a AW^*-module rather than on a Hilbert space as in the case of JBW-algebras. Certainly, these applications only illustrate ability of the Boolean-valued machinery. It is possible to advance far afield and obtain many other results by using the direct Boolean-valued interpretation as in [1–3, 5, 6], or forcing as in [4], or the combined method of [7]. The space limitation of the article does not allow us to expose all these in more detail.

The necessary background of the theory of JB-algebras and Boolean-valued models can be found in [11] and [15].

§1. Preliminaries

We recall some definitions and fix terminology and notation.

1.1. Throughout the article, $\mathbb{B}$ is a fixed complete Boolean algebra, $V^{(\mathbb{B})}$ is the corresponding Boolean-valued model of set theory, $\mathbb{R}$ stands for the conventional field of reals, and $\mathcal{R}$ is the same field in the model $V^{(\mathbb{B})}$. As is known, the descent $\mathcal{R}{\downarrow}$ is a real K-space, i.e., an order-complete vector lattice isomorphic to the vector lattice $C_\infty(Q)$, where Q is the Stone compact space of the algebra $\mathbb{B}$. We denote by $\mathbb{B}(\mathbb{R})$ the bounded part of the K-space $\mathcal{R}{\downarrow}$ that is isomorphic to $C(Q)$, and denote the norm in $\mathbb{B}(\mathbb{R})$ by $\|\cdot\|_\infty$. Note that $\mathbb{B}(\mathbb{R})$ is a commutative and associative Banach algebra. The idempotents of this algebra form a complete Boolean algebra which is isomorphic to $\mathbb{B}$. Every idempotent e determines a positive projection, the operator of multiplication by e.

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Henceforth, we take the liberty of identifying the Boolean algebras of idempotents and projections with $\mathbb{B}$ and write $\mathbb{B} \subset \mathbb{B}(\mathbb{R})$. In particular, the units of these algebras as well as the unity in $\mathbb{B}(\mathbb{R})$ are denoted by the same symbol $\mathbf{1}$.

1.2. Recall that a *JB-algebra* is simultaneously a real Banach space A and a unital Jordan algebra; moreover, A must satisfy the conditions:

(1) $\|xy\| \le \|x\| \cdot \|y\| \quad (x, y \in A)$,

(2) $\|x^2\| = \|x\|^2 \quad (x \in A)$,

(3) $\|x^2\| \le \|x^2 + y^2\| \quad (x, y \in A)$.

The set $A_+ := \{x^2 : x \in A\}$, presenting a salient convex cone, determines the structure of an ordered vector space in A so that the unity $\mathbf{1}$ of the algebra A serves as a strong order unit, and the order interval $[-\mathbf{1}, \mathbf{1}] := \{x \in A : -\mathbf{1} \le x \le \mathbf{1}\}$ serves as the unit ball. Moreover, the inequalities $-\mathbf{1} \le x \le \mathbf{1}$ and $0 \le x^2 \le \mathbf{1}$ are equivalent.

The intersection of all maximal associative subalgebras of A is called the *center* of A and denoted by $\mathcal{Z}(A)$. The element a belongs to $\mathcal{Z}(A)$ if and only if $(ax)y = a(xy)$ for arbitrary $x, y \in A$. If $\mathcal{Z}(A) = \mathbb{R} \cdot \mathbf{1}$, then A is said to be a JB-factor. The center $Z(A)$ is an associative JB-algebra, and any such algebra is isometrically isomorphic to the real Banach algebra $C(K)$ of continuous functions on some compact set K.

1.3. The idempotents of JB-algebras are also called *projections*. The set of all projections that are in the center forms a Boolean algebra which is denoted by $\mathfrak{P}_c(A)$. Assume that $\mathbb{B}$ is a subalgebra of the Boolean algebra $\mathfrak{P}_c(A)$ or, equivalently, that $\mathbb{B}(\mathbb{R})$ is a subalgebra of the center $\mathcal{Z}(A)$. Then we say that A is a *$\mathbb{B}$-JB-algebra* if, for any partition of unity $(e_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ and any family $(x_\xi)_{\xi \in \Xi}$ in A, there exists a unique $\mathbb{B}$-mixing $x := \mathrm{mix}_{\xi \in \Xi}(e_\xi x_\xi)$, i.e., a unique element $x \in A$ such that $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$. If $\mathbb{B}(\mathbb{R}) = \mathcal{Z}(A)$, then a $\mathbb{B}$-JB-algebra is also referred to as *centrally extended JB-algebra.*

1.4. *The unit ball of a $\mathbb{B}$-JB-algebra is closed under $\mathbb{B}$-mixings.*

$\triangleleft$ Since the unit ball of a JB-algebra coincides with the order interval $[-\mathbf{1}, \mathbf{1}]$, the required assertion is equivalent to the following: If an $x \in A$ and a partition of unity $(e_\xi)_{\xi \in \Xi} \subset \mathbb{B}$ are such that $e_\xi x \ge 0$ for all $\xi \in \Xi$, then $x \ge 0$. This follows from the fact that if $e_\xi x = a_\xi^2$ $(\xi \in \Xi)$ for a suitable family (a_ξ) in A, then the element $a = \mathrm{mix}(e_\xi a_\xi)$ satisfies the equality $x = a^2$. $\triangleright$

1.5. Given an arbitrary $a \in A$, we introduce the operators $T_a, U_a : A \longrightarrow A$ by the formulas

$$T_a : x \longmapsto ax, \ U_a : x \longmapsto 2a(ax) - a^2 x \quad (x \in A).$$

We say that elements $a, b \in A$ *commute operatorwise* if $T_a \circ T_b = T_b \circ T_a$. Obviously, the center $\mathcal{Z}(A)$ consists of elements that commute operatorwise with each of the elements of A.

