INTEGRAL OPERATORS DETERMINED BY QUASIELLIPTIC EQUATIONS. II

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In this article we continue the study of properties of the families of integral operators which are connected with quasielliptic equations [1]. The operators under consideration were introduced by the author in [2, 3] while constructing approximate solutions to the following quasielliptic equations in the half-space:

$$\begin{gathered} L(D_x)u = f(x), \quad x \in R_n^+, \\ B_j(D_x)u|_{x_n=0} = 0, \quad j = 1,\dots,\mu, \end{gathered} \tag{1}$$

with boundary operators satisfying the Lopatinskiĭ condition. Study of their properties enables us to obtain a number of new results in the theory of boundary value problems (1).

§1. Definitions and Statement of the Main Results

We shall assume that the operator $L(D_x)$ is quasielliptic and its symbol $L(i\xi)$ is homogeneous with respect to some vector $\alpha = (\alpha_1,\dots,\alpha_n)$, where $1/\alpha_i$ are naturals; i.e., $L(c^\alpha i\xi) = cL(i\xi)$, $c > 0$. Without loss of generality we can suppose that the coefficient of the highest derivative with respect to x_n equals unity. By virtue of quasiellipticity of $L(D_x)$, the equation

$$L(is, i\lambda) = 0, \quad s \in R_{n-1} \setminus \{0\}, \tag{2}$$

has $1/\alpha_n$ roots in λ; moreover, none of them can be real. Denote all roots with positive imaginary part by $\lambda_k^+(s)$, $k = 1,\dots,\mu$, and let

$$M^+(s,\lambda) = \prod_{k=1}^{\mu} (\lambda - \lambda_k^+(s)).$$

We now specify the conditions on the boundary operators $B_j(D_x)$ for $x_n = 0$. Assume that the number of the boundary operators $B_j(D_x)$ equals μ and each of them has the form

$$B_j(D_x) = D_{x_n}^{m_j} + \sum_{k<m_j} B_{j,k}(D_{x'})D_{x_n}^k, \quad x' = (x_1,\dots,x_{n-1});$$

moreover, the symbols $B_j(i\xi)$ are homogeneous with respect to the vector α and homogeneity exponent β_j, $0 \le \beta_j < 1$; i.e.,

$$B_j(c^\alpha i\xi) = c^{\beta_j} B_j(i\xi), \quad c > 0.$$

We let S_n stand for the trace operator on the hyperplane $\{x_n = 0\}$.

We assume that the Lopatinskiĭ condition is satisfied for the operator

$$\{L(D_x),\, S_n \circ B_1(D_x), \dots, S_n \circ B_\mu(D_x)\};$$

i.e., the polynomials $B_j(is, i\lambda)$, $j = 1,\dots,\mu$, are mod $M^+(s,\lambda)$ linearly independent for $s \in R_{n-1} \setminus \{0\}$ as polynomials in λ. This means that $\det\big(b_{j,k}(s)\big) \ne 0$, $s \in R_{n-1} \setminus \{0\}$, where the elements $b_{j,k}(s)$ are determined from the identities

$$\sum_{k=1}^{\mu} b_{j,k}(s)(i\lambda)^{k-1} \equiv B_j(is, i\lambda) \pmod{M^+(s,\lambda)}.$$

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Now we describe the construction of an approximate solution to problem (1) which was presented in [2, 3]. It is based on the use of the following special averaging [4] for the functions $f(x) \in L_p(R_k)$:

$$f(x) = \lim_{h \to 0} \frac{1}{(2\pi)^k} \int_h^{h^{-1}} v^{-|\alpha'|-1} \int_{R_k} \int_{R_k} \exp\left(i \frac{x-y}{v^\alpha} \xi\right) G(\xi) f(y)\, d\xi dy dv, \tag{3}$$

where $G(\xi) = m\langle\xi\rangle^m \exp(-\langle\xi\rangle^m)$, $m = 2l > k+1$, $\langle\xi\rangle^2 = \sum_{i=1}^k \xi_i^{2/\alpha_i}$, $|\alpha'| = \sum_{i=1}^k \alpha_i$.

Let $\Gamma^+(s)$ denote a contour, in the complex plane, which encloses the roots $\lambda_k^+(s)$ of equation (2), and let $\Gamma^-(s)$, be a contour enclosing all roots in the lower half-plane. Define the contour integrals

$$J_+(s, x_n) = \frac{1}{2\pi} \int_{\Gamma^+(s)} \frac{\exp(ix_n\lambda)}{L(is, i\lambda)}\, d\lambda, \quad J_-(s, x_n) = -\frac{1}{2\pi} \int_{\Gamma^-(s)} \frac{\exp(ix_n\lambda)}{L(is, i\lambda)}\, d\lambda,$$

$$J_j(s, x_n) = \frac{1}{2\pi i} \int_{\Gamma^+(s)} \frac{\exp(ix_n\lambda)}{M^+(s, \lambda)} N_j(s, \lambda)\, d\lambda, \quad j = 1, \dots, \mu,$$

where $N_j(s, \lambda)$ are polynomials in λ such that the following equalities are valid [5]:

$$\frac{1}{2\pi i} \int_{\Gamma^+(s)} \frac{B_k(is, i\lambda) N_j(s, \lambda)}{M^+(s, \lambda)}\, d\lambda = \delta_j^k.$$

Using the integral representation (3) for $k = n-1$ together with the contour integrals introduced, we define some linear integral operators R_h^+, R_h^-, $R_{j,h}$, $j = 1, \dots, \mu$, $h \in (0,1)$, as follows. Given an arbitrary function $f(x) \in L_p(R_n^+) \cap L_1(R_n^+)$, we define

$$R_h^+ f(x) = \frac{1}{(2\pi)^{n-1}} \int_h^{h^{-1}} \frac{1}{v} \int_0^{x_n} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x'-y')s) G(sv^{\alpha'}) J_+(s, x_n - y_n) f(y)\, ds dy' dy_n dv,$$

$$x = (x', x_n),\ y = (y', y_n),\ \alpha = (\alpha', \alpha_n),$$

$$R_h^- f(x) = \frac{1}{(2\pi)^{n-1}} \int_h^{h^{-1}} \frac{1}{v} \int_{x_n}^{\infty} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x'-y')s) G(sv^{\alpha'}) J_-(s, x_n - y_n) f(y)\, ds dy' dy_n dv,$$

$$R_{j,h} f(x) = \frac{1}{(2\pi)^{n-1}} \int_h^{h^{-1}} \frac{1}{v} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x'-y')s) G(sv^{\alpha'}) J_j(s, x_n) \int_0^\infty I_j(s, y_n) f(y', y_n)\, dy_n ds dy' dv,$$

where $j = 1, \dots, \mu$ and

$$I_j(s, y_n) = -B_j(is, D_{z_n}) J_-(s, z_n - y_n)\big|_{z_n = 0},$$

$$G(s) = m\langle s\rangle^m \exp(-\langle s\rangle^m), \quad m = 2l > n, \quad \langle s\rangle^2 = \sum_{i=1}^{n-1} s_i^{2/\alpha_i}.$$

From the definition of the integral operators R_h^+, R_h^-, and $R_{j,h}$, $j = 1, \dots, \mu$, it follows (see [2, 3]) that, for small $h > 0$, the function

$$u_h(x) = \left(R_h^+ + R_h^- + \sum_{j=1}^{\mu} R_{j,h} \right) f(x)$$

is an approximate solution to the boundary value problem (1).

Now, by analogy with [1], we define some scale of function spaces in which we shall investigate the action of the operators R_h^+, R_h^-, $R_{j,h}$, $j = 1, \dots, \mu$, $h \in (0, 1)$.

Let $r = (1/\alpha_1, \dots, 1/\alpha_n)$, $1 < p < \infty$, $1/p' + 1/p = 1$, $0 \le \sigma \le 1$, and $\alpha_{\min} = \min\{\alpha_1, \dots, \alpha_n\}$. Introduce the weighted Sobolev space $W^r_{p,\sigma}(R_n^+)$; by definition, $W^r_{p,\sigma}(R_n^+)$ is the completion, of the set of functions in $C^\infty(\overline{R}_n^+)$ vanishing at large $|x|$, in the norm

$$\|u(x),\, W^r_{p,\sigma}(R_n^+)\| = \sum_{0 \le \beta\alpha \le 1} \|(1 + \langle x \rangle)^{-\sigma(1-\beta\alpha)} D_x^\beta u(x),\, L_p(R_n^+)\|,$$

where $\langle x \rangle^2 = \sum_{i=1}^{n} x_i^{2/\alpha_i}$. For $\sigma = 0$, the space indicated is the Sobolev space $W_p^r(R_n^+)$.

Denote by $L_{1,\gamma}(R_n^+)$ the space of summable functions $u(x)$ with the finite norm $\|u(x),\, L_{1,\gamma}(R_n^+)\| = \|(1 + \langle x \rangle)^{-\gamma} u(x),\, L_1(R_n^+)\|$. In particular, for $\gamma = 0$ we have $L_{1,0}(R_n^+) = L_1(R_n^+)$.

Let $\mathcal{L}_{p,\sigma,N}(R_n^+)$ be the subspace of $L_p(R_n^+)$ constituted by the functions $f(x)$ satisfying the conditions

$$(1 + \langle x \rangle)^{\sigma + N|\alpha|} f(x) \in L_1(R_n^+), \tag{4}$$

$$\int_{R_n^+} x^\beta f(x)\, dx = 0, \quad |\beta| = 0, \dots, N - 1. \tag{5}$$

We recall one definition of [1].

DEFINITION. Let V and U be normed vector spaces. We say that a family of linear operators R_h, $h \in (0, 1)$, is *fundamental in the pair of spaces* $\{V, U\}$ *as* $h \to 0$ if, for every $h \in (0, 1)$, the operator $R_h : V \to U$ is bounded; moreover,

$$\sup_{0<h<1} \|R_h\| \le c < \infty$$

and

$$\|R_{h_1} - R_{h_2}\| \to 0, \quad h_1, h_2 \to 0.$$

Now we formulate the main results of the present article.

Theorem 1. *Assume* $|\alpha| > 1$ *and* $|\alpha|/p > \sigma > 1 - |\alpha|/p'$. *Then the operator family* $(R_h^+ + R_h^-)$ *is fundamental in the pair of spaces* $\{L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+),\, W^r_{p,\sigma}(R_n^+)\}$ *as* $h \to 0$.

Theorem 2. *Assume* $|\alpha|/p > \sigma$ *and* $1 \ge |\alpha| > 1 - N\alpha_{\min}$, *where* N *is a natural number and*

$$1 - |\alpha|/p' - (N-1)\alpha_{\min} \ge \sigma > 1 - |\alpha|/p' - N\alpha_{\min}. \tag{6}$$

Then the operator family $(R_h^+ + R_h^-)$ *is fundamental in the pair of spaces* $\{\mathcal{L}_{p,\sigma,N}(R_n^+),\, W^r_{p,\sigma}(R_n^+)\}$ *as* $h \to 0$.

Theorem 3. *Let* $|\alpha| > 1$ *and* $|\alpha|/p > \sigma > 1 - |\alpha|/p'$. *Then the operator family* $R_{j,h}$ *is fundamental in the pair of spaces* $\{L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+),\, W^r_{p,\sigma}(R_n^+)\}$ *as* $h \to 0$.

