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ASYMPTOTIC DECOMPOSITION OF SLOW INTEGRAL MANIFOLDS

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Introduction

We consider the system of differential equations

$$\varepsilon\dot{z} = Z(z,t,\varepsilon), \quad z \in \mathbb{R}^{m+n},\ t \in \mathbb{R}, \tag{1}$$

where $0 < \varepsilon \ll 1$ and the vector-function Z is sufficiently smooth in all variables. Our fundamental hypothesis is as follows: the limit system of equations $Z(z,t,0) = 0$ ($\varepsilon = 0$) admits an m-parametric family of solutions

$$z = \psi(v,t), \quad v \in \mathbb{R}^m,\ t \in \mathbb{R}, \tag{2}$$

where ψ is a sufficiently smooth vector-function. We pose the question of existence of an integral manifold of slow motions (a slow integral manifold)

$$z = P(v,t,\varepsilon) \tag{3}$$

on which the motion is governed by the equation

$$\dot{v} = Q(v,t,\varepsilon). \tag{4}$$

Recall that by an *integral manifold* of system (1) one usually means some set in $\mathbb{R}^{m+n} \times \mathbb{R}$ that consists of integral curves of the system. We confine ourselves to study of smooth integral surfaces in the ε-neighborhood of the surface $z = \psi(v,t)$, i.e., $P(v,t,0) = \psi(v,t)$, on which the motion is governed by differential equations of the form (4) with the right-hand sides dependent on ε in a sufficiently smooth fashion.

Equation (1) describes motions with velocities of order $O(\varepsilon^{-1})$; and equation (4) describes those with velocities of order $O(1)$, i.e., slow motions. Therefore, integral manifold (3) is conventionally called an *integral manifold of slow motions*, or a *slow manifold.*

Foundations of the theory of integral manifolds were laid by N. N. Bogolyubov and Yu. A. Mitropol'skiĭ. Principal results of the theory are exposed in the monograph [1].

The method of integral manifolds for study of singularly-perturbed differential systems was used, for instance, in the articles [2–4].

From the standpoint of the theory of integral manifolds, the most thoroughly studied class of singularly-perturbed systems of differential equations is given by the systems of the form

$$\dot{x} = X(x,y,t,\varepsilon), \quad x \in \mathbb{R}^m,\ t \in \mathbb{R}, \tag{5}$$

$$\varepsilon\dot{y} = Y(x,y,t,\varepsilon), \quad y \in \mathbb{R}^n. \tag{6}$$

If the equation $Y(x,y,t,0) = 0$ has an isolated solution

$$y = \varphi(x,t) \tag{7}$$

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and $\det B(x,t) \neq 0$, $B(x,t) = Y_y(x,\varphi,t,0)$ then, under additional conditions on the eigenvalues of the matrix $B(x,t)$, in the ε-neighborhood of surface (7) system (5), (6) has a slow integral manifold

$$y = h(x,t,\varepsilon) \tag{8}$$

on which the motion is governed by the equation

$$\dot{x} = X(x, h(x,t,\varepsilon), t, \varepsilon). \tag{9}$$

For systems (5), (6), the role of the variable v is played by the vector x; the role of integral manifold (3), by integral manifold (8); and the role of equation (4), by equation (9).

In the application of the method of integral manifolds to solving concrete problems, the central question is that of calculating the functions that describe integral manifolds. Further we consider several methods for constructing integral manifolds in the form of asymptotic expansions in the powers of a small parameter. We precede this with discussing the question of existence of slow integral manifolds for systems of the form (1).

We point out that asymptotic expansions of some solutions to such systems were studied, for instance, in the articles [5–7].

§ 1. Existence of a Slow Manifold

Assume the following conditions to be satisfied: the rank of the matrix $\psi_v(v,t)$ equals m; the rank of the matrix $A(v,t) = Z_z(\psi(v,t),t,0)$ equals n; the matrix $A(v,t)$ has the zero eigenvalue of multiplicity m and the other n eigenvalues $\lambda_i(v,t)$ of $A(v,t)$ meet the condition

$$\operatorname{Re}\lambda_i(v,t) \leq -2\alpha < 0, \quad t \in \mathbb{R}, \ v \in \mathbb{R}^m. \tag{1.1}$$

Differentiating the identity $Z(\psi(v,t),t,0) = 0$ with respect to v, we obtain $Z_z(\psi(v,t),t,0)\psi_v(v,t) = 0$, or $A(v,t)\psi_v(v,t) = 0$. The preceding equality together with the first condition means that the $(m+n)\times(m+n)$-matrix $A(v,t)$ has m linearly independent eigenvectors (the columns of the matrix $\psi_v(v,t)$) corresponding to the multiple zero eigenvalue.

Let D_1^T be an $(m+n)\times n$-matrix whose columns form a basis for the kernel of the operator A and let D_2^T be an $(m+n)\times m$-matrix such that the matrix (D_1^T, D_2^T) is nondegenerate. Then

$$A^T(D_1^T \ D_2^T) = (0 \ B^T)$$

or

$$DA = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}.$$

Thus, the premultiplication of A by the nondegenerate matrix D distinguishes a zero block of order $m\times(m+n)$ and a nonzero $n\times(m+n)$-block B.

The rank of the matrix B equals n. Consequently, without loss of generality, we may assume that under the above-made assumptions system (1) is representable as

$$\varepsilon\dot{x} = f_1(x, y_2, t, \varepsilon), \quad x \in \mathbb{R}^m, \tag{1.2}$$

$$\varepsilon\dot{y}_2 = f_2(x, y_2, t, \varepsilon), \quad y_2 \in \mathbb{R}^n; \tag{1.3}$$

moreover, the following conditions hold:

I. The equation $f_2(x,y_2,t,0) = 0$ has a smooth isolated solution $y_2 = \varphi(x,t)$ for $x \in \mathbb{R}^m$ and $t \in \mathbb{R}$, and $f_1(x,\varphi(x,t),t,0) = 0$.

