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ON MEAN QUASICONFORMAL MAPPINGS†)

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Various classes of mean quasiconformal mappings on the plane were studied in the articles by L. Ahlfors, I. N. Pesin, S. L. Krushkal′, R. Kühnau, V. M. Miklyukov, G. D. Suvorov, and many other authors (see, for instance, [1–8]). One of the major recent achievements in this field is the new (1988) existence and uniqueness theorem for the Beltrami equation which was proven by G. R. David [9]. Study of compactness properties for David homeomorphisms was initiated by P. Tukia [10].

In the present article, we study the compactness problem for homeomorphisms of a Sobolev class with general integral constraints on the dilatation when the integrand is of exponential growth at infinity. In this direction, we have obtained final results. The main among them are Theorem 1 on necessary and sufficient compactness conditions and Theorem 2 on compactification of noncompact classes.

Similar theorems were earlier obtained in the articles [11–13] for quasiconformal mappings with uniformly bounded dilatations and integrands. Further progress became possible due to a new fundamental theorem on convergence which was proven in [14]. The theorem is based on the well-known differentiability lemma by Gehring and Lehto [15, 16].

The most important consequence of the compactness criterion is convexity of the set of complex characteristics for such classes, which makes the approach universal that was first used by Professor V. Ya. Gutlyanskiĭ [17] in constructing variations.

1. Notation and preliminaries. A concise exposition and comparative analysis of results by I. N. Pesin and G. R. David, together with the general definitions and properties of homeomorphisms of the Sobolev class $W^1_{1,\mathrm{loc}}$ on the plane, can be found in the article [14].

We denote by H^Φ the collection of all orientation preserving homemorphisms $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ of the Sobolev class $W^1_{1,\mathrm{loc}}$ with the normalization $f(0)=0$, $f(1)=1$, and $f(\infty)=\infty$ and the integral constraint on the dilatation $p(z)$ of the form

$$\iint\limits_{\mathbb{C}} \Phi(p(z))\,dx\,dy \le 1. \tag{1}$$

Here $\Phi: I\to\overline{\mathbb{R}^+}$ is an arbitrary function; $I=[1,\infty]$; and $p(z)=(1+|\mu(z)|)/(1-|\mu(z)|)$, where $\mu(z)=f_{\bar z}/f_z$ is the complex characteristic of a mapping f.

Proposition 1. *Let $\Phi: I\to\overline{\mathbb{R}^+}$ be an arbitrary function with $\Phi(\infty)=\infty$. Then the class H^Φ is nonempty if and only if*

$$\inf_{t\in I}\Phi(t)=0. \tag{2}$$

PROOF. Necessity of condition (2) is obvious. Sufficiency is immediate. Namely, let $t_n\ge 1$ be an arbitrary sequence of numbers such that $\Phi(t_n)\le 2^{-n}/\pi$, $n=1,2,\dots$. Consider the mappings

$$f(z)=c_n z|z|^{t_n-1},\quad \sqrt{n-1}\le|z|\le\sqrt{n},$$

where $c_1=1$ and the subsequent c_n are found by induction from the agreement condition

$$c_n n^{t_n/2}=c_{n+1}n^{t_{n+1}/2}.$$

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The complex characteristic of the mapping $f(z)$ is easily calculable:

$$\mu(z) = \frac{t_n - 1}{t_n + 1}\frac{z}{\bar{z}}, \quad \sqrt{n-1} < |z| < \sqrt{n};$$

i.e., the dilatation of the mapping f equals $p(z) = t_n$, $\sqrt{n-1} < |z| < \sqrt{n}$, and satisfies (1) by construction. It is easy to see that $f(0) = 0$ and $f(1) = 1$. Moreover, the mapping f takes the circles with center zero into circles with the same center but other radii. Obviously,

$$\frac{f(\sqrt{n+1})}{f(\sqrt{n})} = \left(\frac{\sqrt{n+1}}{\sqrt{n}}\right)^{t_{n+1}} \geq \sqrt{\frac{n+1}{n}}.$$

Inducting, we therefore obtain $f(\sqrt{n}) \geq \sqrt{n}$; i.e., $f(z) \to \infty$ as $z \to \infty$, or $f(\infty) = \infty$.