Theorem 4. *Assume $|\alpha|/p > \sigma$, $1 \geq |\alpha| > 1 - N\alpha_{\min}$, where N is a natural number satisfying (6). Then the operator family $R_{j,h}$ is fundamental in the pair of spaces $\{\mathcal{L}_{p,\sigma,N}(R_n^+),\ W_{p,\sigma}^r(R_n^+)\}$ as $h \to 0$.*

In consequence of the theorems we separately formulate the results on the operator family R_h, $h \in (0,1)$, where

$$R_h = R_h^+ + R_h^- + \sum_{j=1}^{\mu} R_{j,h}.$$

Theorem 5. *Let $|\alpha| > 1$ and $|\alpha|/p > \sigma > 1-|\alpha|/p'$. Then the operator family R_h is fundamental in the pair of spaces $\{L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+),\ W_{p,\sigma}^r(R_n^+)\}$ as $h \to 0$.*

Theorem 6. *Assume $|\alpha|/p > \sigma$ and $1 \geq |\alpha| > 1-N\alpha_{\min}$, where N is a natural number satisfying inequalities (6). Then the operator family R_h is fundamental in the pair of spaces $\{\mathcal{L}_{p,\sigma,N}(R_n^+),$ $W_{p,\sigma}^r(R_n^+)\}$ as $h \to 0$.*

Using Theorems 5 and 6, one can prove the following assertion on well-posedness of the boundary value problem (1).

Theorem 7. *Let $|\alpha| > 1$ and $|\alpha|/p > \sigma > 1 - |\alpha|/p'$. Then, for every function $f(x) \in L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+)$, the boundary value problem (1) has a unique solution $u(x) \in W_{p,\sigma}^r(R_n^+)$; moreover, the following estimate is valid:*

$$\big\|u(x),\ W_{p,\sigma}^r(R_n^+)\big\| \leq c\big(\big\|f(x),\ L_p(R_n^+)\big\| + \big\|f(x),\ L_{1,-\sigma}(R_n^+)\big\|\big) \tag{7}$$

with a constant $c > 0$ independent of $f(x)$.

Theorem 8. *Assume $|\alpha|/p > \sigma$ and $1 \geq |\alpha| > 1 - N\alpha_{\min}$, where N is a natural number satisfying inequalities (6). Then, for every function $f(x) \in \mathcal{L}_{p,\sigma,N}(R_n^+)$, there exists a unique solution $u(x) \in W_{p,\sigma}^r(R_n^+)$ to the boundary value problem (1) which satisfies the inequality*

$$\big\|u(x),\ W_{p,\sigma}^r(R_n^+)\big\| \leq c\big(\big\|f(x),\ L_p(R_n^+)\big\| + \big\|(1+\langle x\rangle)^{\sigma+N|\alpha|}f(x),\ L_1(R_n^+)\big\|\big) \tag{8}$$

with a constant $c > 0$ independent of $f(x)$.

From Theorems 7 and 8 ensues the next assertion on well-posedness of the boundary value problem (1) in the Sobolev space $W_p^r(R_n^+)$.

Theorem 9. *If $|\alpha|/p' > 1$ then, for every function $f(x) \in L_p(R_n^+) \cap L_1(R_n^+)$, the boundary value problem (1) has a unique solution $u(x) \in W_p^r(R_n^+)$ which satisfies inequality (7) for $\sigma = 0$. If $|\alpha|/p' \leq 1$ and if $f(x) \in L_p(R_n^+)$ meets conditions (4), (5) for $\sigma = 0$ and some natural N determined from inequalities (6), then problem (1) is also uniquely solvable in $W_p^r(R_n^+)$ and its solution satisfies estimate (8) for $\sigma = 0$.*

Observe that in the presented theorems we specify sufficient conditions on the right-hand side $f(x)$ under which the boundary value problem (1) is well-posed in $W_{p,\sigma}^r(R_n^+)$. The natural question arises: to what extent are these additional conditions essential? The question is still open; nevertheless, as follows from the theorem formulated below, the orthogonality conditions (5) are close to necessary solvability conditions.

Theorem 10. *Let $\Gamma(s)$ be a contour, in the complex plane, which encloses all roots of the equation $L(is,i\lambda) = 0$, $s \in R_{n-1} \setminus \{0\}$. If $|\alpha|/p' + \alpha_{\min} > 1 \geq |\alpha|/p'$, $1 < p \leq 2$, and*

$$\int\limits_{\Gamma(s)} \frac{B_j(is,i\lambda)}{L(is,i\lambda)}\, d\lambda \not\equiv 0$$

for some $j = 1, \ldots, \mu$, then for solvability of the boundary value problem (1) in $W_p^r(R_n^+)$ it is necessary that

$$\int_{R_n^+} f(x)\,dx = 0.$$

From this theorem it follows in particular that the condition $\int_{R_3^+} f(x)\,dx = 0$ is necessary for the solvability of the Neumann problem

$$\Delta u = f(x),\ x \in R_3^+, \quad D_{x_3}u|_{x_3=0} = 0$$

in $W_p^2(R_3^+)$, $p \leq 2$.

At present, there are many articles devoted to study of boundary value problems for quasielliptic equations (see, for instance, [6–15]), but the theory is not yet sufficiently complete.

REMARK. In the case of a compactly-supported right-hand side $f(x)$, Theorem 7 follows from [15]. The assertion of Theorem 8 for $\sigma = 0$ strengthens the author's result [2, 3]. The main results of the article are announced in [16].

§2. Multipliers

This section is auxiliary. Here we present estimates and some identities for the contour integrals $J_+(s, x_n)$, $J_-(s, x_n)$, $J_j(s, x_n)$, and $I_j(s, x_n)$, $j = 1, \ldots, \mu$, as well as indicate several functions $\mu(\xi)$, $\xi = (s, \xi_n)$, that are multipliers in $L_p(R_n)$. All of these will be essentially used below.

The following two lemmas hold [3]:

Lemma 1. *For $x_n > 0$ and $s \in R_{n-1} \setminus \{0\}$ the estimates*

$$|D_{x_n}^k D_s^\beta J_+(s, x_n)| \leq c\langle s\rangle^{(k+1)\alpha_n - \beta\alpha - 1} \exp(-\delta x_n \langle s\rangle^{\alpha_n}),$$
$$|D_{x_n}^k D_s^\beta J_-(s, -x_n)| \leq c\langle s\rangle^{(k+1)\alpha_n - \beta\alpha - 1} \exp(-\delta x_n \langle s\rangle^{\alpha_n}),$$
$$|D_{x_n}^k D_s^\beta J_j(s, x_n)| \leq c\langle s\rangle^{k\alpha_n - \beta\alpha - \beta_j} \exp(-\delta x_n \langle s\rangle^{\alpha_n})$$

are valid for arbitrary k and $\beta = (\beta_1, \ldots, \beta_{n-1})$, where c and $\delta > 0$ are constants.

Lemma 2. *For $s \in R_{n-1} \setminus \{0\}$, the identities*

$$D_{x_n}^k (J_+(s, x_n) - J_-(s, x_n))\big|_{x_n=0} = \delta_{m-1}^k, \quad k = 0, \ldots, m-1,$$

hold, where δ_{m-1}^k is the Kronecker symbol.

With the help of these lemmas we prove the next three lemmas.

Lemma 3. *For $s \in R_{n-1} \setminus \{0\}$, the identity*

$$\int_{R_1} \exp(-i\lambda x_n)(\theta(x_n)J_+(s, x_n) + \theta(-x_n)J_-(s, x_n))\,dx_n = \frac{1}{L(is, i\lambda)} \tag{9}$$

holds, where $\theta(x_n)$ is the Heaviside function.

PROOF. Let $\varphi(x_n)$ be an arbitrary function in $C_0^\infty(R_1)$. Consider the following boundary value problem on the real axis for an ordinary differential equation with parameter $s \in R_{n-1} \setminus \{0\}$:

$$L(is, D_{x_n})v = \varphi(x_n), \quad |v(s, x_n)| < \infty, \quad x_n \in R_1.$$

Since the characteristic equation $L(is, i\lambda) = 0$ has no real roots in λ, the problem has a unique solution. Using Lemma 2, it is easy to prove that such a solution can be represented as

$$v(s, x_n) = \int_{R_1} (\theta(x_n - y_n) J_+(s, x_n - y_n) + \theta(y_n - x_n) J_-(s, x_n - y_n)) \varphi(y_n)\, dy_n.$$

From Lemma 1 it follows that we can apply the Fourier transform to this function. Taking into account a formula for the Fourier transform of convolution, we have

$$\hat{v}(s, \lambda) = \int_{R_1} \exp(-i\lambda x_n)(\theta(x_n) J_+(s, x_n) + \theta(-x_n) J_-(s, x_n))\, dx_n \widehat{\varphi}(\lambda).$$

On the other hand, $L(is, i\lambda)\hat{v}(s, \lambda) = \widehat{\varphi}(\lambda)$. Hence (9) follows in view of arbitrariness of the function $\varphi(x_n)$. The lemma is proven.

Lemma 4. *For $s \in R_{n-1} \setminus \{0\}$, the following identity is valid:*

$$\int_{R_1} \exp(i\lambda x_n) \frac{1}{L(is, i\lambda)}\, d\lambda = 2\pi(\theta(x_n) J_+(s, x_n) + \theta(-x_n) J_-(s, x_n)).$$

The proof is obtainable from (9) by applying the inverse Fourier transform.

Lemma 5. *Assume $f(x', x_n) \in L_1(R_n^+)$ and let $\hat{f}(s, x_n)$ be the partial Fourier transform with respect to x'. Then, for $s \in R_{n-1} \setminus \{0\}$, the following identities are valid:*

$$J_j(s, x_n) \int_0^\infty I_j(s, y_n) \hat{f}(s, y_n)\, dy_n$$

$$= -\int_0^\infty D_{z_n} \left(J_j(s, x_n + z_n) \int_0^\infty I_j(s, y_n + z_n) \hat{f}(s, y_n)\, dy_n \right) dz_n, \quad j = 1, \ldots, \mu.$$

The proof is straightforward from Lemma 1.

Using the contour integrals $J_+(s, x_n)$, $J_k(s, x_n)$, and $I_k(s, x_n)$, we introduce the function

$$U(s, x_n, y_n) = J_+(s, x_n - y_n) + \sum_{k=1}^{\mu} J_k(s, x_n) I_k(s, y_n)$$

for $x_n > 0$ and $s \in R_{n-1} \setminus \{0\}$. From the definitions of the integrals $J_+(s, x_n)$, $J_k(s, x_n)$, and $I_k(s, x_n)$ immediately follows

Lemma 6. *The identity holds:*

$$U(c^{\alpha'} s, c^{-\alpha_n} x_n, 0) \equiv c^{-1+\alpha_n} U(s, x_n, 0), \quad c > 0.$$

Henceforth we use the following notation:

$$M^+(s, D_{x_n}) = \prod_{k=1}^{\mu} \left(\frac{1}{i} D_{x_n} - \lambda_k^+(s) \right), \qquad \Lambda = \max_{i, s'} |\lambda_i(s')|, \quad \langle s' \rangle = 1,$$

$$\psi_j(s) = \int_{\Gamma(s)} \frac{B_j(is, i\lambda)}{L(is, i\lambda)}\, d\lambda, \quad j = 1, \ldots, \mu, \quad \Gamma(s) = \{\lambda \in C : |\lambda| = 2\Lambda \langle s \rangle^{\alpha_n}\}.$$

Lemma 7. *The function $U(s,x_n,0)$ is a solution to the boundary value problem*

$$M^+(s,D_{x_n})U = 0, \quad x_n > 0,$$
$$B_j(is,D_{x_n})U\big|_{x_n=0} = \psi_j(s), \quad j = 1,\dots,\mu,$$
$$U \to 0 \quad \text{as} \quad x_n \to +\infty.$$

PROOF. From the definitions of the contour integrals $J_+(s,x_n)$ and $J_k(s,x_n)$ and Lemma 1 it is obvious that

$$M^+(s,D_{x_n})U(s,x_n,0) \equiv 0, \quad |U(s,x_n,0)| \to 0 \quad \text{for} \quad x_n \to +\infty.$$

Verify the boundary conditions. Recalling the definitions of the integrals $J_k(s,x_n)$ and $I_k(s,x_n)$, we have

$$B_j(is,D_{x_n})U(s,x_n,0)|_{x_n=0} = B_j(is,D_{x_n})\Big(J_+(s,x_n) + \sum_{k=1}^{\mu} J_k(s,x_n)I_k(s,0)\Big)\Big|_{x_n=0}$$
$$= B_j(is,D_{x_n})J_+(s,x_n)|_{x_n=0} + I_j(s,0) = B_j(is,D_{x_n})(J_+(s,x_n) - J_-(s,x_n))|_{x_n=0}$$
$$= \frac{1}{2\pi}\int\limits_{\Gamma(s)} \frac{B_j(is,i\lambda)}{L(is,i\lambda)}\,d\lambda = \psi_j(s).$$

The lemma is proven.

