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ON CERTAIN CLASSES OF METRIC SPACES

P. V. Chernikov UDC 513.83

The spaces analogous to absolute retracts were studied in the series of articles [1–5]. In particular, the notion of absolute t-retract was defined in [5] which helped to generalize the classical Luzin theorem on approximation of measurable functions by continuous functions and prove the corresponding converse assertion. The notion of absolute t-retract was introduced in [4]. In the present article we establish some properties of absolute (neighborhood) t-retracts.

We now present the relevant definitions.

A closed subset A in a compact metric space X is called a *t-retract* of X if for every Radon measure $\mu \geq 0$ on A and every $\varepsilon > 0$ there exists a compact subset $A_\varepsilon^\mu \subset A$ such that $\mu(A \setminus A_\varepsilon^\mu) \leq \varepsilon$ and a continuous map $r_\varepsilon^\mu : X \to A$ such that $r_\varepsilon^\mu(x) = x$ for all $x \in A\varepsilon^\mu$.

A compact metric space Y is called an *absolute t-retract* (see [5]) if, given an arbitrary compact metric space X, each closed subset A of X homeomorphic to Y is a t-retract. We denote by AR_t the totality of all absolute t-retracts.

A closed subset A in a compact metric space X is called a *neighborhood t-retract* of X if there exists a neighborhood U of A in X satisfying the following condition: for every Radon measure $\mu \geq 0$ on A and every number $\varepsilon > 0$ there exists a compact subset $A_\varepsilon^\mu \subset A$ such that $\mu(A \setminus A_\varepsilon^\mu) \leq \varepsilon$ and a continuous map $r_\varepsilon^\mu : U \to A$ such that $r_\varepsilon^\mu(x) = x$ for all $x \in A_\varepsilon^\mu$.

A compact metric space Y is called an *absolute neighborhood t-retract* (see [4]) if, given an arbitrary compact metric space X, each closed subset A of X homeomorphic to Y is a t-retract.

The notion of absolute σ-retract is introduced in [5]. Following [5], we denote the totality of all absolute σ-retracts by AR_σ.

The following assertions [4] hold:

Lemma 1. *If Y is a connected compact metric space and $Y \in ANR$ then $Y \in AR_\sigma$.*

Lemma 2. *If a compact set Y belongs to AR_t then Y is arcwise connected.*

Lemma 3. *A compact metric space Y belongs to AR_t if and only if $Y \in ANR_t$ and Y is connected.*

Theorem 1. *If $Y \in ANR_t$ then $Y = \bigcup_{i=1}^{n} Y_i$ with $Y_i \cap Y_j = \varnothing$ for $i \neq j$, where $Y_i \in AR_t$, $i = 1, \dots, n$, and $n \in \{1, 2 \dots\}$.*

PROOF. We may assume that $Y \subset Q$. There exists a neighborhood U of Y in Q satisfying the following condition: for every Radon measure $\mu \geq 0$ on Y and every $\varepsilon > 0$ there exists a compact set A_ε^μ such that $\mu(Y \setminus A_\varepsilon^\mu) \leq \varepsilon$ and a continuous map $r_\varepsilon^\mu : U \to Y$ such that $r_\varepsilon^\mu(x) = x$ for all $x \in A_\varepsilon^\mu$.

Let $\{Y_i\}_{i \in S}$ stand for the set of all connected components of Y. Suppose that S is infinite. Choose $x_i \in Y_i$, $i \in S$. There exists a sequence $\{z_k\}_{k=1}^\infty \subset \{x_i\}_{i \in S}$ convergent to some point $z \in Z \in \{Y_i\}_{i \in S}$ and a ball $B(z, \rho) = \{x \in Q : \|x - z\| < \rho\}$, $\rho > 0$, such that $B(z, \rho) \subset U$. There exists a number N such that $z_k \in B(z, \rho)$ for all $k \geq N$. Look at the points z and z_N. We can suppose that $z_N \notin Z$. We define the Radon measure $\delta \geq 0$ on Y as follows: $\delta(\{z\}) = \delta(\{z_N\}) = 1$ and $\delta(Y \setminus \{z, z_N\}) = 0$. There exists a continuous map $r : U \to Y$ such that $r(z) = z$ and $r(z_N) = z_N$. Consider the interval $I = \{sz + (1 - s)z_N : 0 \leq s \leq 1\}$, $I \subset B(z, \rho)$. The set $r(I)$ is included in Y; it is connected, and $r(I) \cap Z \neq \varnothing$. Consequently, the compact set Z is not a maximal connected subset in Y. A contradiction.