II. The Jacobian matrix

$$A(x,t) = \begin{pmatrix} f_{1x} & f_{1y_2} \\ f_{2x} & f_{2y_2} \end{pmatrix}_{y_2=\varphi(x,t),\varepsilon=0}$$

has m zero eigenvalues and an m-dimensional kernel on the surface $y_2 = \varphi(x,t)$; and the matrix $B(x,t) = f_{2y_2}(x, \varphi(x,t), t, 0)$ has n eigenvalues satisfying inequality (1.1) for $v = x$.

III. In the domain

$$\Omega = \{(x, y_2, t, \varepsilon) \mid x \in \mathbb{R}^m,\ \|y_2 - \varphi(x,t)\| \le \rho,\ t \in \mathbb{R},\ 0 \le \varepsilon \le \varepsilon_0\}$$

the functions f_1, f_2, and A are continuously differentiable and bounded together with their partial derivatives with respect to all variables up to the order $(k+2)$ $(k \ge 0)$.

Executing the change of variables $y_2 = y_1 + \varphi(x,t)$ in equations (1.2), (1.3), we obtain the following equations in the variables x and y_1:

$$\varepsilon\dot{x} = C(x,t)y_1 + F_1(x, y_1, t) + \varepsilon X(x, y_1, t, \varepsilon), \tag{1.4}$$

$$\varepsilon\dot{y}_1 = B(x,t)y_1 + F_2(x, y_1, t) + \varepsilon Y(x, y_1, t, \varepsilon); \tag{1.5}$$

here

$$\begin{gathered} C(x,t) = f_{1y_2}(x, \varphi(x,t), t, 0), \quad B(x,t) = f_{2y_2}(x, \varphi(x,t), t, 0), \\ F_1(x, y_1, t) = f_1(x, y_1 + \varphi(x,t), t, 0) - C(x,t), \\ F_2(x, y_1, t) = f_2(x, y_1 + \varphi(x,t), t, 0) - B(x,t), \\ \varepsilon X(x, y_1, t, \varepsilon) = f_1(x, y_1 + \varphi(x,t), t, \varepsilon) - f_1(x, y_1 + \varphi(x,t), t, 0), \\ \varepsilon Y(x, y_1, t, \varepsilon) = f_2(x, y_1 + \varphi(x,t), t, \varepsilon) - f_2(x, y_1 + \varphi(x,t), t, 0). \end{gathered}$$

Observe that the vector-functions F_i $(i = 1, 2)$ meet the relations $\|F_i(x, y_1, t)\| = O(\|y_1\|^2)$. Therefore, the functions $\varepsilon^{-2}F_i(x, \varepsilon y, t)$ are continuous.

The following theorem holds:

Theorem 1. *Under conditions* I–III *there exists an* ε_1, $0 < \varepsilon_1 \le \varepsilon_0$, *such that, for every* $\varepsilon \in (0, \varepsilon_1)$, *system* (1.4), (1.5) *has a unique integral manifold* $y_1 = \varepsilon p(x, t, \varepsilon)$ *on which the motion is governed by the equation*

$$\dot{x} = X_1(x, t, \varepsilon),$$

where $X_1(x, t, \varepsilon) = C(x,t)p(x, t, \varepsilon) + X(x, \varepsilon p, t, \varepsilon) + \varepsilon^{-1}F_1(x, \varepsilon p, t)$ *and the function* $p(x, t, \varepsilon)$ *and the corresponding series are continuously differentiable with respect to* x *and* t.

PROOF. The claim is provable following a standard scheme [1, 2]. Observe that the change of variable $y_1 = \varepsilon y$ reduces system (1.4), (1.5) to the form (5), (6). Also, observe that in this case the role of the parameter v is played by the variable x.

§ 2. Explicit and Implicit Descriptions for Slow Manifolds

In describing integral manifolds of slow motions for systems of the form (5), (6), one usually uses their explicit form (8) [2]. For approximate calculation, one applies the asymptotic expansion of the function $h(x, t, \varepsilon)$ in the powers of the small parameter ε:

$$h(x, t, \varepsilon) = h_0(x,t) + \varepsilon h_1(x,t) + \dots + \varepsilon^k h_k(x,t) + \dots, \quad h_0 = \varphi(x,t). \tag{2.1}$$

The defining equation for the coefficients in the expansion appears after the substitution of h for y in equation (6) and differentiation by force of equation (5):

$$\varepsilon\frac{\partial h}{\partial t} + \varepsilon\frac{\partial h}{\partial x}X(x, h, t, \varepsilon) = Y(x, h, t, \varepsilon). \tag{2.2}$$

Inserting formal expansion (2.1) into equation (2.2), we obtain the equality

$$\varepsilon\sum_{k\ge0}\varepsilon^k\frac{\partial h}{\partial t} + \varepsilon\Big(\sum_{k\ge0}\varepsilon^k\frac{\partial h_k}{\partial x}\Big)X\Big(x, \sum_{k\ge0}\varepsilon^k h_k, t, \varepsilon\Big) = Y\Big(x, \sum_{k\ge0}\varepsilon^k h_k, t, \varepsilon\Big). \tag{2.3}$$

For the functions involved in the preceding equality, we can write down the formal asymptotic expansions

$$X\left(x, \sum_{k\geq 0} \varepsilon^k h_k, t, \varepsilon\right) = \sum_{k\geq 0} \varepsilon^k X_k(x, h_0, \dots, h_k, t),$$

$$Y\left(x, \sum_{k\geq 0} \varepsilon^k h_k, t, \varepsilon\right) = B(x,t)\sum_{k\geq 1} \varepsilon^k h_k + \sum_{k\geq 1} \varepsilon^k Y_k(x, h_0, \dots, h_{k-1}, t).$$