Lemma 8. *The following representation holds:*

$$U(s,x_n,0) = \sum_{k=1}^{\mu} J_k(s,x_n)\psi_k(s).$$

The proof is straightforward from the preceding lemma.

At the conclusion of the section we state a lemma on multipliers.

Lemma 9. *For every vector $\nu = (\nu',\nu_n)$, $\nu\alpha = \nu'\alpha' + \nu_n\alpha_n = 1$, the functions*

$$\mu_\nu^+(\xi) = (is)^{\nu'}\int\limits_0^\infty e^{-i\xi_n x_n} D_{x_n}^{\nu_n} J_+(s,x_n)\,dx_n,$$

$$\mu_\nu^-(\xi) = (is)^{\nu'}\int\limits_{-\infty}^0 e^{i\xi_n x_n} D_{x_n}^{\nu_n} J_-(s,x_n)\,dx_n,$$

$$\mu_{j,\nu}(\xi) = \langle s\rangle^{\beta_j-\nu_n\alpha_n}\int\limits_0^\infty e^{i\xi_n x_n} D_{x_n}^{\nu_n+1} J_j(s,x_n)\,dx_n,$$

$$m_{j,\nu}(\xi) = (is)^{\nu'}\int\limits_0^\infty e^{i\xi_n x_n}\langle s\rangle^{\nu_n\alpha_n-\beta_j} B_j(is,D_{y_n})J_-(s,y_n-x_n)|_{y_n=0}dx_n, \; j = 1,\dots,\mu,$$

are multipliers in $L_p(R_n)$.

PROOF. By Lizorkin's theorem [17], it is sufficient to prove that, for every vector $\gamma = (\gamma_1,\dots,\gamma_n)$, where either $\gamma_i = 0$ or $\gamma_i = 1$, the inequalities

$$\big|\xi^\gamma D_\xi^\gamma \mu_\nu^+(\xi)\big| \le c, \; \big|\xi^\gamma D_\xi^\gamma \mu_\nu^-(\xi)\big| \le c, \; \big|\xi^\gamma D_\xi^\gamma \mu_{j,\nu}(\xi)\big| \le c, \; \big|\xi^\gamma D_\xi^\gamma m_{j,\nu}(\xi)\big| \le c$$

hold for $\xi_l \ne 0$, $l = 1,\dots,n$, with some constant $c > 0$ independent of ξ. The verification of the inequalities causes no difficulties and can be accomplished with the help of Lemma 1.

§3. Proofs of the Theorems Stating that the Operator Family $(R_h^+ + R_h^-)$ is Fundamental as $h \to 0$

In this section we prove the main results on the property of fundamentalness of the operator family $(R_h^+ + R_h^-)$ in the corresponding pairs of weighted spaces as $h \to 0$.

We begin proving Theorem 1. First of all, we observe that σ belongs to the nonempty interval $(1 - |\alpha|/p', |\alpha|/p)$, since $|\alpha| > 1$.

The proof of the theorem is divided into three lemmas. In the first lemma we estimate the higher derivative

$$D_x^\beta (R_h^+ + R_h^-) f(x), \quad \beta\alpha = 1,$$

in the L_p-norm; in the second we estimate all derivatives with $\beta\alpha < 1$ in the corresponding weighted norms; and in the third we prove that the function $u_h(x) = (R_h^+ + R_h^-) f(x)$ can be approximated by functions vanishing for $|x| \gg 0$ in the norm of $W_{p,\sigma}^r(R_n^+)$.

Lemma 1. *Let $|\alpha| > 0$, $f(x) \in L_p(R_n^+) \cap L_1(R_n^+)$, and $u_h(x) = (R_h^+ + R_h^-) f(x)$. Then*

$$\| D_x^\beta u_h(x), L_p(R_n^+) \| \le c \| f(x), L_p(R_n^+) \|, \quad \beta\alpha = 1, \quad 0 < h < 1,$$

where the constant $c > 0$ is independent of h and $f(x)$; moreover,

$$\| D_x^\beta u_{h_1}(x) - D_x^\beta u_{h_2}(x),\ L_p(R_n^+) \| \to 0, \quad h_1, h_2 \to 0.$$

The proof of the lemma for $\beta = (1/\alpha_1, \dots, 1/\alpha_n)$ and a compactly-supported function $f(x)$ is contained in [3] (see Lemma 3). The general case is settled by the same scheme by using Lemma 9 of §2.

Lemma 2. *Let $f(x) \in L_p(R_n^+)$, $(1 + \langle x \rangle)^\sigma f(x) \in L_1(R_n^+)$, $u_h(x) = (R_h^+ + R_h^-) f(x)$ and $|\alpha| > 1$, $|\alpha|/p > \sigma > 1 - |\alpha|/p'$. Then, for $1 > \beta\alpha \ge 0$, the following estimate is valid:*

$$\begin{aligned} &\| (1 + \langle x \rangle)^{-\sigma(1-\beta\alpha)} D_x^\beta u_h(x),\ L_p(R_n^+) \| \\ &\le c \big(\| f(x),\ L_p(R_n^+) \| + \| (1 + \langle x \rangle)^{\sigma(1-\beta\alpha)} f(x),\ L_1(R_n^+) \| \big), \quad 0 < h < 1, \end{aligned} \tag{10}$$

where the constant $c > 0$ is independent of $h \in (0,1)$ and $f(x)$; moreover,

$$\begin{aligned} &\| (1 + \langle x \rangle)^{-\sigma(1-\beta\alpha)} (D_x^\beta u_{h_1}(x) - D_x^\beta u_{h_2}(x)),\ L_p(R_n^+) \| \\ &\le \varepsilon(h_1, h_2) \big(\| f(x),\ L_p(R_n^+) \| + \| (1 + \langle x \rangle)^{\sigma(1-\beta\alpha)} f(x),\ L_1(R_n^+) \| \big) \end{aligned} \tag{11}$$

and $\varepsilon(h_1, h_2) \to 0$, $h_1, h_2 \to 0$.

PROOF. Since $\beta\alpha < 1$, we have $0 \le \beta_n < 1/\alpha_n$. Consequently,

$$D_x^\beta u_h(x) = \varphi_h^+(x) + \varphi_h^-(x),$$

where

$$\varphi_h^+(x) = \frac{1}{(2\pi)^{n-1}} \int_h^{h^{-1}} v^{-1} \int_0^{x_n} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x' - y')s) G(sv^{\alpha'}) (is)^{\beta'} D_{x_n}^{\beta_n} J_+(s, x_n - y_n) f(y)\, ds\, dy'\, dy_n\, dv,$$

$$\varphi_h^-(x) = \frac{1}{(2\pi)^{n-1}} \int_h^{h^{-1}} v^{-1} \int_{x_n}^{\infty} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x' - y')s) G(sv^{\alpha'}) (is)^{\beta'} D_{x_n}^{\beta_n} J_-(s, x_n - y_n) f(y)\, ds\, dy'\, dy_n\, dv.$$

Consider the function $\varphi_h^+(x)$. By Minkowski's inequality, we have

$$\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\varphi_h^+(x),\, L_p(R_n^+)\| \le \int_h^1 \frac{1}{v}\Big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\int_0^{x_n}\int_{R_{n-1}}\int_{R_{n-1}} \exp(i(x'-y')s)G(sv^{\alpha'})$$

$$\times (is)^{\beta'} D_{x_n}^{\beta_n} J_+(s, x_n - y_n) f(y)\, ds dy' dy_n,\, L_p(R_n^+)\Big\| dv$$

$$+\int_1^{h^{-1}} \frac{1}{v}\Big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\int_0^{x_n}\int_{R_{n-1}}\int_{R_{n-1}} \exp(i(x'-y')s)G(sv^{\alpha'})$$

$$\times (is)^{\beta'} D_{x_n}^{\beta_n} J_+(s, x_n - y_n) f(y)\, ds dy' dy_n,\, L_p(R_n^+)\Big\| dv = I_{1,h} + I_{2,h}.$$

Since $\sigma(1-\beta\alpha) \ge 0$ and $0 \le \beta\alpha < 1$, repeating the arguments in the proof of Lemma 5 in [3], we obtain

$$I_{1,h} \le c_1\|f(y),\, L_p(R_n^+)\| \tag{12}$$

with a constant c_1 independent of $f(x)$ and $h \in (0,1)$.

Estimate the summand $I_{2,h}$. On using the function

$$K_+(v, x', x_n) = \int_{R_{n-1}} \exp(ix's)G(sv^{\alpha'})(is)^{\beta'}\theta(x_n)D_{x_n}^{\beta_n}J_+(s, x_n)\, ds, \tag{13}$$

it can be rewritten as

$$I_{2,h} = \int_1^{h^{-1}} v^{-1}\Big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\int_{R_n} K_+(v, x'-y', x_n-y_n)\theta(y_n)f(y)\, dy,\, L_p(R_n^+)\Big\| dv$$

(here and in the sequel we assume $f(y', y_n) = 0$ for $y_n < 0$). By applying the estimate

$$\langle x-y\rangle(1+\langle x\rangle)^{-1} \le a(1+\langle y\rangle) \tag{14}$$

together with Minkowski's inequality and Young's inequality, we obtain

$$I_{2,h} \le a\int_1^{h^{-1}} v^{-1}\|\langle x\rangle^{-\sigma(1-\beta\alpha)}K_+(v,x',x_n),\, L_p(R_n^+)\|\, dv\|(1+\langle y\rangle)^{\sigma(1-\beta\alpha)}\theta(y_n)f(y)\, dy,\, L_1(R_n)\|.$$

Involving the equality

$$K_+(v, x', x_n) = v^{1-\beta\alpha-|\alpha|}K_+(1, x'v^{-\alpha'}, x_nv^{-\alpha_n}), \tag{15}$$

we rewrite the preceding inequality as follows:

$$I_{2,h} \le a\int_1^{h^{-1}} v^{-|\alpha|/p'-\beta\alpha-\sigma(1-\beta\alpha)}\, dv\|\langle z\rangle^{-\sigma(1-\beta\alpha)}$$

$$\times K_+(1, z', z_n),\, L_p(R_n^+)\|\|(1+\langle y\rangle)^{\sigma(1-\beta\alpha)}f(y),\, L_1(R_n^+)\|.$$

Since $|\alpha|/p > \sigma(1-\beta\alpha)$, from the definition (13) of the function $K_+(v, x'x_n)$ and Lemma 1 of §2 it follows that the first norm is finite. Recalling also that $1 \geq \sigma > 1 - |\alpha|/p'$ and $\beta\alpha \geq 0$, we have $|\alpha|/p' + \beta\alpha + \sigma(1-\beta\alpha) > 1$. Consequently, for $h \in (0,1)$, we obtain the inequality

$$I_{2,h} \leq c_2 \big\| (1+\langle x\rangle)^{\sigma(1-\beta\alpha)} f(x),\ L_1(R_n^+) \big\| \tag{16}$$

with a constant $c_2 > 0$ independent of $f(x)$ and h.