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Thus, S is finite. Let $S = \{1, \dots, n\}$. Obviously, $Y_i \in ANR_t$, $i = 1, \dots, n$. By Lemma 3 the compact set Y_i belongs to AR_t, $i = 1, \dots, n$. The theorem is proven.

Theorem 1 was formulated in [4] without proof.

Now we consider Cartesian products of absolute neighborhood t-retracts.

Theorem 2. *The Cartesian product* $X = \prod_{n=1}^{\infty} X_n$ *is an absolute neighborhood t-retract if and only if every* X_n *belongs to* ANR_t *and almost all* X_n *belong to* AR_t.

PROOF. Suppose that $X_n \in ANR_t$ for all $n \geq 1$ and there exists a number N such that $X_n \in AR_t$ for all $n > N$. From Theorem 6 of [6], we have $\prod_{n=1}^{N} X_n \in ANR_t$. By Theorem 1 of [7], $\prod_{n=N+1}^{\infty} X_n \in AR_t$. Hence, by Theorem 6 of [6], $X \in ANR_t$.

Suppose now that $X \in ANR_t$. Each projection $\varphi_j : X \to X_j$ given by the formula $\varphi_j(\{x_n\}_{n=1}^{\infty}) = x_j$ is an r-map. Therefore, from Lemma 4 of [6] and the condition $X \in ANR_t$ it follows that $X_j \in ANR_t$ for all $j \geq 1$.

Demonstrate that there exists a number N_1 such that the compact sets X_j's are connected for $j \geq N_1$. Suppose the contrary. Then there exists a sequence of positive integers $\{j_k\}_{k=1}^{\infty}$, $j_1 < j_2 < \dots < j_k < \dots$, such that the compact sets X_{j_k}'s are not connected ($k = 1, 2, \dots$). We have $X_{j_k} = Y_1^k \cup Y_2^k$, $Y_1^k \cap Y_2^k = \varnothing$, Y_1^k and Y_2^k are closed subsets in X_{j_k}, and $Y_1^k, Y_2^k \neq \varnothing$ ($k = 1, 2, \dots$). Choose points $a_1^k \in Y_1^k$ and $a_2^k \in Y_2^k$ arbitrarily. Put $D_k = \{a_1^k, a_2^k\}$. The set D_k is a retract of the space X_{j_k}. Consequently, the product $D^\omega = \prod_{k=1}^{\infty} D_k$ is a retract of the product $\prod_{k=1}^{\infty} X_{j_k}$. Since $\prod_{k=1}^{\infty} X_{j_k} \in ANR_t$; therefore, by Lemma 4 of [6], the compact set D^ω is an ANR_t-space. But the compact set D^ω is homeomorphic to some perfect Cantor set which obviously is not an ANR_t-space.

Thus, there exists a number N_1 such that X_j is a connected ANR_t-compact set for all $j \geq N_1$. Hence, by Lemma 3, $X_j \in AR_t$ for $j \geq N_1$. The theorem is proven.

