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A STABILITY ESTIMATE IN THE PROBLEM OF DETERMINING THE DISPERSION INDEX AND RELAXATION FOR THE TRANSPORT EQUATION†)

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We consider the inverse problem for the transport equation which consists in finding the relaxation $\sigma(x)$, $x \in \mathbf{R}^2$, and the dispersion index $K(x, \nu(\theta) \cdot \nu(\theta'))$ in the transport equation (see formulas (1.1) and (1.2) below). Their determination is based on some information on a one-parameter family of solutions of the direct problems for the transport equation. A specific feature of the direct problems in question is presetting some incident radiation acute-directed with respect to the angle variable θ and having the form described by the Dirac delta-function $\delta(\theta - \alpha)$, where α is a parameter of the problem. Presetting incident radiation in such a way is convenient for studying the inverse problem and, apparently, is acceptable from the applied viewpoint. It enables us to split the original problem into two inverse problems to be solved successively. The first problem is to find the relaxation $\sigma(x)$ given the singular part of the information on solutions to the direct problems, and the second is to find the dispersion index given the coefficient $\sigma(x)$. This approach makes it possible to obtain a conditional stability estimate for a solution and to prove a uniqueness theorem for the inverse problem.

Observe that the idea of using singularities of incident radiation for studying inverse problems connected with determining the relaxation and the dispersion index in the transport equation was proposed for the first time in the article [1] by D. S. Anikonov and was further developed in the articles [2, 3]. In the article [4], some approach was proposed which is a logical extension of this idea and is based on use of the singular part of the fundamental solution to the Cauchy problem for the transport equation. Moreover, in the above-mentioned articles, the problem of finding the coefficient $\sigma(x)$ (and in the last article also the problem of finding the dispersion index) was reduced to the classical tomography problem of determining a function given its integrals along all straight lines. This reduction yields some uniqueness and stability theorems for a solution to the original problem. The statement of the problem in the present article is new. It uses minimal information on solutions to the direct problems and in this aspect is more attractive than that of [4]. The main result of the article is a stability estimate for the dispersion index.

§ 1. Statement of the Problem and the Main Results

Let $D \subset \mathbf{R}^2$ be an open unit disk, $D = \{x \in \mathbf{R}^2 \mid |x| < 1\}$, and let S be its boundary, $S = \{x \in \mathbf{R}^2 \mid |x| = 1\}$. Consider the transport equation in a function $u = u(x, \theta) = u(x, \theta + 2\pi)$ 2π-periodic in θ:

$$L(\sigma, K)v \equiv \nabla u \cdot \nu(\theta) + \sigma u + Su = 0, \quad (x, \theta) \in G \equiv D \times [0, 2\pi], \tag{1.1}$$

where $\sigma = \sigma(x)$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$, $\nu(\theta) = (\cos\theta, \sin\theta)$, $\nabla u \cdot \nu$ is the inner product of the vectors ∇u and $\nu(\theta)$, and Su is the operator

$$Su \equiv \int_0^{2\pi} K(x, \nu(\theta) \cdot \nu(\theta'))u(x, \theta')\, d\theta' \tag{1.2}$$

describing scattering.

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Denote by $n(x)$ the outward unit normal to S at a point $x \in S$. Also, let $\overline{D} = D \bigcup S$, $\overline{G} = \overline{D} \times [0, 2\pi]$, $\partial_- G \equiv \{(x, \theta) \in S \times [0, 2\pi] \mid \nu(\theta) \cdot n(x) \leq 0\}$, and $\partial_+ G \equiv \{(x, \theta) \in S \times [0, 2\pi] \mid \nu(\theta) \cdot n(x) > 0\}$.

Consider the following boundary value problem for equation (1.1) with data on $\partial_- G$:

$$u(x, \theta) = \delta_p(\theta - \alpha), \quad (x, \theta) \in \partial_- G. \tag{1.3}$$

Here $\delta_p(\theta - \alpha)$ is a 2π-periodic function whose restriction to an arbitrary interval $[\alpha - \varepsilon, \alpha + \varepsilon], \varepsilon \in (0, 2\pi)$, coincides with the Dirac delta-function supported at the point $\theta = \alpha$. Henceforth, α is a parameter of the problem. In this connection, a (distributional) solution to problem (1.1), (1.3) is denoted by $u(x, \theta, \alpha)$ to emphasize the dependence on the parameter α. It is obvious that $u(x, \theta, \alpha + 2\pi) = u(x, \theta, \alpha)$. Therefore, we consider the domain $Q = \{(x, \theta, \alpha) \mid (x, \theta) \in G, \alpha \in [0, 2\pi]\}$ as the main domain of variation of the variables x, θ, and α.

The following lemma is valid for problem (1.1), (1.3) (see § 2 for proof):

Lemma 1.1. *Suppose that the coefficient $\sigma(x)$ belongs to $\mathbf{C}(\overline{D})$, $\sigma(x) \geq 0$, the dispersion index $K(x, \cos\psi)$ belongs to $\mathbf{C}(\overline{G})$, and*

$$K' = \left(\int_0^{2\pi} \sup_{x \in D} K^2(x, \cos\psi)\, d\psi \right)^{1/2} < \frac{1}{\sqrt{8\pi}}. \tag{1.4}$$

Then there is a weak solution to problem (1.1), (1.3) which is representable as

$$u(x, \theta, \alpha) = \delta_p(\theta - \alpha) \exp(-w(x, \alpha)) + v(x, \theta, \alpha). \tag{1.5}$$

Here $w(x, \alpha) \in \mathbf{C}(\overline{G})$, $v(x, \theta, \alpha) \in \mathbf{C}(\overline{Q})$, $\overline{Q} = \overline{G} \times [0, 2\pi]$, and

$$w(x, \alpha) = \int_{L(x, \alpha)} \sigma(\xi)\, ds \equiv \int_0^{s(x, \alpha)} \sigma(x - s\nu(\alpha))\, ds, \tag{1.6}$$

where $L(x, \alpha)$ is the segment of the ray emanating from the point $x \in D$ in the direction $-\nu(\alpha)$ between the point x and the intersection point of the ray with S; $s(x, \alpha)$ is the length of the segment; and ds is the line element.

Representation (1.5), (1.6) gives grounds for stating the following inverse problem:

PROBLEM. *Given the trace of a solution to problem (1.1), (1.3) on $\partial_+ Q = \{(x, \theta, \alpha) \mid (x, \theta) \in \partial_+ G, \alpha \in [0, 2\pi]\}$:*

$$u(x, \theta, \alpha) = f(x, \alpha) \delta_p(\theta - \alpha) + F(x, \theta, \alpha), \quad (x, \theta, \alpha) \in \partial_+ Q,$$

find $\sigma(x) \geq 0$ and $K(x, \cos\psi)$, $(x, \psi) \in \overline{G}$.

Specifying the trace on $\partial_+ Q$ for a solution to problem (1.1), (1.3) is equivalent to specifying the functions $f(x, \alpha)$ and $F(x, \theta, \alpha)$. Moreover, the following equality holds by Lemma 1.1:

$$\int_{L(x, \alpha)} \sigma(\xi)\, ds = -\ln f(x, \alpha) \equiv g(x, \alpha), \quad (x, \alpha) \in \partial_+ G. \tag{1.7}$$

The problem of constructing the function σ inside D given the function $g(x, \alpha)$ is referred to as the tomography problem (the inversion problem for the Radon transform) and was studied by many authors. There are various well-known inversion formulas for equality (1.7) as well as stability estimates for a solution. We will return to this question below.