The operator U_a is positive, i.e., $U_a(A_+) \subset A_+$. If $a \in \mathcal{Z}(A)$, then $U_a = T_{a^2}$.

1.6. Let $\mathbb{B}$ be a subalgebra of $\mathfrak{P}_c(A)$. Then we can define a $\mathbb{B}$-valued semimetric d in A by the formula

$$d(x, y) := \bigvee \{\mathbf{1} - e : e \in \mathbb{B}, ex = ey\} \quad (x, y \in A).$$

This semimetric is a metric (i.e., $d(x, y) = 0 \longrightarrow x = y$) if and only if, for every $a \in A$, there exists some greatest element e in $\mathbb{B}$ such that $ea = 0$. In case the latter proviso is fulfilled, we say that the JB-algebra A is $\mathbb{B}$-metrizable. By applying the operation of $\mathbb{B}$-extension to a $\mathbb{B}$-metrizable JB-algebra (see [8; 3.5.8(2)]), we can derive the following assertion:

Every $\mathbb{B}$-metrizable JB-algebra A can be extended to a $\mathbb{B}$-JB-algebra $\widetilde{A}$ in such a way that $\widetilde{A} = \mathrm{mix}(A)$, i.e., each element of the algebra $\widetilde{A}$ is a mixing of a family in A.

With this circumstance in mind, we shall only handle $\mathbb{B}$-JB-algebras in what follows, although some results could be formulated for a more general case of $\mathbb{B}$-metrizable JB-algebras.

§2. Cyclic Banach Spaces

Here we expose a Boolean-valued approach to studying Banach spaces with complete Boolean algebras of projections. Such spaces appear in different branches of analysis and are especially frequent in the theory of operator algebras.

2.1. Let X be a Banach space and let $\mathcal{L}(X)$ stand for the set of all bounded linear operators on X. Assume that a mapping $\varphi := \varphi_X : \mathbb{B} \longrightarrow \mathcal{L}(X)$ is injective and satisfies the following conditions:

(1) $\varphi(b)$ is a projection with norm one for all $b \in \mathbb{B}$; moreover, $\varphi(\mathbf{1})$ and $\varphi(\mathbf{0})$ coincide with the identity and zero operators respectively;

(2) the projections $\varphi(b)$ and $\varphi(b')$ commute for arbitrary $b, b' \in \mathbb{B}$;

(3) the equalities $\varphi(b \vee b') = \varphi(b) \circ \varphi(b')$ and $\varphi(b^*) = I_X - \varphi(b)$ hold for all b and b'.

In this case the set $\mathcal{B} := \varphi(\mathbb{B})$ is referred to as the *complete Boolean algebra of projections* in the space X. We shall symbolize the above situation by $\mathbb{B} \sqsubset \mathcal{L}(X)$. In the sequel we identify the Boolean algebras $\mathbb{B}$ and $\mathcal{B}$ and shall take the liberty of speaking about the Boolean algebra $\mathbb{B}$ of projections.

If $(e_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$ and $(x_\xi)_{\xi\in\Xi}$ is a family in X, then an element $x \in X$ for which $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$ is called a *mixing* of (x_ξ) with respect to (e_ξ). A Banach space X is said to be $\mathbb{B}$-*cyclic* if $\mathbb{B} \sqsubset \mathcal{L}(X)$ and the following conditions hold:

(4) a mixing of each bounded family in X with respect to every partition of unity in $\mathbb{B}$ (with the same index set) exists and is unique;

(5) the unit ball of X is closed under each mixing.

The simplest example of a $\mathbb{B}$-cyclic Banach space is $\mathbb{B}(\mathbb{R})$. It is seen from 1.4 that every $\mathbb{B}$-JB-algebra is also a $\mathbb{B}$-cyclic Banach space.

2.2. Let X and Y be Banach spaces; moreover, $\mathbb{B} \sqsubset \mathcal{L}(X)$ and $\mathbb{B} \sqsubset \mathcal{L}(Y)$. An operator $T : X \longrightarrow Y$ is called $\mathbb{B}$-*linear* if it is linear and commutes with projections in $\mathbb{B}$, i.e., if $b \circ T = T \circ b$. (Here we certainly bear in mind $\varphi_Y(b) \circ T = T \circ \varphi_X(b)$.) Denote the set of all bounded $\mathbb{B}$-linear operators from X into Y by $\mathcal{L}_{\mathbb{B}}(X, Y)$. Then $Z := \mathcal{L}_{\mathbb{B}}(X, Y)$ is a Banach space and, clearly, $\mathbb{B} \sqsubset Z$. Indeed, the projection $\varphi_Z(b)$ can be determined by the formula $T \longmapsto b \circ T \quad (T \in Z)$. It is easily verified that the space Z is $\mathbb{B}$-cyclic if Y is $\mathbb{B}$-cyclic. The converse assertion is also true if $X \neq \{0\}$. We shall call a bijective $\mathbb{B}$-linear operator a $\mathbb{B}$-*isomorphism* and if, in addition, the operator is norm-preserving, we shall speak about an *isometric* $\mathbb{B}$-*isomorphism*.

