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The Short Distance Behavior of $(\phi^4)_3$

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Abstract: We consider the ϕ_3^4 quantum field theory on a torus and study the short distance behavior. We reproduce the standard result that the singularities can be removed by a simple mass renormalization. For the resulting model we give an L_p bound on the short distance regularity of the correlation functions. To obtain these results we develop a systematic treatment of the generating functional for correlations using a renormalization group method incorporating background fields.

1. Introduction

The renormalization group is not a group, but a technique for isolating the singularities of a quantum field theory. Originally invented by Wilson it has become one of the standard tools used in rigorous work on the subject. Still, its application is far from routine.

In a series of papers starting with a paper by Brydges and Yau [BY90], the authors have developed a systematic version of the technique which we believe has substantial advantages [Bry92, DH91, DH92b, DH92a, DH93]. Until now the Brydges-Yau method has not been applied to ϕ^4 type models, but we have developed a modification (incorporating background fields) which covers this case as well. In this paper we use it to study the short distance problem for the ϕ_3^4 model. We believe it can be used for many other problems. The paper [BDH93] also reviews the general framework of the background field method.

Here is a brief history of rigorous work on the ϕ_3^4 model. The original stability estimate was given by Glimm and Jaffe [GJ73] in a very difficult proof using a phase-cell cluster expansion. The complete construction of the model was finished by Feldman and Osterwalder [FO76] and Magnen and Sénéor [MS77]. Since then it has been worked over by many other authors, usually looking for a simpler proof. Some of the work continued to use a phase-cell cluster expansion, for example Battle and Federbush [BF83] and Williamson [Wil87]. Others used renormalization group

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techniques, for example Benfatto et al [BCG+80] and Bałaban [Ba83]. Each of these techniques was subsequently substantially strengthened to attack more difficult models. For the phase-cell cluster expansion there is the recent work of the Paris school [FMRS86, Riv91, MRS93] and for the renormalization group there is the work of Gawędzki and Kupiainen [GK85, GK86]. The method of Gawędzki and Kupiainen was applied to the ϕ_3^4 model in [Wat89]. We should also mention a third method using a random path representation due to Brydges, Fröhlich, and Sokal [BFS83].

The present paper is intended not only as a discussion of the ϕ_3^4 model, but also as a development of new and general methods. We work in a fixed volume and use the renormalization group with a mass renormalization to obtain stability bounds independent of the ultraviolet cutoff. In §2 we set up the ϕ_3^4 model, and define the renormalization group. The perturbative renormalization problem is solved to second order in §3, as a warmup to non-perturbative problem. Section §4 sets up a general renormalization group for polymer expansions with background fields. Section §5 provides the details of the norms we will work with. The technical heart of the paper is in §6 where we give in a model independent form the basic lemmas which control a single renormalization group step. We return to ϕ_3^4 in §7 where we set up and prove the main theorem giving uniform bounds on the polymer expansions at each step of the renormalization group. This implies the ultraviolet stability of the generating functional for correlations. The final section §8 derives in a straightforward fashion new bounds on all correlation functions. These say, for example, that the test functions can be taken to be in L_p for any $p > 3$. All these results carry over to the theory with no ultraviolet cutoff. The ultraviolet limit could be taken using our techniques, but this requires further technical results we do not include. (See however [DH93] where this step is carried out for the sine-Gordon model).

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2. The Model and the Renormalization Group

We define the ϕ_d^4 model on the unit torus $\Lambda = \mathbf{R}^d/\mathbf{Z}^d$. (We could as well take any finite torus.) The fields are real valued functions ϕ on Λ and the model is defined by a measure on these functions.

As a reference point we take the massless free theory is defined by the Gaussian measure with covariance $\tilde{v} = (-\triangle)^{-1}$, denoted $d\mu_{\tilde{v}}(\phi)$. (We could as well take a massive theory with covariance $(-\triangle + m^2)^{-1}$.) We also use a regularized version of the free measure, where $\tilde{v}$ is replaced by a covariance $\tilde{v}_N$ with kernel:

$$\tilde{v}_N(x,y) = |\Lambda|^{-1} \sum_{\substack{p\in\Lambda^* \\ p\neq 0}} e^{ip(x-y)} p^{-2} e^{-L^{-2N}p^2}. \tag{1}$$

Here $\Lambda^* = (2\pi\mathbf{Z})^d$. This function approximates $\tilde{v}$ at distances larger than $\mathcal{O}(L^{-N})$, and converges to $\tilde{v}$ as $N \to \infty$. The kernel is now smooth and the corresponding measure $d\mu_{\tilde{v}_N}(\phi)$ can be realized on a suitable Sobolev space $\mathcal{H}(\Lambda)$ of smooth functions whose integral is zero.

The regularized full theory is defined by a measure

$$\tilde{Z}^N(\phi) d\mu_{\tilde{v}_N}(\phi) \tag{2}$$

on $\mathscr{H}(\Lambda)$ with

$$\tilde{Z}^N(\phi) = e^{-\tilde{V}^N(\phi)}. \tag{3}$$

The potential is of the form

$$\tilde{V}^N(\phi) = \lambda V_4(\phi, \Lambda; \tilde{v}_N) + \mu V_2(\phi, \Lambda; \tilde{v}_N) \tag{4}$$

where we use the notation

$$V_n(\phi, \Lambda; v) = \int_\Lambda : \phi(x)^n :_v \; dx. \tag{5}$$

Here λ is the coupling constant and μ is a possible adjustment in the mass.

The correlation functions of this measure have the generating functional

$$S^N(\rho) = \left\langle e^{i(\rho,\phi)} \right\rangle_N = \int e^{i(\rho,\phi)} \tilde{Z}^N(\phi) d\mu_{\tilde{v}_N}(\phi) \tag{6}$$

where $(\rho, \phi) = \int_\Lambda \rho(x)\phi(x)dx$.

The goal is to show that a normalized generating functional has a limit as $N \to \infty$. For $d = 2$ this is well-known. For our case $d = 3$ there is a limit provided we renormalize the mass (i.e. let μ depend on N). For $d \geq 4$ it may be that all limits are trivial no matter how one renormalizes.

If we make a contour shift in the functional integral replacing ϕ by $\phi + i\tilde{v}_N\rho$ we obtain the formula

$$\begin{aligned} S^N(\rho) &= e^{-1/2(\rho,\tilde{v}_N\rho)} \int \tilde{Z}^N(\phi + i\tilde{v}_N\rho) d\mu_{\tilde{v}_N}(\phi) \\ &= e^{-1/2(\rho,\tilde{v}_N\rho)} (\mu_{\tilde{v}_N} * \tilde{Z}^N)(i\tilde{v}_N\rho) \end{aligned} \tag{7}$$

where $\mu_v *$ denotes convolution by the Gaussian measure $d\mu_v$. Since $\tilde{v}_N$ has a limit as $N \to \infty$, this shows that it suffices to find a limit for $\mu_{\tilde{v}_N} * \tilde{Z}^N$.

We break the integral with respect to $\mu_{\tilde{v}_N}$ into pieces as follows. For any $0 \leq i \leq N$, we define the fluctuation covariance $\tilde{w}_i^N = \tilde{v}_N - \tilde{v}_i$, where $\tilde{v}_i$ is given by (1) with $N \to i$ when $i > 0$ and $\tilde{v}_0 = 0$. For $i > 0$, this has Fourier transform

$$\tilde{w}_i^N(p) = p^{-2}[e^{-L^{-2N}p^2} - e^{-L^{-2i}p^2}] \tag{8}$$

and $\tilde{w}_0^N = \tilde{v}_N$. For each i, there is a decomposition of Gaussian measures

$$\mu_{\tilde{v}_N} = \mu_{\tilde{v}_i} * \mu_{\tilde{w}_i^N}. \tag{9}$$

If we define

$$\tilde{Z}_i^N = \mu_{\tilde{w}_i^N} * \tilde{Z}^N \tag{10}$$

for any $1 \leq i \leq N$ then we have

$$\mu_{\tilde{v}_N} * \tilde{Z}^N = \mu_{\tilde{v}_i} * \tilde{Z}_i^N. \tag{11}$$

This further isolates the N dependence. The family $\{\tilde{Z}_i^N\}$ interpolates between $\tilde{Z}^N$ and $\tilde{Z}_0^N \equiv \mu_{\tilde{v}_N} * \tilde{Z}^N$. Each density $\tilde{Z}_i^N$ is supposed to capture the behaviour of the original measure on length scales greater than $\mathscr{O}(L^{-i})$.

To control the $\tilde{Z}_i^N$ it is also advantageous to give an iterative definition:

$$\tilde{Z}_{i-1}^N = \mu_{\tilde{C}_i} * \tilde{Z}_i^N \tag{12}$$

with a single-slice fluctuation covariance $\tilde{C}_i = \tilde{v}_i - \tilde{v}_{i-1}$ (note the special case $\tilde{C}_1 = \tilde{v}_1$).

As it stands each fluctuation integral comes on a different momentum scale. To really understand the iteration we need to scale the problem so each fluctuation step comes on the same scale, say a unit scale. This will make it easier to identify the most important terms in the functionals $\tilde{Z}_i^N$ (the relevant variables) and follow the true renormalization group flow.