The function $v(x,\theta,\alpha)$ involved in representation (1.5) is a solution to the problem

$$L(\sigma,K)v + K(x,\nu(\theta)\cdot\nu(\alpha))\exp(-w(x,\alpha)) = 0, \quad (x,\theta,\alpha)\in Q, \tag{1.8}$$

$$v(x,\theta,\alpha) = 0, \quad (x,\theta,\alpha)\in \partial_-Q \equiv \partial_-G\times[0,2\pi], \tag{1.9}$$

and is connected with the function $F(x,\theta,\alpha)$ by the relation

$$v(x,\theta,\alpha) = F(x,\theta,\alpha), \quad (x,\theta,\alpha)\in \partial_+Q \equiv \partial_+G\times[0,2\pi]. \tag{1.10}$$

Hence, we arrive at the nonlinear tomography problem that consists in finding the function $K(x,\cos\psi)$, $\psi=\theta-\alpha$, from relations (1.8)–(1.10). Moreover, the function $w(x,\alpha)$ is calculated by formula (1.6), once the coefficient σ is determined from integral equation (1.7). The question of solvability of the problem remains open. We discuss the question of stability and uniqueness for a solution to the problem.

Let (σ_j,K_j) be a solution to the inverse problem with data $f_j(x,\alpha)$ and $F_j(x,\theta,\alpha)$, $j=1,2$. Denote by $w_j(x,\alpha)$ and $v_j(x,\theta,\alpha)$ the functions w and v that correspond to $\sigma=\sigma_j$ and $K=K_j$. Moreover, let

$$\tilde\sigma=\sigma_1-\sigma_2,\ \widetilde K = K_1-K_2,\ \widetilde w = w_1-w_2,\ \tilde v = v_1-v_2,\ \tilde g = \ln f_2-\ln f_1,\ \widetilde F = F_1-F_2.$$

Then the functions $\widetilde w(x,\alpha)$ and $\tilde v(x,\theta,\alpha)$ are connected with the functions $\tilde\sigma(x)$ and $\widetilde K(x,\cos\psi)$ by the relations

$$\widetilde w(x,\alpha) = \int\limits_{L(x,\alpha)} \tilde\sigma(\xi)\,ds, \tag{1.11}$$

$$\int\limits_{L(x,\alpha)} \tilde\sigma(\xi)\,ds = \tilde g(x,\alpha), \quad (x,\alpha)\in\partial_+G, \tag{1.12}$$

$$L(\sigma_1,K_1)\tilde v + \tilde\sigma(x)v_2(x,\theta,\alpha) + \int\limits_0^{2\pi} \widetilde K(x,\cos\theta')v_2(x,\theta'+\theta,\alpha)\,d\theta'$$

$$+\widetilde K(x,\cos(\theta-\alpha))\exp(-w_1(x,\alpha)) + K_2(x,\cos(\theta-\alpha))R(x,\alpha)\widetilde w(x,\alpha) = 0, \tag{1.13}$$

$$\tilde v|_{\partial_-Q} = 0, \quad \tilde v|_{\partial_+Q} = \widetilde F(x,\theta,\alpha), \tag{1.14}$$

where

$$R(x,\alpha) = -\int\limits_0^1 \exp\{-[w_1(x,\alpha)t + w_2(x,\alpha)(1-t)]\}\,dt. \tag{1.15}$$

We will make use of the stability estimate for a solution to equation (1.12) which ensues from the results of the article [5].

Lemma 1.2. *If $\sigma_j\in \mathrm{C}^1(D)$, $j=1,2$, then*

$$\int\limits_D \tilde\sigma^2(x)\,dx \le -\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{S_+(\alpha)} \frac{\partial\tilde g(x,\alpha)}{\partial l}\frac{\partial \tilde g(x,\alpha)}{\partial\alpha}\,dl\,d\alpha \equiv J_0(\tilde g), \tag{1.16}$$

where $S_+(\alpha)=\{x\in S \mid x\cdot\nu(\alpha)>0\}$, $\partial\tilde g(x,\alpha)/\partial l$ is the derivative in the direction l tangent to $S_+(\alpha)$, and dl is the line element on $S_+(\alpha)$.

The estimate for the function $K(x, \cos\psi)$ relies upon a priori estimates for the functions w, v, and $\tilde{v}$ that hold on condition that the functions σ and K belong to some fixed function class. We now describe this class. Suppose that $\sigma(x) \in \mathbf{C}^1(\overline{D})$, $K(x, \cos\psi) \in \mathbf{C}^1(\overline{G})$, and the following inequalities hold:

$$0 \le \sigma(x) \le \sigma_0, \quad |\nabla\sigma| \le \sigma_{01}, \quad \left(\int_0^{2\pi} \sup_{x \in D} K^2(x, \cos\psi)\, d\psi \right)^{1/2} \le K_0 < \frac{1}{\sqrt{8\pi}},$$
$$\left(\int_0^{2\pi} \sup_{x \in D} |\nabla K(x, \cos\psi)|^2\, d\psi \right)^{1/2} \le K_{01}, \tag{1.17}$$

where σ_0, σ_{01}, K_0, and K_{01} are fixed constants. Denote the set of functions (σ, K) satisfying these conditions by $\mathcal{M}$.

The following theorem is valid:

Theorem 1.1. *Suppose that $\sigma_j, K_j \in \mathcal{M}$, $j = 1, 2$. Then there exists $\delta > 0$ such that if $\sigma_0^2 + \sigma_{01}^2 + K_0^2 + K_{01}^2 < \delta^2$ then the function $\widetilde{K} = K_1 - K_2$ satisfies the estimate*

$$\int_G \widetilde{K}^2(x, \cos\psi)\, dx d\psi \le C[J(\widetilde{F}) + J_0(\tilde{g})], \tag{1.18}$$

where

$$J(\widetilde{F}) = -\int_0^{2\pi} \int_0^{2\pi} \int_{S_+(\theta)} \frac{\partial \widetilde{F}(x, \theta, \alpha)}{\partial l} \left(\frac{\partial \widetilde{F}(x, \theta, \alpha)}{\partial \theta} + \frac{\partial \widetilde{F}(x, \theta, \alpha)}{\partial \alpha} \right) dl d\theta d\alpha, \tag{1.19}$$

and the constant C depends only on σ_0, σ_{01}, K_0, and K_{01}.

The proof of the theorem is given in § 4. It is based on a priori estimates for the functions w, v, and $\tilde{v}$ in the class $(\sigma, K) \in \mathcal{M}$ which are established in § 3.

The following uniqueness theorem for the original problem is a simple consequence of Theorem 1.1.

Theorem 1.2. *The inverse problem has at most one solution in the set of functions $(\sigma, K) \in \mathcal{M}$ provided that σ, σ_{01}, K_0, and K_{01} are sufficiently small.*

The results of the present article can be generalized to the case of spaces of higher dimension.

§ 2. Proof of Lemma 1.1

Represent a solution to problem (1.1)–(1.3) in the form (1.5), with the function $w(x, \alpha)$ defined by equality (1.6), and substitute the function $u(x, \theta, \alpha)$ into equalities (1.1) and (1.3). As a result of this, we infer that the function $v(x, \theta, \alpha)$ is a solution to problem (1.8), (1.9). Taking the inverse of the differential operator $\nabla v \cdot \nu + \sigma(x) v$ by using boundary condition (1.9), we obtain an integral equation in the function $v(x, \theta, \alpha)$ as follows:

$$v(x, \theta, \alpha) = \int_{L(x,\theta)} \left\{ \int_0^{2\pi} K(\xi, \nu(\theta) \cdot \nu(\theta')) v(\xi, \theta', \alpha)\, d\theta' \right.$$
$$\left. + K(\xi, \nu(\theta) \cdot \nu(\alpha)) \exp(-w(\xi, \alpha)) \right\} \exp[w(\xi, \theta) - w(x, \theta)]\, ds, \tag{2.1}$$

where $\xi = (x - s\nu(\theta)) \in L(x, \theta)$ and $L(x, \theta)$ is the segment of the ray which was defined in § 1.