The space $X^\sharp := \mathcal{L}_{\mathbb{B}}(X, \mathbb{B}(\mathbb{R}))$ is called the $\mathbb{B}$-*dual* of X. If the spaces Y and $X^\sharp$ are isometrically $\mathbb{B}$-isomorphic, then Y is said to be a $\mathbb{B}$-*dual space*, and X is a $\mathbb{B}$-*predual space* of it.

2.3. Theorem. *The restricted descent of a Banach space from the model $V^{(\mathbb{B})}$ is a $\mathbb{B}$-cyclic Banach space. Conversely, if X is a $\mathbb{B}$-cyclic Banach space, then in the model $V^{(\mathbb{B})}$ there exists a Banach space $\mathcal{X}$ which is unique up to isometric isomorphism and whose restricted descent is isometrically $\mathbb{B}$-isomorphic to X.*

◁ The first part of the theorem follows immediately from [16, Theorem 4.8] or [6, Theorem 4.1]. Now, assume X to be a $\mathbb{B}$-cyclic Banach space. It is demonstrated in [16, Theorem 4.8] that X can be transformed into a Banach-Kantorovich space with some $\mathbb{B}(\mathbb{R})$-valued norm $|\cdot|$; moreover, the scalar norm in X is mixed, i.e., $\|x\| = \|\,|x|\,\|_\infty$. According to Gordon's theorem, we can identify $\mathbb{B}(\mathbb{R})$ with the bounded part of the descent $\mathcal{R}\!\downarrow$. By [17, Theorem 3.4.4], X can be represented as a Banach space $\mathcal{X}$ in the model $V^{(\mathbb{B})}$; moreover, the descent $\mathcal{X}\!\downarrow$ serves as universal completion of X; i.e., $\mathcal{X}\!\downarrow$ coincides with the set $\operatorname{mix}(X)$ of all mixings of elements in X. An element $x \in \mathcal{X}\!\downarrow$ belongs to X if and only if its vector norm $|x|$ is contained in $\mathbb{B}(\mathbb{R})$. However, since $[\![\,|x| = \|x\|_{\mathcal{X}}]\!] = \mathbf{1}$, we conclude that X is the bounded part of $\mathcal{X}\!\downarrow$. ▷

2.4. An element $\mathcal{X} \in V^{(\mathbb{B})}$ mentioned in Theorem 2.3 is referred to as the *Boolean-valued representation* of X. Let $\mathcal{X}$ and $\mathcal{Y}$ be the Boolean-valued representations of $\mathbb{B}$-cyclic Banach spaces X and Y respectively. We let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the element in $V^{(\mathbb{B})}$ that represents the space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$.

Theorem. *The restricted descent of the Banach space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and the $\mathbb{B}$-cyclic Banach space $\mathcal{L}_{\mathbb{B}}(X, Y)$ are isometrically $\mathbb{B}$-isomorphic. Such an isomorphism is executed by assigning to any bounded $\mathbb{B}$-linear operator $T : X \longrightarrow Y$ the element $t := T\!\uparrow$ that is defined by the relations $[\![t : \mathcal{X} \longrightarrow \mathcal{Y}]\!] = \mathbf{1}$ and $[\![t(x) = T(x)]\!] = \mathbf{1} \quad (x \in X)$.*

◁ Let $X_0 = \mathcal{X}\!\downarrow$ and $Y_0 = \mathcal{Y}\!\downarrow$. By [17, Theorem 3.4.8], $\mathcal{L}(\mathcal{X}, \mathcal{Y})\!\downarrow$ and $\mathcal{L}_b(X_0, Y_0)$ are linearly isometric as regards vector norms. Here $\mathcal{L}_b(X_0, Y_0)$ consists of the linear operators $T_0 : X_0 \longrightarrow Y_0$ that satisfy the inequality $|T_0 x| \le c\,|x| \quad (x \in X_0)$ for some $c \in \mathcal{R}\!\downarrow$. The smallest element c with the indicated property is exactly the vector norm $|T_0|$ of the operator T_0. The bounded part $\mathcal{L}_b(X_0, Y_0)$

consists of those operators T_0 for which $|T_0| \in \mathbb{B}(\mathbb{R})$. Since the norms in X and Y are mixed (see [16, Theorem 4.8]), it follows that $\mathcal{L}_b(X,Y) = \mathcal{L}_{\mathbb{B}}(X,Y)$. Let an operator $t : \mathcal{X} \longrightarrow \mathcal{Y}$ inside $V^{(\mathbb{B})}$ be connected with the operator T_0 by the relation $[\![t(x) = T_0(x)]\!] = \mathbf{1}$ $(x \in X)$. Then $[\![\|t\| = |T_0|]\!] = \mathbf{1}$; therefore, t belongs to the bounded part of $\mathcal{L}(\mathcal{X},\mathcal{Y})$ if and only if T is contained in the bounded part of $\mathcal{L}_b(X_0, Y_0)$. It remains to observe that every operator $T_0 \in \mathcal{L}_b(X_0, Y_0)$ with the norm $|T_0| \in \mathbb{B}(\mathbb{R})$ is uniquely determined by its restriction to X, while every operator T in $\mathcal{L}_b(X,Y)$ has exactly one norm-preserving extension $T_0 \in \mathcal{L}_b(X_0, Y_0)$. ▷

2.5. Corollary. *Let $\mathcal{X}^*$ be the dual Banach space of $\mathcal{X}$ and let $\simeq$ and $\cong$ denote isometric isomorphy and isometric $\mathbb{B}$-isomorphy respectively. Then*

$$X^{\#} \cong Y \longleftrightarrow [\![\mathcal{X}^* \simeq \mathcal{Y}]\!] = \mathbf{1}.$$

2.6. Consider the category Ban($\mathbb{B}$) whose objects are Banach spaces X such that $\mathbb{B} \sqsubset \mathcal{L}(X)$. The morphisms of this category are bounded $\mathbb{B}$-linear operators. Let C-Ban($\mathbb{B}$) stand for the complete subcategory of Ban($\mathbb{B}$) whose objects are $\mathbb{B}$-cyclic Banach spaces.