The basic rescaling is a transformation from $\mathscr{H}(\Lambda)$ to $\mathscr{H}(\Lambda_N)$ where $\Lambda_N = \mathbf{R}^d/(L^N\mathbf{Z})^d$. We define Z^N on $\mathscr{H}(\Lambda_N)$ by

$$Z^N(\phi) = \tilde{Z}^N(\phi_{L^N}) \tag{13}$$

where in general

$$\phi_L(x) = L^{(d-2)/2}\phi(Lx). \tag{14}$$

After this change of variables we find

$$(\mu_{\tilde{v}_N} * \tilde{Z}^N)(\phi) = (\mu_{v_N} * Z^N)(\phi_{L^{-N}}) \tag{15}$$

where the new covariance v_N has unit cutoff and is given by

$$v_N(x,y) = |\Lambda_N|^{-1} \sum_{\substack{p\in\Lambda_N^* \\ p\neq 0}} e^{ip(x-y)}p^{-2}e^{-p^2}, \tag{16}$$

where $\Lambda_N^* = (2\pi L^{-N}\mathscr{Z})^d$. The interaction density is now

$$Z^N(\phi) = e^{-V^N(\phi)} \tag{17}$$

with

$$V^N(\phi) = \lambda_N V_4(\phi, \Lambda_N; v_N) + \mu_N V_2(\phi, \Lambda_N; v_N)), \tag{18}$$
$$\lambda_N = L^{-(4-d)N}\lambda \qquad \mu_N = L^{-2N}\mu.$$

We also define effective densities Z_i^N on $\mathscr{H}(\Lambda_i)$ by

$$Z_i^N(\phi) = \tilde{Z}_i^N(\phi_{L^i}) \tag{19}$$

and find that (12) becomes

$$Z_{i-1}^N(\phi) = [\mu_{C_i} * Z_i^N](\phi_{L^{-1}}) \tag{20}$$

where now (taking $v_0 = 0$),

$$\begin{aligned} C_i(x-y) &= v_i(x-y) - L^{2-d}v_{i-1}((x-y)/L) \\ &= \begin{cases} |\Lambda_i|^{-1}\sum_{\substack{p\in\Lambda_i^* \\ p\neq 0}} e^{ip(x-y)}p^{-2}(e^{-p^2} - e^{-L^2p^2}) & i>1 \\ |\Lambda_1|^{-1}\sum_{\substack{p\in\Lambda_1^* \\ p\neq 0}} e^{ip(x-y)}p^{-2}e^{-p^2} & i=1 \end{cases}. \end{aligned} \tag{21}$$

The dependence on i is weak (except for $i = 1$) and hereafter we write C for C_i.

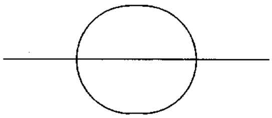
Note that (11) can now be written $(\mu_{\tilde{v}_N} * \tilde{Z}^N)(\phi) = (\mu_{v_i} * Z_i^N)(\phi_{L^{-i}})$ and this gives an expression for $S^N(\rho)$ on the volume Λ_i for any i. Specializing to $i = 0$ we have

$$S^N(\rho) = e^{-1/2(\rho,\tilde{v}_N\rho)} Z_0^N(i\tilde{v}_N\rho). \tag{22}$$

Thus to gain control over the generating functional $S^N(\rho)$ it suffices to have control over Z_0^N.

3. Perturbative Renormalization

Now we specialize to d=3 and discuss renormalization. If one calculates the physical mass in perturbation theory one finds that the shift represented by the diagram



diverges like $\mathcal{O}(N)$ as $N \to \infty$. This turns out to be the only serious divergence and one renormalizes by subtracting it off, choosing $\mu_N = \mu_N^N$ in (18) to be defined by

$$\mu_N^N = 48\lambda_N^2 \int_{\Lambda_N} v_N(x-y)^3 \, dy. \tag{23}$$

It corresponds to choosing $\mu = L^{2N}\mu_N^N = \mathcal{O}(N)$ in (4). Note that μ_N^N is very small $(\mathcal{O}(\lambda_N^{2-\epsilon}))$. It is characteristic of superrenormalizable models that the rescaled coupling constants like λ_N and μ_N are exponentially small.

We now show how this does the job in perturbation theory. We consider the effective potentials V_i^N defined by

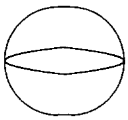
$$Z_i^N = \exp(-V_i^N)$$

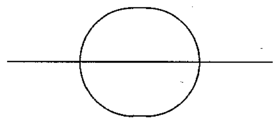
and show that they stay bounded to second order in λ as $N \to \infty$. This will be a guide to the complete flow which we study later on. Our discussion parallels that of [GK86].

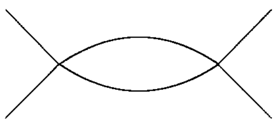
We focus our attention on the relevant terms: those which grow under the iteration of the renormalization group map (20). In addition to terms $\int :\phi^2:$ and $\int :\phi^4:$ which grow respectively like L^2 and L, we also consider nonlocal polynomials of the form

$$Q^{2n}(v, w; \phi) = \int :\phi(x)^n:_v \, w(x-y) :\phi(y)^n:_v \, dxdy. \tag{24}$$

These have the diagrammatic representation:

$Q_0 =$


$Q_2 =$


$Q_4 =$


$Q_6 =$


Ignoring constants and $(\partial\phi)^2$ terms we now assert that the effective potential on $\mathscr{H}(\Lambda_i)$ has the form

$$V_i = \lambda_i V_4(v_i) + \mu_i^N V_2(v_i) - Q_i + \mathscr{O}(\lambda_i^3) \tag{25}$$

where

$$Q_i = \lambda_i^2(8Q^6(v_i, w_i^N) + 36Q^4(v_i, (w_i^N)^2)). \tag{26}$$

Here $\lambda_i = L^{-i}\lambda$,

$$\begin{aligned} w_i^N(x-y) &= \sum_{k=i+1}^{N} L^{k-i} C_k(L^{k-i}x, L^{k-i}y) \\ &= L^{N-i} v_N\left(L^{N-i}(x-y)\right) - v_i(x-y) \\ &= |\Lambda_i|^{-1} \sum_{\substack{p\in\Lambda_i^* \\ p\neq 0}} e^{ip(x-y)} p^{-2}(e^{-L^{-2(N-i)}p^2} - e^{-p^2}) \end{aligned} \tag{27}$$

(except for $i = 0$ where the e^{-p^2} term is omitted). Also

$$\begin{aligned} \mu_i^N &= L^{2(N-i)}\mu_N^N - 48\lambda_i^2 \int \left(w_i^N(x-y)\right)^3 dx \\ &= 48\lambda_i^2 \int \left(w_i^N(x-y) + v_i(x-y)\right)^3 - \left(w_i^N(x-y)\right)^3 dx. \end{aligned} \tag{28}$$

This establishes the boundedness as $N \to \infty$ since in the expression for μ_i^N the two $\mathscr{O}(N)$ divergences cancel and give a finite result.

To see that the assertion is true we proceed by induction and compute V_{i-1} from V_i. We have

$$\begin{aligned} V_{i-1}(\phi) &= V^\#(\phi_{L^{-1}}) \\ V^\# &= -\log\big(\mu_{C_i} * \exp(-V_i)\big) \end{aligned}$$

where now $\phi_{L^{-1}}(x) = L^{-1/2}\phi(x/L)$. Perturbation theory can be generated by expanding in powers of V and has the form

$$V^\#(\phi) = (\mu_{C_i} * V_i) + 1/2\big(-\mu_{C_i} * V_i^2 + (\mu_{C_i} * V_i)^2\big) + \dots .$$

To evaluate the convolutions it is helpful to know that on polynomials F, $\mu_C * F = e^{\Delta_C} F$ where Δ_C is defined in (34). Also one has : $F :_C = e^{-\Delta_C} F$.

For the first order term we find the contributions

$$\begin{aligned} \mu_C * V_i &= \lambda_i V_4(v^\#) + \mu_i^N V_2(v^\#) \\ &\quad -\lambda_i^2\big(8Q^6(v^\#, w_i) + 72Q^4(v^\#, w_i C) + 144Q^2(v^\#, w_i C^2)\big) \\ &\quad -\lambda_i^2\big(36Q^4(v^\#, w_i^2) + 144Q^2(v^\#, w_i^2 C)\big) + \mathcal{O}(\lambda_i^3) \end{aligned}$$

where $v^\# = v_i - C$ and we abbreviate $w_i = w_i^N$. For the second order term we find (cf. [GK86], Eq. 2.33)

$$-\lambda_i^2\big(8Q^6(v^\#, C) + 36Q^4(v^\#, C^2) + 48Q^2(v^\#, C^3)\big) + \mathcal{O}(\lambda_i^3).$$

Adding these all together and defining $w^\# = w_i + C$ we find

$$\begin{aligned} V^\# &= \lambda_i V_4(v^\#) + \mu_i^N V_2(v^\#) \\ &\quad -\lambda_i^2\big(8Q^6(v^\#, w^\#) + 36Q^4(v^\#, (w^\#)^2) + 48Q^2(v^\#, (w^\#)^3 - w_i^3)\big) \\ &\quad +\mathcal{O}(\lambda_i^3). \end{aligned}$$

Now in the term $-48\lambda_i^2 Q^2(v^\#, (w^\#)^3 - w_i^3)$ replace $\phi(x)\phi(y)$ by $1/2\big(\phi(x)^2 + \phi(y)^2\big)$. The difference depends only on $\partial\phi$. What is left (modulo constants and $(\partial\phi)^2$ terms) is $\delta\mu_i^N V_2(v^\#)$ where

$$\delta\mu_i^N = -48\lambda_i^2 \int \big(w^\#(x-y)^3 - w_i^N(x-y)^3\big)dx \tag{29}$$

and we combine this with the other quadratic term.

Now do the scaling and compute V_{i-1}. We use that $v_{i-1}(x-y) = Lv^\#(L(x-y))$, that $w_{i-1}(x-y) = Lw^\#\big(L(x-y)\big)$, that $\lambda_{i-1} = L\lambda_i$, and that

$$\mu_{i-1}^N = L^2(\mu_i^N + \delta\mu_i^N). \tag{30}$$

We obtain (25) for $i-1$ and so our assertion is correct.

When we come to this step for the full non-perturbative theory there will be a number of modifications. There will be explicit bounds on the errors. The interaction terms will be broken up into local pieces. In each step we will only pick out local contributions to the mass and so there will be some residual non-local Q^2. Nevertheless the core idea is the same.

4. Polymer Expansions

The starting point in an analysis such as this is the representation of each $Z = Z_i^N$ by a polymer expansion in the d-dimensional torus $\Lambda = \Lambda_i$. In the original Brydges-Yau treatment [BY90] polymers are closed so that adjacent blocks overlap and thus are excluded from occurring in a single term in the expansion. This has some definite advantages, particularly for infrared problems. Nevertheless, in the present paper we find it convenient to return to a more standard formulation in which the basic blocks are open (or localized by their centers), so that adjacent blocks do not overlap. It seems that the open approach we adopt will work for most ultraviolet problems.