As is known, the union of two $ANR(\mathfrak{M})$-spaces closed in the union is an $ANR(\mathfrak{M})$-space if their intersection is an $ANR(\mathfrak{M})$-space too. As far as the union of two AR_t-compact sets is concerned, we have the next

Theorem 3. *Let a metric space Z be the union of two compact subsets X and Y that belong to AR_t and intersect at a finite number of points. Then $Z \in AR_t$.*

PROOF. Let $a_1, \dots, a_n \in X$, $b_1, \dots, b_n \in Y$, and $X \cap Y = \{a_1 = b_1, \dots, a_n = b_n\}$. Let Q_1 and Q_2 be two copies of the Hilbert cube Q intersecting at a finite number of points $c_1, \dots, c_n$. Embed the compact set X into the Hilbert cube Q_1. By Theorem 11.1 of [8], we can assume that $a_i = c_i$, $i = 1, \dots, n$. Embed the compact set Y into Q_2. We can assume that $b_i = c_i$, $i = 1, \dots, n$.

Let $\mu \geq 0$ be a Radon measure on Z. We define some Radon measure $\delta \geq 0$ on Z as follows: $\delta(\{c_i\}) = 1$, $i = 1, \dots, n$, and $\delta(Z \setminus \{c_1, \dots, c_n\}) = 0$. Put $\nu_1 = \mu|X + \delta|X$ and $\nu_2 = \mu|Y + \delta|Y$. Given $\varepsilon > 0$, there exist a compact subset $A_\varepsilon^1 \subset X$ and a continuous map $r_\varepsilon^1 : Q_1 \to X$ such that $\nu_1(X \setminus A_\varepsilon^1) \leq \varepsilon/2$ and $r_\varepsilon^1(x) = x$ for all $x \in A_\varepsilon^1$. Also, there exist a compact subset $A_\varepsilon^2 \subset Y$ and a continuous map $r_\varepsilon^2 : Q_2 \to Y$ such that $\nu_2(Y \setminus A_\varepsilon^2) \leq \varepsilon/2$ and $r_\varepsilon^2(x) = x$ for all $x \in A_\varepsilon^2$. If $\varepsilon > 0$ is sufficiently small then $A_\varepsilon^1 \cap A_\varepsilon^2 = \{c_1, \dots, c_n\}$. Put $A_\varepsilon^\mu = A_\varepsilon^1 \cup A\varepsilon^2$. Then $\mu(X \cup Y \setminus A_\varepsilon^\mu) \leq \varepsilon$. We define a continuous map $r_\varepsilon^\mu : Q_1 \cup Q_2 \to Z$ as follows:

$$r_\varepsilon^\mu(x) = \begin{cases} r_\varepsilon^1(x), & x \in Q_1, \\ r_\varepsilon^2(x), & x \in Q_2. \end{cases}$$

If $x \in A_\varepsilon^\mu$ then $r_\varepsilon^\mu(x) = x$. By Theorem 9.1 of [9, p. 132], $Q_1 \cup Q_2 \in ANR$; hence, $Q_1 \cup Q_2 \in AR_\sigma$. The compact set Z is a t-retract of the compact set $Q_1 \cup Q_2$. Therefore, $Z \in AR_t$. The theorem is proven.

Corollary 1. *Every ANR_t-compact set can be transformed into an absolute t-retract by adjoining a finite number of one-dimensional simplices.*

Corollary 2. *Let a metric space Z be the union of two compact subsets X and Y such that $X, Y \in ANR_t$ with intersection either empty or consisting of finitely many points. Then $Z \in ANR_t$.*

Corollaries 1 and 2 follow from Theorems 1 and 3.

REMARK. As is known [8], every ANR-compact space can be transformed into an absolute retract by adjoining a finite number of cells. By adjoining a finite number of one-dimensional cells to X, we can obtain a connected compact ANR-space X_1. By Lemma 1, X_1 is an absolute σ-retract.

Now we state one assertion that concerns extension of maps.

Theorem 4. *Let X be a compact Hausdorff space such that, given a closed subset $A \subset X$, a continuous map $f : A \to K$ from A into a CW-complex K, and any finite open covering ω of K, there exists a continuous map $g : X \to K$ such that the maps f and $g|A$ are ω-close. Then there exists a continuous extension $f_0 : X \to K$ of $f : A \to K$.*

PROOF. There exists a locally finite CW-complex L homotopically equivalent to K. Let $\varphi : K \to L$ and $\psi : L \to K$ be continuous maps such that $\psi\varphi \simeq \mathrm{id}_K$. Then there exists a finite CW-complex $L_0 \subset L$ such that

$$\varphi f(A) \subset \operatorname{int} L_0.$$

The compact set L_0 is metrizable and belongs to ANR. There is a $\delta > 0$ such that any two continuous maps g_1 and g_2 from a topological space $\widetilde{X}$ into L_0 satisfying the condition

$$\rho(g_1(x), g_2(x)) \leq \delta$$

for all $x \in \widetilde{X}$ are homotopic on $\widetilde{X}$.