Introduce one more category $\mathrm{Ban}_\infty^{(\mathbb{B})}$. Its objects are the elements $\mathcal{X} \in V^{(\mathbb{B})}$ for which $[\![\mathcal{X}$ is a Banach space$]\!] = \mathbf{1}$. The morphisms of this category are the elements $\alpha \in V^{(\mathbb{B})}$ such that $[\![\alpha$ is a bounded linear operator$]\!] = \mathbf{1}$ and $[\![\|\alpha\| \leq \lambda^\wedge]\!] = \mathbf{1}$ for some number $\lambda \in \mathbb{R}$.

Theorem. *The categories C-Ban($\mathbb{B}$) and $\mathrm{Ban}_\infty^{(\mathbb{B})}$ are equivalent. Such an equivalence is established by the pair of adjoint functors, the functor of immersion and the functor of restricted descent.*

◁ See [17, Theorem 3.2.11] and Subsections 2.3 and 2.4. ▷

§3. Boolean-Valued Representation of JB-Algebras

Since any $\mathbb{B}$-JB-algebra is a $\mathbb{B}$-cyclic Banach space, the results of the previous section apply to them too.

3.1. Theorem. *The restricted descent of a JB-algebra in the model $V^{(\mathbb{B})}$ is a $\mathbb{B}$-JB-algebra. Conversely, for any $\mathbb{B}$-JB-algebra A there exists a unique (up to isomorphism) JB-algebra $\mathcal{A}$ whose restricted descent is isometrically $\mathbb{B}$-isomorphic to A. Moreover, $[\![\mathcal{A}$ is a JB-factor$]\!] = \mathbf{1}$ if and only if $\mathbb{B}(\mathbb{R}) = \mathcal{Z}(A)$.*

◁ Take an arbitrary $\mathbb{B}$-JB-algebra A. According to Theorem 2.3, we can suppose that, as a Banach space A, coincides with the restricted descent of some Banach space $\mathcal{A} \in V^{(\mathbb{B})}$. Let us introduce the structure of a Jordan algebra in $\mathcal{A}$. To this end, we must verify that the multiplication in A is extensional. Take $x, y, x', y' \in A$ and put $e := [\![x = x']\!] \wedge [\![y = y']\!]$. Since the relations $e \leq [\![u = v]\!]$ and $eu = ev$ are equivalent, it follows that $ex = ex'$ and $ey = ey'$. Since e is a central projection, we can write

$$e(xy) = (ex)y = (ex')y = (ey)x' = (ey')x' = e(x'y').$$

Hence

$$[\![x = x']\!] \wedge [\![y = y']\!] = e \leq [\![xy = x'y']\!];$$

i.e., the multiplication of A is extensional.

Now we define a binary operation $(x,y) \mapsto x \circ y$ in $\mathcal{A}$ as the ascent of the multiplication of A. It means that, for any $x, y \in A$, there exists exactly one element $x \circ y \in V^{(\mathbb{B})}$ such that $[\![x \circ y \in \mathcal{A}]\!] = [\![x \circ y = xy]\!] = \mathbf{1}$. Demonstrate that $(\mathcal{A}, \circ)$ is a JB-algebra inside $V^{(\mathbb{B})}$. From what was said above it is seen that the operator T_a is extensional. If $\mathcal{T}_a$ is the operator $x \longmapsto a \circ x$ $(x \in \mathcal{A})$ inside $V^{(\mathbb{B})}$, then, obviously, $[\![\mathcal{T}_a = T_a\uparrow]\!] = \mathbf{1}$. Therefore the operators T_x and T_y commute if and only if the operators $\mathcal{T}_x$ and $\mathcal{T}_y$ commute inside $V^{(\mathbb{B})}$. For $y = x^2$, it follows in particular that the Jordan identity $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ holds in $\mathcal{A}$. Moreover, it is clear that an element $x \in A$ belongs to the center $\mathcal{Z}(A)$ if and only if $[\![x \in \mathcal{Z}(\mathcal{A})]\!] = \mathbf{1}$. But this amounts to the equality

$$[\![\mathcal{Z}(A)\uparrow = \mathcal{Z}(\mathcal{A})]\!] = \mathbf{1}.$$

It remains to show that conditions 1.2(1)–(3) hold in $\mathcal{A}$. To this end, it suffices to establish that the vector norm in A satisfies conditions that are analogous to 1.2(1)–(3). First, observe that the following equivalences hold:

$$\|x\| \leq 1 \longleftrightarrow \| \,|x|\, \|_\infty \leq 1 \longleftrightarrow |x| \leq \mathbf{1}.$$