From estimates (12) and (16) we infer the estimate

$$\big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\varphi_h^+(x),\ L_p(R_n^+)\big\| \leq c_1 \big\|f(x),\ L_p(R_n^+)\big\| + c_2\big\|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)} f(x),\ L_1(R_n^+)\big\|.$$

In exactly the same way one can establish the inequality

$$\big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\varphi_h^-(x),\ L_p(R_n^+)\big\| \leq c\big(\big\|f(x),\ L_p(R_n^+)\big\| + \big\|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)} f(x),\ L_1(R_n^+)\big\|\big)$$

with a constant $c > 0$ independent of $f(x)$ and h.

The estimates written down yield inequality (10). Inequality (11) can be proven analogously. The lemma is proven.

Consider the function $\chi(s)$, $\chi(s) \in C^\infty(\overline{R}_1^+)$, $0 \leq \chi(s) \leq 1$,

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq 1, \\ 0 & \text{for } s \geq 2. \end{cases}$$

Lemma 3. *Assume that the conditions of Theorem 1 are satisfied. Then, for every $h \in (0,1)$ and $1 \geq \beta\alpha \geq 0$, the limit relation*

$$\left\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)} D_x^\beta\left(u_h(x) - u_h(x)\chi\left(\frac{\langle x\rangle^2}{\rho^2}\right)\right),\ L_p(R_n^+)\right\| \to 0 \tag{17}$$

holds as $\rho \to \infty$.

The lemma can be proven in exactly the same manner as Lemma 3 in [1].

From Lemmas 1–3 it follows that, for every $f(x) \in L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+)$, the function $u_h(x) = (R_h^+ + R_h^-)f(x)$ belongs to the space $W_{p,\sigma}^r(R_n^+)$; moreover, the estimate

$$\big\|u_h(x), W_{p,\sigma}^r(R_n^+)\big\| \leq c\big(\big\|f(x),\ L_p(R_n^+)\big\| + \big\|(1+\langle x\rangle)^\sigma f(x),\ L_1(R_n^+)\big\|\big)$$

holds with a constant $c > 0$ independent of $h \in (0,1)$ and $f(x)$, and

$$\big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\big(D_x^\beta u_{h_1}(x) - D_x^\beta u_{h_2}(x)\big),\ L_p(R_n^+)\big\| \to 0$$

as $h_1, h_2 \to 0$. Consequently, the operator family $(R_h^+ + R_h^-)$ is fundamental in the pair of spaces $\{L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+), W_{p,\sigma}^r(R_n^+)\}$ as $h \to 0$. Theorem 1 is proven.

The proof of Theorem 2 can be carried out by the scheme exposed above. We expatiate on major differences.

Recall that the assertion of Lemma 1 is valid for every $|\alpha| > 0$. We formulate an analog of Lemma 2.

Lemma 2°. *Let the conditions of Lemma 2 be satisfied. Then, for every $f(x) \in \mathcal{L}_{p,\sigma,N}(R_n^+)$ and $1 > \beta\alpha \geq 0$, the estimate*

$$\begin{aligned} &\big\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)} D_x^\beta u_h(x),\ L_p(R_n^+)\big\| \\ &\leq c\big(\big\|f(x),\ L_p(R_n^+)\big\| + \big\|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)+N|\alpha|} f(x), L_1(R_n^+)\big\|\big),\ 0 < h < 1, \end{aligned} \tag{18}$$

holds for the function $u_h(x) = (R_h^+ + R_h^-)f(x)$, *where the constant* $c > 0$ *is independent of* $h \in (0,1)$ *and* $f(x)$; *moreover,*

$$\begin{aligned}&\left\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}\left(D_x^\beta u_{h_1}(x) - D_x^\beta u_{h_2}(x)\right),\ L_p(R_n^+)\right\| \\ &\le \varepsilon(h_1,h_2)\left(\left\|f(x),\ L_p(R_n^+)\right\| + \left\|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)+N|\alpha|}f(x),\ L_1(R_n^+)\right\|\right)\end{aligned} \tag{19}$$

and $\varepsilon(h_1,h_2) \to 0$, $h_1, h_2 \to 0$.

PROOF. From the arguments given in the proof of Lemma 2, it is seen that the main difficulty in proving estimates (18) and (19) relates to the case in which all β_i's equal zero. Consider the case in more detail.

Assume $N = 1$ under the conditions of Theorem 2, i.e., assume

$$1 \ge |\alpha| > 1 - \alpha_{\min}, \quad 1 - |a|/p' \ge \sigma > 1 - |\alpha|/p' - \alpha_{\min}, \quad |\alpha|/p > \sigma,$$
$$(1+\langle x\rangle)^{\sigma+|\alpha|}f(x) \in L_1(R_n^+), \quad \int\limits_{R_n^+} f(x)\,dx = 0.$$

Then

$$\int\limits_0^\infty \hat{f}(s,x_n)\big|_{s=0}\,dx_n = 0. \tag{20}$$

Represent the function $u_h(x)$ as follows:

$$u_h(x) = \left(\frac{1}{(2\pi)^{n-1}}\int\limits_1^{h^{-1}}\frac{1}{v}\int\limits_0^{x_n}\int\limits_{R_{n-1}}\int\limits_{R_{n-1}} \exp(i(x'-y')s)G(sv^{\alpha'})J_+(s,x_n-y_n)f(y)\,ds\,dy'\,dy_n\,dv\right.$$
$$\left.+\frac{1}{(2\pi)^{n-1}}\int\limits_1^{h^{-1}}\frac{1}{v}\int\limits_{x_n}^{\infty}\int\limits_{R_{n-1}}\int\limits_{R_{n-1}} \exp(i(x'-y')s)G(sv^{\alpha'})J_-(s,x_n-y_n)f(y)\,ds\,dy'\,dy_n\,dv\right)$$
$$+\left(\frac{1}{(2\pi)^{n-1}}\int\limits_h^{1}\frac{1}{v}\int\limits_0^{x_n}\int\limits_{R_{n-1}}\int\limits_{R_{n-1}} \exp(i(x'-y')s)G(sv^{\alpha'})J_+(s,x_n-y_n)f(y)\,ds\,dy'\,dy_n\,dv\right.$$
$$\left.+\frac{1}{(2\pi)^{n-1}}\int\limits_h^{1}\frac{1}{v}\int\limits_{x_n}^{\infty}\int\limits_{R_{n-1}}\int\limits_{R_{n-1}} \exp(i(x'-y')s)G(sv^{\alpha'})J_-(s,x_n-y_n)f(y)\,ds\,dy'\,dy_n\,dv\right) = \psi_h^1(x) + \psi_h^2(x).$$

Consider the function $\psi_h^1(x)$. Using the Heaviside function, rewrite the former as

$$\psi_h^1(x) = \frac{1}{(2\pi)^{(n-1)/2}}\int\limits_1^{h^{-1}}\frac{1}{v}\int\limits_{R_{n-1}} \exp(ix's)G(sv^{\alpha'})\left(\int\limits_{R_1}\left(\theta(x_n-y_n)J_+(s,x_n-y_n)\right.\right.$$
$$\left.\left.+\theta(y_n-x_n)J_-(s,x_n-y_n)\right)\theta(y_n)\hat{f}(s,y_n)\,dy_n\right)ds\,dv.$$

By the formula for the Fourier transform of convolution and Lemma 3 of §2, the preceding equality can be rewritten as

$$\psi_h^1(x) = \frac{1}{(2\pi)^{n/2}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_n} \exp(ix\xi) G(sv^{\alpha'})(L(is, i\xi_n))^{-1} \hat{f}_\theta(s, \xi_n)\, d\xi dv, \ \xi = (s, \xi_n).$$

Taking condition (20) into account, by Hadamard's lemma we have

$$\psi_h^1(x) = \frac{1}{(2\pi)^{n/2}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_n} \exp(ix\xi) G(sv^{\alpha'})(L(is, i\xi_n))^{-1} \left(\hat{f}_\theta(s, \xi_n) - \hat{f}_\theta(0, 0)\right) d\xi dv$$

$$= \sum_{k=1}^{n} \frac{1}{(2\pi)^n} \int_0^1 \int_1^{h^{-1}} \frac{1}{v} \int_{R_n} \int_{R_n} \exp(i(x - \lambda y)\xi) G(sv^{\alpha'})(-i\xi_k)(L(is, i\xi_n))^{-1} y_k \theta(y_n) f(y)\, d\xi dy dv d\lambda.$$

Grounding on the representation obtained, we estimate the function $\psi_h^1(x)$.
Applying Minkowski's inequality, Young's inequality, and an estimate of the form (14),

$$\langle x - \lambda y\rangle (1 + \langle x\rangle)^{-1} \leq a(1 + \langle \lambda y\rangle),$$

we obtain

$$\left\|(1 + \langle x\rangle)^{-\sigma} \psi_h^1(x),\ L_p(R_n^+)\right\| \leq \sum_{k=1}^{n} \frac{1}{(2\pi)^n} \int_1^{h^{-1}} v^{-|\alpha|/p' - \sigma - \alpha_k}\, dv$$

$$\times \left\| \langle x\rangle^{-\sigma} \int_{R_n} \exp(i(x's + x_n\xi_n)) G(s) \frac{(-i\xi_k)}{L(is, i\xi_n)}\, d\xi,\ L_p(R_n^+)\right\| \left\|(1 + \langle y\rangle)^{\sigma + |\alpha|} f(y),\ L_1(R_n^+)\right\|.$$

We have $|\alpha|/p' + \alpha_{\min} + \sigma > 1$; therefore, for proving the estimate

$$\left\|(1 + \langle x\rangle)^{-\sigma} \psi_h^1(x),\ L_p(R_n^+)\right\| \leq c\left\|(1 + \langle x\rangle)^{\sigma + |\alpha|} f(x),\ L_1(R_n^+)\right\| \tag{21}$$

with some constant $c > 0$ independent of $f(x)$, $h \in (0, 1)$, it suffices to demonstrate that

$$B_k = \left\| \langle x\rangle^{-\sigma} \int_{R_n} \exp(i(x's + x_n\xi_n)) G(s) \frac{(-i\xi_k)}{L(is, i\xi_n)}\, d\xi,\ L_p(R_n)\right\| \leq c_k < \infty.$$

To this end we use Lemma 4 of §2. For instance, for $k = 1$ it implies

$$B_1 = 2\pi \left\| \langle x\rangle^{-\sigma} \int_{R_{n-1}} \exp(ix's) G(s) s_1 (\theta(x_n) J_+(s, x_n) + \theta(-x_n) J_-(s, x_n))\, ds,\ L_p(R_n^+)\right\|.$$