Every continuous solution to integral equation (2.1) is a weak solution to problem (1.8), (1.9). The expression $\nabla v \cdot \nu(\theta)$ is understood to be the derivative of the function v in the direction $\nu(\theta)$.

Lemma 2.1. *If $v(x,\theta,\alpha) \in \mathbf{C}(\overline{Q})$ and condition* (1.4) *is satisfied then*

$$\int_0^{2\pi} \sup_{x\in D} v^2(x,\theta,\alpha)\,d\theta \le \frac{(2K')^2}{(1-K'\sqrt{8\pi})^2}, \tag{2.2}$$

where the number K' is defined by formula (1.4).

PROOF. Since the function σ is nonnegative, we have the inequalities $w(\xi,\alpha) \ge 0$ and $w(\xi,\theta) - w(x,\theta) \le 0$. Therefore, the exponential factors in formula (2.1) do not exceed unity. Taking this fact into account, we square both sides of equality (2.1) and use the inequality $(a+b)^2 \le (1+\lambda)a^2 + (1+\lambda^{-1})b^2$, $\lambda > 0$, and the Cauchy–Bunyakovskiĭ inequality to obtain

$$\begin{aligned} v^2(x,\theta,\alpha) &\le (1+\lambda) \int\limits_{L(x,\theta)} \int_0^{2\pi} K^2(\xi,\nu(\theta)\cdot\nu(\theta'))\,d\theta'\,ds \int\limits_{L(x,\theta)} \int_0^{2\pi} v^2(\xi,\theta',\alpha)\,d\theta'\,ds \\ &\quad + (1+\lambda^{-1})s(x,\theta) \int\limits_{L(x,\theta)} K^2(\xi,\nu(\theta)\cdot\nu(\alpha))\,ds \\ &\le 4(1+\lambda)(K')^2 \int_0^{2\pi} \sup_{x\in D} v^2(x,\theta,\alpha)\,d\theta + 4(1+\lambda^{-1}) \sup_{x\in D} K^2(x,\cos(\theta-\alpha)). \end{aligned}$$

Integrating the inequality with respect to the variable θ from 0 to 2π, we find

$$\int_0^{2\pi} \sup_{x\in D} v^2(x,\theta,\alpha)\,d\theta \le 8\pi(K')^2(1+\lambda) \int_0^{2\pi} \sup_{x\in D} v^2(x,\theta,\alpha)\,d\theta + 4(1+\lambda^{-1}) \int_0^{2\pi} \sup_{x\in D} K^2(x,\cos\psi)\,d\psi.$$

Hence, if the condition $1 - 2\pi(2K')^2(1+\lambda) > 0$ is satisfied then

$$\int_0^{2\pi} \sup_{x\in D} v^2(x,\theta,\alpha)\,d\theta \le \frac{(1+\lambda^{-1})(2K')^2}{1-2\pi(2K')^2(1+\lambda)}.$$

Putting $\lambda = -1 + 1/K'\sqrt{8\pi}$, we obtain an optimal estimate which coincides with (2.2).

Now, we demonstrate that, under condition (1.4), equation (2.1) has a unique solution in the class $\mathbf{C}(\overline{Q})$. Rewrite (2.1) as the operator equation

$$v = Av, \tag{2.3}$$

where the operator A is defined by the right-hand side of (2.1). Consider equation (2.3) in the space $\mathbf{C}(G,\mathbf{L}_2[0,2\pi])$ comprising the functions continuous in the variables x and α and square summable in θ. Endow this space with the norm

$$\|v\| = \sup_{(x,\alpha)\in G} \left(\int_0^{2\pi} v^2(x,\theta,\alpha)\,d\theta \right)^{1/2}.$$

The operator A carries the space $\mathbf{C}(G,\mathbf{L}_2[0,2\pi])$ onto $\mathbf{C}(\overline{Q})$ which is embedded in $\mathbf{C}(G,\mathbf{L}_2[0,2\pi])$ and is a contraction. Indeed, as follows from the estimates obtained in the proof of Lemma 2.1, the inequality

$$\|Av' - Av''\|^2 \le 2\pi(2K')^2\|v'-v''\|^2$$

holds for every $(v',v'') \in \mathbf{C}(G,\mathbf{L}_2[0,2\pi])$. Since $K'\sqrt{8\pi} < 1$; therefore, A is a contraction on $\mathbf{C}(G,\mathbf{L}_2[0,2\pi])$. Then, by the Banach principle, there exists a unique element $v \in \mathbf{C}(G,\mathbf{L}_2[0,2\pi])$ satisfying (2.3). Recalling that $A : \mathbf{C}(G,\mathbf{L}_2[0,2\pi]) \to \mathbf{C}(\overline{Q})$, we have $v \in \mathbf{C}(\overline{Q})$. Lemma 1.1 is proven.

§ 3. A Priori Estimates

This section is preparatory to proving Theorem 1.1. It contains estimates for the functions w, v, and $\tilde{v}$ which are necessary for obtaining a stability estimate in the problem of determining the dispersion index. Moreover, we suppose that (σ, K) and (σ_j, K_j), $j = 1, 2$, belong to the set $\mathcal{M}$. The method for obtaining a priori estimates uses the approaches of the articles [5–7].

Lemma 3.1. *The function $w(x;\alpha)$ satisfies the inequalities*

$$\begin{gathered} 0 \le w(x,\alpha) \le 2\sigma_0, \quad |w_\alpha(x,\alpha)| \le 2(\sigma_0 + \sigma_{01}), \\ \sqrt{1 - |x|^2 + \big(x\cdot\nu(\alpha)\big)^2}\,|\nabla w(x,\alpha)| \le \sigma_0 + 2\sigma_{01}, \quad (x,\alpha) \in G, \end{gathered} \tag{3.1}$$

$$\int\limits_G w^2(x,\alpha)\,dx d\alpha \le 4\pi J_0(g), \qquad \int\limits_G w_\alpha^2(x,\alpha)\,dx d\alpha \le 4\pi J_0(g). \tag{3.2}$$

Here

$$J_0(g) = -\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{S_+(\alpha)} \frac{\partial g(x,\alpha)}{\partial l}\frac{\partial g(x,\alpha)}{\partial\alpha}\,dl d\alpha \ge 0, \tag{3.3}$$

where $g(x,\alpha) = -\ln f(x,\alpha)$, $\partial g(x,\alpha)/\partial l$ is the derivative of the function $g(x,\alpha)$ at a point $x \in S_+(\alpha) = \{x \in S \mid n(x)\cdot\nu(x) > 0\}$ in the direction l tangent to S, and dl is the line element on $S_+(\alpha)$.