Since analytic arcs are removable [18, p. 47], the mapping f is locally quasiconformal, yielding $f \in ACL$. Hence, $f \in W^1_{1,\mathrm{loc}}$ [19, p. 42], and therefore $f \in H^\Phi$.

Denote by $\mathfrak{M}^\Phi$ the class of all measurable functions $\mu(z) : \mathbb{C} \to \mathbb{C}$, $|\mu(z)| \leq 1$, with $p(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$ satisfying inequality (1). Owing to the result of [9, pp. 27 and 55], we obtain

Proposition 2. *Let $\Phi : I \to \overline{\mathbb{R}^+}$ be of exponential growth at infinity. Then, given a mapping $\mu \in \mathfrak{M}^\Phi$, there is a unique mapping $f \in H^\Phi$ with complex characteristic μ.*

Here, by definition, a function Φ *is of exponential growth at infinity* if

$$\Phi(t) \geq \beta e^{\gamma t} \tag{3}$$

for all $t \geq T$ and some $T \geq 1$, $\beta > 0$, and $\gamma > 0$. The following notion turns out to be useful in the description of closure for the classes H^Φ. The *lower envelope* of a function Φ is defined as the function

$$\Phi_0(t) = \sup_{\varphi \in \Psi} \varphi(t), \quad t \in I,$$

where Ψ is the family of all continuous (downward) convex functions $\varphi : I \to \overline{\mathbb{R}^+}$ such that $\varphi(t) \leq \Phi(t)$.

The general properties of convex functions [20, pp. 56–66] imply that the lower envelope of a function Φ is the greatest nondecreasing convex function Φ_0 which is left continuous with respect to $\overline{\mathbb{R}^+}$ at the point

$$Q = \sup_{\Phi(t) < \infty} t \tag{4}$$

and whose graph is situated lower than the graph of Φ. Furthermore, $\Phi_0(t) \equiv \infty$ for $t > Q$ and $\Phi_0(t) < \infty$ for $t < Q$. A more constructive description for the lower envelope is given in [13, Lemma 1].

Proposition 3. *Let a function $\Phi : I \to \overline{\mathbb{R}^+}$ be of exponential growth at infinity. Then its lower envelope $\Phi_0 : I \to \overline{\mathbb{R}^+}$ too is of exponential growth at infinity. Moreover, inequality (3) holds for Φ_0 with the same $\beta > 0$ and $\gamma > 0$ as for Φ, provided $t \geq T^* = T + 1/\gamma$.*

Indeed, for $t = T^* = T + 1/\gamma$ the tangent to the graph of $\varphi_0(t) = \beta e^{\gamma t}$ passes through the point $(\varphi, t) = (0, T)$, which amounts to the equality $\varphi_0(T^*) = \varphi_0'(T^*)(T^* - T)$. Hence, the function

$$\varphi(t) = \begin{cases} 0, & t \in [1, T], \\ \varphi_0'(T^*)(t - T), & t \in [T, T^*], \\ \varphi_0(t), & t \in [T^*, \infty], \end{cases}$$

belongs to the class Ψ determining the lower envelope Φ_0.

2. Statement of the main results. We begin with a compactness criterion for classes H^Φ.

Theorem 1. *Let $\Phi : I \to \overline{\mathbb{R}^+}$, $I = [1, \infty]$, be an arbitrary function of exponential growth at infinity and let* $\inf \Phi = 0$. *Then the following assertions are equivalent:*

(1) H^Φ *is closed;*
(2) H^Φ *is compact (bicompact);*
(3) H^Φ *is countably compact;*
(4) H^Φ *is sequentially compact;*
(5) Φ *is nondecreasing, convex, and left continuous with respect to* $\overline{\mathbb{R}^+}$ *at the point Q in* (4).

The equivalence of assertions (1)–(4) ensues from the general topological arguments presented in Section 3 below.

We point out that any polynomial growth of Φ at infinity is insufficient for the sequential compactness of H^Φ even though all conditions on Φ listed in item (5) of the theorem are satisfied (cf. [7, 9]).

When Φ is of exponential growth at infinity, sufficiency of condition (5) is a straightforward corollary to Theorem 1, Proposition 1, and Remark 1 of the article [14], wherein under these conditions the semicontinuity of the dilatation in the mean was established:

$$\iint \Phi(p(z))\,dxdy \le \varliminf_{n\to\infty} \iint \Phi(p_n(z))\,dxdy;$$

David's result on equicontinuity and openness [9, p. 27]; and the well-known Arzelà–Ascoli theorem [21, p. 289].