A *polymer* X is defined to be a union of blocks, where a *block* is an open unit cube centered on a point of the lattice $\mathbf{Z}^d$. Every set is a polymer unless otherwise specified. For example, Λ is now regarded as a union of open blocks. We consider polymer activities which are real valued functions $A(X)$ defined on polymers and possibly other variables. There is a commutative product

$$(A \circ B)(X) = \sum_{Y \subset X} A(Y)B(X \setminus Y)$$

and an exponential

$$\mathscr{E}xp(A) = \mathscr{I} + A + 1/2 A \circ A + ...$$

where $\mathscr{I}(\emptyset) = 1$ and otherwise $\mathscr{I}(X) = 0$. Note that one can also write

$$\big(\mathscr{E}xp(A)\big)(X) = \sum_{\{X_j\}} \prod_j A(X_j)$$

where the sum is over partitions of X into collections of polymers $\{X_j\}$.

In our formalism each interaction density Z on some torus Λ is expressed as a polymer expansion

$$Z(\phi) = \text{const } \big(\mathscr{E}xp(A)\big)(\Lambda, \phi) \tag{31}$$

with polymer activities $A(X, \phi)$ depending on $\phi(x)$ for $x \in X$ (in a sense made precise in chapter 5). It will further be convenient to write polymer activities A in the form of a background term and a deviation. The simplest choice for the background is $\square$ where $\square(X)$ defined to be 1 if $|X| = 1$ (i.e. X is a single block) and to be 0 otherwise. More generally we take the form $\square e^{-V}$ and assume that $V(X, \phi)$ is *additive* in X:

$$V(X, \phi) = \sum_{\Delta \subset X} V(\Delta, \phi) \tag{32}$$

where the sum is over single blocks Δ contained in X. The deviation is represented by polymer activities $K(X, \phi)$ so that

$$A = \square e^{-V} + K. \tag{33}$$

For any polymer Z one finds that

$$(\mathscr{E}xp(\square e^{-V} + K))(Z) = \sum_{\{X_j\}} \exp(-V(Z \setminus X)) \prod_j K(X_j)$$

where now the sum is over sets $\{X_j\}$ of disjoint polymers in Z (possibly empty) and $X = \cup_j X_j$.

We now discuss how the representation (31) - (33) changes under the action of the renormalization group.

4.1. The Fluctuation Step. Suppose that μ_C is a Gaussian measure on $\mathscr{H}(\Lambda)$ with covariance C and suppose that we have polymer activities A. We want to find new activities $\bar{\mathscr{F}}(A)$ so that

$$\mu_C * (\mathscr{E}xpA) = \mathscr{E}xp(\bar{\mathscr{F}}(A)).$$

We reformulate this as asking for activities $A(t)$ such that

$$\mu_{tC} * (\mathscr{E}xpA) = \mathscr{E}xp(A(t))$$

and then taking $\bar{\mathscr{F}}(A) = A(1)$. Now $\mathscr{E}xp$ has an inverse $\mathscr{L}$og defined on activities which have the value 1 on the empty set. Thus if $Z(t) = \mu_{tC} * (\mathscr{E}xpA)$ then $A(t) = \mathscr{L}\mathrm{og}(Z(t))$.

The function $Z(t)$ is the solution of the infinite dimensional diffusion equation

$$\partial Z/\partial t - \triangle_C Z = 0$$

where

$$\triangle_C = 1/2 \int dxdy C(x,y) \frac{\partial^2}{\partial\phi(x)\partial\phi(y)}. \tag{34}$$

(Functional derivatives are discussed in chapter 5.) It follows that $A(t) = \mathscr{L}\mathrm{og}(Z(t))$ satisfies the equation

$$\begin{aligned} \frac{\partial A}{\partial t} &= \frac{\partial Z}{\partial t} \circ Z^{-1} \\ &= \triangle_C Z \circ Z^{-1} \\ &= \triangle_C A + 1/2 C\left(\frac{\partial A}{\partial \phi} \underset{,}{\circ} \frac{\partial A}{\partial \phi}\right) \end{aligned} \tag{35}$$

with the initial condition $A(0) = A$. Here

$$C\left(\frac{\partial A}{\partial \phi} \underset{,}{\circ} \frac{\partial A}{\partial \phi}\right) = \int dxdy C(x,y) \frac{\partial A}{\partial\phi(x)} \circ \frac{\partial A}{\partial\phi(y)}.$$

This is equivalent to the integral equation

$$A(t) = \mu_{tC} * A + 1/2 \int_0^t \mu_{(t-s)C} * C\left(\frac{\partial A(s)}{\partial \phi} \underset{,}{\circ} \frac{\partial A(s)}{\partial \phi}\right) ds. \tag{36}$$

Note that a finite iteration of this equation yields a closed form expression for $A(t, X)$.

In the background version (33) we can write the result as

$$\mu_C * \mathscr{E}xp(\Box e^{-V} + K) = \mathscr{E}xp(\Box e^{-V_1} + \mathscr{F}(K)) \tag{37}$$

for any additive V_1 provided we define

$$\mathscr{F}(K) = \bar{\mathscr{F}}(\Box e^{-V} + K) - \Box e^{-V_1}.$$

4.2. The Extraction Step. This is a rearrangement that is helpful in keeping track of the leading terms. In this step one removes a piece $F(X,\phi)$ from the activities $A(X,\phi)$. Typically F is a local version of the low order terms in A. We suppose that

$$F(X,\phi) = F_0(X,\phi) + F_1(X,\phi)$$

where for $a = 0, 1$ we have

$$F_a(X,\phi) = \alpha_a(X)P_a(X,\phi)$$

and assume that P_a is additive in X. The extraction operation factors the F_0 terms out of the $\mathscr{E}xp$, but incorporates the F_1 terms into a change in the potential V. (For ϕ_3^4, F_0 will be constant and F_1 will be quadratic in ϕ).

Given the activities $A = \Box e^{-V} + K$ we seek new activities $\bar{\mathscr{E}}(A) = \Box e^{-V'} + \mathscr{E}(K)$ so that:

$$\mathscr{E}xp(\Box e^{-V} + K)(\Lambda) = \exp(\sum_{Y\subset\Lambda} F_0(Y))\mathscr{E}xp(\Box e^{-V'} + \mathscr{E}(K))(\Lambda), \tag{38}$$

where the potential is changed by

$$V'(\Delta) = V(\Delta) - [\sum_{Y\supset\Delta} \alpha_1(Y)][P_1(\Delta)] \tag{39}$$

and the linearization in K and F is

$$\mathscr{E}_1(K,F) = K - Fe^{-V}. \tag{40}$$

See Eq. (52) in the appendix to this section for the formula for $\mathscr{E}$.

4.3. The Scaling Step. Here a polymer expansion $\mathscr{E}xp(A)$ on Λ_i is scaled to a polymer expansion on Λ_{i-1}. To keep the basic blocks the same size one must also incorporate a reblocking operation.

The new activities $\bar{\mathscr{S}}(A)$ are chosen so that

$$\mathscr{E}xp(\bar{\mathscr{S}}(A))(\Lambda_{i-1},\phi) = \mathscr{E}xp(A)(\Lambda_i,\phi_{L^{-1}}).$$

We find that

$$\bar{\mathscr{S}}(A)(Z,\phi) = \sum_{\{X_j\}\to LZ} \prod_j A(X_j,\phi_{L^{-1}}). \tag{41}$$

The sum in (41) is over sets of disjoint polymers $\{X_j\}$ with the property that $\{\bar{X}_j^L\}$ is overlap connected and the union of the $\{\bar{X}_j^L\}$ is LZ, where $\bar{X}^L$ denotes the smallest L-polymer containing X. A set $\{X_i\}$ of polymers is called *overlap connected* if the graph on $\{X_i\}$ consisting of bonds (ij) such that $X_i \cap X_j \neq \emptyset$ is connected.

In the background version we define $\mathscr{S}(K)$, so as to satisfy

$$\mathscr{E}xp(\Box e^{-V'} + \mathscr{S}(K))(\Lambda_{i-1},\phi) = \mathscr{E}xp(\Box e^{-V} + K)(\Lambda_i,\phi_{L^{-1}}),$$

by

$$\Box e^{-V'} + \mathscr{S}(K) = \bar{\mathscr{S}}(\Box e^{-V} + K)$$

where $V'(X,\phi) = V(LX, \phi_{L^{-1}})$. Then, after some rearrangement, we find

$$\mathscr{S}(K)(Z,\phi) = \sum_{\{X_j\}\to LZ} \exp(-V(LZ\setminus X, \phi_{L^{-1}})) \prod_j K(X_j, \phi_{L^{-1}}) \tag{42}$$

where $X = \cup X_j$. Now the sum is over sets of disjoint polymers $\{X_j\}$ with the property that $\{\bar{X}_j^L\}$ is overlap connected and the union of the $\{\bar{X}_j^L\}$ is LZ.

4.4. Appendix: The Equation for $\mathscr{E}(K)$. Given a polymer activity J define

$$J^+(X) = \sum_{\{X_i\}\to X} \prod_i J(X_i) \tag{43}$$

where the sum is over overlap connected sets of distinct polymers whose union is X.

Lemma 1.

$$\sum_{\{X_i\}} \prod_i J(X_i) = \mathscr{E}xp(\square + J^+)(X), \tag{44}$$

where the sum is over sets of distinct polymers contained in X.

Proof. Group the $\{X_i\}$ into disjoint overlap connected sets.

□

Lemma 2. *Let F be any polymer activity and let*

$$\Omega(X) = \sum_{Y\subset X} F(Y). \tag{45}$$

Then

$$e^{\Omega} = \mathscr{E}xp(\square + (e^F - 1)^+). \tag{46}$$

Proof. Write $e^{\Omega}(X) = \prod_{Y\subset X}(e^{F(Y)} - 1 + 1)$, expand the product and use Lemma 1 with $J = e^F - 1$.