Hence,

$$B_1 \leq \sum_{|\beta| + |\gamma| \leq 2k} c_{\beta,\gamma} \Bigg\| \langle x\rangle^{-\sigma} (1 + |x'|^{2k})^{-1} \int_{R_{n-1}} \left|D_s^\beta (G(s) s_1)\right|$$

$$\times \left|D_s^\gamma (\theta(x_n) J_+(s, x_n) + \theta(-x_n) J_-(s, x_n))\right| ds,\ L_p(R_n^+)\Bigg\|.$$

Taking note of the estimates for contour integrals given by in Lemma 1 of §2, we obtain

$$B_1 \le \sum_{|\beta|+|\gamma|\le 2k} c'_{\beta,\gamma} \Big\| \langle x\rangle^{-\sigma}(1+|x'|^{2k})^{-1} \int\limits_{R_{n-1}} \big|D_s^\beta(G(s)s_1)\big| \langle s\rangle^{\alpha_n-\gamma\alpha-1} \exp(-\delta x_n \langle s\rangle^{\alpha_n})\, ds,\ L_p(R_n^+)\Big\|$$

$$\le \sum_{|\beta|+|\gamma|\le 2k} c_{\beta,\gamma} \Big\| \langle x\rangle^{-\sigma}(1+|x'|^{2k})^{-1}(1+|x_n|)^{-1} \int\limits_{R_{n-1}} \big|D_s^\beta(G(s)s_1)\big| \langle s\rangle^{-\gamma\alpha-1}(1+\langle s\rangle^{\alpha_n})\, ds,\ L_p(R_n^+)\Big\|.$$

Letting $k \ge [(n-1)/2p]+1$, we obtain $B_1 \le c_1 < \infty$ from the definition of the kernel $G(s)$, since $|\alpha|/p > \sigma$. In exactly the same fashion we can estimate the other norms B_k. Inequality (21) is thus established.

Consider the function $\psi_h^2(x)$. Since $\sigma \ge 0$, arguing as in the proof of Lemma 5 of [3], we can easily demonstrate that the estimate

$$\big\|(1+\langle x\rangle)^{-\sigma}\psi_h^2(x),\ L_p(R_n^+)\big\| \le c\big\|f(x),\ L_p(R_n^+)\big\| \tag{22}$$

holds with a constant $c > 0$ independent of $f(x)$ and $h \in (0,1)$.

From inequalities (21) and (22), estimate (18) is straightforward in the case when $\beta\alpha = 0$ and $N = 1$. Inequality (19) can be proven in the same way.

Observe that, in the case considered, the orthogonality condition (5) written down in the form (20) is essentially used in estimating the function $\psi_h^1(x)$, whereas the condition is not required in estimating $\psi_h^2(x)$ (cf. [3]). Therefore, while dealing with the general case $N \ge 2$, we are to represent the function $\hat{f}_\theta(s,\xi_n)$ as

$$\hat{f}_\theta(s,\xi_n) = \frac{1}{(2\pi)^{n/2}} \int_0^1 \cdots \int_0^1 \Big(\int\limits_{R_n} \exp(-i\lambda_1 \ldots \lambda_N(y's + y_n\xi_n))$$
$$\times(-iy's - iy_n\xi_n)^N \theta(y_n) f(y)\, dy \Big) \lambda_1^{N-1} \ldots \lambda_{N-2}^2 \lambda_{N-1}\, d\lambda_1 \ldots d\lambda_N, \tag{23}$$

by using Hadamard's lemma and afterward repeating the above arguments for $\psi_h^1(x)$. We omit these easy calculations. The lemma is proven.

We also have an analog of Lemma 3:

Lemma 3°. *Assume that the conditions of Theorem 2 are satisfied. Then, for every $h \in (0,1)$ and $1 \ge \beta\alpha \ge 0$, the limit relation* (17) *holds as $\rho \to \infty$.*

The lemma can be proven in exactly the same way as Lemma 3 in [1].

From Lemmas 1, 2°, and 3° it follows that, for every $f(x) \in \mathcal{L}_{p,\sigma,N}(R_n^+)$, the function $u_h(x) = (R_h^+ + R_h^-)f(x)$ belongs to the space $W_{p,\sigma}^r(R_n^+)$; moreover, the estimate

$$\big\|u_h(x),\ W_{p,\sigma}^r(R_n^+)\big\| \le c\Big(\big\|f(x),\ L_p(R_n^+)\big\| + \big\|(1+\langle x\rangle)^{\sigma+N|\alpha|} f(x),\ L_1(R_n^+)\big\|\Big)$$

holds with a constant $c > 0$ independent of $h \in (0,1)$ and $f(x)$, and

$$\big\|u_{h_1}(x) - u_{h_2}(x),\ W_{p,\sigma}^r(R_n^+)\big\| \to 0$$

as $h_1, h_2 \to 0$. Consequently, the operator family $(R_h^+ + R_h^-)$ is fundamental in the pair of spaces $\{\mathcal{L}_{p,\sigma,N}(R_n^+), W_{p,\sigma}^r(R_n^+)\}$ as $h \to 0$. Theorem 2 is proven.

§4. Proofs of the Theorems Stating that the Operator Family $R_{j,h}$ is Fundamental as $h \to 0$

In this section we prove Theorems 3 and 4.

The proof of Theorem 3 is carried out by the same scheme as in §3; we divide it into three lemmas. In the first lemma we give an estimate for the highest derivatives $D_x^{\beta} R_{j,h} f(x)$, $\beta\alpha = 1$, in the L_p-norm; in the second we estimate all derivatives with $\beta\alpha < 1$ in the corresponding weighted norms; and in the third we demonstrate that the function $u_h(x) = R_{j,h} f(x)$ can be approximated in the norm of $W^r_{p,\sigma}(R_n^+)$ by functions vanishing at $|x| \gg 0$.

Lemma 1. *Let $|\alpha| > 0$, $f(x) \in L_p(R_n^+) \cap L_1(R_n^+)$, and $u_h(x) = R_{j,h} f(x)$. Then*

$$\left\| D_x^{\beta} u_h(x),\, L_p(R_n^+) \right\| \le c \left\| f(x),\, L_p(R_n^+) \right\|, \quad \beta\alpha = 1, \quad 0 < h < 1,$$

with some constant $c > 0$ independent of h and $f(x)$; moreover,

$$\left\| D_x^{\beta} u_{h_1}(x) - D_x^{\beta} u_{h_2}(x),\, L_p(R_n^+) \right\| \to 0, \quad h_1, h_2 \to 0.$$

A proof of the lemma for $\beta = (1/\alpha_1, \dots, 1/\alpha_n)$ and $f(x)$ compactly-supported is contained in [3] (see Lemma 4). The general case can be considered along the same lines by using Lemma 9 of §2.

Lemma 2. *Let $f(x) \in L_p(R_n^+)$, $(1 + \langle x \rangle)^{\sigma} f(x) \in L_1(R_n^+)$, $u_h(x) = R_{j,h} f(x)$ and $|\alpha| > 1$, $|\alpha|/p > \sigma > 1 - |\alpha|/p'$. Then, for $1 > \beta\alpha \ge 0$, the following estimate is valid:*

$$\begin{gathered} \left\| (1 + \langle x \rangle)^{-\sigma(1-\beta\alpha)} D_x^{\beta} u_h(x),\, L_p(R_n^+) \right\| \\ \le c\left(\left\| f(x),\, L_p(R_n^+) \right\| + \left\| (1 + \langle x \rangle)^{\sigma(1-\beta\alpha)} f(x),\, L_1(R_n^+) \right\| \right), \quad 0 < h < 1, \end{gathered} \tag{24}$$

where the constant $c > 0$ is independent of $h \in (0,1)$ and $f(x)$; moreover,

$$\begin{gathered} \left\| (1 + \langle x \rangle)^{-\sigma(1-\beta\alpha)} \left(D_x^{\beta} u_{h_1}(x) - u_{h_2}(x) \right),\, L_p(R_n^+) \right\| \\ \le \varepsilon(h_1, h_2) \left(\left\| f(x),\, L_p(R_n^+) \right\| + \left\| (1 + \langle x \rangle)^{\sigma(1-\beta\alpha)} f(x),\, L_1(R_n^+) \right\| \right) \end{gathered} \tag{25}$$

and $\varepsilon(h_1, h_2) \to 0$, $h_1, h_2 \to 0$.

PROOF. Represent $D_x^{\beta} u_h(x)$ as

$$\begin{aligned} D_x^{\beta} u_h(x) = {} & \frac{1}{(2\pi)^{n-1}} \int_h^1 \frac{1}{v} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x' - y')s) G(s v^{\alpha'}) \\ & \times (is)^{\beta'} D_{x_n}^{\beta_n} J_j(s, x_n) \int_0^{\infty} I_j(s, y_n) f(y', y_n)\, dy_n ds dy' dv \\ & + \frac{1}{(2\pi)^{n-1}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_{n-1}} \int_{R_{n-1}} \exp(i(x' - y')s) G(s v^{\alpha'}) (is)^{\beta'} \\ & \times D_{x_n}^{\beta_n} J_j(s, x_n) \int_0^{\infty} I_j(s, y_n) f(y', y_n)\, dy_n ds dy' dv = F_{1,h}(x) + F_{2,h}(x). \end{aligned} \tag{26}$$

Consider the function $F_{1,h}(x)$. Since $\sigma(1-\beta\alpha)\geq 0$ and $0\leq\beta\alpha<1$, by repeating the arguments in the proof of Lemma 6 of [3], we obtain

$$\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}F_{1,h}(x),\, L_p(R_n^+)\|\leq c_1\|f(x),\, L_p(R_n^+)\|, \tag{27}$$

with some constant $c_1>0$ independent of $f(x)$ and $h\in(0,1)$.