In the asymptotic expansion of the function Y we have used the relation $Y(x, \varphi(x,t), t, 0) \equiv 0$. As above, $B(x,t)$ denotes the matrix $Y_y(x, \varphi(x,t), t, 0)$. Inserting these formal expansions into (2.3) and successively equating the coefficients of the same powers of the small parameter, we can obtain a chain of equalities of the form

$$\frac{\partial h_{k-1}}{\partial t} + \sum_{i=0}^{k-1} \frac{\partial h_i}{\partial x} X_{k-i-1} = B h_k + Y_k.$$

Basing on nondegeneracy of the matrix B, for h_k we obtain

$$h_k = B^{-1}\left(\frac{\partial h_{k-1}}{\partial t} + \sum_{i=0}^{k-1} \frac{\partial h_i}{\partial x} X_{k-i-1} - Y_k\right), \quad k = 1, 2 \dots. \tag{2.4}$$

Expansion (2.1) is of asymptotic character. More precisely, the function h is representable as

$$h(x,t,\varepsilon) = h_0(x,t) + \varepsilon h_1(x,t) + \dots + \varepsilon^k h_k(x,t) + \varepsilon^{k+1} p_{k+1}(x,t,\varepsilon),$$

where $p(x,t,\varepsilon)$ is a bounded continuous function.

It is clear that in analysis of many problems a solution to the equation

$$Y(x,y,t,0) = 0$$

is obtainable in neither explicit nor parametric form. In this event, to describe a slow surface and the behavior of solutions on it, one can use an implicit equation of the surface.

In the zero approximation, the behavior of solutions on the slow manifold is described by the following system of differential equations:

$$\dot{x} = X(x,y,t,0), \tag{2.5}$$
$$0 = Y(x,y,t,0). \tag{2.6}$$

To obtain the first approximation, we differentiate the function $Y(x,y,t,\varepsilon)$ with respect to the time. In virtue of system (5), (6), we obtain

$$\varepsilon \frac{d}{dt} Y = Y_y Y + \varepsilon Y_t + \varepsilon Y_x X.$$

The behavior of solutions on the slow manifold is described in the first approximation by the algebraic-differential system of equations of the form

$$\dot{x} = X(x,y,t,\varepsilon), \tag{2.7}$$
$$Y_y Y + \varepsilon Y_t + \varepsilon Y_x X = 0, \tag{2.8}$$

where all terms of order $o(\varepsilon)$ should be discarded.

To obtain the equations of the second approximation, we differentiate the function $Y(x,y,t,\varepsilon)$ twice with respect to the time by using system (5), (6). Even for the second approximation, we obtain a rather bulky expression for the implicitly-defined slow integral manifold. By this reason, we confine ourselves to the case of an autonomous system. Then the equation of the second approximation takes the form (2.7) and

$$Y+\varepsilon(Y_y)^{-1}Y_xX+\varepsilon^2Y_y^{-2}\{Y_xX_x+Y_{xx}X-Y_{xy}(Y_y)^{-1}Y_xX$$

$$-Y_xX_y(Y_y)^{-1}Y_x-Y_{yx}X(Y_y)^{-1}Y_x+Y_{yy}(Y_y)^{-1}Y_xX(Y_y)^{-1}Y_xX\}=0. \tag{2.9}$$

In equalities (2.7), (2.8), we should discard the terms that involve the powers of the small parameter which are greater than two. To obtain the kth approximation, we should differentiate the function $Y(x,y,t,\varepsilon)$ k times with respect to t by force of system (5), (6).

To validate the above formulas, it suffices to observe that, when we seek the function $h(x,t,\varepsilon)$ in the form of asymptotic expansions

$$h=h_0+O(\varepsilon),\quad h=h_0+\varepsilon h_1+O(\varepsilon^2),\quad h=h_0+\varepsilon h_1+\varepsilon^2h_2+O(\varepsilon^3),$$

the application of equations (2.5)–(2.9) gives the same result as the application of equation (2.3).

EXAMPLE. Consider the system of equations

$$\dot{x}=y,\quad \varepsilon\dot{y}=x^2+y^2-a,\quad a>0.$$

The first approximation to the integral manifold looks like

$$y^2+x^2-a+\varepsilon x=0.$$

The second approximation of the form

$$y^2+(x+\varepsilon/2)^2=a-\varepsilon^2/4$$

coincides with all subsequent ones and yields an exact equation for the slow integral manifold.

§ 3. Parametric Description for Integral Manifolds

First, we study the question of constructing slow integral manifolds for systems of the form (5), (6). As was already observed, in many problems it is impossible to find a root of the equation $Y(x,y,t,0)=0$ in the form $y=\varphi(x,t)$, since the equation turns out to be either transcendental or a polynomial of high degree in y. The implicit description has obvious imperfections in comparison with the explicit description. One frequently manages to write down a solution to the equation $Y(x,y,t,0)=0$ in parametric form

$$x=\chi_0(v,t),\quad y=\varphi_0(v,t), \tag{3.1}$$

where $v\in\mathbb{R}^m$ and the identity

$$Y(\chi_0(v,t),\varphi_0(v,t),t,0)\equiv 0,\quad t\in\mathbb{R},\ v\in\mathbb{R}^m, \tag{3.2}$$

holds. In this case it is reasonable to seek slow manifolds in parametric form too:

$$x=\chi(v,t,\varepsilon),\quad y=\varphi(v,t,\varepsilon), \tag{3.3}$$

where $t\in\mathbb{R}$, $v\in\mathbb{R}^m$, $\chi(v,t,0)=\chi_0$, and $\varphi(v,t,0)=\varphi_0$. The motion on a slow manifold is governed by the equation

$$\dot{v}=F(v,t,\varepsilon), \tag{3.4}$$

where the function $F(v,t,\varepsilon)$ will be defined below. We shall seek the functions χ, φ, and F in the form of asymptotic expansions

$$\begin{aligned} \chi(v,t,\varepsilon) &= \chi_0(v,t) + \varepsilon\chi_1(v,t) + \dots + \varepsilon^k\chi_k(v,t) + \dots, \\ \varphi(v,t,\varepsilon) &= \varphi_0(v,t) + \varepsilon\varphi_1(v,t) + \dots + \varepsilon^k\varphi_k(v,t) + \dots, \\ F(v,t,\varepsilon) &= F_0(v,t) + \varepsilon F_1(v,t) + \dots + \varepsilon^k F_k(v,t) + \dots \end{aligned} \tag{3.5}$$

in accordance with (3.4) from the equations

$$\frac{\partial\chi}{\partial t} + \frac{\partial\chi}{\partial v}F = X(\chi,\varphi,t,\varepsilon), \tag{3.6}$$

$$\varepsilon\frac{\partial\varphi}{\partial t} + \varepsilon\frac{\partial\varphi}{\partial v}F = Y(\chi,\varphi,t,\varepsilon). \tag{3.7}$$