□

Lemma 3. *Let K, F be any polymer activities and let*

$$\tilde{K}(X) = K(X) - (e^F - 1)^+(X)e^{-V(X)}. \tag{47}$$

Then

$$e^{-V} \circ \mathscr{E}xp(K) = e^{-V+\Omega} \circ \mathscr{E}xp(\tilde{K}) \tag{48}$$

with Ω as in Lemma 2.

Proof. $e^{-V} \circ \mathscr{E}xp(K) = \mathscr{E}xp(\Box e^{-V} + K)$ by the additivity of V (see Eq. 32). $\mathscr{E}xp(\Box e^{-V} + K) = \mathscr{E}xp(\Box e^{-V} + (e^F - 1)^+ e^{-V}) \circ \mathscr{E}xp(\tilde{K})$ by the definition of $\tilde{K}$. By Lemma 2, $\mathscr{E}xp(\Box e^{-V} + (e^F - 1)^+ e^{-V}) = e^{-V}\mathscr{E}xp(\Box + (e^F - 1)^+) = e^{-V+\Omega}$. □

Since Ω is not additive, we cannot immediately rewrite $e^{-V+\Omega} \circ \mathscr{E}xp(\tilde{K})$ in the form $\mathscr{E}xp(\Box e^{-V'} + \tilde{K})$ for some V'. We are now going to absorb this non-additivity by reorganizing $e^{-V+\Omega} \circ \mathscr{E}xp(\tilde{K})$ into new polymers.

Let $\Omega_\alpha(X) = \sum_{Y \subset X} F_\alpha(Y)$ and let $X^c = \Lambda \setminus X$. We have

$$\begin{aligned} \Omega_1(X^c) &= \sum_{Z \subset X^c} F_1(Z) \\ &= \sum_{Z \subset X^c} \sum_{\Delta \subset Z} \alpha_1(Z) P_1(\Delta) \\ &= \sum_{\Delta \subset X^c} \left\{ \sum_{Z \supset \Delta} - \sum_{Z \supset \Delta, Z \not\subset X^c} \right\} \alpha_1(Z) P_1(\Delta). \end{aligned}$$

Add $V(X^c) = \sum_{\Delta \subset X^c} V(\Delta)$ to both sides. Recalling the definition of V', Eq. (39) we find

$$\begin{aligned} (V - \Omega_1)(X^c) &= V'(X^c) + \sum_{\Delta \subset X^c} \sum_{Z \supset \Delta, Z \not\subset X^c} \alpha_1(Z) P_1(\Delta) \\ &= V'(X^c) + \sum_{Z \not\subset X, Z \not\subset X^c} \alpha_1(Z) P_1(Z \setminus X). \end{aligned}$$

Therefore

$$\begin{aligned} e^{-V+\Omega_1}(X^c) &= e^{-V'}(X^c) \cdot \prod_{Z \not\subset X, Z \not\subset X^c} e^{-\alpha_1(Z) P_1(Z \setminus X)} \\ &= e^{-V'}(X^c) \cdot \sum_{\{Z_k\}} \prod_k (e^{-\alpha_1(Z_k) P_1(Z_k \setminus X)} - 1) \end{aligned} \tag{49}$$

with $Z \in \{Z_j\}$ required to intersect X and X^c. We also have

$$\begin{aligned} e^{\Omega_0}(X^c) &= e^{\Omega_0}(\Lambda) \prod_{Y : Y \cap X \neq \emptyset} (e^{-F_0(Y)} - 1 + 1) \\ &= e^{\Omega_0}(\Lambda) \sum_{\{Y_j\}} \prod_j (e^{-F_0(Y_j)} - 1) \end{aligned} \tag{50}$$

where the sum is over sets $\{Y_j\}$ of distinct polymers intersecting X. Substitute Eqs. (49,50) and the definition of $\mathscr{E}xp(\tilde{K})$ into

$$e^{-V+\Omega} \circ \mathscr{E}xp(\tilde{K})(\Lambda) = \sum_{X \subset \Lambda} e^{\Omega_0}(X^c) e^{-V+\Omega_1}(X^c) \mathscr{E}xp(\tilde{K})(X). \tag{51}$$

Then group the polymers in the sum over $\{X_i\}, \{Y_j\}, \{Z_k\}$ into disjoint overlap connected sets. One finds that $e^{-V+\Omega} \circ \mathscr{E}xp(\tilde{K})(\Lambda) = e^{\Omega_0}(\Lambda)\mathscr{E}xp(\Box e^{-V'} + \mathscr{E}(K))(\Lambda)$ with $\mathscr{E}(K) = \mathscr{E}(K, F)$ given by

$$\mathscr{E}(K)(W) = \sum_{\{X_i\},\{Y_j\},\{Z_k\}\to W} \exp(-V'(W\setminus X)) \tag{52}$$
$$\prod_i \tilde{K}(X_i)\prod_j(\exp(-F_0(Y_j))-1)\prod_k(\exp(-\alpha_1(Z_k)P_1(Z_k\setminus X))-1).$$

Here $X=\cup_i X_i$, and the sum is over collections of disjoint subsets $\{X_i\}$ and pairs of collections $\{Y_j\},\{Z_k\}$ of distinct subsets so that

1. the union is W;
2. each Y_j intersects X;
3. each Z_k intersects both X and $X^c=\Lambda\setminus X$;
4. the polymers $\{X_i\},\{Y_j\},\{Z_k\}$ are overlap connected.

5. Norms

In this section we define the weighted norms on the polymer activities which will enable us to control the activities K_i in the renormalization group flow. For more details see [BY90].

As a preliminary step we make a modification in the definition of the activities. Typical functionals such as $\int(\partial\phi)^2$ have functional derivatives with respect to ϕ which are derivatives of measures. We prefer to avoid this by treating $\partial\phi$ as a new field. This is formalized as follows. Let $\Lambda'=\Lambda\times(\emptyset,1,\dots,d)$. Every differentiable function ϕ on Λ determines a function ψ_ϕ on Λ' by

$$\psi_\phi(\xi)=\phi(x) \quad \text{if} \quad \xi=(x,\emptyset)$$
$$\psi_\phi(\xi)=\partial_k\phi(x) \quad \text{if} \quad \xi=(x,k).$$

We consider continuous complex valued functions ψ on Λ' and at each stage of our analysis will construct functionals $K(X,\psi)$ with the property that they reduce to the $K(X,\phi)$ when $\psi=\psi_\phi$. This is possible because all the elementary operations of Chapter 4 (including $\mathscr{F}$) have natural generalizations to functionals on ψ. We require that the $K(X,\psi)$ are $\mathscr{C}^\infty$ functions on the Banach space $C(\Lambda')$. The derivatives at $\psi=\psi_\phi$ are measures written as

$$K_N(X,\phi;\xi_1,\dots,\xi_N)=\left[\frac{\delta^N K(X,\psi)}{\delta\psi(\xi_1)\cdots\delta\psi(\xi_N)}\right]_{\psi=\psi_\phi}.$$

We also require that the support of this measure is $\bar{X}'\times\dots\times\bar{X}'$.

Since we do want to keep track of the field and its derivatives separately, we define $\Lambda_0'=\Lambda\times\emptyset$ and $\Lambda_1'=\Lambda\times(1,\dots,d)$. Then we have the decomposition into components $\Lambda'=\Lambda_0'\cup\Lambda_1'$. For $n=(n_0,n_1)$ and $|n|=n_0+n_1$ let $K_n(X,\phi)$ be the restriction of $K_{|n|}(X,\phi)$ to $(\Lambda_0')^{n_0}\times(\Lambda_1')^{n_1}$. These partial derivatives determine the full derivative. For each X,ϕ,n, let $\|K_n(X,\phi)\|$ be the total variation norm of $K_n(X,\phi)$.

Next, dependence on the variable ϕ is dominated by a large field regulator

$$G=G(\epsilon_0,\epsilon_1;X,\phi)=\exp(\epsilon_0\int_X\phi^2+\epsilon_1\int_X\sum_{1\le|\alpha|\le s}|\partial^\alpha\phi|^2). \tag{53}$$

Here we usually chose $s>d/2+1$ so that $\phi\in C^1(X)$ when G is finite. More generally, we say that $G(X,\phi)$ is a regulator iff for all polymers X,Y,

- (G1) $G(X, \phi = 0) \geq 1$
- (G2) $G(X \cup Y, \phi) \geq G(X, \phi)G(Y, \phi)$ whenever $X \cap Y = \emptyset$.

We introduce a partition of unity into products of blocks $\Delta = \Delta_1 \times \ldots \times \Delta_n$ and define

$$\|K_n(X)\|_G = \sum_{\Delta} \sup_{\phi \in \mathscr{H}(\Lambda)} \|K_n(X, \phi)1_\Delta\| G(X, \phi)^{-1}. \tag{54}$$

This choice for G lacks terms from the boundary of X which were necessary in previous papers to keep G from growing too rapidly under repeated convolution with the Gaussian measures (cf. the property (78). For ultraviolet problems one can allow ϵ to start off very small and hence allow the more rapid growth. In any case our open set formalism does not allow boundary terms, since condition (G2) would be violated.

Dependence on the set X is controlled by a large set regulator which we will choose to be either of the form $\gamma(X) = 2^{|X|}$ or of the form

$$\Gamma(X) = A^{|X|}\Theta(X) \tag{55}$$

$$\Theta(X) = \inf_{\tau} \prod_{b \in \tau} \theta(|b|) \tag{56}$$

for some large constant $A \geq L^{d+1}$. Here the infimum is over trees τ composed of bonds b connecting the centres of the blocks in X. Lengths such as $|b|$ are measured in an l^∞ metric on $\mathbf{R}^d$. The function θ is defined so that $\theta(s) = 1$ for $s = 0, 1$ and

$$\theta(\{s/L\}) = L^{-d-1}\theta(s), \qquad s \geq 2 \tag{57}$$

where $\{x\}$ denotes the smallest integer greater than or equal to x.