Estimate the function $F_{2,h}(x)$. Since $\sigma(1-\beta\alpha)\geq 0$, from inequality (14) we obtain

$$\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}F_{2,h}(x),\, L_p(R_n^+)\|\leq a\int\limits_1^{h^{-1}}\frac{1}{v}\Big\|\int\limits_{R_n^+}\Big|\langle x-y\rangle^{-\sigma(1-\beta\alpha)}\int\limits_{R_{n-1}}\exp(i(x'-y')s)$$
$$\times G(sv^{\alpha'})(is)^{\beta'}D_{x_n}^{\beta_n}J_j(s,x_n)I_j(s,y_n)\,ds\Big|(1+\langle y\rangle)^{\sigma(1-\beta\alpha)}|f(y)|\,dy,\, L_p(R_n^+)\Big\|\,dv$$
$$=a\int\limits_1^{h^{-1}}\frac{1}{v}\Big\|\int\limits_{R_n^+}K_j(v,x'-y',x_n,y_n)((1+\langle y\rangle)^{\sigma(1-\beta\alpha)}$$
$$\times|f(y)|)^{1/p}((1+\langle y\rangle)^{\sigma(1-\beta\alpha)}|f(y)|)^{1/p'}\,dy,\, L_p(R_n^+)\Big\|\,dv.$$

Applying Young's inequality, we obtain the estimate

$$\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}F_{2,h}(x),\, L_p(R_n^+)\|$$
$$\leq a\int\limits_1^{h^{-1}}\frac{1}{v}\bigg(\Big\|\int\limits_{R_n^+}|K_j(v,x'-y',x_n,y_n)|^p(1+\langle y\rangle)^{\sigma(1-\beta\alpha)}|f(y)|\,dy\bigg)^{1/p},\, L_p(R_n^+)\Big\|\,dv$$
$$\times\|(1+\langle z\rangle)^{\sigma(1-\beta\alpha)}f(z),\, L_1(R_n^+)\|^{1/p'}.$$

By the Tonelli theorem, the inequality can be rewritten as

$$\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}F_{2,h}(x),\, L_p(R_n^+)\|\leq a\int\limits_1^{h^{-1}}\frac{1}{v}\bigg(\int\limits_{R_n^+}|f(y)|$$
$$\times(1+\langle y\rangle)^{\sigma(1-\beta\alpha)}\bigg(\int\limits_{R_n^+}|K_j(v,x'-y',x_n,y_n)|^p\,dx\bigg)\,dy\bigg)^{1/p}dv\|(1+\langle z\rangle)^{\sigma(1-\beta\alpha)}f(z),\, L_1(R_n^+)\|^{1/p'}$$
$$=a\int\limits_1^{h^{-1}}\frac{1}{v}\bigg(\int\limits_{R_n^+}(1+\langle y\rangle)^{\sigma(1-\beta\alpha)}|f(y)|A_j(v,y)\,dy\bigg)^{1/p}dv\|(1+\langle z\rangle)^{\sigma(1-\beta\alpha)}f(z),\, L_1(R_n^+)\|^{1/p'}. \tag{28}$$

Estimate the function $A_j(v,y)$. By definition, it is obvious that

$$A_j(v,y) = \int\limits_{R_n} |K_j(v,x'-y',x_n,y_n)|^p\theta(x_n)\theta(y_n)\,dx = \int\limits_{R_n} |K_j(v,z',z_n+y_n,y_n)|^p\theta(z_n+y_n)\theta(y_n)\,dz$$

$$= \int\limits_{R_n} \langle z\rangle^{-p\sigma(1-\beta\alpha)}\left|\int\limits_{R_{n-1}} \exp(iz's)G(sv^{\alpha'})(is)^{\beta'}D_{z_n}^{\beta_n}J_j(s,z_n+y_n)\theta(z_n+y_n)\theta(y_n)I_j(s,y_n)\,ds\right|^p dz$$

$$= v^{(-|\alpha|/p'+(1-\sigma)(1-\beta\alpha))p}\int\limits_{R_n}\langle x\rangle^{-p\sigma(1-\beta\alpha)}\left|\int\limits_{R_{n-1}}\exp(ix'\xi)\right.$$

$$\left.\times G(\xi)(i\xi)^{\beta'}D_{x_n}^{\beta_n}J_j(\xi,x_n+y_nv^{-\alpha_n})\theta(x_nv^{\alpha_n}+y_n)\theta(y_n)I_j(\xi,y_nv^{-\alpha_n})\,ds\right|^p dx.$$

Put $\Delta = -|\alpha|/p' + (1-\sigma)(1-\beta\alpha)$ and represent $A_j(v,y)$ as

$$A_j(v,y) = v^{\Delta p}\int\limits_{R_n}\langle x\rangle^{-p\sigma(1-\beta\alpha)}(1+|x'|^{2k})^{-p}\left|\int\limits_{R_{n-1}} (((1+(-1)^k\Delta^k)\exp(ix'\xi))G(\xi)\right.$$

$$\left.\times(i\xi)^{\beta'}D_{x_n}^{\beta_n}J_j(\xi,x_n+y_nv^{-\alpha_n})\theta(x_nv^{\alpha_n}+y_n)\theta(y_n)I_j(\xi,y_nv^{-\alpha_n})\,ds\right|^p dx$$

$$\le \sum_{|\nu|+|\gamma|\le 2k} c_{\nu,\gamma}v^{\Delta p}\int\limits_{R_n}\langle x\rangle^{-p\sigma(1-\beta\alpha)}(1+|x'|^{2k})^{-p}$$

$$\times\left|\int\limits_{R_{n-1}} |D_\xi^\nu G(\xi)|\left|D_\xi^\gamma\left(\xi^{\beta'}D_{x_n}^{\beta_n}J_j(\xi,x_n+y_nv^{-\alpha_n})\theta(x_nv^{\alpha_n}+y_n)\theta(y_n)I_j(\xi,y_nv^{-\alpha_n})\right)\right|ds\right|^p dx.$$

Now, by Lemma 2 of §2, we obtain

$$A_j(v,y) \le \sum_{|\nu|+|\gamma|\le 2k} c'_{\nu,\gamma}v^{\Delta p}\int\limits_{R_n}\langle x\rangle^{-p\sigma(1-\beta\alpha)}(1+|x'|^{2k})^{-p}$$

$$\times\left|\int\limits_{R_{n-1}} |D_\xi^\nu G(\xi)|\langle\xi\rangle^{\beta\alpha+\alpha_n-1}\exp(-\delta(x_n+y_nv^{-\alpha_n})\langle\xi\rangle^{\alpha_n})\right.$$

$$\left.\times\theta(x_nv^{\alpha_n}+y_n)\theta(y_n)\exp(-\delta y_nv^{-\alpha_n}\langle\xi\rangle^{\alpha_n})\,ds\right|^p dx.$$

Since $|\alpha|/p > \sigma$, we have

$$A_j(v,y) \le cv^{\Delta p}\theta(y_n) \tag{29}$$

with some constant $c > 0$ independent of v, y. Inserting (29) into (28), we obtain

$$\left\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}F_{2,h}(x),\, L_p(R_n^+)\right\| \le ac^{1/p}\int\limits_1^{h^{-1}} v^{-1+\Delta}\,dv\left\|(1+\langle z\rangle)^{\sigma(1-\beta\alpha)}f(z),\, L_1(R_n^+)\right\|.$$

By the conditions of the lemma, we have $\Delta < 0$. Therefore,

$$\left\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}F_{2,h}(x),\, L_p(R_n^+)\right\| \le c_2\left\|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)}f(x),\, L_1(R_n^+)\right\| \tag{30}$$

for $h\in(0,1)$, where the constant $c_2 > 0$ is independent of $f(x)$ and h.

Estimates (27) and (30) yield (24). Inequality (25) can be established analogously. The lemma is proven.

Lemma 3. *Assume that the conditions of Theorem 3 are satisfied. Then, for every $h \in (0,1)$ and $1 \geq \beta\alpha \geq 0$, the limit relation (17) holds as $\rho \to \infty$.*

The lemma can be proven in exactly the same way as Lemma 3 in [1].

From Lemmas 1–3 it follows that, for every function $f(x) \in L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+)$, the function $u_h(x) = R_{j,h}f(x)$ belongs to $W_{p,\sigma}^{r}(R_n^+)$; moreover, the following estimate is valid:

$$\|u_h(x),\, W_{p,\sigma}^{r}(R_n^+)\| \leq c(\|f(x),\, L_p(R_n^+)\| + \|(1+\langle x\rangle)^{\sigma} f(x),\, L_1(R_n^+)\|),$$

with a constant $c > 0$ independent of $h \in (0,1)$ and $f(x)$, and

$$\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}(D_x^{\beta}u_{h_1}(x) - D_x^{\beta}u_{h_2}(x)),\, L_p(R_n^+)\| \to 0$$

as $h_1, h_2 \to 0$. Consequently, the operator family $R_{j,h}$, $j = 1, \dots, \mu$, is fundamental in the pair of spaces $\{L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+),\, W_{p,\sigma}^{r}(R_n^+)\}$ as $h \to 0$. Theorem 3 is proven.

The proof of Theorem 4 can be carried out by the scheme described above. We expatiate on major differences.

Recall that the assertion of Lemma 1 is valid for every $|\alpha| > 0$. We formulate an analog of Lemma 2:

Lemma 2°. *Let the conditions of Theorem 4 be satisfied. Then, for every $f(x) \in \mathcal{L}_{p,\sigma,N}(R_n^+)$ and $1 > \beta\alpha \geq 0$, the estimate*

$$\begin{aligned}&\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)} D_x^{\beta} u_h(x),\, L_p(R_n^+)\| \\ &\leq c(\|f(x),\, L_p(R_n^+)\| + \|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)+N|\alpha|} f(x),\, L_1(R_n^+)\|),\quad 0 < h < 1,\end{aligned} \tag{31}$$

holds for the function $u_h(x) = R_{j,h}f(x)$, where the constant $c > 0$ is independent of $h \in (0,1)$ and $f(x)$; moreover,

$$\begin{aligned}&\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}(D_x^{\beta}u_{h_1}(x) - D_x^{\beta}u_{h_2}(x)),\, L_p(R_n^+)\| \\ &\leq \varepsilon(h_1,h_2)(\|f(x),\, L_p(R_n^+)\| + \|(1+\langle x\rangle)^{\sigma(1-\beta\alpha)+N|\alpha|} f(x),\, L_1(R_n^+)\|)\end{aligned} \tag{32}$$

and $\varepsilon(h_1,h_2) \to 0$, $h_1, h_2 \to 0$.

PROOF. From the arguments presented while we prove Lemma 2 it is seen that the main difficulty in demonstrating estimates (30) and (31) is in the case $\beta\alpha = 0$. We analyze this case in more detail.

Assume $N = 1$ in Theorem 4, i.e., assume similarly as in the proof of Theorem 2 that equality (20) holds for the function $f(x)$.

While proving the preceding lemma, we expressed the function $u_h(x)$ in the form (26), where $\beta = (0, \dots, 0)$, i.e.,

$$u_h(x) = F_{1,h}(x) + F_{2,h}(x).$$

From the arguments presented while we estimate the summands $F_{1,h}(x)$ and $F_{2,h}(x)$ it follows that inequality (27) is satisfied for all $N \geq 0$. Consequently, to prove estimate (31) for $N = 1$ and $\beta\alpha = 0$ it suffices to establish the inequality

$$\|(1+\langle x\rangle)^{-\sigma} F_{2,h}(x),\, L_p(R_n^+)\| \leq c\|(1+\langle x\rangle)^{\sigma+|\alpha|} f(x), L_1(R_n^+)\| \tag{33}$$

with some constant $c > 0$ independent of $f(x)$ and $h \in (0,1)$.