Equating the coefficients of the same powers of the small parameter, we acquire

$$\begin{gathered} \frac{\partial\chi_0}{\partial t} + \frac{\partial\chi_0}{\partial v}F_0 = X(\chi_0,\varphi_0,t,0), \quad Y(\chi_0,\varphi_0,t,0) = 0, \\ \frac{\partial\chi_1}{\partial t} + \frac{\partial\chi_1}{\partial v}F_0 + \frac{\partial\chi_0}{\partial v}F_1 = X_x(\chi_0,\varphi_0,t,0)\chi_1 + X_y(\chi_0,\varphi_0,t,0)\varphi_1 + X_1, \\ \frac{\partial\varphi_0}{\partial t} + \frac{\partial\varphi_0}{\partial v}F_0 = Y_x(\chi_0,\varphi_0,t,0)\chi_1 + Y_y(\chi_0,\varphi_0,t,0)\varphi_1 + Y_1, \\ X_1 = X_\varepsilon(\chi_0,\varphi_0,t,0), \quad Y_1 = Y_\varepsilon(\chi_0,\varphi_0,t,0). \end{gathered}$$

Equations (3.6), (3.7) involve the unknown functions χ, φ, and F. Therefore, in dependence on the concrete problem, we may assume one of the functions or several m components of the functions χ, φ, and F to be known and find the remaining ones from equations (3.6), (3.7). Moreover, at different stages of determining the coefficients in expansion (3.5) we can assume the coefficients to be known of expansions of different functions or different m components of the coefficients. If the right-hand side has a fixed structure, and F may consequently be considered as known, then we can find the coefficients in the expansions of χ and φ from equations (3.5), (3.6). For instance, if χ is given in advance, then from these equations we can find the coefficients in the expansions of F and φ. In the case of a slow manifold $y = h(x,t,\varepsilon)$ we obtain the relations

$$v = x, \quad \chi = v, \quad \varphi = h(v,t,\varepsilon), \quad F = X(v,h(v,t,\varepsilon),t,\varepsilon),$$

and (3.5) takes the form

$$\varepsilon\frac{\partial h}{\partial t} + \varepsilon\frac{\partial h}{\partial v}X(v,h,t,\varepsilon) = Y(v,h,t,\varepsilon), \quad h = h(v,t,\varepsilon).$$

If $\dim x = \dim y$ and the vector y is taken as the parameter v, we obtain the relations $\varphi = v$ and

$$\frac{\partial\chi}{\partial t} + \frac{\partial\chi}{\partial v}F = X(\chi,v,t,\varepsilon), \quad \varepsilon F = Y(\chi,v,t,\varepsilon). \tag{3.8}$$

Whence we in turn infer the equation for χ:

$$\varepsilon\frac{\partial\chi}{\partial t} + \frac{\partial\chi}{\partial v}Y(\chi,v,t,\varepsilon) = \varepsilon X(\chi,v,t,\varepsilon), \tag{3.9}$$

from which we uniquely determine the coefficients in the asymptotic expansion of χ in case $\det(\frac{\partial\chi_0}{\partial v}) \neq 0$. Observe that $Y(\chi_0,\varphi_0,t,0) = 0$. Consequently, equation (3.4) is always regularly-perturbed.

Consider the question of constructing a slow integral manifold for equation (1). We seek the manifold and the equation of motion on it in parametric form

$$z = P(v,t,\varepsilon), \quad \dot{v} = Q(v,t,\varepsilon). \tag{3.10}$$

We seek the functions P and Q in the form of asymptotic expansions

$$\begin{aligned} P(v,t,\varepsilon) &= P_0(v,t) + \varepsilon P_1(v,t) + \cdots + \varepsilon^k P_k(v,t) + \dots, \\ Q(v,t,\varepsilon) &= Q_0(v,t) + \varepsilon Q_1(v,t) + \cdots + \varepsilon^k Q_k(v,t) + \dots. \end{aligned} \tag{3.11}$$

Differentiating P with respect to the time, in virtue of (1), (4) we obtain

$$\varepsilon\frac{\partial P}{\partial t} + \varepsilon\frac{\partial P}{\partial v}Q = Z(P,t,\varepsilon). \tag{3.12}$$

Expanding the function $Z(P,t,\varepsilon)$ in a formal series in the powers of the small parameter

$$Z(P,t,\varepsilon) = Z(P_0,t,0) + \varepsilon Z_1(P_0,P_1,t) + \cdots + \varepsilon^k Z_k(P_0,P_1,\dots,P_k,t)\dots,$$

we represent the functions Z_k $(k \geq 1)$ as

$$Z_k(P_0,\dots,P_k,t) = Z_2(P_0,t,0)P_k + R_k(P_0,P_1,\dots,P_{k-1},t).$$

In particular, $Z_1(P_0,P_1,t) = Z_2(P_0,t,0)P_1 + Z_\varepsilon(P_0,t,0)$. Using these formulas, equate the coefficients of the same powers of the small parameter in (3.12). At $\varepsilon = 0$ we obtain

$$Z(P_0,t,0) = 0.$$

In correspondence with formula (2), we assign $P_0(v,t) = \psi(v,t)$. With the notation $A(v,t) = Z_z(\psi(v,t),t,0)$ introduced in § 1, at the first power of ε we obtain

$$\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial v}Q_0 = AP_1 + R_1. \tag{3.13}$$