This regulator has been constructed to satisfy certain bounds which relate a polymer X to $\bar{X}^L$, the smallest union of L-blocks containing X. The polymer X is called a *small set* if its closure is connected and if it has volume $|X| \leq 2^d$. Otherwise it is a *large set*. For any set X, there is a constant c such that

$$(\gamma\Gamma)(L^{-1}\bar{X}^L) \leq c(\gamma^{-3}\Gamma)(X). \tag{58}$$

For any large set X, there is a stronger bound

$$(\gamma\Gamma)(L^{-1}\bar{X}^L) \leq cL^{-d-1}(\gamma^{-3}\Gamma)(X). \tag{59}$$

These bounds are needed to control the scaling step (41), and are proved in [BY90].

Next for each n define the norm

$$\|K_n\|_{G,\Gamma} = \sup_{\Delta} \sum_{X \supset \Delta} \Gamma(X)\|K_n(X)\|_G. \tag{60}$$

If the function is translation invariant one can drop the supremum.

Finally, for $h = (h_0, h_1)$, $h^n = h_0^{n_0}h_1^{n_1}$ and $n! = n_0!n_1!$ we define

$$\|K\|_{G,\Gamma,h} = \sum_n \left(h^n/n!\right) \|K_n\|_{G,\Gamma}. \tag{61}$$

A functional for which this norm is finite is analytic in ψ. In the translation invariant case this can also be written

$$\|K\|_{G,\Gamma,h} = \sum_{X\supset\Delta} \Gamma(X)\|K(X)\|_{G,h}. \tag{62}$$

We will find it necessary to have extra control over the low order ϕ and $\partial\phi$ derivatives at $\phi = 0$. This control is provided by an additional norm defined in the translation invariant case by

$$\begin{aligned} |K(X)|_h &= \sum_n \frac{h^n}{n!}\|K_n(X,\phi=0)\| \\ |K|_{\Gamma,h} &= \sum_{X\supset\Delta} \Gamma(X)|K(X)|_h. \end{aligned} \tag{63}$$

The kernel norm $|K|_{\Gamma,h}$ can be thought of as a limiting case of the norms $\|K\|_{G,\Gamma,h}$ in which G is concentrated at $\phi = 0$.

The following multiplicative properties can be derived:

$$\|K_1(X)K_2(X)\|_{G_1G_2,h} \leq \|K_1(X)\|_{G_1,h}\ \|K_2(X)\|_{G_2,h}, \tag{64}$$

$$|K_1(X)K_2(X)|_h \leq |K_1(X)|_h\ |K_2(X)|_h. \tag{65}$$

We now estimate the norms of certain classes of functionals which will arise later. First, we consider polynomials of degree r of the form

$$P(X,\phi) = \sum_{k=0}^{r} 1/k! \int_{X^k} \phi(x_1)...\phi(x_k)p_k(X,x_1,...,x_k)dx_1...dx_k \tag{66}$$

where $p_k(X,x_1,...,x_k)dx_1...dx_k$ is a symmetric measure supported on X^k, and $\phi(x)$ means $\psi(x,\emptyset)$.

Lemma 4. *For some constant* c *and* $\epsilon > 0$

$$\|P\|_{G(\epsilon,\epsilon),\Gamma,h} \leq |P|_{\Gamma,h+\sqrt{cr/\epsilon}} \tag{67}$$

$$\leq \left(1+\sqrt{cr/\epsilon h^2}\right)^r |P|_{\Gamma,h}. \tag{68}$$

Remark. The norm $|P|_{\Gamma,h}$ is generally easy to estimate. This lemma also has straightforward generalizations to polynomials depending on gradients.

Proof. Computing the derivatives and taking the norm of the measure yields

$$\|P_n(X,\phi)1_\Delta\| \leq \sum_{k=n}^{r} 1/(k-n)!\|\phi\|_{X,\infty}^{k-n}\ \|p_k(X)1_\Delta\|.$$

But by a Sobolev inequality

$$\|\phi\|_{X,\infty} \leq \sqrt{cr/\epsilon}G(\epsilon/r,\epsilon/r,X,\phi)$$

which leads to

$$\|P_n\|_{G(\epsilon,\epsilon),\Gamma} \leq \sum_k 1/(k-n)!\,(cr/\epsilon)^{(k-n)/2}\ \|p_k\|_\Gamma.$$

Multiplying by $h^n/n!$ and summing over n gives the first bound and the second follows directly.

□

For the next example, we estimate e^{-V} where $V(X,\phi) = \lambda V_4(X,\phi;v) + \mu V_2(X,\phi;v)$ with $\lambda > 0$ and μ possibly complex.

Theorem 1. *Let λh^4, ϵ^2/λ, and $|\mu|^2/\lambda$ be sufficiently small and let $h^{-2}v(0) \leq 1$. Then for any polymer X:*

$$\|e^{-V(X)}\|_{G(-\epsilon,0),h} \leq 2^{|X|}, \tag{69}$$

$$|e^{-V(X)}|_h \leq 2^{|X|}. \tag{70}$$

Remark. If X is a subset of a unit block Δ, then the same proof gives:

$$\|e^{-V(X)}\|_{G(-\epsilon,0),h} \leq 2. \tag{71}$$

This fact will be needed when we verify the hypotheses of Theorem 6.

Proof. We first prove the result when X is a single block Δ. We compute the derivatives of e^{-V} by

$$\frac{h^n}{n!}(e^{-V(\phi)})_n(x_1,...,x_n) = \frac{h^n}{n!}\sum_{\pi}(-1)^{|\pi|}\prod_j V_{n_j}(\phi, x_{\pi_j})e^{-V(\phi)}.$$

Here $\pi = \{\pi_j\}$ is any partition of $1,\dots,n$ and $n_j = |\pi_j|$, and x_{π_j} denotes the set of points x_i with $i \in \pi_j$. Now take the total variation norm. Furthermore classify the partitions by the number of elements r and order the elements in the partition which overcounts by a factor of $r!$. Finally use the fact that there are $n!/n_1!...n_r!$ ordered partitions with given n_j. This yields

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\| \leq \sum_r \frac{1}{r!}\sum_{\mathbf{n}}\prod_{j=1,\dots,r}\left[\frac{h^{n_j}}{n_j!}\|V_{n_j}(\phi)\|\right]e^{-\Re V(\phi)}.$$

Dropping the constraint $\sum_j n_j = n$ gives

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\| \leq \exp(-\Re V(\phi) + \sum_{n\geq 1}\frac{h^n}{n!}\|V_n(\phi)\|).$$

Next we note that

$$-\Re V(\phi) + \sum_{n\geq 1}\frac{h^n}{n!}\|V_n(\phi)\| \leq -\lambda h^4\int_\Delta p(h^{-1}|\phi|) + \mathcal{O}(|\mu|h^2)\int_\Delta q(h^{-1}|\phi|). \tag{72}$$

Here $p(t)$ is a polynomial whose coefficients are integers times non-negative powers of $h^{-2}v(0)$ and $p(t) = t^4$+terms of lower degree in t. Also q is a polynomial of the same type with $q(t) = t^2 + \dots$ Since $\epsilon h^2 = (\epsilon\lambda^{-1/2})(\lambda^{1/2}h^2)$ and $|\mu|h^2 = (|\mu|\lambda^{-1/2})(\lambda^{1/2}h^2)$ it follows that

$$-\Re V(\phi) + \sum_{n\geq 1}\frac{h^n}{n!}\|V_n(\phi)\| + \epsilon h^2\int_\Delta h^{-2}\phi^2 \leq \mathcal{O}(\lambda h^4) + \mathcal{O}((\epsilon^2 + |\mu|^2)/\lambda) \tag{73}$$

for all ϕ. From this we conclude

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\|_{G(-\epsilon,0)} \le \exp\left(\mathscr{O}(\lambda h^4) + \mathscr{O}((\epsilon^2 + |\mu|^2)/\lambda)\right). \tag{74}$$

This argument was valid for arbitrary $h \ge \sqrt{v(0)}$. Therefore we can replace h by $4h$ and conclude that

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\|_{G(-\epsilon,0)} \le 4^{-n}\exp\left(\mathscr{O}(\lambda h^4) + \mathscr{O}((\epsilon^2 + |\mu|^2)/\lambda)\right). \tag{75}$$

If we now take the parameters sufficiently small the sum over n is bounded by 2 as required.

In the general case we write

$$\begin{aligned} e^{-V(X)} &= \prod_{\Delta\in X} e^{-V(\Delta)}, \\ G(X) &= \prod_{\Delta\in X} G(\Delta). \end{aligned}$$

By the multiplicative property (64),

$$\|e^{-V(X)}\|_{G,h} \le \prod_{\Delta\in X}\|e^{-V(\Delta)}\|_{G,h} \le 2^{|X|}.$$

The kernel bound follows similarly.

□

Corollary 1. *Under the hypotheses of the theorem there is a constant* $0 < a < 1$ *such that if* $\epsilon' \ge a\lambda^{1/2}$, *and* P *is a polynomial of degree* r *then*

$$\|Pe^{-V}\|_{G(0,\epsilon'),\Gamma,h} \le (1 + \sqrt{cr/a\lambda^{1/2}h^2})^r |P|_{\gamma\Gamma,h}, \tag{76}$$

$$|Pe^{-V}|_{\Gamma,h} \le |P|_{\gamma\Gamma,h}. \tag{77}$$

Proof. Choose a so that the theorem holds for $\epsilon \le a\lambda^{1/2}$. We prove the bound for $\epsilon' = \epsilon \equiv a\lambda^{1/2}$. Combining the theorem and Lemma 4 we have

$$\begin{aligned} \|Pe^{-V}\|_{G(0,\epsilon),\Gamma,h} &\le \|P\|_{G(\epsilon,\epsilon),\gamma\Gamma,h}\sup_X\left[\gamma(X)^{-1}\|e^{-V(X)}\|_{G(-\epsilon,0),h}\right] \\ &\le (1 + \sqrt{cr/\epsilon h^2})^r |P|_{\gamma\Gamma,h}. \end{aligned}$$

The result (76) follows for $\epsilon' > \epsilon$. The kernel bound (77) is similar.