Now, take arbitrary $x, y \in A$ and $0 < \varepsilon \in \mathbb{R}$. Put $x_0 := \alpha^{-1}x$ and $y_0 := \beta^{-1}y$, where $\alpha := |x| + \varepsilon\mathbf{1}$ and $\beta := |y| + \varepsilon\mathbf{1}$. Since $|x_0| = |\alpha^{-1}|\,|x| \leq \mathbf{1}$, it follows that $\|x_0\| \leq 1$. Similarly, $\|y_0\| \leq 1$. Hence $\|x_0y_0\| \leq 1$ or $|x_0y_0| \leq \mathbf{1}$. From this we obtain

$$|xy| \leq |x| \cdot |y|\, \varepsilon(|x| + |y|) + \varepsilon^2\mathbf{1}.$$

By tending ε to zero, we find $|xy| \leq |x| \cdot |y|$. Further, put $\gamma^2 := |x^2| + \varepsilon\mathbf{1}$ and $x' := \gamma^{-1}x$. Then $|x'^2| = |\gamma^{-2}|\,|x^2|$; whence $\|x'\|^2 = \|x'^2\| \leq 1$ or $\|x'\| \leq 1$. Therefore, $|x'| \leq \mathbf{1}$, and also $|x'|^2 \leq \mathbf{1}$ and $|x|^2 \leq \gamma^2$. Consequently, $|x|^2 \leq |x^2| + \varepsilon\mathbf{1}$, and as $\varepsilon \longrightarrow 0$ we get $|x|^2 \leq |x^2|$. The reverse inequality follows from what was proved already; therefore, $|x|^2 = |x^2|$. Finally, by putting $\delta^2 := |x^2 + y^2| + \varepsilon\mathbf{1}$, we easily observe that $|\delta^{-2}x^2| \leq \mathbf{1}$ since

$$\|\delta^{-2}x^2\| \leq \|\delta^{-2}x^2 + \delta^{-2}y^2\| = \|\delta^{-2}\,|x^2 + y^2|\,\|_\infty \leq 1.$$

But then $|x^2| \leq \delta^2$, and as $\varepsilon \longrightarrow 0$ we come to the inequality $|x^2| \leq |x^2 + y^2|$.

From the relation $[\![\|x\|_{\mathcal{A}} = |x|]\!] = \mathbf{1}$ and the above-proven properties of the vector norm, we can derive the assertion: $[\![$ the norm in $\mathcal{A}$ satisfies conditions 1.2(1)–(3) $]\!] = \mathbf{1}$ by straightforward calculation of Boolean truth values.

Denote $\Lambda := \mathbb{B}(\mathbb{R})$. If $\Lambda = \mathcal{Z}(A)$, then

$$\mathbf{1} = [\![\mathcal{Z}(A){\uparrow} = \Lambda{\uparrow} = \mathcal{R} \cdot 1]\!] \wedge [\![(A) = \mathcal{Z}(\mathcal{A})]\!] \leq [\![\mathcal{Z}(\mathcal{A}) = \mathcal{R} \cdot 1]\!].$$

Consequently, $[\![\mathcal{A}$ is a JB-factor $]\!] = \mathbf{1}$. Conversely, suppose that $[\![\mathcal{Z}(\mathcal{A}) = \mathcal{R} \cdot 1]\!] = \mathbf{1}$. Then $[\![\mathcal{Z}(A){\uparrow} = \mathcal{R} \cdot 1]\!] = \mathbf{1}$; therefore,

$$\operatorname{mix}(\mathcal{Z}(A)) = \mathcal{Z}(A){\uparrow\downarrow} = \mathcal{R}{\downarrow} \cdot \mathbf{1} = \operatorname{mix}(\Lambda).$$

Distinguishing the bounded parts, we obtain $\Lambda = \mathcal{Z}(A)$.▷

3.2. Let A and B be some $\mathbb{B}$-JB-algebras. A $\mathbb{B}$-linear operator that is a homomorphism of the algebras is referred to as $\mathbb{B}$-*homomorphism*.

Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be the Boolean-valued representations of $\mathbb{B}$-JB-algebras A and B respectively. Let Φ be a $\mathbb{B}$-linear operator from A into B and $\varphi := \Phi{\uparrow}$. Then the following assertions hold:*

(1) *Φ is a $\mathbb{B}$-homomorphism $\longleftrightarrow$ $[\![\varphi$ is a homomorphism$]\!] = \mathbf{1}$;*
(2) *Φ is positive $\longleftrightarrow$ $[\![\varphi$ is positive$]\!] = \mathbf{1}$;*
(3) *Φ is normal $\longleftrightarrow$ $[\![\varphi$ is normal$]\!] = \mathbf{1}$.*

§4. Some Applications

Now we give several applications of the results on Boolean-valued representation to the structure of $\mathbb{B}$-JB-algebras. The theorems appear by transfer of the corresponding facts from the theory of JB-algebras.

4.1. Let A be a $\mathbb{B}$-JB-algebra and let $\Lambda := \mathbb{B}(\mathbb{R})$. An operator $\Phi \in A^\sharp$ is called a Λ-*valued state* if $\Phi \geq 0$ and $\Phi(\mathbf{1}) = \mathbf{1}$. A state Φ is said to be *normal* if, for any increasing net (x_α) in A with the least upper bound $x := \sup x_\alpha$, we have $\Phi(x) = o\text{-}\lim \Phi(x_\alpha)$. If $\mathcal{A}$ is the Boolean-valued representation of the algebra A, then the ascent $\varphi := \Phi{\uparrow}$ is a bounded linear functional on $\mathcal{A}$ as it follows from 2.4. Moreover, φ is positive and o-continuous; i.e., φ is a normal state on $\mathcal{A}$. The converse is also true: if

$[\![\varphi$ is a normal state on $\mathcal{A}]\!] = 1$, then the restriction of the operator $\varphi\downarrow$ to A is a Λ-valued normal state. Now we will characterize $\mathbb{B}$-JB-algebras that are $\mathbb{B}$-dual spaces. Toward this end, it suffices to give Boolean-valued interpretation for Theorem 2.3 of [14] that declares a JB-algebra to be a dual Banach space if and only if it is monotone complete and has a separating family of normal states.