Necessity of condition (5) for the sequential compactness of the class H^Φ will be proven in Section 5 on the grounds of the following closure theorem which is of interest in its own right.

Theorem 2. *Let $\Phi : I \to \overline{\mathbb{R}^+}$ be of exponential growth at infinity. Then in the topology of locally uniform convergence*

$$\overline{H^\Phi} = H^{\Phi_0}, \tag{5}$$

where $\Phi_0 : I \to \overline{\mathbb{R}^+}$ is the lower envelope of the function Φ. Moreover, the class H^{Φ_0} is sequentially compact.

We point out at once that the sequential compactness of the class H^{Φ_0} follows from Proposition 3, the properties of Φ_0, and the above arguments. A proof of relation (5) is given in Section 7. This is the most laborious part of the proof.

Denote by $P_{\delta,\mathrm{M}}$, $\delta > 0$ and $\mathrm{M} > 0$, the Pesin class of all orientation preserving ACL homeomorphisms g of the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ onto itself with the normalization $g(0) = 0$, $g(1) = 1$ and the following condition on the dilatation:

$$\iint_\Delta e^{[p(z)]^{1+\delta}}\,dxdy \le \mathrm{M}.$$

Observe that if $\mathrm{M} \ge \pi e$ then the class $P_{\delta,\mathrm{M}}$ is certainly nonempty, and if $\mathrm{M} > \pi e$ then it is nontrivial.

Corollary 1 [7]. *The class $P_{\delta,\mathrm{M}}$ is sequentially compact for arbitrary $\delta > 0$ and $\mathrm{M} > 0$.*

Indeed, suppose that $\mathrm{M} > \pi e$ and consider the class H^Φ with $\Phi(t) = (\exp t^{1+\delta} - e)/(\mathrm{M} - \pi e)$. By the uniqueness theorem of David [9, p. 55], the mapping $g \in P_{\delta,\mathrm{M}}$ with complex characteristic $\nu : \Delta \to \Delta$ has the representation $g = A_f \circ f$, where f is a single mapping in the class H^Φ with complex characteristic $\mu : \mathbb{C} \to \Delta$,

$$\mu(z) = \begin{cases} \nu(z), & z \in \Delta, \\ 0, & z \in \mathbb{C} \setminus \Delta, \end{cases}$$

and $A_f : f(\Delta) \to \Delta$ is a single conformal mapping with normalization $A_f(0) = 0$ and $A_f(1) = 1$.

By using Carathéodory's theorem and Rado's theorem [22, pp. 56 and 60], we can easily show that $A_{f_n} \to A_f$ as $f_n \to f$. Therefore, Pesin's result follows from Theorem 1.

Denoting by H_Δ^Φ the class of all orientation preserving ACL automorphisms $g : \Delta \to \Delta$ with $g(0) = 0$ and $d(1) = 1$ which are distinguished by the inequality

$$\iint\limits_\Delta \Phi(p(z))\, dx\, dy \leq 1,$$

we by analogy obtain the following generalization:

Corollary 2. *Let a function $\Phi : I \to \overline{\mathbb{R}^+}$ be of exponential growth at infinity, nondecreasing, convex, and continuous with respect to $\overline{\mathbb{R}^+}$. Then the class H_Δ^Φ is sequentially compact.*

Corollary 3. *The classes H^Φ with $\Phi(t) = \beta(e^{\gamma(t-1)} - 1)$ are sequentially compact for arbitrary $\beta > 0$ and $\gamma > 0$.*

Corollary 4. *The classes of all normalized Q-quasiconformal mappings distinguished by the integral constraints on the dilatation of the form*

$$\iint\limits_{\mathbb{C}} [p(z) - 1]^\alpha\, dxdy \leq \mathrm{M}, \quad \mathrm{M} > 0,$$

are sequentially compact only if $\alpha \geq 1$.