□

6. Estimates on $\mathscr{F}, \mathscr{E}, \mathscr{S}$.

In this chapter we obtain general estimates on the three functionals $\mathscr{F}, \mathscr{E}$ and $\mathscr{S}$ which make up the renormalization group transformation for any space dimension d.

For greater generality, we treat $\tilde{\mathscr{F}}$ rather than $\mathscr{F}$. For the same reason, we will treat $\mathscr{E}$ and $\mathscr{S}$ with hypotheses for a general background V.

6.1. Estimates on $\mathcal{F}$. The basic result bounds $\bar{\mathcal{F}}(A)$ for A not too large, provided we allow a deterioration of the regulators G and h. Let $G(t)$ be a family of large field regulators which satisfies the "homotopy" property

$$\mu_{(t-s)C} * G(s) \leq G(t) \text{ for } 0 \leq s < t \leq 1 \tag{78}$$

and let $\mathbf{h} = (h, h)$, $\mathbf{h}' = (h', h')$, with $h' < h$. We denote "before" and "after" norms by $\|\cdot\|_0 = \|\cdot\|_{G_0,\Gamma,\mathbf{h}}$ and $\|\cdot\|_1 = \|\cdot\|_{G_1,\Gamma,\mathbf{h}'}$. The size of the fluctuation covariance C is measured by a norm

$$\|C\|_\theta = \sup_{\Delta_1} \sum_{\Delta_2} C(\Delta_1, \Delta_2)\theta(d(\Delta_1, \Delta_2))$$

where $C(\Delta_1, \Delta_2) = \sup_{\xi_1 \in \Delta_1, \xi_2 \in \Delta_2} |C(\xi_1, \xi_2)|$. We suppose that A is not too large in relation to the above choices:

$$\|A\|_0 \leq D \equiv (16\|C\|_\theta)^{-1}(h - h')^2.$$

Theorem 2. *[BY90] Under these assumptions,*

$$\|\bar{\mathcal{F}}(A)\|_1 \leq \|A\|_0$$

and the map $A \to \bar{\mathcal{F}}(A)$ is Frechet analytic.

Remark. Analyticity is reviewed in [DH93].

We can obtain sharper control over the fluctuation step if we can find approximate solutions of the flow equation (35) for $\bar{\mathcal{F}}_t(A) = A(t)$. Suppose $B(t)$ satisfies

$$\frac{\partial B}{\partial t} = \Delta_C B + \frac{1}{2}C\left(\frac{\partial B}{\partial \phi} ; \frac{\partial B}{\partial \phi}\right) + E \tag{79}$$

where the error term $E(t)$ is to be thought of as small. Let $(\bar{\mathcal{F}}_t)_1(A; B)$ denote the derivative of the fluctuation operator evaluated at A, namely

$$(\bar{\mathcal{F}}_t)_1(A; B) = \frac{d}{d\beta}\bar{\mathcal{F}}_t(A + \beta B)|_{\beta=0}. \tag{80}$$

The following formula can be used to show that if $A(0)$ is close to $B(0)$, then $A(t)$ remains close to $B(t)$.

Theorem 3. *Suppose $A(0) = B(0) + R(0)$. Then $\bar{\mathcal{F}}_t(A(0)) \equiv A(t) = B(t) + R(t)$ where*

$$R(t) = \int_0^1 (\bar{\mathcal{F}}_t)_1(B(0) + sR(0); R(0))ds - \int_0^t \left(\bar{\mathcal{F}}_{t-s}\right)_1 (B(s); E(s))ds. \tag{81}$$

Proof.

$$\begin{aligned} R(t) &= A(t) - B(t) \\ &= \left[\bar{\mathcal{F}}_t(B(0) + R(0)) - \bar{\mathcal{F}}_t(B(0))\right] + \left[\bar{\mathcal{F}}_t(B(0)) - B(t)\right] \\ &= \int_0^1 (\bar{\mathcal{F}}_t)_1(B(0) + sR(0); R(0))ds - \int_0^t \frac{d}{ds}\bar{\mathcal{F}}_{t-s}(B(s))ds. \end{aligned}$$

Now

$$\begin{aligned}\frac{d}{ds}\bar{\mathscr{F}}_{t-s}(B(s)) &= -\frac{d}{dr}\bar{\mathscr{F}}_{t-s+r}(B(s-r))|_{r=0} \\ &= -\frac{d}{dr}\left[\bar{\mathscr{F}}_{t-s}\left(\bar{\mathscr{F}}_r(B(s))\right) + \bar{\mathscr{F}}_{t-s}(B(s-r))\right]|_{r=0} \\ &= \left(\bar{\mathscr{F}}_{t-s}\right)_1 (B(s); E(s))\end{aligned}$$

where $E(s) = -\frac{d}{dr}\bar{\mathscr{F}}_r(B(s)) + \frac{\partial B(s)}{\partial s}$.

□

This formula will be used in conjunction with the following bounds on the linearized fluctuation operator:

Theorem 4. *Assume the hypotheses of Theorem 2 hold with the family $G(t)$ satisfying $G(t, X, \phi = 0) \leq \gamma(X)$ for all $t \in [0, 1]$.*

1. *If $\|A\|_0 \leq \frac{1}{2}D$ then*

$$\|\bar{\mathscr{F}}_1(A; B)\|_1 \leq 2\|B\|_0. \tag{82}$$

2. *Let $\|A\|_{G(0),\gamma\Gamma,h} \leq 4$. For any $\eta = (\eta, \eta)$ with $\eta \in [0, 1)$ and any M there exists $C = C(\eta, M, \|C\|_\theta)$ such that for all sufficiently large h,*

$$|\bar{\mathscr{F}}_1(A; B)|_{\Gamma,\eta} \leq C(|B|_{\Gamma,1} + h^{-M}\|B\|_{G(0),\gamma\Gamma,h}). \tag{83}$$

Remark. The idea is that $\|B\|$ enters the kernel estimates with a large negative power of h to reduce its contribution.

Proof.

1. The first bound is a consequence of the Cauchy integral formula:

$$\bar{\mathscr{F}}_1(A; B) = (2\pi i)^{-1}\oint \frac{d\beta}{\beta^2}\bar{\mathscr{F}}(A + \beta B).$$

   We integrate over the contour $|\beta| = \frac{1}{2}D\|B\|_0^{-1}$ and use the bound $\|\bar{\mathscr{F}}(A + \beta B)\|_1 \leq \|A + \beta B\|_0 \leq \frac{D}{2} + \frac{D}{2}$ which follows by Theorem 2.
2. The difficulty here is that there is no straightforward version of Theorem 2 for the kernel norm. We consider the Taylor expansion of $\bar{\mathscr{F}}_1(\alpha A; B)$ about $\alpha = 0$:

$$\bar{\mathscr{F}}_1(A; B) = \sum_{j=1}^{N}\frac{1}{(j-1)!}\bar{\mathscr{F}}_j(B, A, \ldots, A) + \frac{1}{2\pi i}\oint\frac{d\alpha}{\alpha^N(\alpha - 1)}\bar{\mathscr{F}}_1(\alpha A; B), \tag{84}$$

   and take the $|\cdot|_{\Gamma,\eta}$ norm of both sides. For the error term, we take the contour $|\alpha| = R \equiv \frac{1}{2}D\|A\|^{-1}_{G(0),\gamma\Gamma,h}$ (we may assume that $R \geq 2$ since h is large) and by (82)

$$|\bar{\mathscr{F}}_1(\alpha A; B)|_{\Gamma,\eta} \leq \|\bar{\mathscr{F}}_1(\alpha A; B)\|_{G(1),\gamma\Gamma,h'} \leq 2\|B\|_{G(0),\gamma\Gamma,h}.$$

   For the terms in the sum over j, we use the bound $|A|_{\Gamma,1} \leq \|A\|_{G(0),\gamma\Gamma,h} \leq 4$ and apply the following lemma with $h' = h/2$. We obtain

$$|\bar{\mathscr{F}}_1(A;B)|_{\Gamma,\eta} \leq C(N)\|C\|_\theta^{N-1}(|B|_{\Gamma,1}+h'^{-M}\|B\|_{G(0),\gamma\Gamma,h}) + \frac{4}{R^N}\|B\|_{G(0),\gamma\Gamma,h}.$$

Since $R = \mathscr{O}(h^2)$, (83) follows from this bound by choosing N large enough.

□

The technical lemma we need gives bounds on the derivatives of $\bar{\mathscr{F}}$ at $A = 0$:

Lemma 5. *Assume the hypotheses of Theorem 2 hold with the family $G(t)$ satisfying $G(t, X, \phi = 0) \leq \gamma(X)$ for all $t \in [0, 1]$. Then*

1. *For any $n \geq 1$*

$$\|\bar{\mathscr{F}}_n(A_1,\ldots,A_n)\|_1 \leq D\left(\frac{n}{D}\right)^n \prod_j \|A_j\|_0. \tag{85}$$

2. *For any $\eta < 1$ and any integer M there is a constant $\mathscr{O}(1)$ such that for $h \geq 4$, $\bar{\mathscr{F}}_1(A) = \mu_C * A$ is bounded by*

$$|\bar{\mathscr{F}}_1(A)|_{\Gamma,\eta} \leq \mathscr{O}(1)(|A|_{\Gamma,1} + h^{-M}\|A\|_{G(0),\gamma\Gamma,h}). \tag{86}$$

For $n > 1$

$$|\bar{\mathscr{F}}_n(A_1,\ldots,A_n)|_{\Gamma,\eta^n} \leq \mathscr{O}(1)\|C\|_\theta^{n-1}\prod_j(|A_j|_{\Gamma,1,\geq 1}+h'^{-M}\|A_j\|_{G(0),\gamma\Gamma,h}) \tag{87}$$

where $|\cdot|_{\Gamma,h,\geq 1} = |\cdot|_{\Gamma,h} - |\cdot|_{\Gamma,h=0}$.

Proof.