4.2. Theorem. *For a $\mathbb{B}$-JB-algebra A, the following assertions are equivalent:*

(1) *A is a $\mathbb{B}$-dual space;*

(2) *A is monotone complete and admits a separating set of Λ-valued normal states.*

If one of these conditions holds, then the part of $A^\sharp$ consisting of order-continuous operators serves as a $\mathbb{B}$-predual space of A.

◁ By Theorem 3.1, we can assume that A coincides with the restricted descent of a JB-algebra $\mathcal{A}$ in the model $V^{(\mathbb{B})}$. According to the transfer principle, Theorem 2.3 of [14], and Corollary 2.5, it suffices to demonstrate that:

(a) the algebras A and $\mathcal{A}$ are simultaneously either monotone complete or not;

(b) A has a separating set of normal Λ-valued states if and only if $[\![\mathcal{A}$ has a separating set of normal states$]\!] = \mathbf{1}$.

The first claim follows from the fact that the operations of descent and ascent preserve polars (see [15, Theorems 3.2.13 and 3.3.12]). Moreover, the fact should be taken into consideration that the polar $\pi_{\leq}(M)$ with respect to $\leq$ (where $\leq$ stands for the order relation in $\mathcal{A}$ or A) is the set of upper bounds of the set M, and if there exists $\sup M$, then $\{\sup M\} = \pi_{\leq}(M) \cap \pi_{\leq}^{-1}(\pi_{\leq}(M))$ (cf. [15, Theorems 4.2.9 and 4.4.10(2)]).

Prove claim (b). Let $\mathcal{S}(\mathcal{A})$ denote the set of all states on $\mathcal{A}$ inside $V^{(\mathbb{B})}$ and let $\mathcal{S}_{\mathbb{B}}(A)$ be the set of all Λ-valued states on A. Since any state $\Phi \in \mathcal{S}_{\mathbb{B}}(A)$ is $\mathbb{B}$-linear, it is extensional and admits of the ascent $\varphi := \Phi\uparrow$ that represents a functional $\varphi : \mathcal{A} \to \mathcal{R}$. The ascent preserves linearity and positivity and, as was already noted in 2.4, $[\![\|\varphi\| = |\Phi|]\!] = \mathbf{1}$. Therefore, the correspondence $\Phi \longmapsto \varphi$ is a bijection between $\mathcal{S}_{\mathbb{B}}(A)$ and $\mathcal{S}(\mathcal{A})\downarrow$. Moreover, a state Φ will be normal if and only if $[\![\varphi$ is a normal state$]\!] = \mathbf{1}$ (see 3.2). Suppose that $\mathcal{S}_{\mathbb{B}}(A)$ is a separating set of states. Take a nonzero element $x \in A$. Choose a state $\Phi_0 \in \mathcal{S}_{\mathbb{B}}(A)$ so as to have $\Phi_0(x) \neq 0$. Since Φ is extensional, we have $[\![x \neq 0]\!] \leq [\![\Phi_0(x) \neq 0]\!]$. Using the calculation rules for Boolean truth values and involving the above facts, we can write

$$[\![\mathcal{S}(\mathcal{A}) \text{ is a separating set of states}]\!] = [\![(\forall x \in \mathcal{A})(x \neq 0 \longrightarrow (\exists \varphi \in \mathcal{S}(\mathcal{A}))\varphi(x) \neq 0]\!]$$
$$= \bigwedge_{x \in A} [\![x \neq 0]\!] \Rightarrow \bigvee_{\Phi \in \mathcal{S}_{\mathbb{B}}(A)} [\![\Phi\uparrow(x) \neq 0]\!] \geq \bigwedge_{x \in A} [\![x \neq 0]\!] \Rightarrow [\![\varphi_0(x) \neq 0]\!] = \mathbf{1}.$$

Thus $\mathcal{S}(\mathcal{A})$ is a separating set of states inside $V^{(\mathbb{B})}$. Conversely, assume that the preceding condition holds. Then, for every $0 \neq x \in A$, we have $b := [\![x \neq 0]\!] > 0$; therefore, by the maximum principle, there exists $\varphi \in \mathcal{S}(\mathcal{A})\downarrow$ such that $b \leq [\![\varphi(x) \neq 0]\!]$. Denote by Φ the restriction of $\varphi\downarrow$ to $A \subset \mathcal{A}\downarrow$. Then $\Phi \in \mathcal{S}_{\mathbb{B}}(A)$ and $b \leq [\![\Phi(x) \neq 0]\!]$. From here it is seen that the support of the element $\Phi(x)$ is greater than or equal to b; hence $\Phi(x) \neq 0$. ▷

4.3. An algebra A satisfying one of the equivalent conditions 4.2(1), (2) is called a $\mathbb{B}$-*JBW-algebra*. If, moreover, $\mathbb{B}$ coincides with the set of all central projections, then A is said to be a $\mathbb{B}$-*JBW-factor*. It follows from Theorems 2.1 and 4.2 that A is a $\mathbb{B}$-JBW-algebra ($\mathbb{B}$-JBW-factor) if and only if its Boolean-valued representation $\mathcal{A} \in V^{(\mathbb{B})}$ is a JBW-algebra (JBW-factor).