In the last case we deal with the classes H^Φ, where

$$\Phi(t) = \begin{cases} (t-1)^\alpha, & 1 \leq t \leq Q, \\ \infty, & t > 0. \end{cases}$$

In general, if $Q < \infty$ then (4) automatically implies validity of the condition of exponential growth at infinity. Hence, we have

Corollary 5. *Let $\Phi : I \to \overline{\mathbb{R}^+}$ be an arbitrary function with $\inf \Phi = 0$ and let $Q < \infty$. Then the class H^Φ is sequentially compact if and only if the function Φ is nondecreasing, convex, and continuous on the interval $[1, Q]$.*

In particular, this yields the most interesting example of a noncompact class:

Corollary 6. *The class of all Q-quasiconformal mappings $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $1 < Q < \infty$, with the normalization $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$ which is distinguished by the condition*

$$\iint\limits_{\mathbb{C}} |\mu(z)|\, dxdy \leq 1$$

on the complex characteristic is not sequentially compact.

Indeed, in accordance with our notation, the above class is the class H^Φ with

$$\Phi(t) = \begin{cases} (t-1)/(t+1), & 1 \leq t \leq Q, \\ \infty, & t > Q. \end{cases}$$

It is easy to see that the function $\varphi(t) = (t-1)/(t+1)$ is not convex, for $\varphi''(t) = -4/(t+1)^3 < 0$. Thus, one of the compactness conditions of classes H^Φ is violated.

Finally, we indicate one consequence most interesting from the standpoint of the theory of the variational method.

Corollary 7. *Let $\Phi : I \to \overline{\mathbb{R}^+}$ be an arbitrary function of exponential growth at infinity. If the class H^Φ is sequentially compact then the class of $\mathfrak{M}^\Phi$ of complex characteristics is a convex set.*

Indeed, by straightforward calculation of the first and second derivatives we check that the function

$$\varkappa(\tau) = \frac{1+\tau}{1-\tau} = \frac{2}{1-\tau} - 1$$

is increasing and convex. Considering that Φ is increasing and convex and applying the triangle inequality, for arbitrary $\lambda \in [0,1]$ we obtain

$$\Phi(\varkappa(|\lambda\mu(z) + (1-\lambda)\nu(z)|)) \le \lambda\Phi(\varkappa(|\mu(z)|)) + (1-\lambda)\Phi(\varkappa(|\nu(z)|)).$$

3. Topological remarks. Observe that the space H of all homeomorphisms of the plane, endowed with the topology of locally uniform convergence, is metrizable. One of the metrics compatible with this convergence is [23, p. 243]

$$\rho(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\rho_j(f,g)}{1+\rho_j(f,g)}, \tag{6}$$

where ρ_j, $j = 1, 2, \dots$, are the pseudometrics $\rho_j(f,g) = \max_{|z|\le j} |f(z) - g(z)|$.

Thus, the following holds in H:

Proposition 4 [24, pp. 7–9]. *Let $H' \subseteq H$ be an arbitrary subspace of homeomorphisms of the plane, endowed with the topology of locally uniform convergence. Then the following statements are equivalent:*

(1) *H' is compact (bicompact);*
(2) *H' is countably compact;*
(3) *H' is sequentially compact.*

It is seen from Proposition 4 that the equivalence of assertions (2)–(4) of Theorem 1 is a direct consequence of metrizability of H.

If H' is embedded in a compact subspace $H_0 \subseteq H$ then we can say more.

Proposition 5 [25, p. 129]. *Suppose that $H' \subseteq H_0$, where H_0 is a compact (bicompact) subset of H. Then the following statements are equivalent:*

(1) *H' is closed;*
(2) *H' is compact (bicompact).*

Thus, the equivalence of assertions (1)–(4) of Theorem 1 ensues from metrizability of H and compactness of $H_0 = H^{\Phi_0}$.

Recall that the *compactness* (*bicompactness*) of H' means that the Borel–Lebesgue condition is satisfied: every open covering of H' has a finite subcovering.

The *countable compactness* of H' reduces to the Borel condition: every countable open covering of H' has a finite subcovering.

The *sequential compactness* of H' reduces to the Bolzano–Weierstrass condition: a convergent subsequence $f_{n_k} \to f \in H'$ can be extracted from an arbitrary sequence $f_n \in H'$.

Henceforth we denote by $H(K)$, $1 \le K < \infty$, the subclass of H constituted by all orientation preserving ACL homeomorphisms of the plane with the dilatation $p(z) \le K$ almost everywhere [16, p. 28], i.e. the set of all K-quasiconformal mappings $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ normalized by the conditions $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$. The class $H(K)$ is well known to be sequentially compact under locally uniform convergence [18, p. 76].