Demonstrate that there exists a continuous map $f_\delta : X \to K$ such that $\varphi f_\delta(A) \subset L_0$ and

$$\rho(\varphi f(x), \varphi f_\delta(x)) \leq \delta$$

for all $x \in A$. Given any point $y \in \varphi f(A)$, choose neighborhoods $U_y, V_y \subset \operatorname{int} L_0$ so as to have $\overline{V}_y \subset U_y$ and $\operatorname{diam} U_y \leq \delta$. There exist a finite number of points $y_1, \dots, y_m \in \varphi f(A)$ such that $\varphi f(A) \subset \bigcup_{j=1}^{m} V_{y_j}$. Put

$$W = L \setminus \bigcup_{j=1}^{m} \overline{V}_{y_j}.$$

The finite family of the sets $\{U_{y_1}, \dots, U_{y_m}; W\}$ forms an open covering of the space L. Consequently, the family $\omega_0 = \{\varphi^{-1}(U_{y_1}), \dots, \varphi^{-1}(U_{y_m}); \varphi^{-1}(W)\}$ forms an open covering of the complex K. By assumption, for the map $f : A \to K$ and covering ω_0 there exists a continuous map $f_\delta : X \to K$ such that the maps f and $f_\delta|A$ are ω_0-close. If $x \in A$ then, for a suitable number $s \in \{1, \dots, m\}$, we have $\varphi f(x) \in U_{y_s}$ and $\varphi f_\delta(x) \in U_{y_s}$. Therefore, $\rho(\varphi f(x), \varphi f_\delta(x)) \leq \delta$. Hence $\varphi f \simeq \varphi f_\delta|A$ and consequently $\psi\varphi f \simeq \psi\varphi f_\delta|A$; i.e., $f \simeq f_\delta|A$.

There exists a homotopy $F : A \times I \to K$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f_\delta(x)$ for all $x \in A$. It is possible to find a finite subcomplex $K_0 \subset K$ such that $F(A \times I) \subset K_0$ and $f_\delta(X) \subset K_0$. By the Borsuk theorem on extension of a homotopy, there exists a continuous map $f_0 : X \to K_0$ for which $f_0|A = f$. The theorem is proven.

Corollary 1. *If, under the assumptions of Theorem 4, K is a CW-complex $K(G, n)$ then $c\text{-}\dim_G X \leq n$ (here $K(G, n)$ is an Eilenberg-MacLane space [10]).*

Corollary 2. *If, under the assumptions of Corollary 1, $G = Z$, X is a compact metric space, and $FdX < \infty$ then $FdX \leq n$.*

The proof of Corollary 2 results from applying a lemma of [11].

We point out that the condition $FdX < \infty$ in Corollary 2 is essential. This follows from Theorem 2 of [11] and the fact that there exists a compact metric space Y with $c\text{-}\dim_Z Y = 3$ and $\dim Y = \infty$ [12].

In [13] there was defined the notion of absolute h-retract in the class of compact Hausdorff spaces. We rephrase the definition as follows:

A closed subset A of a compact space X is called an *h-retract* of X if there exists a continuous map $r : X \to A$ such that $r|A \simeq \mathrm{id}_A$.

A compact space Y is called an ***absolute h-retract*** if, given an arbitrary compact metric space X, each closed subset A of X homeomorphic to Y is an h-retract of X. We denote the class of all absolute h-retracts by AHR.

A closed subset A of a compact space X is called a ***neighborhood h-retract*** (see [14]) of X if there exist a neighborhood U of A in X and a continuous map $r : U \to A$ such that $r|A \simeq \mathrm{id}_A$.