Consider an example. Let X be an AW^*-module over the algebra $\overline{\Lambda} := \mathbb{B}(\mathbb{C})$ (see [4, 7]). Then X is a $\mathbb{B}$-cyclic Banach space and $\mathcal{L}_{\mathbb{B}}(X)$ is a type 1 AW^*-algebra. For arbitrary $x, y \in X$, define the seminorm

$$p_{x,y}(a) := \|\langle ax, y\rangle\|_\infty \quad (a \in \mathcal{L}_{\mathbb{B}}(X)),$$

where $\langle\cdot,\cdot\rangle$ is the inner product in X with values in $\overline{\Lambda}$. Denote by σ_∞ the topology on $\mathcal{L}_{\mathbb{B}}(X)$ that is generated by the system of seminorms $p_{x,y}$. It is possible to demonstrate (see the proof in 4.4 below) that a σ_∞-closed $\mathbb{B}$-JB-algebra of selfadjoint operators in $\mathcal{L}_{\mathbb{B}}(X)$ is a monotone closed subalgebra in $\mathcal{L}_{\mathbb{B}}(X)_{sa}$. At the same time, the latter algebra is monotone complete and admits a separating set of

Λ-valued normal states. Thus, a σ_∞-closed $\mathbb{B}$-JB-algebra of selfadjoint operators presents an example of a $\mathbb{B}$-JBW-algebra.

4.4. Theorem. *A special $\mathbb{B}$-JB-algebra A is a $\mathbb{B}$-JBW-algebra if and only if it is isomorphic to a σ_∞-closed $\mathbb{B}$-JB-subalgebra of $\mathcal{L}_\mathbb{B}(X)_{sa}$ for some AW^*-module X.*

◁ Again we can assume that A coincides with the restricted descent of a JB-algebra $\mathcal{A}$ in the model $V^{(\mathbb{B})}$. Moreover, it is easy to show that $\mathcal{A}$ is special too.

Let $\mathcal{X}$ be a complex Hilbert space inside $V^{(\mathbb{B})}$. The restricted descent X of the space $\mathcal{X}$ is an AW^*-module and, conversely, any AW^*-module has such a representation (see [4, 7]). Furthermore, it is known from [14] that a special JBW-algebra is a JW-algebra, i.e., it is isomorphic to a weakly closed subalgebra of the algebra $\mathcal{L}(\mathcal{X})_{sa}$. Thus, we can suppose that $\mathcal{A}$ is a uniformly closed Jordan subalgebra in $\mathcal{L}(\mathcal{X})_{sa}$, and it now suffices to prove that A is a σ_∞-closed subalgebra in $\mathcal{L}_\mathbb{B}(X)_{sa}$ if and only if $V^{(\mathbb{B})} \models$ "$\mathcal{A}$ is a weakly closed subalgebra in $\mathcal{L}(\mathcal{X})_{sa}$."

The algebraic part of the claim is obvious. Let the formula $\psi(\mathcal{A}, u)$ formalize the sentence: an operator u belongs to the weak closure of $\mathcal{A}$. Then the formula can be written as

$$(\forall n \in \omega)(\forall \theta_1, \theta_2 \in \mathcal{P}_{fin}(\mathcal{X}))(\exists v \in \mathcal{A})(\forall x \in \theta_1)(\forall y \in \theta_2)\,|\langle u(x) - v(x), y\rangle| \leq n^{-1},$$

where ω is the set of natural numbers, $\langle\cdot\,,\cdot\rangle$ is the inner product in $\mathcal{X}$, and $\mathcal{P}_{\mathrm{fin}}(\mathcal{X})$ is the set of all finite subsets of X. Suppose that $[\![\psi(\mathcal{A}, u)]\!] = \mathbf{1}$. Calculation of Boolean truth values with the help of the maximum principle and the relation (cf. [15, 3.1.11])

$$\mathcal{P}_{fin}(\mathcal{X}) = \{\theta{\uparrow} : \theta \in \mathcal{P}_{fin}(X)\}{\uparrow}$$

yields the following assertion: For any $n \in \omega$ and any finite collections $\theta_1 := \{x_1, \dots, x_n\} \subset X$ and $\theta_2 := \{y_1, \dots, y_m\} \subset X$, there exists $v \in \mathcal{A}{\downarrow}$ such that

$$[\![(\forall x \in \theta_1)(\forall y \in \theta_2)|\langle u(x) - v(x), y\rangle| \leq n^{-1}]\!] = \mathbf{1}.$$

According to the Kaplansky density theorem (see [10, Theorem 4.5.12]), we can choose v so that the extra condition $[\![\|v\| \leq \|u\|]\!] = \mathbf{1}$ holds. If U and V are the respective restrictions of the operators $u{\downarrow}$ and $v{\downarrow}$ to X, then

$$|U| \leq |V|,$$
$$|\langle (U - V)(x_k), y_l\rangle| < n^{-1}\mathbf{1} \quad (k := 1, \dots, n;\ l := 1, \dots, m).$$

There exists a fixed partition of unity $(e_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ which depends only on u and is such that $e_\xi\,|U| \in \Lambda$ for all ξ. From here it is seen that $e_\xi U \in A$ and $e_\xi V \in A$. Moreover,

$$\|\langle e_\xi (U - V)(x_k), y_l\rangle\|_\infty < n^{-1} \quad (k := 1, \dots, n;\ l := 1, \dots, m).$$

Repeating the above argument in the opposite direction, we come to the following conclusion: The formula $\psi(\mathcal{A}, u)$ is true inside $V^{(\mathbb{B})}$ if and only if there exist a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(U_\xi)_{\xi\in\Xi}$, where U_ξ belongs to the σ_∞-closure of A, such that $e_\xi \leq [\![u = U_\xi{\uparrow}]\!]$ for all ξ, i.e., $u = \mathrm{mix}(e_\xi U_\xi{\uparrow})$.