1. This follows immediately from Theorem 2 via Cauchy bounds.
2. We prove these results for $\bar{\mathscr{F}}_t$ for all $0 \leq t \leq 1$.
   For (86), we let $\tilde{A}(t) = \mu_{tC} * A(0)$ and make a Taylor expansion of $\tilde{A}_m(t) = \mu_{(t-s)C} * \tilde{A}_m(s)$ around $t = s$. We find

$$\begin{aligned}\tilde{A}_m(t,X) &= \sum_{j=0}^{l-1}\frac{(t-s)^j \Delta_C^j \tilde{A}_m(s,X)}{j!} \\ &+ \frac{1}{(l-1)!}\int_s^t (t-\tau)^{l-1}\mu_{(\tau-s)C} * (\Delta_C^l\tilde{A}_m)(s,X)d\tau.\end{aligned}$$

Now evaluate at $\phi = 0$, and take the variation norm, $\|\cdot\|$. Then use

$$\|(\Delta_C^j\tilde{A})_m(t,X,\phi=0)\| \leq C_1(m,j)\ \|C/2\|_\infty^j \sum_{p\geq m:|p-m|=2j}\|\tilde{A}_p(t,X,\phi=0)\|. \tag{88}$$

Recall that $|\tilde{A}|_{\Gamma,\eta} = \sum_m \frac{\eta^m}{m!}\|\tilde{A}_m(\phi=0)\|_\Gamma$. By multiplying by $\Gamma(X)$ and summing over X, we obtain

$$\begin{aligned}\|(\Delta_C^j\tilde{A})_m(t,\phi=0)\|_\Gamma &\leq C_1(m,j)\ \|C\|_\infty^j \sum_{p\geq m:|p-m|=2j}\|\tilde{A}_p(t,\phi=0)\|_\Gamma \\ &\leq C_2(m,j)\ \|C\|_\infty^j\ \eta^{-|m|-2j}\ |\tilde{A}(t)|_{\Gamma,\eta}.\end{aligned}$$

By the hypotheses on G

$$\begin{aligned} \|[\mu_{(\tau-s)C} * (\Delta_C^l \tilde{A}(s))_m](\phi=0)\|_\Gamma & \\ \leq\ & \|\mu_{(\tau-s)C} * (\Delta_C^l \tilde{A}(s))_m\|_{G(\tau),\gamma\Gamma} \\ \leq\ & \|\Delta_C^l \tilde{A}_m(s)\|_{G(s),\gamma\Gamma} \\ \leq\ & C_3(l,m)\ \|C\|_\infty^l \sum_{p\geq m:|p-m|=2l} \|\tilde{A}_p(s)\|_{G(s),\gamma\Gamma} \\ \leq\ & C_4(l,m)\ \|C\|_\infty^l\ h^{-|m|-2l}\ \|\tilde{A}(s)\|_{G(s),\gamma\Gamma,h}. \end{aligned} \tag{89}$$

Therefore

$$\|\tilde{A}_m(t,\phi=0)\|_\Gamma \leq C_5(l,m,\eta)\left\{ |\tilde{A}(s)|_{\Gamma,\eta} + h^{-|m|-2l}\ \|\tilde{A}(s)\|_{G(s),\gamma\Gamma,h}\right\}, \tag{90}$$

$$\|\tilde{A}_m(t,\phi=0)\|_\Gamma \leq \frac{m!}{h^m}\ \|\tilde{A}(s)\|_{G(s),\gamma\Gamma,h}. \tag{91}$$

The second inequality is obtained from $\tilde{A}_m(t) = \mu_{(t-s)C} * \tilde{A}_m(s)$ and the hypotheses on G using (89) with $l=0$. Estimate the terms in the sum $|\tilde{A}(t)|_{\Gamma,\eta} = \sum_m \frac{\eta^m}{m!}\|\tilde{A}_m(t,\phi=0)\|_\Gamma$ with $|m| < M$ by using (90) with l the least integer such that $|m|+2l \geq M$. For $|m| \geq M$ use (91). We obtain the bound

$$|\tilde{A}(t)|_{\Gamma,\eta} \leq \mathscr{O}(1)(|\tilde{A}(s)|_{\Gamma,1} + h^{-M}\|\tilde{A}(s)\|_{G(s),\gamma\Gamma,h}) \tag{92}$$

from which the special case (86) follows.
We prove (87) for n, assuming it is true for i with $1 < i < n$ (there is no assumption on $n=2$). Taking derivatives in (36) with respect to A at $A=0$ we find

$$(\bar{\mathscr{F}}_t)_n(A_1,...,A_n) = \frac{1}{2}\sum_{I,J}\int_0^t \mu_{(t-s)C} * C_{IJ}ds \tag{93}$$

where

$$C_{IJ} = C\left(\frac{(\bar{\mathscr{F}}_s)_i(A_I)}{\partial\phi}\,;\frac{(\bar{\mathscr{F}}_s)_j(A_J)}{\partial\phi}\right). \tag{94}$$

The sum is over partitions of $(1,....n)$ into two proper subsets I,J and we define $A_I = \{A_i\}_{i\in I}$ and $i=|I|$. Neither set can be empty since $(\bar{\mathscr{F}}_s)_0 = 0$. Estimate the norm of $\mu_{(t-s)C} * C_{IJ}$ using (92) to obtain

$$|(\bar{\mathscr{F}}_t)_n(A_1,...A_n)|_{\Gamma,\eta^n} \leq \tag{95}$$

$$\mathscr{O}(1)\sum_{I,J}\int_0^t \left[|C_{IJ}|_{\Gamma,\eta^n} + (h')^{-nM}\|C_{IJ}\|_{G(s),\gamma\Gamma,h'/2}\right]ds. \tag{96}$$

Now

$$\begin{aligned} |C_{I,J}|_{\Gamma,h} &\leq 2\|C\|_\theta \frac{\partial}{\partial h}|(\bar{\mathscr{F}}_s)_i(A_I)|_{\Gamma,h}\ \frac{\partial}{\partial h}|(\bar{\mathscr{F}}_s)_j(A_J)|_{\Gamma,h} \\ \frac{\partial}{\partial h}\|\cdot\|_{\Gamma,h} &\leq \mathscr{O}(1)\|\cdot\|_{\Gamma,\eta^{-1}h,\geq 1}, \end{aligned}$$

and by the inductive hypothesis if $1 < i, j < n$ and (86) if $i = 1$ or $j = 1$, we have

$$\begin{aligned} |C_{IJ}|_{\Gamma,\eta^n} &\leq \mathscr{O}(1)\|C\|_\theta \, |(\bar{\mathscr{F}}_s)_i(A_I)|_{\Gamma,\eta^i,\geq 1}|(\bar{\mathscr{F}}_s)_j(A_J)|_{\Gamma,\eta^j,\geq 1} \\ &\leq \mathscr{O}(1)\|C\|_\theta^{n-1}\prod_j(|A_j|_{\Gamma,1,\geq 1} + h'^{-M}\|A_j\|_{G(0),\gamma\Gamma,h}). \end{aligned}$$

By similar estimates followed by (85) for $\bar{\mathscr{F}}_s$ we have

$$\begin{aligned} \|C_{IJ}\|_{G(s),\gamma\Gamma,h'/2} &\leq \mathscr{O}(1)\|C\|_\theta\|(\bar{\mathscr{F}}_s)_i(A_I)\|_{G(s),\gamma\Gamma,h'}\|(\bar{\mathscr{F}}_s)_j(A_J)\|_{G(s),\gamma\Gamma,h'} \\ &\leq \mathscr{O}(1)\|C\|_\theta^{n-1}\prod_j\|A_j\|_{G(0),\gamma\Gamma,h}. \end{aligned}$$

These suffice to prove the bound (87) for $n > 1$.

□

6.2. Estimates on $\mathscr{E}$. An important aspect of the definition of $\mathscr{E}$ is finiteness of the geometric constant τ defined as the largest number of distinct small sets that can intersect, i.e.,

$$\tau = \sup_x |\{X : x \in X \text{ and } X \text{ is small}\}|. \tag{97}$$

Recall that large and small sets were defined in Chapter 5.

Theorem 5. *Let G be any regulator. Let F, α_i, P_i, V, V' and $\mathscr{E}(K)$ be defined as in Sect. 4.2 and suppose F, V are translation invariant. Assume that for some $r > 0$*

$$\|e^{-\frac{V}{\tau}-\alpha P_1}(\Delta)\|_{G^{\frac{1}{\tau}},h} \leq 2 \tag{98}$$

for all complex α with $|\alpha| \leq r$. Δ is a unit block. Assume in addition that

- $P_0 = 1$;
- $F_0(Y)$, $F_1(Y) = 0$ *if Y is not a small set;*
- $\|\alpha_0\|_\Gamma$, $r^{-1}\|\alpha_1\|_\Gamma$, *and* $\|K\|_{G,\gamma\Gamma,h}$ *are sufficiently small.*

Then $\mathscr{E}$ is jointly analytic in K, F_0, F_1 and there is $\mathscr{O}(1)$ such that

1. $\|\mathscr{E}(K)\|_{G,\Gamma,h} \leq \mathscr{O}(1)(\|K\|_{G,h,\gamma\Gamma} + \|\alpha_0\|_\Gamma + r^{-1}\|\alpha_1\|_\Gamma)$;
2. $|\mathscr{E}(K)|_\Gamma \leq \mathscr{O}(1)(|K|_{\gamma\Gamma} + \|\alpha_0\|_\Gamma + r^{-1}\|\alpha_1\|_\Gamma)$.

Proof. We prove the first bound. The second bound is a variation in which the large field regulator G^{-1} is concentrated at $\phi = 0$. Also we only give the proof for the case where K is translation invariant.