PROOF. The first inequality in (3.1) is an obvious consequence of the inequalities $0 \le \sigma(x) \le \sigma_0$ and $s(x,\alpha) \le 2$. Differentiating (1.6) with respect to α, we find

$$\begin{gathered} w_\alpha(x,\alpha) = \sigma(x - s(x,\alpha)\nu(\alpha))\frac{\partial s(x,\alpha)}{\partial\alpha} - \int\limits_0^{s(x,\alpha)} s\nabla\sigma(x - s\nu(\alpha))\cdot\nu'(\alpha)\,ds, \\ \nabla w(x,\alpha) = \sigma(x - s(x,\alpha)\nu(\alpha))\nabla s(x,\alpha) + \int\limits_0^{s(x,\alpha)} \nabla\sigma(x - s\nu(\alpha))\,ds, \end{gathered} \tag{3.4}$$

where $\nu'(\alpha) \equiv \partial\nu(\alpha)/\partial\alpha = (-\sin\alpha, \cos\alpha)$. Since the length of the segment $L(x,\theta)$ is calculated by the formula $s(x,\alpha) = (x\cdot\nu(\alpha)) + \sqrt{1 - |x|^2 + (x\cdot\nu(\alpha))^2}$, we have

$$\left|\frac{\partial s(x,\alpha)}{\partial\alpha}\right| = \frac{|(x\cdot\nu'(\alpha))|s(x,\alpha)}{\sqrt{1 - |x|^2 + (x\cdot\nu(\alpha))^2}} \le 2,$$

$$|\nabla s(x,\alpha)| = \frac{|x - \nu\cdot s(x,\alpha)|}{\sqrt{1 - |x|^2 + (x\cdot\nu(\alpha))^2}} = \frac{1}{\sqrt{1 - |x|^2 + (x\cdot\nu(\alpha))^2}}.$$

The above inequalities and formulas (3.4) imply the second and third inequalities of (3.1).

To obtain inequalities (3.2), we make use of the differential relation

$$\nabla w\cdot\nu(\alpha) = \sigma(x) \tag{3.5}$$

between the functions $w(x,\alpha)$ and $\sigma(x)$ together with the following identity of the article [5]:

$$2(\nabla w\cdot\nu'(\alpha))\frac{\partial}{\partial\alpha}(\nabla w\cdot\nu(\alpha)) \equiv \frac{\partial}{\partial\alpha}[(\nabla w\cdot\nu(\alpha))(\nabla w\cdot\nu'(\alpha))] + \frac{\partial}{\partial x_1}(w_\alpha w_{x_2}) - \frac{\partial}{\partial x_2}(w_\alpha w_{x_1}) + |\nabla w|^2.$$

The left-hand side of the identity vanishes on solutions to equation (3.5). Integrating the resultant equality over the domain G and using periodicity of $w(x,\alpha)$ in α and the Green formula, we find that

$$\int\limits_G |\nabla w|^2(x,\alpha)\,dxd\alpha \leq -\int\limits_0^{2\pi}\int\limits_S w_\alpha(x,\alpha)\frac{\partial w(x,\alpha)}{\partial l}\,dld\alpha = 2\pi J_0(g). \tag{3.6}$$

Whence we infer that $J_0(g)$ is nonnegative, which fact is a necessary condition for solvability of integral equation (1.7).

On the other hand,

$$w(x,\alpha) = \int\limits_{L(x,\alpha)} \nabla w \cdot \nu(\alpha)\,ds.$$

Hence,

$$w^2(x,\alpha) \leq s(x,\alpha) \int\limits_{L(x,\alpha)} (\nabla w \cdot \nu(\alpha))^2\,ds. \tag{3.7}$$

Integrating inequality (3.7) over G, we find

$$\int\limits_G w^2(x,\alpha)\,dxd\alpha \leq \int\limits_G s(x,\alpha) \int\limits_{L(x,\alpha)} (\nabla w \cdot \nu(\alpha))^2\,dsdxd\alpha$$

$$= \int\limits_G (\nabla w \cdot \nu(\alpha))^2 \int\limits_{L^+(x,\alpha)} s\,dsdxd\alpha \leq 2\int\limits_G (\nabla w \cdot \nu(\alpha))^2\,dxd\alpha \leq 4\pi J_0(g). \tag{3.8}$$

In these intermediate calculations, $L^+(x,\alpha)$ denotes the segment of the ray emanating from the point $x \in D$ in the direction $\nu(\alpha)$ between the point x and the point of intersection of the ray with S.

Since formula (3.5) implies that

$$\nabla w_\alpha \cdot \nu(\alpha) = -\nabla w \cdot \nu'(\alpha),$$

we have

$$w_\alpha(x,\alpha) = \int\limits_{L(x,\alpha)} \nabla w_\alpha \cdot \nu(\alpha)\,ds = -\int\limits_{L(x,\alpha)} \nabla w \cdot \nu'(\alpha)\,ds.$$

Therefore, arguments analogous to those in the previous case lead to the inequality

$$\int\limits_G w_\alpha^2(x,\alpha)\,dxd\alpha \leq 2\int\limits_G (\nabla w \cdot \nu'(\alpha))^2\,dxd\alpha \leq 4\pi J_0(g).$$

The lemma is proven.

Lemma 3.2. *Under the condition $K_0\sqrt{2\pi^3} < 1$, a solution $v(x,\theta,\alpha)$ to problem (1.8), (1.9) satisfies the inequalities*

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} \sup_{x\in D} v^2(x,\theta,\alpha)\,d\theta d\alpha \leq \frac{8\pi K_0^2}{(1-K_0\sqrt{8\pi})^2} \equiv C_0,$$

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} \sup_{x\in D}[(1-|x|^2+(x\cdot\nu(\alpha))^2)|\nabla v(x,\theta,\alpha)|^2]\,d\theta d\alpha \leq C_{01}, \tag{3.9}$$

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} \sup_{x\in D}\left|\frac{\partial}{\partial\alpha}v(x,\alpha+\psi,\alpha)\right|^2\,d\psi d\alpha \leq C_{02},$$

where the constants C_{01} and C_{02} depend only on σ_0, σ_{01}, K_0, and K_{01}; moreover, $(C_{01}, C_{02}) \to 0$ as $(K_0, K_{01}) \to 0$.

PROOF. The first inequality in (3.9) follows from inequality (2.2) and the hypothesis $K' \le K_0 < 1/\sqrt{8\pi}$.

We now prove the second inequality in (3.9). First of all, observe that the presence of the factor $1 - |x|^2 + (x \cdot \nu(\alpha))^2$ under the integral sign is due to the fact that the function $|\nabla v(x, \theta, \alpha)|$ increases unboundedly as the variable x approaches the endpoints of the half-circle $S_+(\theta) = \{x \in S \mid x \cdot \nu(\theta) > 0\}$, whereas the product $(1 - |x|^2 + (x \cdot \nu(\alpha))^2)|\nabla v(x, \theta, \alpha)|$ remains a continuous function (see Lemma 2.3 of [6]).

Differentiating equality (2.1) with respect to the variable x_j, $j = 1, 2$, and introducing the notation

$$v_j(x, \theta, \alpha) = \frac{\partial}{\partial x_j} v(x, \theta, \alpha), \quad w_j(x, \alpha) = \frac{\partial}{\partial x_j} w(x, \alpha),$$

$$K_j(x, \cos(\theta - \alpha)) = \frac{\partial}{\partial x_j} K(x, \nu(\theta) \cdot \nu(\alpha)), \quad j = 1, 2,$$

we obtain the following integral equation in the functions v_j, $j = 1, 2$:

$$\begin{aligned} v_j(x, \theta, \alpha) = \Bigg[& K(\xi^*, \cos(\theta - \alpha)) \exp(-w(\xi^*, \alpha)) \\ & + \int_0^{2\pi} K(\xi^*, \cos\theta') v(\xi^*, \theta + \theta', \alpha)\, d\theta' \Bigg] (\exp[w(\xi^*, \theta) - w(x, \theta)]) \frac{\partial s(x, \alpha)}{\partial x_j} \\ & + \int_{L(x,\theta)} \Bigg\{ [K_j(\xi, \cos(\theta - \alpha)) + K(\xi, \cos(\theta - \alpha))(w_j(\xi, \theta) - w_j(x, \theta) - w_j(\xi, \alpha))] \\ & \times \exp(-w(\xi, \alpha)) + \int_0^{2\pi} [K_j(\xi, \cos\theta') v(\xi, \theta + \theta', \alpha) + K(\xi, \cos\theta') v_j(\xi, \theta + \theta', \alpha) \\ & + (w_j(\xi, \theta) - w_j(x, \theta)) K(\xi, \cos\theta') v(\xi, \theta + \theta', \alpha)] \Bigg\}\, d\theta' \exp[w(\xi, \theta) - w(x, \theta)]\, ds \end{aligned}$$