Now, assume that A is σ_∞-closed and the formula $\psi(\mathcal{A}, u)$ is true inside $V^{(\mathbb{B})}$. Then U_ξ is contained in A by assumption and $[\![U_\xi{\uparrow} \in \mathcal{A}]\!] = \mathbf{1}$. Hence $e_\xi \leq [\![u \in \mathcal{A}]\!]$ for all ξ, i.e., $[\![u \in \mathcal{A}]\!] = \mathbf{1}$. Therefore,

$$V^{(\mathbb{B})} \models (\forall u \in \mathcal{L}(\mathcal{X}))\psi(\mathcal{A}, u) \to u \in \mathcal{A}.$$

Conversely, assume $\mathcal{A}$ to be weakly closed. If U belongs to the σ_∞-closure of A, then $u = U{\uparrow}$ is contained in the weak closure of $\mathcal{A}$. By assumption, $[\![u \in \mathcal{A}]\!] = \mathbf{1}$ or $u \in \mathcal{A}{\downarrow}$. The restriction of the operator $u{\downarrow}$ to X, coinciding with U, belongs to A. ▷

4.5. Let $M_3^8 := M_3(\mathbb{O})$ be the algebra of Hermitian (3×3)-matrices over the Cayley numbers $\mathbb{O}$. If $(\cdot)^\wedge$ denotes the canonical embedding into the Boolean-valued universe $V^{(\mathbb{B})}$, then $[\![\mathbb{O}^\wedge$ is a normed algebra over the field $\mathbb{R}]\!] = \mathbf{1}$ and $[\![(M_3^8)^\wedge$ is an $\mathbb{R}^\wedge$-algebra of Hermitian (3×3)—matrices over $\mathbb{O}]\!] = \mathbf{1}$. Let $\mathcal{O}$ and $\mathcal{M}_3^8$ be the norm completions of the algebras $\mathbb{O}$ and $(M_3^8)^\wedge$ inside $V^{(\mathbb{B})}$ respectively. It is easy to show (using, for example, the Hurwitz theorem) that $[\![\mathcal{O}$ is an algebra of the Cayley numbers $]\!] = \mathbf{1}$ and $[\![\mathcal{M}_3^8$ is an algebra of Hermitian (3×3)-matrices over the Cayley numbers $]\!] = \mathbf{1}$. By Theorem 2.1, the restricted descent $\mathcal{M}_3^8$ is a $\mathbb{B}$-JB-algebra. At the same time, the restricted descent of the JB-algebra $\mathcal{M}_3^8$ is isomorphic to the algebra $C(Q, M_3^8)$ of continuous vector-functions, where Q is the Stone compact space of the Boolean algebra $\mathbb{B}$. Actually $\mathcal{M}_3^8$ and $C(Q, M_3^8)$ are isometrically isomorphic (see [18]). Taking all the above into account, we give a Boolean-valued interpretation to the following fact (see [12, Theorem 8.6]): Each JBW-factor is isomorphic either to M_3^8 or to a JC-algebra.

4.6. Theorem. *Each $\mathbb{B}$-JBW-factor A admits a unique decomposition $A = eA \oplus e^*A$ with a central projection $e \in \mathbb{B}, e^* := \mathbf{1} - e$, such that the algebra eA is special and the algebra e^*A is purely exceptional. Moreover, eA is $\mathbb{B}$-isomorphic to a σ_∞-closed subalgebra of selfadjoint endomorphisms of some AW^*-module and e^*A is isomorphic to $C(Q, M_3^8)$, where Q is the Stone compact space of the Boolean algebra $e^*\mathbb{B} := [0, e^*]$.*

$\triangleleft$ If $\mathcal{A}$ is the Boolean-valued representation of the algebra A, then $[\![\mathcal{A}$ is a JBW-factor $]\!] = \mathbf{1}$ (see 4.3). Consequently, by the transfer principle, $[\![\mathcal{A}$ is isomorphic either to JC-algebra or to $M_3^8]\!] = 1$. Put $e := [\![\mathcal{A}$ is special $]\!]$. Then $e^* = [\![\mathcal{A}$ is isomorphic to $\mathcal{M}_3^8]\!]$. Moreover, the following assertions hold:

$$V^{(e\mathbb{B})} \models \text{“}e\mathcal{A} \text{ is a special } JBW\text{-factor;”}$$
$$V^{(e^*\mathbb{B})} \models \text{“}e^*\mathcal{A} \text{ is an algebra isomorphic to } \mathcal{M}_3^8.\text{”}$$

The restricted descent of $e\mathcal{A}$ presents a special $e\mathbb{B}$-algebra. It remains to apply Theorem 4.4 and a remark in 4.5. $\triangleright$

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