4. A lemma on equality. With the notation of Section 1, the following holds:

Lemma 1. *Let Φ_1 and $\Phi_2 : I \to \overline{\mathbb{R}^+}$ be arbitrary functions with $\inf \Phi_1 = \inf \Phi_2 = 0$ and $\Phi_1(\infty) = \Phi_2(\infty) = \infty$. Then the equality*

$$\mathfrak{M}^{\Phi_1} = \mathfrak{M}^{\Phi_2} \tag{7}$$

holds if and only if

$$\Phi_1(t) \equiv \Phi_2(t), \quad t \in I. \tag{8}$$

This lemma enables us to reduce the proof of Theorem 1 to Theorem 2 straightforwardly.

PROOF. Sufficiency of condition (8) for equality (7) is tautological.

Assume that (8) fails. For definiteness, assume that $\varphi_1 = \Phi_1(t_0) > \Phi_2(t_0) = \varphi_2$ at some point $t_0 \in I \setminus \{\infty\}$. Then there are positive numbers ε, r, and c such that $\varepsilon < \sqrt{\varphi_1/\varphi_2} - 1$, $(1+\varepsilon)/\sqrt{\pi\varphi_1} < r < 1/\sqrt{\pi\varphi_2}$, $\pi r^2(\varphi_2 + c) \leq 1$.

Since $\inf \Phi_2 = 0$ and $\Phi_2(\infty) = \infty$, there is a sequence $t_n \in I \setminus \{\infty\}$ such that $\Phi_2(t_n) \leq c2^{-n}$.

Further, putting $\mu(z) = (t_n - 1)/(t_n + 1)$ for $r\sqrt{n} \leq |z| < r\sqrt{n+1}$ and $n = 0, 1, 2, \ldots$, we arrive at the inequalities

$$\iint\limits_{\mathbb{C}} \Phi_1(p(z))\, dxdy \geq \pi r^2 \varphi_1 > (1+\varepsilon)^2 > 1;$$

i.e., $\mu \notin \mathfrak{M}^{\Phi_1}$. On the other hand,

$$\iint\limits_{\mathbb{C}} \Phi_2(p(z))\, dxdy \leq \pi r^2 \varphi_2 + \pi r^2 c \leq 1;$$

i.e., $\mu \in \mathfrak{M}^{\Phi_2}$.

Thus, equality (7) fails certainly, and necessity of condition (8) is proven.

5. Proof of Theorem 1. In view of remarks made in Sections 2 and 3, it suffices to prove the implication (1) $\Rightarrow$ (5).

The closure of H^Φ means that $\overline{H^\Phi} = H^\Phi$, while by Theorem 2 $\overline{H^\Phi} = H^{\Phi_0}$. By virtue of Propositions 2 and 3, whence we infer the equality $\mathfrak{M}^\Phi = \mathfrak{M}^{\Phi_0}$. Lemma 1 yields $\Phi(t) \equiv \Phi_0(t)$. Thus, Φ satisfies all conditions listed in item (5) of Theorem 1.

6. A density lemma. The following lemma holds:

Lemma 2. *Let $\Phi : I \to \overline{R^+}$, $I = [1, \infty]$, be of exponential growth at infinity and let $K \in I \setminus \{\infty\}$. Then the class $H(K) \cap H^{\Phi_0}$, endowed with the topology of locally uniform convergence, contains an everywhere dense subset of homeomorphisms whose complex characteristics take at most finitely many values.*

PROOF. Let $f \in H(K) \cap H^{\Phi_0}$ be a mapping with complex characteristic $\mu(z) = q(z)e^{i\nu(z)}$. Put $\mu_n(z) = q_n(z)e^{i\nu_n(z)}$, with $q_n(z) = m2^{-n}$ and $\nu_n(z) = l2^{-n}$ on each of the sets

$$E^n_{ml} = \{z \in \mathbb{C} : m2^{-n} \leq q(z) < (m+1)2^{-n},\ l2^{-n} \leq \nu(z) < (l+1)2^{-n}\},$$
$$m, l = 0, \pm 1, \pm 2, \ldots; \quad n = 1, 2, \ldots.$$

Observe that all the sets E^n_{ml} are measurable [26, p. 28]. Therefore, the functions $\mu_n(z)$, $\Phi_0(p_n(z))$, and $p_n(z) = (1+q_n(z))/(1-q_n(z))$, $n = 1, 2, \ldots$, are measurable too [26, p. 29]. Moreover, $0 \leq q_n(z) \leq q(z)$. Consequently, Propositions 2 and 3 imply existence of a unique homeomorphism $f_n \in H^{\Phi_0}$ with complex characteristic $\mu_n \in \mathfrak{M}^{\Phi_0}$.