Equation (3.13) contains two unknown functions P_1 and Q_0. With respect to P_1, relation (3.13) represents an inhomogeneous linear system of algebraic equations which has determinant zero, $\det A(v,t) \equiv 0$, $v \in \mathbb{R}^m$, $t \in \mathbb{R}$. Thus, the choice of the function Q_0 is conditioned by the requirement that the system be compatible. It is clear that we have some freedom of choice while defining the functions Q_0 and P_1. One of the ways to define the functions in a one-to-one fashion is as follows: Apply the matrix D that is introduced in § 1 to equation (3.13) so as to obtain

$$D_1\frac{\partial\psi}{\partial t} + D_1\frac{\partial\psi}{\partial v}Q_0 = D_1R_1, \tag{3.14}$$

$$D_2\frac{\partial\psi}{\partial t} + D_2\frac{\partial\psi}{\partial v}Q_0 = BP_1 + D_2R_1. \tag{3.15}$$

If we additionally assume the matrix $D_1 = \partial\psi/\partial v$ invertible, then from equation (3.14) we can find $Q_0 = (D_1\psi_v)^{-1}D_1(R_1 - \psi_t)$, which enables us to determine P_1 uniquely from equation (3.15); namely, $P_1 = B^{-1}D_2(\psi_t + \psi_vQ_0 - R_1)$. Determination of the next pairs P_k and Q_{k-1} for the coefficients in the asymptotic expansions is carried out analogously. Equating the coefficients of ε^k, we arrive at the equation

$$\frac{\partial P_{k-1}}{\partial t} + \frac{\partial\varphi}{\partial v}Q_{k-1} + \sum_{i=1}^{k-1}\frac{\partial P_i}{\partial v}Q_{k-i-1} = AP_k + R_k$$

which, being premultiplied by the matrix D, splits into two equations

$$\left(D_1\frac{\partial\psi}{\partial v}\right)Q_{k-1}+D_1\left(\frac{\partial P_{k-1}}{\partial t}+\sum_{i=1}^{k-1}\frac{\partial P_i}{\partial v}Q_{k-i-1}\right)=D_1R_k,$$

$$D_2\left[\frac{\partial P_{k-1}}{\partial t}+\frac{\partial\psi}{\partial v}Q_{k-1}+\sum_{i=1}^{k-1}\frac{\partial P_i}{\partial v}Q_{k-i-1}\right]=BP_k+D_2R_k$$

from which we find

$$Q_{k-1}=(D_1\psi_v)^{-1}D_1\left[R_k-\sum_{i=1}^{k-1}\frac{\partial P_i}{\partial v}Q_{k-i-1}-\frac{\partial P_{k-1}}{\partial t}\right];$$

$$P_k=B^{-1}D_2\left[\frac{\partial P_{k-1}}{\partial t}+\psi_vQ_{k-1}+\sum_{i=1}^{k-1}\frac{\partial P_i}{\partial v}Q_{k-i-1}-R_k\right].$$

The estimate for the remainder in the asymptotic expansion of the slow integral manifold is carried out by the same scheme as the proof of existence of an integral manifold [2].

§ 4. Stability of Integral Manifolds

In majority of problems, of most interest are attracting (stable) integral manifolds. Stability is guaranteed by validity of conditions (1.1). Each trajectory, of the differential system under consideration, with the beginning near the integral manifold unboundedly approaches some trajectory on the manifold with the increase of t. The reduction principle is valid [2] for a stable integral manifold. We formulate it in the case when the initial system admits the zero solution; i.e., $Z(0,t,\varepsilon)=0$.

The zero solution to equation (1) is stable (asymptotically stable, unstable) if and only such is the zero solution to the equation $\dot v=Q(v,t,\varepsilon)$ on the integral manifold.

In study of the behavior of solutions to the initial equation near an integral manifold, stability of the integral manifold allows us to restrict ourselves to analysis of the equation on the manifold.

We now consider equation (1) for which condition (1.1) is replaced with the condition

$$\operatorname{Re}\lambda_i(v,t)\geq 2\alpha>0. \tag{4.1}$$

If we pass in equation (1) to the "backward" time $t\to -t$ then as a result we obtain an equation that meets the conditions guaranteeing existence for a stable slow integral manifold. Consequently, each equation of the form (1) has an unstable slow manifold for which the reduction principle holds for $t\to-\infty$. Unstable integral manifolds play an important role in the theory of thermal explosion [3, 8–10].

In the case when, for part of eigenvalues, some inequalities of the form (1.1) are valid and, for the other part, inequality (4.1) holds, equation (1) possesses a stable slow integral manifold. Such manifolds are typical of optimal control problems with singular perturbations [11, 12].

The formalism of constructing slow integral manifolds is in no way connected with their stability; it is based exclusively on invertibility of the matrix B. At the same time, to justify the asymptotic behavior for the expansions of the functions describing the integral manifolds, it suffices to require validity for the condition of separation of eigenvalues from the imaginary axis which guarantees existence of a stable, unstable, or conditionally stable manifold.

Observe that in the example exhibited in § 2 the upper semicircle ($y\geq 0$) of the one-dimensional slow integral manifold is unstable, whereas the lower one ($y<0$) is stable.

The cycles having as their parts unstable one-dimensional integral manifolds succeeding immediately stable ones exhibit particular instances of duck-trajectories that possess the same property [13–15]; in literature they are referred to as duck-cycles. The duck-trajectories play an important role in various problems of the theory of thermal explosion [9, 10].