By definition, the function $F_{2,h}(x)$ can be written down as

$$F_{2,h}(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_{n-1}} \exp(ix's) G(sv^{\alpha'}) J_j(s, x_n) \int_0^{\infty} I_j(s, y_n) \hat{f}(s, y_n)\, dy_n\, ds\, dv$$

and, on account of (20), as

$$f : U \to \mathbb{R}^n F_{2,h}(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_{n-1}} \exp(ix's)G(sv^{\alpha'})$$

$$\times J_j(s, x_n) \int_0^\infty I_j(s, y_n)(\hat{f}(s, y_n) - \hat{f}(0, y_n))\, dy_n ds dv$$

$$+\frac{1}{(2\pi)^{(n-1)/2}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_{n-1}} \exp(ix's)G(sv^{\alpha'})J_j(s, x_n)$$

$$\times \int_0^\infty (I_j(s, y_n) - I_j(s, 0))\hat{f}(0, y_n)\, dy_n ds dv = F^1_{2,h}(x) + F^2_{2,h}(x).$$

First, we consider the function $F^2_{2,h}(x)$. Represent it as follows:

$$F^2_{2,h}(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_1^{h^{-1}} \frac{1}{v} \int_{R_{n-1}} \exp(ix's)G(sv^{\alpha'})J_j(s, x_n)$$

$$\times \int_0^\infty \int_0^1 D_{z_n} I_j(s, z_n)\big|_{z_n=\lambda y_n} y_n \hat{f}(0, y_n)\, d\lambda dy_n ds dv$$

$$= \frac{1}{(2\pi)^{(n-1)/2}} \int_1^{h^{-1}} \frac{1}{v^{|\alpha|+\alpha_n}} \int_{R_{n-1}} \exp\left(i\frac{x'}{v^{\alpha'}}\xi\right) G(\xi)J_j(\xi, x_n v^{-\alpha_n})$$

$$\times \int_0^\infty \int_0^1 D_{z_n} I_j(\xi, z_n)\big|_{z_n=\lambda y_n v^{-\alpha_n}} y_n \hat{f}(0, y_n)\, d\lambda dy_n ds dv$$

(in the last equality we use the substitution $\xi_k = s_k v^{\alpha_k}$, $k = 1, \ldots, n-1$). Estimate the function $F^2_{2,h}(x)$. Since $\sigma \geq 0$, we obviously have

$$\|(1 + \langle x\rangle)^{-\sigma} F^2_{2,h}(x),\, L_p(R^+_n)\| \leq \frac{1}{(2\pi)^{n-1}} \int_0^1 \int_1^{h^{-1}} v^{-|\alpha|/p'-\alpha_n-\sigma}$$

$$\times \left\| \int_{R^+_n} \langle x\rangle^{-\sigma} \left| \int_{R_{n-1}} \exp(ix'\xi)G(\xi)J_j(\xi, x_n)D_{z_n} I_j(\xi, z_n)\big|_{z_n=\lambda y_n v^{-\alpha_n}}\, d\xi \right| |y_n f(y)|\, dy,\, L_p(R^+_n) \right\| dv d\lambda.$$

Represent the function $|y_n f(y)|$ as

$$|y_n f(y)| = |y_n f(y)|^{1/p} |y_n f(y)|^{1/p'}.$$

By using Hölder's inequality, we obtain

$$\left\|(1+\langle x\rangle)^{-\sigma}F^2_{2,h}(x),\ L_p(R^+_n)\right\| \le \frac{1}{(2\pi)^{n-1}}\int\limits_0^1\int\limits_1^{h^{-1}} v^{-|\alpha|/p'-\alpha_n-\sigma}$$
$$\times\Bigg(\int\limits_{R^+_n}\int\limits_{R^+_n}\langle x\rangle^{-p\sigma}\Bigg|\int\limits_{R_{n-1}}\exp(ix'\xi)G(\xi)J_j(\xi,x_n)$$
$$\times D_{z_n}I_j(\xi,z_n)\big|_{z_n=\lambda y_n v^{-\alpha_n}}d\xi\Bigg|^p|y_nf(y)|\,dy\,dx\Bigg)^{1/p}dv\,d\lambda\left\|z_nf(z),\ L_1(R^+_n)\right\|^{1/p'}. \tag{34}$$

Introduce the notation

$$A_j(\lambda,v,y)=\int\limits_{R^+_n}\langle x\rangle^{-p\sigma}\Bigg|\int\limits_{R_{n-1}}\exp(ix'\xi)G(\xi)J_j(\xi,x_n)\theta(y_n)D_{z_n}I_j(\xi,z_n)\big|_{z_n=\lambda y_n v^{-\alpha_n}}\,d\xi\Bigg|^p\,dx$$

and demonstrate that the estimate

$$|A_j(\lambda,v,y)|\le c^p<\infty \tag{35}$$

holds for $\lambda>0$ and $v>0$, with a constant $c>0$ independent of λ, v, and y. Represent $A_j(\lambda,v,y)$ in the form

$$A_j(\lambda,v,y)=\int\limits_{R^+_n}\langle x\rangle^{-p\sigma}(1+|x'|^{2k})^{-p}\Bigg|\int\limits_{R_{n-1}}((1+(-1)^k\Delta^k)\exp(ix'\xi))$$
$$\times G(\xi)J_j(\xi,x_n)\theta(y_n)D_{z_n}I_j(\xi,z_n)\big|_{z_n=\lambda y_n v^{-\alpha_n}}\,d\xi\Bigg|^p\,dx.$$

Now, integrating by parts and involving the definition of the function $G(\xi)$, we obtain

$$A_j(\lambda,v,y)\le\sum_{|\nu|+|\gamma|\le 2k}c_{\nu,\gamma}\int\limits_{R^+_n}\langle x\rangle^{-p\sigma}(1+|x'|^{2k})^{-p}\theta(y_n)$$
$$\times\Bigg|\int\limits_{R_{n-1}}\left|D^\nu_\xi G(\xi)D^\gamma_\xi\left(J_j(\xi,x_n)D_{z_n}I_j(\xi,z_n)\big|_{z_n=\lambda y_n v^{-\alpha_n}}\right)\right|\,d\xi\Bigg|^p\,dx.$$

From the estimates for contour integrals indicated in Lemma 1 of §2, it follows

$$A_j(\lambda,v,y)\le\sum_{|\nu|+|\gamma|\le 2k}c'_{\nu,\gamma}\int\limits_{R^+_n}\langle x\rangle^{-p\sigma}(1+|x'|^{2k})^{-p}\theta(y_n)$$
$$\times\Bigg|\int\limits_{R_{n-1}}|D^\nu_\xi G(\xi)|\langle\xi\rangle^{\gamma\alpha+\alpha_n-1}\exp(-\delta(x_n+\lambda y_n v^{-\alpha_n})\langle\xi\rangle^{\alpha_n})\,d\xi\Bigg|^p\,dx$$
$$\le\sum_{|\nu|+|\gamma|\le 2k}c'_{\nu,\gamma}\int\limits_{R^+_n}\langle x\rangle^{-p\sigma}(1+|x'|^{2k})^{-p}\Bigg|\int\limits_{R_{n-1}}|D^\nu_\xi G(\xi)|$$
$$\times\langle\xi\rangle^{\gamma\alpha+\alpha_n-1}\exp(-\delta x_n\langle\xi\rangle^{\alpha_n})\,d\xi\Bigg|^p\,dx.$$

Since $|\alpha|/p > \sigma$, from the definition of the kernel $G(s)$ we derive estimate (35).

Inserting (35) into (34), we obtain

$$\|(1+\langle x\rangle)^{-\sigma}F^2_{2,h}(x),\ L_p(R^+_n)\| \le c\int_1^{h^{-1}} v^{-|\alpha|/p'-\alpha_n-\sigma}\,dv\|z_nf(z),\ L_1(R^+_n)\|.$$

Hence,

$$\|(1+\langle x\rangle)^{-\sigma}F^2_{2,h}(x),\ L_p(R^+_n)\| \le c^2_2\|z_nf(z),\ L_1(R^+_n)\| \tag{36}$$

with a constant $c^2_2 > 0$ independent of $f(x)$ and $h \in (0,1)$.

Consider the function $F^1_{2,h}(x)$. Using Hadamard's lemma, we write the function in the form

$$F^1_{2,h}(x) = \sum_{k=1}^{n-1}\frac{1}{(2\pi)^{n-1}}\int_0^1\int_1^{h^{-1}}\frac{1}{v}\int_{R_{n-1}}\int_{R^+_n}\exp((ix'-\lambda y')s)$$
$$\times G(sv^{\alpha'})(-is_k)J_j(s,x_n)I_j(s,y_n)y_kf(y',y_n)\,dyds dv d\lambda.$$

Repeating the calculations of the proof of estimate (30), we obtain the inequality

$$\|(1+\langle x\rangle)^{-\sigma}F^1_{2,h}(x),\ L_p(R^+_n)\| \le c^1_2\|(1+\langle x\rangle)^{\sigma+|\alpha|}f(x),\ L_1(R^+_n)\| \tag{37}$$

with a constant $c^1_2 > 0$ independent of $f(x)$ and $h \in (0,1)$.

Estimates (36) and (37) imply (33).

By virtue of representation (26) and inequalities (27) and (33), we obtain inequality (31) in the case $\beta\alpha = 0$ and $N = 1$. Inequality (32) can be proven in exactly the same manner.

The general case $N \ge 2$ can be considered similarly. The lemma is proven.

We have the following analog of Lemma 3:

Lemma 3°. *Assume that the conditions of Theorem* 4 *are satisfied. Then, for every* $h \in (0,1)$ *and* $1 \ge \beta\alpha \ge 0$, *the limit relation* (17) *holds as* $\rho \to \infty$.

The lemma can be proven as Lemma 3 in [1].

From Lemmas 1, 2°, and 3° it follows that, for every $f(x) \in \mathcal{L}_{p,\sigma,N}(R^+_n)$, the function $u_h(x) = R_{j,h}f(x)$ belongs to the space $W^r_{p,\sigma}(R^+_n)$; moreover, the following estimate is valid:

$$\|u_h(x),\ W^r_{p,\sigma}(R^+_n)\| \le c(\|f(x),\ L_p(R^+_n)\| + \|(1+\langle x\rangle)^{\sigma+N|\alpha|}f(x),\ L_1(R^+_n)\|)$$

with a constant $c > 0$ independent of $h \in (0,1)$ and $f(x)$, and

$$\|u_{h_1}(x) - u_{h_2}(x),\ W^r_{p,\sigma}(R^+_n)\| \to 0$$

as $h_1, h_2 \to 0$. Thus, the operator family $R_{j,h}$ is fundamental in the pair of spaces $\{\mathcal{L}_{p,\sigma,N}(R^+_n),\ W^r_{p,\sigma}(R^+_n)\}$ as $h \to 0$. Theorem 4 is proven.

§5. Boundary Value Problems for Quasielliptic Equations in the Half-Space

We outline the scheme of the proof of Theorems 7 and 8.

The proof of solvability of the boundary value problem (1) is based on the use of the properties of the family of integral operators R_h, $h \in (0,1)$.

From the definition of the operators R_h^+, R_h^-, and $R_{j,h}$ it follows (see [2, 3]) that, for the function $u_k(x) = R_{h_k} f(x)$, $h_k > 0$, the equalities

$$L(D_x)u_k(x) = \frac{1}{(2\pi)^{n-1}} \int\limits_{h_k}^{h_k^{-1}} \frac{1}{v^{|\alpha'|+1}} \int\limits_{R_{n-1}} \int\limits_{R_{n-1}} \exp\left(i\frac{x'-y'}{v^{\alpha'}}s\right) G(s) f(y', x_n)\, ds dy' dv,$$

$$B_j(D_x)u_k(x)\big|_{x_n=0} = 0, \quad j = 1, \dots, \mu,$$

hold. Consequently, taking (3) into account, we can consider the function $u_k(x)$ as an approximate solution to the boundary value problem (1), and the problem of existence of a solution is reduced to the proof of convergence of the sequence $\{u_k(x)\}$ as $h_k \to 0$ in the space $W^r_{p,\sigma}(R_n^+)$.

From Theorems 5 and 6 it follows that the sequence of functions $\{u_k(x)\}$ is fundamental in the space $W^r_{p,\sigma}(R_n^+)$ for all $|\alpha| > 0$. Therefore, by virtue of completeness of $W^r_{p,\sigma}(R_n^+)$, there exists a function $u(x) \in W^r_{p,\sigma}(R_n^+)$ such that

$$\big\| u_k(x) - u(x), W^r_{p,\sigma}(R_n^+) \big\| \to 0, \quad k \to \infty.$$

Moreover, if $|\alpha| > 1$ then $u(x)$ meets estimate (7), and if $|\alpha| \le 1$ then estimate (8) holds. It is clear that $u(x)$ is a solution to the boundary value problem (1). Uniqueness of the solution can be easily established (see [2, 3]).