Furthermore, by construction $|q(z) - q_n(z)| \le 2^{-n}$ and $|\nu(z) - \nu_n(z)| \le 2^{-n}$; i.e., the sequence $\mu_n(z)$ converges pointwise to $\mu(z)$ as $n \to \infty$. Since $f_n \in H(K)$, the Bers–Bojarsky theorem [18, 27] implies that f_n converges locally uniformly to f as $n \to \infty$.

7. Proof of Theorem 2. 1. The inclusion

$$\overline{H^{\Phi}} \subseteq H^{\Phi_0} \tag{9}$$

is easily provable. Indeed, by the definition of a lower envelope, we have $\Phi_0(t) \le \Phi(t)$, $t \in I$. Moreover, $\Phi_0(t)$ is monotone and continuous on the interval $[1, Q]$, with Q given by (4). Therefore, the function Φ_0 is Borel measurable, i.e. the inverse image of every Borel set is a Borel set. Consequently, Φ_0 is superposition measurable.

Thus, if $p(z)$ is the dilatation of a mapping $f \in H^{\Phi}$, then the superposition $\Phi_0(p(z))$ is a measurable function and $\Phi_0(p(z)) \le \Phi(p(z))$; i.e., $f \in H^{\Phi_0}$. In other words, $H^{\Phi} \subseteq H^{\Phi_0}$, and hence $\overline{H^{\Phi}} \subseteq \overline{H^{\Phi_0}}$. Finally, from sequential compactness we have $H^{\Phi_0} = \overline{H^{\Phi_0}}$, and inclusion (9) follows.

2. To prove the reverse inclusion, we first prove that $H' = H(K) \cap H^{\Phi_0} \subseteq \overline{H^{\Phi}}$ for ever $K < Q$, where Q is taken from (4). Moreover, we shall assume that $\inf \Phi = 0$, i.e., $\Phi_0(1) = 0$, for Proposition 1 would otherwise imply that the classes are empty and we would have nothing to prove.

By Lemma 2, there is a subset $H_0 \subseteq H'$ of homeomorphisms with step-like characteristics, such that $H' \subseteq \overline{H_0}$. Since [23, p. 44]

$$\overline{\overline{H^{\Phi}}} = \overline{H^{\Phi}}, \tag{10}$$

it suffices to demonstrate that $H_0 \subseteq \overline{H^{\Phi}}$.

Thus, let a mapping f belong to H_0 which makes its complex characteristic take the form $\mu(z) = \nu_l$, $z \in \mathcal{E}_l$, $l = 1, 2, \ldots$, where $\cup \mathcal{E}_l = \mathbb{C}$, $\nu_l = \tau_l \eta_l$, $\tau_l \in [0, k]$, $k = (K-1)/(K+1) < 1$, and $\eta_l \in \mathbb{C}$, $|\eta_l| = 1$. Denote $\varphi_l = \Phi_0(t_l)$, $t_l = (1+\tau_l)/(1-\tau_l)$. Without loss of generality we may assume that

$$\iint\limits_{\mathbb{C}} \Phi_0(p(z))\, dxdy < 1. \tag{11}$$

Indeed, let $\mathfrak{M}_0$ denote the class of the characteristics μ of the mappings $f \in H_0$ and let H_Θ stand for the class of normalized homeomorphisms f_Θ of the plane with the characteristics $\mu_\Theta(z) = \Theta\mu(z)$, $\mu \in \mathfrak{M}_0$, $\Theta \in (0, 1)$. Since Φ_0 is nondecreasing, we have $H_\Theta \subseteq H'$. As $\Theta \to 1$, we have $\mu_\Theta(z) \to \mu(z)$, and by the Bers–Bojarsky theorem [18, 27] $f_\Theta \to f$ locally uniformly. Thus,

$$H_0 \subseteq \overline{\bigcup_{\Theta \in (0,1)} H_\Theta}$$

and consequently

$$\bigcup_{\Theta \in (0,1)} H_\Theta \subseteq H' \subseteq \overline{\bigcup_{\Theta \in (0,1)} H_\Theta}$$

This allows us, if need be, to replace H_0 by $\cup H_\Theta$, $\Theta \in (0, 1)$. However, in the classes H_Θ, $\Theta \in (0, 1)$, we certainly have strict inequality (11), because $\Phi_0(1) = 0$ and Φ_0 is convex and nondecreasing.