A compact space Y is called an absolute neighborhood h-retract if, given an arbitrary compact space X, each closed subset of X homeomorphic to Y is a neighborhood h-retract of X. We denote the totality of all absolute neighborhood h-retracts by $ANHR$.

Theorem 5. *Let X_n, $n \geq 1$, be compact metric spaces such that the product $X = \prod_{n=1}^{\infty} X_n$ belongs to $ANHR$ and has a trivial shape. Then $X_n \in AHR$, $n \geq 1$.*

PROOF. We can assume that $X \subset Q$. There exist a neighborhood U of X in Q and a continuous map $r : U \to X$ such that $r|X \simeq \mathrm{id}_X$. By Theorem 27.1 of [15] there exists a continuous map $f_U : Q \to U$ such that $f_U(x) = x$ for all $x \in X$. Put $\varphi(x) = r(f_U(x))$, $x \in Q$. The map $\varphi : Q \to X$ is continuous and $\varphi|X \simeq \mathrm{id}_X$; i.e., $X \in AHR$. Since any retract of an AHR-space is an AHR-space; therefore, $X_n \in AHR$, $n \geq 1$. The theorem is proven.

Theorem 6. *If X is an ANR-compact space then X is an AR-space if and only if the fundamental group $\pi_1(X)$ and all the homology groups $\widetilde{H}_i(X; Z)$, $i \geq 0$, are trivial.*

PROOF. By use made of embedding the compact space X into the Tychonoff cube I^τ, we easily see that X is homotopically dominated by a suitable compact polyhedron. Then, by [10], there exists a CW-complex K homotopically equivalent to X. Consequently, $\pi_1(K) = 0$ and $\widetilde{H}_i(K; Z) = 0$, $i \geq 0$. By the Whitehead theorem, the CW-complex K is contractible. Hence, the ANR-compact space X is contractible and, therefore, $X \in AR$. The theorem is proven.

Theorem 6 generalizes Theorem 10.8 of [9].

In a way analogous to the one used in the proof of Theorem 6, we can prove

Theorem 7. *If X is an $ANHR$-compact set then X is contractible if and only if the fundamental group $\pi_1(X)$ and all the homology groups $\widetilde{H}_i(X; Z)$, $i \geq 0$, are trivial.*

We now exhibit some extra examples. It is known [4] that if X is a connected metrizable ANR-compact space then $X \in AR_t$. The homology groups $H_n(X; Z)$, $n \geq 0$, are finitely generated. We will demonstrate that, in general, this is not true for AR_t-compact spaces.

EXAMPLE 1. Assign $X_i = S^2$, $i = 1, 2, \ldots$; $S = \prod_{i=1}^{\infty} X_i$. Since $S^2 \in ANR$, we have $S^2 \in AR_\sigma$ and hence $S^2 \in AR_t$. From Theorem 1 of [7] we obtain $S \in AR_t$. Obviously, the homology group $H_2(S; Z)$ is not finitely generated.

Generally speaking, the homotopy groups $\pi_i(S)$, $i \geq 0$, are not finitely generated either. Since $\pi_1(S) = 0$; therefore, by the Hurewicz theorem the groups $H_2(S; Z)$ and $\pi_2(S)$ are isomorphic and hence the group $\pi_2(S)$ is not finitely generated.

In [6] the Cartesian products are considered of absolute ε-retracts (ε-AR-spaces). We will show that the homology groups of an absolute ε-retract can fail to be finitely generated.

EXAMPLE 2. Denote by C the closure in the plane $\mathbb{R}^2$ of the graph of the function $y = \sin(1/x)$, where $0 < x \leq 1$. We have $C \in \varepsilon\text{-}AR$. Put $X_i = C$, $i = 1, 2, \ldots$, and $X = \prod_{i=1}^{\infty} X_i$. By Theorem 1 of [6] the compact space X is an absolute ε-retract. The homology group $H_0(X; Z)$ is not finitely generated.

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