In the case $|\alpha|/p' > 1$ Theorems 7 and 9 can be easily transferred to the case of boundary values problems for quasielliptic equations with slowly-varying continuous coefficients

$$L(x, D_x)u = \sum_{\beta\alpha=1} a_\beta(x) D_x^\beta u = f(x), \quad x \in R_n^+,$$
$$B_j(D_x)u\big|_{x_n=0} = 0, \quad j = 1, \dots, \mu. \tag{38}$$

At this juncture we assume the Lopatinskiĭ condition to be satisfied by the operator

$$\{L(x^0, D_x), S_n \circ B_1(D_x), \dots, S_n \circ B_\mu(D_x)\}$$

at any fixed point $x^0 \in R_n^+$.

For simplicity we shall assume that the coefficients $a_\beta(x)$ are constant outside some ball $\{|x| \le r\}$.

Theorem 11. *Let $|\alpha| > 1$, $|\alpha|/p > \sigma > 1 - |\alpha|/p'$, and $f(x) \in L_p(R_n^+) \cap L_{1,-\sigma}(R_n^+)$. Then there exists an $\varepsilon > 0$ such that if the coefficients of the operator $L(x, D_x)$ satisfy inequality*

$$\max_x |a_\beta(x) - a_\beta(x^0)| \le \varepsilon, \quad |x^0| \ge r, \tag{39}$$

then problem (38) has a unique solution $u(x) \in W^r_{p,\sigma}(R_n^+)$ which enjoys estimate (7).

Theorem 12. *Let $|\alpha|/p' > 1$ and $f(x) \in L_p(R_n^+) \cap L_1(R_n^+)$. Then there exists an $\varepsilon > 0$ such that if the coefficients $a_\beta(x)$ satisfy inequality (39), then the boundary value problem (38) is well-posed in $W^r_p(R_n^+)$.*

These theorems can be proven by the perturbation method.

We turn to the proof of Theorem 10. We wish to demonstrate that if

$$\psi_j(s) = \int\limits_{\Gamma(s)} \frac{B_j(is, i\lambda)}{L(is, i\lambda)}\, d\lambda \not\equiv 0, \quad s \in R_{n-1} \setminus \{0\}, \tag{40}$$

then, even in the case of compactly-supported infinitely differentiable functions $f(x)$, for the solvability of the boundary value problem (1) in $W_p^r(R_n^+)$ with

$$|\alpha|/p' + \alpha_{\min} > 1 \geq |\alpha|/p', \quad 1 < p \leq 2, \tag{41}$$

it is necessary that

$$\int\limits_{R_n^+} f(x)\, dx = 0. \tag{42}$$

We carry out the proof by way of contradiction. Assume that, for $f(x) \in C_0^\infty(R_n^+)$ not satisfying conditions (42), problem (1) has a solution $u(x) \in W_p^r(R_n^+)$, $1 < p \leq 2$. Then, by virtue of the Hausdorff-Young inequality, the following estimate is valid:

$$\big\|\|\hat{u}(s, x_n),\, L_{p'}(R_{n-1})\|,\, L_p(R_1^+)\big\| \leq c\big\|u(x),\, L_p(R_n^+)\big\|, \tag{43}$$

where $\hat{u}(s, x_n)$ is the partial Fourier transform of $u(x', x_n)$ with respect to x'.

For $s \in R_{n-1} \setminus \{0\}$, the function $\hat{u}(s, x_n)$ is a solution to the boundary value problem

$$\begin{gathered} L(is, D_{x_n})\hat{u} = \hat{f}(s, x_n), \quad x_n > 0, \\ B_j(is, D_{x_n})\hat{u}\big|_{x_n=0} = 0, \quad j = 1, \dots, \mu, \\ \hat{u} \to 0 \text{ as } x_n \to +\infty, \end{gathered}$$

and, since the Lopatinskiĭ condition is satisfied, $\hat{u}(s, x_n)$ has the representation

$$\hat{u}(s, x_n) = \int\limits_0^{x_n} J_+(s, x_n - y_n)\hat{f}(s, y_n)\, dy_n$$

$$+ \int\limits_{x_n}^{\infty} J_-(s, x_n - y_n)\hat{f}(s, y_n)\, dy_n + \sum_{j=1}^{\mu} \int\limits_0^{\infty} J_j(s, x_n) I_j(s, y_n)\hat{f}(s, y_n)\, dy_n.$$

Using the function $U(s, x_n, y_n)$ of §2, for $x_n \geq d = \operatorname{diam}(\operatorname{supp} f(x))$ we obtain

$$\hat{u}(s, x_n) = \int\limits_0^{\infty} J_+(s, x_n - y_n)\hat{f}(s, y_n)\, dy_n$$

$$+ \sum_{j=1}^{\mu} \int\limits_0^{\infty} J_j(s, x_n) I_j(s, y_n)\hat{f}(s, y_n)\, dy_n = \int\limits_0^{\infty} U(s, x_n, y_n)\hat{f}(s, y_n)\, dy_n.$$

With the help of the preceding representation, from inequality (43) we obtain

$$\left\|\left\|\int\limits_0^{\infty} U(s, x_n, y_n)\hat{f}(s, y_n)\, dy_n,\, L_{p'}(\{\langle s\rangle^{\alpha_n} < 1\})\right\|,\, L_p(\{x_n > 2d\})\right\| \leq c\big\|u(x),\, L_p(R_n^+)\big\|.$$

Consequently, by virtue of Minkowski's inequality, we have

$$\left\| \left\| \int_0^\infty U(s,x_n,0)\hat f(0,y_n)\,dy_n,\ L_{p'}(\{\langle s\rangle^{\alpha_n}<1\})\right\|,\ L_p(\{x_n>2d\})\right\|$$

$$\leq \left\| \left\| \int_0^\infty (U(s,x_n,y_n)-U(s,x_n,0))\hat f(s,y_n)\,dy_n,\ L_{p'}(\{\langle s\rangle^{\alpha_n}<1\})\right\|,\ L_p(\{x_n>2d\})\right\|$$

$$+\left\| \left\| \int_0^\infty U(s,x_n,0)(\hat f(s,y_n)-\hat f(0,y_n))\,dy_n,\ L_{p'}(\{\langle s\rangle^{\alpha_n}<1\})\right\|,\ L_p(\{x_n>2d\})\right\|$$

$$+c\big\|u(x),\ L_p(R_n^+)\big\| = F_1+F_2+c\big\|u(x),\ L_p(R_n^+)\big\|. \tag{44}$$

Prove the estimate

$$F_1+F_2\leq c(f)<\infty. \tag{45}$$

First, we consider the norm F_1. From Lemma 1 of §2, for $0<y_n<d$ and $2d<x_n$ we have

$$|U(s,x_n,y_n)-U(s,x_n,0)| = \left| y_n\int_0^1 D_zU(s,x_n,z)\big|_{z=\lambda y_n}\,d\lambda\right| \leq c_1\langle s\rangle^{2\alpha_n-1}\exp(-\delta x_n\langle s\rangle^{\alpha_n}/2).$$

Consequently,

$$F_1\leq c_1(f)\|\ \|\langle s\rangle^{2\alpha_n-1}\exp(-\delta x_n\langle s\rangle^{\alpha_n}/2),\ L_{p'}(\{\langle s\rangle^{\alpha_n}<1\})\|,\ L_p(\{x_n>2d\})\|.$$

Similarly,

$$F_2\leq c_2(f)\sum_{k=1}^{n-1}\|\ \|\langle s\rangle^{\alpha_n+\alpha_k-1}\exp(-\delta x_n\langle s\rangle^{\alpha_n}/2),\ L_{p'}(\{\langle s\rangle^{\alpha_n}<1\})\|,\ L_p(\{x_n>2d\})\|.$$

Define the domains

$$\omega_i=\{s\in R_{n-1}:2^{-i-1}<\langle s\rangle^{\alpha_n}<2^{-i}\},\quad i=0,1,2,\ldots.$$

Then, in view of the estimates obtained, we have

$$F_1+F_2\leq (c_1(f)+c_2(f))\sum_{i\geq 0}\sum_{k=1}^{n}\|\,\|\langle s\rangle^{\alpha_n+\alpha_k-1}\exp(-\delta x_n\langle s\rangle^{\alpha_n}/2),\ L_{p'}(\omega_i)\|,\ L_p(\{x_n>2d\})\|$$

$$\leq (c_1(f)+c_2(f))\sum_{i\geq 0}\sum_{k=1}^{n}\|\langle s\rangle^{\alpha_n+\alpha_k-1},\ L_{p'}(\omega_i)\|\,\|\exp(-\delta x_n2^{-i-2}),\ L_p(\{x_n>2d\})\|$$

$$=c_3(f)\sum_{i\geq 0}2^{i/p}\sum_{k=1}^{n}\|\langle s\rangle^{\alpha_n+\alpha_k-1},\ L_{p'}(\omega_i)\|\leq c_4(f)\sum_{i\geq 0}\sum_{k=1}^{n}2^{i(1-\alpha_k-|\alpha|/p')/\alpha_n}.$$

But, by (41), $(1-\alpha_k-|\alpha|/p')<0$, and we obtain estimate (45).

From inequalities (44) and (45) we have

$$\left\| \left\| \int_0^\infty U(s,x_n,0)\hat f(0,y_n)\,dy_n,\ L_{p'}(\{\langle s\rangle^{\alpha_n}<1\})\right\|,\ L_p(\{x_n>2d\})\right\|\leq c(f)+c\big\|u(x),\ L_p(R_n^+)\big\|<\infty,$$

and since we supposed that the orthogonality condition is not satisfied, i.e., $\int\limits_0^\infty \hat{f}(0,y_n)\,dy_n \neq 0$, we obtain the estimate

$$\| \, \|U(s,x_n,0),\, L_{p'}(\{\langle s\rangle^{\alpha_n} < 1\})\|,\, L_p(\{x_n > 2d\})\| \leq b < \infty. \tag{46}$$

Introduce the notation

$$V(d,\varepsilon) = \| \, \|U(s,x_n,0),\, L_{p'}(\{\langle s\rangle^{\alpha_n} < \varepsilon\})\|,\, L_p(\{x_n > 2d\})\|, \quad \varepsilon < 1.$$

By virtue of Lemma 6 in §2, for every $c > 0$ we have

$$c^{(1-|a|/p')} V(d,\varepsilon) = V(c^{\alpha_n} d, c^{-\alpha_n}\varepsilon).$$

Observe that condition (40) implies, via Lemma 8,

$$U(s,x_n,0) \not\equiv 0, \quad s \in \omega_i, \quad i = 0,1,2,\dots;$$

therefore, the preceding relation can be written down as

$$c^{(1-|a|/p')} = V(c^{\alpha_n} d, c^{-\alpha_n}\varepsilon)/V(d,\varepsilon).$$

Now, taking (41) into account, for every $c > 1$ we obtain the estimate

$$V(c^{\alpha_n} d, c^{-\alpha_n}\varepsilon)/V(d,\varepsilon) \geq 1.$$

Since inequality (46) holds, we on the other hand have

$$\lim_{c\to\infty} V(c^{\alpha_n} d, c^{-\alpha_n}\varepsilon) = 0.$$

A contradiction.

Thus, under the hypotheses of Theorem 10, the orthogonality condition (42) is necessary for solvability of the boundary value problem (1) in the space $W_p^r(R_n^+)$. The theorem is proven.

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