where $\xi^* = [x - \nu(\theta) s(x, \theta)] \in S$. This equation implies the inequality

$$\begin{aligned} \sup_{x \in D} \big[\sqrt{1 - |x|^2 + (x \cdot \nu(\alpha))^2} |v_j(x, \theta, \alpha)|\big] \le & \sup_{x \in D} |K(x, \cos(\theta - \alpha))| \\ & + \Big(K_0 + 2K_{01} + 2\pi K_0 \sup_{(x,\alpha) \in G} \big[\sqrt{1 - |x|^2 + (x \cdot \nu(\alpha))^2} |w_j(x, \alpha)|\big]\Big) \\ & \times \left(\int_0^{2\pi} \sup_{x \in D} |v(x, \theta, \alpha)|^2\, d\theta \right)^{1/2} + 2 \sup_{x \in D} |K_j(x, \cos(\theta - \alpha))| \\ & + 3\pi \sup_{x \in D} |K(x, \cos(\theta - \alpha))| \sup_{(x,\alpha) \in G} \big[\sqrt{1 - |x|^2 + (x \cdot \nu(\alpha))^2} |w_j(x, \alpha)|\big] \\ & + \pi K_0 \left(\int_0^{2\pi} \sup_{x \in D} \big[\sqrt{1 - |x|^2 + (x \cdot \nu(\alpha))^2} |v_j(x, \theta, \alpha)|\big]^2\, d\theta \right)^{1/2}. \end{aligned}$$

Here we have used the relations

$$\sup_{\theta'\in[0,2\pi]}(1-|\xi|^2+(\xi\cdot\nu(\theta+\theta'))^2)^{-1/2}=(1-|\xi|^2)^{-1/2},$$

$$\int\limits_{L(x,\theta)}\frac{ds}{\sqrt{1-|\xi|^2}}=\int\limits_0^{s(x,\theta)}\frac{ds}{\sqrt{1-|x-s\nu(\theta)|^2}}\le\pi.$$

Squaring both sides of the preceding inequality and integrating the result with respect to θ and α, we find that

$$I_j\equiv\int\limits_0^{2\pi}\int\limits_0^{2\pi}\sup_{x\in D}\left[\sqrt{1-|x|^2+(x\cdot\nu(\alpha))^2}|v_j(x,\theta,\alpha)|\right]^2 d\theta d\alpha\le 2\pi^3K_0^2(1+\lambda)I_j$$

$$+12\pi(1+\lambda^{-1})\Big\{K_0^2+4K_{01}^2+9\pi^2K_0^2\sup_{(x,\alpha)\in G}\left(\sqrt{1-|x|^2+(x\cdot\nu(\alpha))^2}|w_j(x,\alpha)|\right)^2$$

$$+\left[K_0^2+4K_{01}^2+4\pi^2K_0^2\sup_{(x,\alpha)\in G}\left(\sqrt{1-|x|^2+(x\cdot\nu(\alpha))^2}|w_j(x,\alpha)|\right)^2\right]\int\limits_0^{2\pi}\int\limits_0^{2\pi}\sup_{x\in D}v^2(x,\theta,\alpha)\,d\theta d\alpha\Big\}$$

for every $\lambda>0$. Putting $\lambda=-1+1/K_0\sqrt{2\pi^3}$ and using inequality (3.1) and the first inequality in (3.9), we obtain the following estimate for I_j:

$$I_j\le\frac{1}{2}C_{01},\quad j=1,2.$$

Here the constant C_{01} is defined by the formula

$$C_{01}=\frac{24\pi}{(1-K_0\sqrt{2\pi^3})^2}(1+C_0)\left[K_0^2+4K_{01}^2+9\pi^2K_0^2(\sigma_0+2\sigma_{01})^2\right].$$

Hence, the second inequality of (3.9) ensues.

To prove the third inequality in (3.9), put $\theta=\alpha+\psi$ in equation (1.8), denote $v(x,\alpha+\psi,\alpha)=\hat v(x,\psi,\alpha)$, and differentiate the so-obtained equality with respect to α. Write down the result of differentiation as follows:

$$\nabla\hat v_\alpha(x,\psi,\alpha)\cdot\nu(\alpha+\psi)+\sigma(x)\hat v_\alpha(x,\psi,\alpha)$$

$$=-\Big[\nabla\hat v(x,\psi,\alpha)\cdot\nu'(\alpha+\psi)+\int\limits_0^{2\pi}K(x,\cos\theta')\hat v_\alpha(x,\theta'+\psi,\alpha)\,d\theta'$$

$$-K(x,\cos\psi)w_\alpha(x,\alpha)\exp(-w(x,\alpha))\Big]\equiv h(x,\psi,\alpha).$$

Here $\hat v_\alpha(x,\psi,\alpha)=\partial\hat v(x,\psi,\alpha)/\partial\alpha$, $\nu'(\alpha)=(-\sin\alpha,\cos\alpha)$. Formula (1.9) implies that the function $\hat v_\alpha(x,\psi,\alpha)$ satisfies the condition

$$\hat v_\alpha(x,\psi,\alpha)=0,\quad (x,\theta,\alpha)\in\partial_-Q.$$

Consequently, the following representation holds:

$$\hat{v}_\alpha(x,\psi,\alpha) = \int\limits_{L(x,\alpha+\psi)} h(\xi,\psi,\alpha)\exp[w(\xi,\alpha+\psi) - w(x,\alpha+\psi)]\,ds,$$

where $\xi = x - s\nu(\alpha+\psi)$. Since

$$\sqrt{1-|\xi|^2+(\xi\cdot\nu(\theta))^2}\,|_{\xi=x-s\nu(\theta)} = \sqrt{1-|x|^2+(x\cdot\nu(\theta))^2},$$

$$s(x,\theta) \le 2\sqrt{1-|x|^2+(x\cdot\nu(\theta))^2} \le 2,$$

the representation for the function $\hat{v}_\alpha(x,\psi,\alpha)$ implies the estimate

$$\begin{aligned}|\hat{v}_\alpha(x,\psi,\alpha)| &\le \frac{s(x,\alpha+\psi)}{\sqrt{1-|x|^2+(x\cdot\nu(\alpha+\psi))^2}}\sup_{x\in D}\big[\sqrt{1-|x|^2+(x\cdot\nu(\alpha+\psi))^2}|h(x,\psi,\alpha)|\big]\\ &\le 2\sup_{x\in D}\big[\sqrt{1-|x|^2+(x\cdot\nu(\alpha+\psi))^2}|h(x,\psi,\alpha)|\big].\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{x\in D}|\hat{v}_\alpha(x,\psi,\alpha)| &\le 2\sup_{x\in D}\Big[\sqrt{1-|x|^2+\big(x\cdot\nu(\alpha+\psi)\big)^2}\,|\nabla\hat{v}(x,\psi,\alpha)|\Big]\\ &+2K_0\Bigg(\int\limits_0^{2\pi}\sup_{x\in D}|\hat{v}_\alpha(x,\psi,\alpha)|^2\,d\psi\Bigg)^{1/2} + 2\sup_{x\in D}|K(x,\cos\psi)|\sup_{(x,\alpha)\in G}|w_\alpha(x,\alpha)|.\end{aligned}$$