The duck-solution found in the example of § 2 possesses one more remarkable property. For the system of two scalar equations of the form $\dot{x} = f(x, y)$, $\varepsilon\dot{y} = g(x, y, a)$, under certain assumptions valid, for example, for the van der Pol equation, duck-trajectories exist for the values of the parameter a in an interval of length of order $O(e^{-1/c\varepsilon})$, $c > 0$. A routine theorem of the theory of ducks reads literally as follows: "The life of a duck is short." In the above-considered example, the duck "lives for ever": for all $a > \varepsilon^2/4$.

§ 5. Degenerate Systems

In the literature on singular perturbations, systems of the form (1) are conventionally called singularly-perturbed (see, for instance, [7]) in contradistinction to the systems of the form (5), (6) with a nondegenerate matrix $B(x, t)$. In essence, the systems differ only by choice of variables. Some conditions like I–III are usually required. At the same time, in a number of problems the conditions can be violated. We consider several simple typical situations.

The assumption can be violated that the multiplicity of the zero root of the characteristic equation for the matrix $A(v, t)$ agrees with the number of the corresponding eigenvectors [3, 16]. For example, for the system of three vector equations

$$\begin{gathered} \varepsilon\dot{x}_1 = \varepsilon f_1(x_1, x_2, x_3, t, \varepsilon), \quad \varepsilon\dot{x}_2 = \varepsilon f_2(x_1, x_2, x_3, t, \varepsilon), \\ \varepsilon\dot{x}_3 = D(x_1, t)x_2 + \varepsilon f_3(x_1, x_2, x_3, t, \varepsilon), \quad x_i \in \mathbb{R}^{n_i}, \ i = 1, 2, 3, \end{gathered} \tag{5.1}$$

the matrix $A = A(x_1, t)$ has the structure

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D & 0 \end{pmatrix}$$

and, for $D \neq 0$, possesses the zero eigenvalue of multiplicity $n_1 + n_2 + n_3 = n$ to which there correspond $n - k$ eigenvectors and $k > 0$ adjoint vectors. Introducing the new variable $\bar{x}_2 = \varepsilon^{-1/2}x_2$, we arrive at a system of the form

$$\begin{gathered} \dot{x}_1 = f_1(x_1, \sqrt{\varepsilon}\bar{x}_2, x_3, t, \varepsilon), \quad \sqrt{\varepsilon}\dot{\bar{x}}_2 = f_2(x_1, \sqrt{\varepsilon}\bar{x}_2, x_3, t, \varepsilon), \\ \sqrt{\varepsilon}\dot{x}_3 = D(x_1, t)\bar{x}_2 + \sqrt{\varepsilon} f_3(x_1, \sqrt{\varepsilon}\bar{x}_2, x_3, t, \varepsilon). \end{gathered} \tag{5.2}$$

Let $x_3 = \varphi(x_1, t)$ be an isolated root of the equation $f_2(x_1, 0, x_3, t, 0) = 0$. System (5.2) has the form (5), (6) with small parameter $\sqrt{\varepsilon}$. In this case the role of the matrix B is played by the matrix $\overline{B}$:

$$\overline{B} = \begin{pmatrix} O & C(x_1, t) \\ D(x_1, t) & O \end{pmatrix},$$

where $C(x_1, t) = f_{2x_2}(x_1, 0, \varphi(x_1, t), t, 0)$. If $\det \overline{B} \not\equiv 0$ then the following possibilities are most typical: (a) all eigenvalues of the matrix $\overline{B}$ have nonzero real parts and system (5.2) admits an n_1-dimensional conditionally stable slow integral manifold (such situation is often encountered in optimal control problems [11, 12]); (b) the eigenvalues of the matrix $\overline{B}$ are purely imaginary (such systems are encountered in modeling gyroscopic systems and satellites with double rotation [2]). In each of the two cases the slow integral manifold is sought in the form of expansion in the powers of $\sqrt{\varepsilon}$, i.e., in fractional powers of the parameter ε.

Fractional powers of the small parameter arise not only owing to the presence of adjoint vectors.

Consider an autonomous system of the form (5), (6), with the functions X and Y vector-functions of the variables x and y homogeneous of degree r; i.e.,

$$X(\lambda x, \lambda y, \varepsilon) = \lambda^r X(x, y, \varepsilon); \quad Y(\lambda x, \lambda y, \varepsilon) = \lambda^r Y(x, y, \varepsilon)$$

for every number λ and all x, y, and ε in the domain under consideration. In this case the slow manifolds are described by equations of the form $y = L(\varepsilon)x$, where $L(\varepsilon) = (l_{ij})_{n\times m}$.

For L, we have the equation $\varepsilon LX(x, Lx, \varepsilon) = Y(x, Lx, \varepsilon)$. Equating the coefficients of the corresponding terms, we obtain a ramification equation [17] of the form

$$\Phi_r(L, \varepsilon) + \varepsilon\Phi_{r+1}(L, \varepsilon) = 0, \tag{5.3}$$

where Φ_k is an $(n+m)$-dimensional vector-function whose entries are polynomial functions of degree at most k in the variables l_{ij}, $i = 1, \dots, n$; $j = 1, \dots, n$. On assuming $Y(x, 0, 0) = 0$, we are interested in small solutions to equation (5.3), i.e. such solutions for which $L(0) = 0$. To each such solution $L_q(\varepsilon)$ there corresponds an integral manifold $y = L_q(\varepsilon)x$. If there are several such solutions, this means that we have ramification of slow manifolds. The problem of ramification of solutions to equations of the form (1.5) was considered, for instance, in [17].

EXAMPLE. Consider the system

$$\dot{x} = 3x^3, \quad \varepsilon\dot{y} = y^3 + \varepsilon x^3.$$

For it, the equation $Y(x, y, 0) = 0$ takes the form $y^3 = 0$, and naturally the results of the previous sections are inapplicable.