Extend the function Φ and Φ_0 from the interval $I_0 = [1, Q]$ to the interval $[Q^{-1}, 1]$ by means of the relations $\Phi(t) = \Phi(t^{-1})$ and $\Phi_0(t) = \Phi_0(t^{-1})$. In accordance to [13, Lemma 1], the points (t_l, φ_l), $l = 1, 2, \ldots$, belong to segments of support lines to the graph of the function Φ. Let $(t_l^{(1)}, \varphi_l^{(1)})$ and $(t_l^{(2)}, \varphi_l^{(2)})$, $l = 1, 2, \ldots$, denote the ends of these segments and let $\lambda_l \in [0, 1]$ be the numbers determined by the equalities $t_l = \lambda_l t_l^{(1)} + (1 - \lambda_l) t_l^{(2)}$, $l = 1, 2, \ldots$. Then $\varphi_l = \lambda_l \varphi_l^{(1)} + (1 - \lambda_l)\varphi_l^{(2)}$,

$l = 1, 2, \ldots$, and by [13, Lemma 1] there are sequences $t_{lm}^{(1)} \to t_l^{(1)}$ and $t_{lm}^{(2)} \to t_l^{(2)}$ such that $\varphi_{lm}^{(1)} = \Phi(t_{lm}^{(1)}) \to \varphi_l^{(1)}$ and $\varphi_{lm}^{(2)} = \Phi(t_{lm}^{(2)}) \to \varphi_l^{(2)}$ as $m \to \infty$. Thus,

$$t_{lm}^{(0)} = \lambda_l t_{lm}^{(1)} + (1 - \lambda_l) t_{lm}^{(2)} \to t_l, \tag{12}$$

$$\varphi_{lm}^{(0)} = \lambda_l \varphi_{lm}^{(1)} + (1 - \lambda_l) \varphi_{lm}^{(2)} \to \varphi_l. \tag{13}$$

In what follows, it is important that the indicated sequences can always be chosen so that $t_{lm}^{(j)} \leq Q' < \infty$ and $\varphi_{lm}^{(j)} \leq \varphi' < \infty$ for all $j = 0, 1, 2$ and $l, m = 1, 2, \ldots$, where for a fixed Φ the choice of $Q' \leq Q$ and φ' depends only on K.

Allowing for an additional partition of the plane, we can always achieve validity of the inequality $\operatorname{mes} \mathcal{E}_l < \infty$, $l = 1, 2, \ldots$. Moreover, by construction we may assume that

$$\left|\varphi_{lm}^{(0)} - \varphi_l\right| < 2^{-(l+m)} / \operatorname{mes} \mathcal{E}_l \tag{14}$$

for all $l = 1, 2, \ldots$.

We put $\nu_{lm}^{(j)} = \tau_{lm}^{(j)} \eta_l$, where $\tau_{lm}^{(j)} = (t_{lm}^{(j)} - 1)/(t_{lm}^{(j)} + 1)$, $j = 0, 1, 2$; $l, m = 1, 2, \ldots$. Then, in accordance with [12, Lemma 1] (also see [11]), there is a sequence $F_{lm}^{(n)} \in H(Q')$, $n = 1, 2, \ldots$, with characteristics

$$\mu_{lm}^{(n)}(z) = \begin{cases} \nu_{lm}^{(1)}, & z \in E_l^{(n)}, \\ \nu_{lm}^{(2)}, & z \in \mathbb{C} \setminus E_l^{(n)} \end{cases}$$

which converges to a mapping $F_{lm} \in H(Q')$ with characteristic $\mu_{lm}(z) \equiv \nu_{lm}^{(0)}$, $z \in \mathbb{C}$. Furthermore, for each measurable set $\mathcal{E} \subseteq \mathbb{C}$ with $\operatorname{mes} \mathcal{E} < \infty$, the limit

$$\lim_{n \to \infty} \iint_{\mathcal{E}} \Phi(p_{lm}^{(n)}(z))\, dx dy = \varphi_{lm}^{(0)} \operatorname{mes} \mathcal{E}$$

exists, where $p_{lm}^{(n)}(z)$ are the dilatations of the mappings $F_{lm}^{(n)}$.