Hence, we find that

$$\begin{aligned}J \equiv \int\limits_0^{2\pi}\int\limits_0^{2\pi}\sup_{x\in D}|\hat{v}_\alpha(x,\psi,\alpha)|^2\,d\psi d\alpha &\le 8\pi K_0^2(1+\lambda)J + 8(1+\lambda^{-1})\\ \times\Bigg\{8\pi K_0^2(\sigma_0+\sigma_{01})^2 &+ \int\limits_0^{2\pi}\sup_{x\in D}\big\{\big[\sqrt{1-|x|^2+(x\cdot\nu(\theta))^2}|\nabla v(x,\theta,\alpha)|\big]^2\,d\theta\,d\alpha\Bigg\}.\end{aligned}$$

Putting $\lambda = -1 + 1/K_0\sqrt{8\pi}$ and using the second inequality of (3.9), we validate the last inequality of (3.9), with the constant C_{02} determined by the equality

$$C_{02} = \frac{8}{(1-K_0\sqrt{8\pi})^2}\big[C_{01} + 8\pi K_0^2(\sigma_0+\sigma_{01})^2\big].$$

It is obvious that $(C_{01}, C_{02}) \to 0$ as $(K_0, K_{01}) \to 0$. The lemma is proven.

The estimates of the forthcoming two lemmas are needed for the proof of Lemma 3.5 and are based on the relation

$$\begin{aligned}&\nabla\left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha)\right)\cdot\nu(\alpha+\psi) + \nabla\tilde{v}(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi)\\ &+\sigma_1(x)\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha) + \int\limits_0^{2\pi}K_1(x,\cos\theta')\frac{\partial}{\partial\alpha}\tilde{v}(x,\theta'+\alpha+\psi,\alpha)\,d\theta' + N(x,\psi,\alpha) = 0, \quad (3.10)\end{aligned}$$

where

$$N(x,\psi,\alpha) = \tilde{\sigma}(x)\frac{\partial}{\partial\alpha}v_2(x,\alpha+\psi,\alpha) - \widetilde{K}(x,\cos\psi)w_{1\alpha}(x,\alpha)\exp(-w_1(x,\alpha))$$
$$+K_2(x,\cos\psi)[R_\alpha(x,\alpha)\widetilde{w}(x,\alpha) + R(x,\alpha)\widetilde{w}_\alpha(x,\alpha)], \tag{3.11}$$

$$R_\alpha(x,\alpha) = \int_0^1 [w_{1\alpha}(x,\alpha)t + w_{2\alpha}(x,\alpha)(1-t)]\exp\{-[w_1(x,\alpha)t + w_2(x,\alpha)(1-t)]\}\,dt. \tag{3.12}$$

Relation (3.10) is obtained from equality (1.13) by substituting $\alpha+\psi$ for θ and differentiating the equality with respect to α.

Lemma 3.3. *If the function $K(x,\cos\psi)$ satisfies the condition $2K_0\sqrt{\pi} < 1$ then the following estimate holds for every $\mu > 0$:*

$$\int_Q \left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha)\right)^2 dx\,d\psi\,d\alpha$$
$$\le C_1\left[(1+\mu)\int_Q (\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu'(\theta))^2\,dx\,d\theta\,d\alpha + (1+\mu^{-1})\int_Q N^2(x,\psi,\alpha)\,dx\,d\psi\,d\alpha\right], \tag{3.13}$$

where $C_1 = 2/(1-2K_0\sqrt{\pi})^2$.

PROOF. Taking the inverse of the differential operator $\nabla\cdot\nu + \sigma_1$ in (3.10), we find that

$$\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha) = -\int_{L(x,\alpha+\psi)} \Bigg\{\nabla\tilde{v}(\xi,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi)$$
$$+\int_0^{2\pi} K_1(x,\cos\theta')\frac{\partial}{\partial\alpha}\tilde{v}(\xi,\theta'+\alpha+\psi,\alpha)\,d\theta' + N(\xi,\psi,\alpha)\Bigg\}\exp[w_1(\xi,\alpha+\psi) - w_1(x,\alpha+\psi)]\,ds.$$

Hence,

$$\left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha)\right)^2$$
$$\le s(x,\alpha+\psi)\int_{L(x,\alpha+\psi)} \Bigg\{(1+\lambda)K_0^2\int_0^{2\pi}\left(\frac{\partial}{\partial\alpha}\tilde{v}(\xi,\theta'+\alpha+\psi,\alpha)\right)^2 d\theta'$$
$$+(1+\lambda^{-1})[(1+\mu)(\nabla\tilde{v}(\xi,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))^2 + (1+\mu^{-1})N^2(\xi,\psi,\alpha)]\Bigg\}\,ds. \tag{3.14}$$

Integrating inequality (3.14) over Q and changing the order of integration, we infer that

$$\int\limits_Q \left(\frac{\partial}{\partial\alpha}\tilde v(x,\alpha+\psi,\alpha)\right)^2 dxd\psi d\alpha \le \int\limits_Q \left\{(1+\lambda)2\pi K_0^2\left(\frac{\partial}{\partial\alpha}\tilde v(\xi,\alpha+\psi,\alpha)\right)^2\right.$$

$$\left.+(1+\lambda^{-1})[(1+\mu)(\nabla\tilde v(\xi,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))^2+(1+\mu^{-1})N^2(\xi,\psi,\alpha)]\int\limits_{L^+(x,\alpha+\psi)} s\,ds\right\} d\xi d\psi d\alpha$$

$$\le 2\int\limits_Q\left\{(1+\lambda)2\pi K_0^2\left(\frac{\partial}{\partial\alpha}\tilde v(x,\alpha+\psi,\alpha)\right)^2\right.$$

$$\left.+(1+\lambda^{-1})[(1+\mu)(\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))^2+(1+\mu^{-1})N^2(x,\psi,\alpha)]\right\} dxd\psi d\alpha.$$

Putting $\lambda=-1+1/2K_0\sqrt{\pi}$, we arrive at inequality (3.13).

Lemma 3.4. *Under the conditions of Theorem 1.1, the following inequality holds for arbitrary $\lambda>0$ and $\mu>0$:*

$$\int\limits_Q[(\nabla\tilde v\cdot\nu(\theta))^2+(1-\lambda)(\nabla\tilde v\cdot\nu(\theta))^2]\,dxd\theta d\alpha \le \lambda^{-1}(1+\mu^{-1})\int\limits_Q N^2(x,\psi,\alpha)\,dxd\psi d\alpha$$

$$+2\lambda^{-1}(1+\mu)(\sigma_0^2+2\pi K_0^2)\int\limits_Q\left(\frac{\partial}{\partial\alpha}\tilde v(x,\alpha+\psi,\alpha)\right)^2 dxd\psi d\alpha + J(\widetilde F), \tag{3.15}$$

where the expression $J(\widetilde F)$ is defined by formula (1.19).