Using homogeneity of the system, we attempt to seek an integral manifold of the form $y = l(\varepsilon)x$. Then we obtain the following equation for l:

$$l^3 - 3al + \varepsilon^3 = 0. \tag{5.4}$$

It plays the role of the ramification equation (5.3). It is demonstrated in [18] that equation (5.4) has three small solutions:

$$l_1 = \frac{1}{3}\varepsilon^2 + \frac{1}{81}\varepsilon^5 + o(\varepsilon^5), \quad l_{2,3} = \pm\sqrt{3\varepsilon} - \frac{1}{6}\varepsilon^2 + o(\varepsilon^2).$$

Consequently, the system in our example has three integral manifolds of the form $y = l_q(x)$, $q = 1, 2, 3$, on each of which the motion is governed by the equation $\dot{x} = 3x^3$.

Now, consider a system of the form (5), (6) with

$$Y = Y_r(x, y, t) + Y_{r+1}(x, y, t) + \varepsilon Y_1(x, y, t, \varepsilon),$$

where the vector-function $Y_r(x, y, t)$ is homogeneous of degree r with respect to y; i.e., the equality $Y_r(x, \lambda y, t) = \lambda^r Y_r(x, y, t)$ holds for every number λ and all x, y, and t in the domain under consideration. For the function $Y_{r+1}(x, y, t)$, we assume $Y_{r+1} = O(\|y\|^{r+1})$. Denote by μ the quantity $\varepsilon^{1/r}$. The change of variable $y = \mu z$ reduces system (5), (6) to the form

$$\dot{x} = X(x, \mu z, t, \mu^r), \quad \mu\dot{z} = Y_r(x, z, t) + Y_1(x, 0, t, 0) + \mu Y_2(x, z, t, \mu), \tag{5.5}$$

where $\mu Y_2 = \mu^{-r} Y_{r+1}(x, \mu z, t, \mu^r) + Y_1(x, \mu z, t, \mu^r) - Y_1(x, 0, t, 0)$.

For the initial system, the generating equation takes the form

$$Y_r(x, y, t) + Y_{r+1}(x, y, t) = 0$$

and has the zero root $y = 0$ of multiplicity r. In this case the matrix B is identically zero. If the equation

$$Y_r(x, z, t) + Y_1(x, 0, t, 0) = 0$$

admits an isolated root $z = h_0(x, t)$ such that all eigenvalues of the matrix $Y_{ry}(x, h_0, t)$ have nonzero real parts, then it follows from the results of § 1 that system (5.5) possesses an integral manifold $z = h(x, t, \mu)$ for which the asymptotic expansion in the powers of the small parameter μ is valid. If there are several such roots then the system possesses several integral manifolds. Returning to the initial variable, we infer that in the case under consideration system (5), (6) possesses several slow integral manifolds of the form

$$y = \varepsilon^{1/r} h_0(x, t) + \varepsilon^{2/r} h_1(x, t) + \dots.$$

It means that in a neighborhood about the multiple zero root $y = 0$ we have a ramification of slow manifolds and each branch is representable in the form of an asymptotic expansion in fractional powers of the small parameter ε. In the case settled, asymptotic expansions involve integral powers of the parameter μ.

§ 6. Some Applications

Systems of the form (1) are encountered in modeling and studying objects of different nature whose characteristic feature consists in their ability to make fast and slow motions simultaneously.

The motion of a system of solid bodies presents an intricate composite of slow and fast motions. In the problems of dynamics of satellites, this phenomenon can relate to the presence of damping devices or elastic elements with small mass. For gyroscopic instruments and systems, the presence of fast (nutation) and slow (precision) oscillations is well known and observed practically always [2].

A sharp distinction between the rates of transformation of substances is typical of a wide range of processes; also, the rates of thermal and concentration fluctuations differ sharply. For instance, in catalytic systems the rates of reactions on the surface of a catalyst have order of magnitude several times higher than those in a gas phase [19].

For the combustion systems, a high rate of heat release is natural under a comparatively low rate of consumption of the combustible substance. For the gas-phase systems, the distinction is so radical that the phenomenon of self-ignition of a gas mixture has been acquired the name of "thermal explosion" [3, 8].

In the theory of automatic control over models, the described singularly-perturbed differential equations arise due to a number of causes. First, such situation is typical of the problems of control over systems whose dynamics objectively comprise motions with different rates: gyroscopic, electromechanical, and similar systems. Second, the appearance of singular perturbations can relate to the specific of the involved control methods also in systems with a single rate. This is exemplified by problems that use the penalty method with small penalty coefficient for control, "small-gain control," or the problems of stochastic filtration with noise degeneration in the observation channel.

In the present article, we restrict ourselves to applying the results to two control problems. The first of them is a control problem with large amplification factor. Control problems admitting an unbounded increase of the amplification factor were analyzed by many authors in several settings. In the first place we mention M. V. Meerov's articles. From the standpoint of singular perturbations, such problems were treated in the articles [20, 21].

Consider a system of the form

$$\dot{x} = f(x,t) + B_1(x,t)u, \quad x(0) = x_0,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, and $t \in [0,1]$. We assume as usual that the vector-function f and the matrix-function B_1 are bounded and continuous together with partial derivatives of sufficiently high order with respect to x and t. The problem consists in finding a control u so as to transfer the system from the initial state $x = x_0$ into a small neighborhood of a given smooth surface $S(x) = 0$ of dimension m rather quickly. In the article [20], it was proposed to use as control the expression

$$u = -\frac{1}{\varepsilon} K S(x),$$

where K is some constant $(k \times m)$-matrix and ε is a positive small parameter. Introducing the new variable $y = S(x)$, rewrite the extended system as

$$\begin{aligned} \varepsilon\dot{x} &= \varepsilon f(x,t) - B_1(x,t)Ky, & x(0) &= x_0, \\ \varepsilon\dot{y} &= \varepsilon G(x)f(x,t) - G(x)B_1(x,t)Ky, & y(0) &= y_0 = S(x_0), \end{aligned}$$

where $G(x) = \partial S/\partial x$. Suppose that, for the matrices G and B_1, it is possible to pick a matrix K so that the matrix $-N = -GB_1K$ be Hurwitz while N^{-1} be bounded.