By construction, as $n \to \infty$ we have $\rho(F_{lm}^{(n)}, F_{lm}) \to 0$, where ρ is metric (6). Therefore, for arbitrary fixed l and $m = 1, 2, \ldots$, there is an index $N = N(l, m)$ such that $\rho(F^{lm}, F_{lm}) < 1/m$. Here

$$F^{lm}(z) = F_{lm}^{(N)}(z), \qquad \left| \iint_{\mathcal{E}_l} \Phi(p^{lm}(z))\, dx\, dy - \varphi_{lm}^{(0)} \operatorname{mes} \mathcal{E}_l \right| < 2^{-(l+m)},$$

with p^{lm} standing for the dilatation of the mapping F^{lm}.

Let $F_l \in H(Q')$ be a mapping with characteristic $\mu_l(z) \equiv \nu_l$, $z \in \mathbb{C}$, and let $p_l(z) \equiv t_l$. Then the Bers–Bojarsky theorem and relations (12) and (13) yield

$$\lim_{m \to \infty} \rho(F_{lm}, F_l) = 0, \qquad \lim_{m \to \infty} \varphi_{lm}^{(0)} \operatorname{mes} \mathcal{E}_l = \iint_{\mathcal{E}_l} \Phi_0(p_l(z))\, dx dy,$$

and consequently

$$\lim_{m \to \infty} \rho(F^{lm}, F_l) = 0, \tag{15}$$

$$\lim_{m \to \infty} \iint_{\mathcal{E}_l} \Phi(p^{lm}(z))\, dx dy = \iint_{\mathcal{E}_l} \Phi_0(p_l(z))\, dx dy.$$

On putting

$$\mu^m(z) = \mu^{lm}(z), \quad z \in \mathcal{E}_l, \quad l = 1, 2, \ldots, \tag{16}$$

for the corresponding dilatations, from here and (14) we deduce

$$\left| \iint_{\mathbb{C}} \Phi(p^m(z))\, dxdy - \iint_{\mathbb{C}} \Phi_0(p(z))\, dxdy \right| \le 2^{-m}.$$

Thus,

$$\lim_{m\to\infty} \iint_{\mathbb{C}} \Phi(p^m(z))\, dxdy = \iint_{\mathbb{C}} \Phi_0(p(z))\, dxdy.$$

Let $F^m \in H(Q')$ be a mapping with characteristic μ^m, $m = 1, 2, \ldots$. By virtue of (11), $F^m \in H^\Phi$ for m large enough.

Since $H(Q')$ is a sequentially compact class, from (15), (16), Urysohn's lemma, and Lemma 5 of [13, pp. 90–91] we conclude that F^m converges locally uniformly to f as $m \to \infty$.

3. Finally, we prove the reverse inclusion $H^{\Phi_0} \subseteq \overline{H^\Phi}$. Let a mapping f belong to the class H^{Φ_0}, let $\mu(z)$ be its complex characteristic and $p(z)$, the dilatation. Put

$$\mu_m(z) = \begin{cases} \mu(z), & z \in \mathcal{E}_m, \\ 0, & z \in \mathbb{C} \setminus \mathcal{E}_m, \end{cases}$$

where $\mathcal{E}_m = \{z \in \mathbb{C} : p(z) \le m\}$, $m = 1, 2, \ldots$. Denote by f_m a mapping in the class $H(m) \cap H^{\Phi_0}$ with characteristic μ_m. By construction, $\mu_m(z)$ converges pointwise to $\mu(z)$ as $m \to \infty$.

By Proposition 3, Φ_0 is of exponential growth at infinity. Therefore, Tukia's theorem [10] implies that $f_m \to f$ locally uniformly as $m \to \infty$. However, by item 2 of the proof, we have $f_m \in \overline{H^\Phi}$. In view of (10), we then have $f \in \overline{H^\Phi}$ as well, completing the proof of the theorem.

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