Proof. We use equality (3.10). We leave the first two summands of the equality on the left-hand side, transpose the others to the right-hand side, and afterwards multiply both sides of the equality by $2\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi)$. We write down the result as follows:

$$T(x,\psi,\alpha)\equiv 2(\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))\frac{\partial}{\partial\alpha}(\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu(\alpha+\psi))$$

$$=-2(\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))\left[\sigma_1(x)\frac{\partial}{\partial\alpha}\tilde v(x,\alpha+\psi,\alpha)\right.$$

$$\left.+\int_0^{2\pi}K_1(x,\cos\theta')\frac{\partial}{\partial\alpha}\tilde v(x,\theta'+\alpha+\psi,\alpha)\,d\theta'+N(x,\psi,\alpha)\right]. \tag{3.16}$$

Transform the left-hand side of (3.16) to the form

$$T(x,\psi,\alpha)=\frac{\partial}{\partial\alpha}[(\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))(\nabla\tilde v(x,\alpha+\psi,\alpha)\cdot\nu(\alpha+\psi))]$$

$$+\frac{\partial}{\partial x_1}\left[\left(\frac{\partial}{\partial\alpha}\tilde v(x,\alpha+\psi,\alpha)\right)\tilde v_{x_2}(x,\alpha+\psi,\alpha)\right]$$

$$-\frac{\partial}{\partial x_2}\left[\left(\frac{\partial}{\partial\alpha}\tilde v(x,\alpha+\psi,\alpha)\right)\tilde v_{x_1}(x,\alpha+\psi,\alpha)\right]+|\nabla\tilde v(x,\alpha+\psi,\alpha)|^2.$$

Whence, by periodicity of the function $\tilde{v}(x,\alpha+\psi,\alpha)$ in α, we find that

$$\int_Q T(x,\psi,\alpha)\,dx\,d\psi\,d\alpha = \int_Q |\nabla\tilde{v}(x,\alpha+\psi,\alpha)|^2\,dxd\psi d\alpha$$

$$+\int_{\partial_+Q\bigcup\partial_-Q}\left(\frac{\partial}{\partial l}\tilde{v}(x,\alpha+\psi,\alpha)\right)\left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha)\right)dld\psi d\alpha$$

$$=\int_Q[(\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2+(\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu'(\theta))^2]\,dxd\theta d\alpha - J(\widetilde{F}). \tag{3.17}$$

On the other hand, for arbitrary $\lambda>0$ and $\mu>0$ we have

$$T(x,\psi,\alpha)\le\lambda(\nabla\tilde{v}(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))^2$$

$$+\lambda^{-1}\left\{2(1+\mu)\left[\sigma_0^2\left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha)\right)^2+K_0^2\int_0^{2\pi}\left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\theta+\alpha,\alpha)\right)^2d\theta\right]\right.$$

$$\left.+(1+\mu^{-1})N^2(x,\psi,\alpha)\right\}.$$

Hence,

$$\int_Q T(x,\psi,\alpha)\,dxd\psi d\alpha\le\lambda\int_Q(\nabla\tilde{v}(x,\alpha+\psi,\alpha)\cdot\nu'(\alpha+\psi))^2\,dxd\psi d\alpha$$

$$+\lambda^{-1}\left\{2(1+\mu)\left[(\sigma_0^2+2\pi K_0^2)\int_Q\left(\frac{\partial}{\partial\alpha}\tilde{v}(x,\alpha+\psi,\alpha)\right)^2d\theta\right]dxd\psi d\alpha\right.$$

$$\left.+(1+\mu^{-1})\int_Q N^2(x,\psi,\alpha)\,dx\,d\psi\,d\alpha\right\}. \tag{3.18}$$

Relations (3.17) and (3.18) imply the assertion of the lemma.

Lemma 3.5. *Suppose that σ_0 and K_0 satisfy the inequality $8C_1(\sigma_0^2+2\pi K_0^2)<1$ and the hypothesis of Lemma 3.3. Then the following estimate is valid:*

$$\int_Q(\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2\,dxd\theta d\alpha\le C_2\int_Q N^2(x,\psi,\alpha)\,dxd\psi d\alpha+J(\widetilde{F}), \tag{3.19}$$

where $C_2=\left[4+\sqrt{8C_1(\sigma_0^2+2\pi K_0^2)}\right]/2\left[1-\sqrt{8C_1(\sigma_0^2+2\pi K_0^2)}\right]$.

PROOF. Substituting estimate (3.13) for the integral of the derivative of $\tilde{v}(x,\alpha+\psi,\alpha)$ with respect to α in the right-hand side of (3.15), we find that

$$\int_Q(\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2\,dxd\theta d\alpha$$

$$\le\lambda^{-1}(1+\mu^{-1})\left[1+2C_1(\sigma_0^2+2\pi K_0^2)(1+\mu)\right]\int_Q N^2(x,\psi,\alpha)\,dxd\psi d\alpha+J(\widetilde{F}), \tag{3.20}$$

provided that $1-\lambda-2\lambda^{-1}C_1(\sigma_0^2+2\pi K_0^2)(1+\mu)^2 \geq 0$. The last inequality may hold only if $(1+\mu)^2 \leq 1/8C_1(\sigma_0^2+2\pi K_0^2)$. Under the conditions of the lemma, we have $8C_1(\sigma_0^2+2\pi K_0^2) < 1$; therefore, it is possible to choose appropriate positive parameters λ and μ. We take $\lambda = 1/2$ and $\mu = -1+1/\sqrt{8C_1(\sigma_0^2+2\pi K_0^2)}$. Then formula (3.20) implies inequality (3.19).

Lemma 3.6. *Under the condition of Lemma 3.2, the following inequality holds for every $\lambda > 0$:*

$$\int_Q N^2(x,\psi,\alpha)\,dxd\psi d\alpha \leq 8\pi(\sigma_0+\sigma_{01})^2(1+\lambda)\int_G \widetilde{K}^2(x,\cos\psi)\,dxd\psi + (1+\lambda^{-1})C_{03}J_0(\tilde{g}), \tag{3.21}$$

where $C_{03} = 3\big[C_{01}+4\pi K_0^2(1+4(\sigma_0+\sigma_{01}))^2\big]$ and the expression $J_0(\tilde{g})$ is defined by formula (1.16).

PROOF. Inequalities (3.1) and formula (3.12) imply that

$$0 < R(x,\alpha) \leq 1, \quad |R_\alpha(x,\alpha)| \leq 2(\sigma_0+\sigma_{01}). \tag{3.22}$$

Squaring both sides of (3.11) and integrating the result over Q, we arrive at the inequality

$$\begin{aligned}\int_Q N^2(x,\psi,\alpha)\,dxd\psi d\alpha &\leq 8\pi(\sigma_0+\sigma_{01})^2(1+\lambda)\int_G \widetilde{K}^2(x,\cos\psi)\,dxd\psi \\ &+3(1+\lambda^{-1})\Bigg\{\int_D \tilde{\sigma}^2(x)\,dx\int_0^{2\pi}\int_0^{2\pi}\sup_{x\in D}\left|\frac{\partial}{\partial\alpha}v_2(x,\alpha+\psi,\alpha)\right|^2 d\psi d\alpha \\ &+K_0^2\Bigg[\sup_{(x,\alpha)\in G} R_\alpha^2(x,\alpha)\int_G \widetilde{w}^2(x,\alpha)\,dxd\alpha + \sup_{(x,\alpha)\in G} R^2(x,\alpha)\int_G \widetilde{w}_\alpha^2(x,\alpha)\,dxd\alpha\Bigg]\Bigg\}.\end{aligned}$$

Using inequalities (3.9) and (3.22) and estimates for $\widetilde{w}$ and $\widetilde{w}_\alpha$ similar to (3.2), we obtain inequality (3.21).

The last two lemmas yield the next lemma which closes the section:

Lemma 3.7. *Under the conditions of Lemmas 3.2 and 3.5, the following estimate holds for every $\lambda > 0$:*

$$\begin{aligned}&\int_Q (\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2\,dxd\theta d\alpha \\ &\leq J(\widetilde{F}) + (1+\lambda^{-1})C_{04}J_0(\tilde{g}) + (1+\lambda)C_{05}\int_G \widetilde{K}^2(x,\cos\psi)\,dxd\psi,\end{aligned} \tag{3.23}$$

where $C_{04} = C_2C_{03}$ and $C_{05} = 8\pi C_2(\sigma_0+\sigma_{01})^2$.