The generating problem admits an n-parametric family of solutions $x = v$, $y = 0$. The role of the matrix A is played by the matrix $\begin{pmatrix} 0 & -B_1K \\ 0 & -N \end{pmatrix}$ whose determinant differs from zero.

The above-written differential system possesses an n-dimensional slow integral manifold that is representable as

$$x = v, \quad y = \varepsilon N^{-1}(v,t)G(v)f(v,t) + O(\varepsilon^2).$$

The motion on this manifold is governed by the equation

$$\dot{v} = [I - B_1(v,t)KN^{-1}(v,t)G(v)]f(v,t) + O(\varepsilon^2).$$

Introduce new variables by the formulas

$$x = v + B_1(v,t)KN^{-1}(v,t)z; \quad y = z + \varepsilon N^{-1}(x,t)G(x)f(x,t).$$

Then we obtain the following system in the variables v and z:

$$\dot{v} = (I - B_1KN^{-1}G)f + O(\varepsilon), \quad \varepsilon\dot{z} = -(N + O(\varepsilon))z.$$

The relations imply that

$$x = v + O(e^{-\nu t/\varepsilon}), \quad y = \varepsilon\varphi(v,t,\varepsilon) + O(e^{-\nu t/\varepsilon}), \quad \varphi = N^{-1}Gf + O(\varepsilon)$$

for some $\nu > 0$ and arbitrary $t > 0$. Thus, with the above choice of the control $u = \varepsilon^{-1}KS(x)$, the trajectory of a solution enters rapidly into the ε-neighborhood of the surface $S(x) = 0$. Obviously, the modified control

$$u = -\varepsilon^{-1}K[S(x) - \varepsilon N^{-1}(x,t)G(x)f(x,t)]$$

is preferable since, with the same choice of control, the trajectory enters the $e^{-\nu\Delta t/\varepsilon}$-neighborhood of the surface $S(x) = 0$ at the same time Δt. With this choice of control, we obtain the equation

$$\varepsilon\dot{x} = \varepsilon[I - B_1(x,t)K(GB_1K)^{-1}G(x)]f - B_1(x,t)KS(x)$$

in x and the equation

$$\varepsilon\dot{y} = -N(x,t)y$$

in $y = S(x)$; i.e., $y = O(e^{-\nu t/\varepsilon})$, $\nu = 0$, $t > 0$, $\varepsilon \to 0$.

Under modified control over the trajectory of a solution $x = x(t)$, the initial point $x(0) = x_0$ makes a jump into a neighborhood of order $o(\varepsilon^k)$, with k standing for an arbitrary natural number, about the surface $S(x) = 0$.

In conclusion we consider the problem with the so-called "small-gain control."

Look at the linear-quadratic problem

$$\dot{x} = A(t,\varepsilon)x + B(t,\varepsilon)u;$$
$$J = \frac{1}{2}x'(1)Fx(1) + \frac{1}{2}\int_0^1 [x'(t)Q(t,\varepsilon)x(t) + \varepsilon^2u'(t)R(t,\varepsilon)u(t)]\,dt,$$

where $Q = Q' \geq 0$, $F = F' \geq 0$, $R = R' > 0$, $t \in [0,1]$, and ε is a small parameter. Since the quality functional involves a small parameter in control, we consider the problem with a "small" gain in control [21]. The optimal control is given by a formula of the form

$$u = -\varepsilon^{-2}R^{-1}B'Kx,$$

where K is a solution to the Ricatti equation

$$\varepsilon^2(\check{K} + A'K + KA + Q) = KSK, \quad S = BR^{-1}B'; \quad K(1) = F. \tag{6.1}$$

For $\varepsilon = 0$, the generating equation has a multiple root $K = 0$. Asymptotic expansions of solutions to equation (6.1) with use of fractional powers of the small parameter were constructed, for instance, in the article [21]. We restrict ourselves to a certain particular example.

Assume that we have to minimize the quality functional

$$J = \frac{1}{2}\int_0^1 [q(t)y^2(t) + \varepsilon^2 u^2(t)]\,dt + \frac{1}{2}y^2(1),$$

where y meets the equation $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = u(t)$.

Let $K = (\mu^{i+j-1}K_{ij})_{i,j=1,\dots,n}$ and $\mu = \varepsilon^{1/n}$. Then, for the matrix $X = (K_{ij})_{i,j=1,\dots,n}$, we obtain the singularly-perturbed Ricatti equation

$$\mu\dot{X} + (A_0 + \mu A_1(t,\mu))'X + X(A_0 + \mu A_1(t,\mu)) + Q = XSX, \tag{6.2}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The Lurie equation

$$A_0'X + XA_0 - XSX + Q = 0 \tag{6.3}$$

has several real solutions to each of which corresponds a zero-dimensional slow integral manifold of equation (6.2). To each such manifold there corresponds a slow integral manifold of the Ricatti equation in the form (6.1). However, all integral manifolds of this equation equal zero at $\mu = 0$; i.e., in the example we observe a ramification of slow integral manifolds of equation (6.1) and the role of the ramification equation in zero approximation is played by Lurie equation (6.3). Observe that Lurie equation (6.3) in the problem under consideration admits a unique positive definite solution $X = C_0$ for which the matrix $A - SC_0$ is Hurwitz. The corresponding slow integral manifold $X = C(t,\mu)$ is easily calculable in the form of an asymptotic expansion in the powers of the small parameter μ.

Let $C(t,\mu) = (C_{i,j}(t,\mu))_{i,j=1,\dots,n}$. Then, for suboptimal control, we obtain the formula

$$u = -\mu^{-n}(\mu^{n-1}C_{nn}y^{(n-1)} + \cdots + \mu C_{n1}y).$$

The error in the value of the quality functional is a quantity of order $O(e^{-1/\mu})$ [22].

It is worth noting that, in correspondence with the results of [22], the value of the terminal component of the quality functional does not influence the amplification coefficient.

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