Observe that while proving Lemmas 3.3–3.5 we estimated the integrals without substantiating their existence. Under the conditions of Theorem 1.1, the derivatives of the function $\tilde{v}(x,\theta,\alpha)$ with respect to x_i, $i=1,2$, behave like the corresponding derivatives of the function $v(x,\theta,\alpha)$, i.e., have singularities of the type $(1-|x|^2+(x\cdot\nu(\alpha))^2)^{-1/2}$ in a neighborhood of each endpoint x of the half-circle $S_+(\theta)$. These singularities are however weak, not destroying the existence of the integrals in question.

§ 4. Proof of Theorem 1.1

We make use of equality (1.13). Multiply the equality by $\exp(w_1(x,\alpha))$, transpose the term $\widetilde{K}(x,\cos(\theta-\alpha))$ to the other side of the equality, and square the resultant equality. Using estimate (3.1) and the Cauchy–Bunyakovskiĭ inequality, we obtain the inequality

$$\widetilde{K}^2(x,\cos(\theta-\alpha)) \le \exp(4\sigma_0)\Big\{(1+\lambda)\int_0^{2\pi}\widetilde{K}^2(x,\cos\psi)\,d\psi \sup_{x\in D}\int_0^{2\pi} v_2^2(x,\theta',\alpha)\,d\theta'$$
$$+(1+\lambda^{-1})[L(\sigma_1,K_1)\tilde{v}(x,\theta,\alpha)+\tilde{\sigma}(x)v_2(x,\theta,\alpha)$$
$$+K_2(x,\cos(\theta-\alpha))R(x,\alpha)\widetilde{w}(x,\alpha)]^2\Big\}$$

which is valid for every $\lambda > 0$. Integrate the inequality over Q and use inequality (3.9) for estimating the integral of $v_2(x,\theta,\alpha)$. After simple transformations, we obtain

$$2\pi\int_G \widetilde{K}^2(x,\cos\psi)\,dxd\psi \le \exp(4\sigma_0)\Big\{(1+\lambda)2\pi C_0\int_G\widetilde{K}^2(x,\cos\psi)\,dxd\psi$$
$$+(1+\lambda^{-1})\int_Q[L(\sigma_1,K_1)\tilde{v}(x,\theta,\alpha)+\tilde{\sigma}(x)v_2(x,\theta,\alpha)$$
$$+K_2(x,\cos(\theta-\alpha))R(x,\alpha)\widetilde{w}(x,\alpha)]^2\,dxd\theta\,d\alpha\Big\}.$$

Suppose that σ_0 and K_0 are so small that $C_0\exp(4\sigma_0) < 1$. Putting $\lambda = -1 + 1/\sqrt{C_0}\exp(2\sigma_0)$, we then obtain the estimate

$$\int_G \widetilde{K}^2(x,\cos\psi)\,dxd\psi \le C_3\int_Q[L(\sigma_1,K_1)\tilde{v}(x,\theta,\alpha)$$
$$+\tilde{\sigma}(x)v_2(x,\theta,\alpha)+K_2(x,\cos(\theta-\alpha))R(x,\alpha)\widetilde{w}(x,\alpha)]^2\,dxd\theta d\alpha, \tag{4.1}$$

with $C_3 = \exp(4\sigma_0)/2\pi[1-\sqrt{C_0}\exp(2\sigma_0)]^2$.

Estimate the right-hand side of (4.1). Since $R(x,\alpha) \le 1$, the inequality

$$\int_Q[L(\sigma_1,K_1)\tilde{v}(x,\theta,\alpha)+\tilde{\sigma}(x)v_2(x,\theta,\alpha)+K_2(x,\cos(\theta-\alpha)R(x,\alpha)\widetilde{w}(x,\alpha)]^2\,dxd\theta d\alpha$$

$$\le\int_Q\Big\{(1+\mu)(\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2+4(1+\mu^{-1})\Big[\sigma_0^2\tilde{v}^2(x,\theta,\alpha)$$
$$+K_0^2\int_0^{2\pi}\tilde{v}^2(x,\theta',\alpha)\,d\theta'+\tilde{\sigma}^2(x)v_2^2(x,\theta,\alpha)+K_0^2\widetilde{w}^2(x,\alpha)\Big]\Big\}\,dxd\theta d\alpha$$

is valid for every $\mu > 0$. Use estimates (1.16) and (3.9) and the inequalities

$$\int_Q\tilde{v}^2(x,\theta,\alpha)\,dxd\theta d\alpha \le 2\int_Q\Big(\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta)\Big)^2\,dxd\theta d\alpha,$$

$$\int_G \widetilde{w}^2(x,\alpha)\,dx d\alpha \le 4\pi J_0(\tilde{g})$$

similar to inequalities (3.8) and (3.2). Then formula (4.1) implies that

$$\int_G \widetilde{K}^2(x,\cos\psi)\,dx d\psi \le 4C_3(1+\mu^{-1})\big[C_0+8\pi^2K_0^2\big]\,J_0(\tilde{g})$$

$$+C_3\big[(1+\mu)+8(1+\mu^{-1})(\sigma_0^2+2\pi K_0^2)\big]\int_Q (\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2\,dx d\theta d\alpha. \tag{4.2}$$

Putting $\mu=\mu^*=\sqrt{8(\sigma_0^2+2\pi K_0^2)}$, we obtain the inequality

$$\int_G \widetilde{K}^2(x,\cos\psi)\,dx d\psi \le 4C_3[1+(\mu^*)^{-1}]\big[C_0+8\pi^2K_0^2\big]\,J_0(\tilde{g})$$

$$+C_3(1+\mu^*)^2\int_Q (\nabla\tilde{v}(x,\theta,\alpha)\cdot\nu(\theta))^2\,dx d\theta d\alpha. \tag{4.3}$$

Suppose that the inequalities $\sqrt{2\pi^3}K_0<1$ and $8C_1(\sigma_0^2+2\pi K_0^2)<1$ are satisfied. In this case Lemma 3.7 and inequality (3.23) are valid. Using the latter, we arrive at the inequality

$$\int_G \widetilde{K}^2(x,\cos\psi)\,dx d\psi \le C_4(1+\lambda)\int_G \widetilde{K}^2(x,\cos\psi)\,dx d\psi + C_5 J(\widetilde{F}) + C_6(\lambda)J_0(\tilde{g}), \tag{4.4}$$

where $C_4=(1+\mu^*)^2C_3C_{05}$, $C_5=(1+\mu^*)^2C_3$, and $C_6(\lambda)=(1+\lambda^{-1})C_3C_{04}(1+\mu^*)^2+4C_3[1+(\mu^*)^{-1}](C_0+8\pi^2K_0^2)$.

The constants C_0, C_1, C_2, and C_3 have finite limits as $(\sigma_0,\sigma_{01},K_0,K_{01})\to 0$. Moreover, $C_0\to 0$ and $C_4\to 0$. Therefore, there exists a number $\delta>0$ such that, for arbitrary (σ_0, σ_{01}, K_0, K_{01}) satisfying the inequality $\sigma_0^2+\sigma_{01}^2+K_0^2+K_{01}^2<\delta^2$, the inequalities

$$\sqrt{2\pi^3}K_0<1,\quad C_0\exp(4\sigma_0)<1,\quad 8C_1(\sigma_0^2+2\pi K_0^2)<1,\quad C_4<1$$

hold simultaneously. Putting $\lambda=\lambda^*=-1+1/\sqrt{C_4}$ in inequality (4.4) and denoting

$$C=\max(C_5,C_6(\lambda^*))\,/(1-\sqrt{C_4}),$$

we now arrive at estimate (1.18).

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