We write $e^{V'(W\setminus X)} = \prod_{\Delta\subset W\setminus X} g^\tau(\Delta)$ where $g(X) = e^{-V'(X)/\tau}$. Then we can redistribute factors $g(\Delta)$ to rewrite Eq. (52) as

$$\begin{aligned} \mathscr{E}(K)(W) = &\sum_{\{X_i\},\{Y_j\},\{Z_k\}\to W}\ \prod_{\Delta\subset W\setminus X}(g^{n(\Delta)}(\Delta)) \\ &\prod_i \tilde{K}(X_i)\prod_j(\exp(-F_0(Y_j)) - 1)\prod_k J(Z_k, Z_k\setminus X) \end{aligned}$$

where $n(\Delta) = \tau - |\{Z_k : Z_k \supset \Delta\}|$ and $J(Z, Z\setminus X) = (e^{-\alpha_1(Z)P_1(Z\setminus X)} - 1)g(Z\setminus X)$. Since Z_k are small sets, $n(\Delta) \geq 0$. We take the norm using the multiplicative property (64) and obtain the result

$$\|\mathscr{E}(K)(W)\|_{G,h} \leq \sum_{\{X_i\},\{Y_j\},\{Z_k\}\to W} \prod_{\Delta\subset W\setminus X} (\gamma(\Delta))^{n(\Delta)} \tag{99}$$
$$\prod_i \|\tilde{K}(X_i)\|_{G,h} \prod_j |\exp(-F_0(Y_j)) - 1| \prod_k \|J(Z, Z\setminus X)\|_{G^{\frac{1}{\tau}}(Z\setminus X),h}.$$

We have used $\|g(\Delta)\|_{G(\Delta)^{1/\tau},h} \leq \gamma(\Delta)$ which follows from (98).

With the contour $|t| = r' \equiv 2r(3|\alpha_1(Z)|)^{-1}$:

$$\begin{aligned}
\|J(Z, Z\setminus X)\|_{G^{\frac{1}{\tau}}(Z\setminus X),h} &\leq \frac{1}{2\pi}\oint \left|\frac{dt}{t(t-1)}\right| \\
&\qquad \Big\| \prod_{\Delta\subset Z\setminus X} e^{t\alpha_1(Z)P_1(\Delta) - \frac{V'(\Delta)}{\tau}} \Big\|_{G^{\frac{1}{\tau}}(Z\setminus X),h} \\
&\leq \frac{1}{r'-1} \sup_t \prod_{\Delta\subset Z\setminus X} \|e^{t\alpha_1(Z)P_1(\Delta) - \frac{V'(\Delta)}{\tau}}\|_{G^{\frac{1}{\tau}}(\Delta),h} \\
&\leq 3r^{-1}|\alpha_1(Z)|\gamma(Z\setminus X).
\end{aligned}$$

We have used (98) and assumed $\sum_{Y\supset\Delta} |\alpha_1(Y)| \leq \|\alpha_1\|_\Gamma \leq \frac{1}{3}r$.

Next we write

$$\sum_{\{X_i\}} = \sum_N \frac{1}{N!} \sum_{(X_1,\ldots,X_N)}$$

where the sum is over ordered sets, but otherwise the restrictions apply. Similarly $\{Y_j\}$ and $\{Z_k\}$ are replaced by sums over $(Y_1,\ldots,Y_M)$ and $(Z_1,\ldots,Z_L)$.

The factors $\gamma^{n(\Delta)}(\Delta)$ and $\gamma(Z_k\setminus X)$ in (99) combine to give $\gamma^\tau(W\setminus X)$. Since $W\setminus X$ is a union of sets $Y_j\setminus X$, $Z_k\setminus X$ and these are small sets, we have $|W\setminus X| \leq 2^d(M+L)$. Therefore, we can overestimate $\gamma^\tau(W\setminus X)$ by $2^{2^d\tau(M+L)} = \mathscr{O}(1)^{M+L}$. Next we multiply by $\Gamma(W)$ and use $\Gamma(W) \leq \prod_i \Gamma(X_i)\prod_j \Gamma(Y_j)\prod_k \Gamma(Z_k)$ which follows from the overlap connectedness. Then sum over W with a pin, and use a spanning tree argument[1] and the small norm hypotheses to obtain

$$\begin{aligned}
&\|\mathscr{E}(K)\|_{G,h,\Gamma} \\
&\quad\leq \sum_{\substack{N,M,L\\N+M+L\geq 1}} \frac{(N+M+L)!}{N!M!L!} \|\tilde{K}\|^N_{G,h,\gamma\Gamma} (\mathscr{O}(1)|\exp(-F_0) - 1|_{\gamma\Gamma})^M \\
&\qquad\qquad \cdot \left(\frac{\mathscr{O}(1)\|\alpha_1\|_{\gamma\Gamma}}{r}\right)^L \\
&\quad\leq \mathscr{O}(1)\left(\|\tilde{K}\|_{G,h,\gamma\Gamma} + \|\alpha_0\|_\Gamma + r^{-1}\|\alpha_1\|_\Gamma\right).
\end{aligned}$$

Since F lives on small sets we have dropped the γ in the norms of F_0, α_1 at the cost of increasing the $\mathscr{O}(1)$.

[1] described in the proof of Lemma 5.1 of [BY90]

In the case of translation invariance $\|\cdot\|_{G,\Gamma,h} = \|\cdot\|_{G,h,\Gamma}$, so the proof is complete once we show that $\|\tilde{K}\|_{G,h,\gamma\Gamma} \leq \mathcal{O}(1)\{\|K\|_{G,h,\gamma\Gamma} + \|\alpha_0\|_\Gamma + r^{-1}\|\alpha_1\|_\Gamma\}$ for $\tilde{K} = K + (e^{-F} - 1)^+ e^{-V}$. The norm of

$$(e^{-F} - 1)^+ e^{-V}(X) = e^{-V(X)} \sum_{\{X_i\}} \prod_i (e^{-F} - 1)(X_i) \tag{100}$$

is estimated by the same argument we just used. By introducing a Cauchy integral over a circular contour $|t| = r'$ chosen so that

$$\log 2/|\alpha_0(X)| \geq r' \geq 1 + \left(2r^{-1}|\alpha_1(X)| + 2|\alpha_0(X)|\right)^{-1}$$

and using the hypothesis on r, one finds

$$\begin{aligned}
&\|(e^{-F} - 1)(X)e^{-V(X)/\tau}\|_{G^{1/\tau}(X),h} \\
&\quad \leq \frac{1}{2\pi} \oint \left|\frac{dt}{t(t-1)}\right| \prod_{\Delta \subset X} \|e^{-t[\alpha_0(X)+\alpha_1(X)P_1(\Delta)] - \frac{V(\Delta)}{\tau}}\|_{G^{\frac{1}{\tau}}(\Delta),h} \\
&\quad \leq \frac{1}{r' - 1} \sup_t \prod_{\Delta \subset X} \left[e^{-t\alpha_0(X)} \|e^{-t\alpha_1(X)P_1(\Delta) - \frac{V(\Delta)}{\tau}}\|_{G^{\frac{1}{\tau}}(\Delta),h} \right] \\
&\quad \leq \mathcal{O}(1)\left(r^{-1}|\alpha_1(X)| + |\alpha_0(X)|\right).
\end{aligned}$$

(for $\mathcal{O}(1)$ here we need that X is small). Now we use

$$\begin{aligned}
&\|((e^{-F} - 1)^+ e^{-V})(X)\|_{G,h} \\
&\quad \leq \sum_{\{X_i\}} \prod_{\Delta \subset X} (\gamma(\Delta))^{n(\Delta)} \prod_i \|(e^{-F} - 1)(X_i)e^{-V(X_i)/\tau}\|_{G^{1/\tau}(X_i),h}.
\end{aligned}$$

This is obtained by writing $e^{-V(X)} = \prod_{\Delta \subset X} g^\tau(\Delta)$ where $g(\Delta) = e^{-V(\Delta)/\tau}$ and distributing the factors of $g(\Delta)$. Now one inserts the first bound into the second and continues as before to obtain the desired bound on $\tilde{K}$.

□

Corollary 2. *Assume the hypotheses of Theorem 5. Let*

$$\begin{aligned}
\mathcal{E}_1(K,F) &= K - Fe^V \\
\mathcal{E}_{\geq 2}(K,F) &= \mathcal{E}(K,F) - \mathcal{E}_1(K,F),
\end{aligned}$$

where $\mathcal{E}(K,F) \equiv \mathcal{E}(K)$. *Then*

$$\begin{aligned}
\|\mathcal{E}_{\geq 2}(K,F)\|_{G,\Gamma,h} &\leq \mathcal{O}(1)\|K\|_{G,h,\gamma\Gamma}\|\alpha\|_\Gamma \\
|\mathcal{E}_{\geq 2}(K,F)|_\Gamma &\leq \mathcal{O}(1)|K|_{\gamma\Gamma}\|\alpha\|_\Gamma
\end{aligned}$$

where $\|\alpha\|_\Gamma = \|\alpha_0\|_\Gamma + \frac{1}{r}\|\alpha_1\|_\Gamma$.

Proof. Since, by construction, $\mathscr{E}_{\geq 2}(uK, vF)$ vanishes if either $u = 0$ or $v = 0$ and $\mathscr{E}(K, F)$ is analytic, we have the Cauchy representation

$$\mathscr{E}_{\geq 2}(K, F) = \int \frac{du}{2\pi i u(u-1)} \int \frac{dv}{2\pi i v(v-1)} \mathscr{E}(uK, vF). \tag{101}$$

The Corollary follows by choosing $|v|$ proportional to $\|\alpha\|_{\Gamma}^{-1}$ and $|u|$ proportional to the inverse norm of K and taking norms.

□

6.3. Estimates on $\mathscr{S}$. Given a functional $F(X, \phi)$ we define the rescaled functional $F_{L^{-1}}(X, \phi) = F(LX, \phi_{L^{-1}})$ where $\phi_{L^{-1}}(x) = L^{1-d/2}\phi(\frac{x}{L})$. Also for $h = (h_0, h_1)$ define $h_L = (L^{1-d/2}h_0, L^{-d/2}h_1)$.

Theorem 6. *Let $G := G(\epsilon_0, \epsilon_1)$. Let V be additive and translation invariant and suppose, for some h, it satisfies: $\forall L^{-1}$scale polymers $X \subset$ some unit block Δ*

$$\|(e^{-V})_{L^{-1}}(X)\|_{G,h} \leq 2. \tag{102}$$

If $\|K\|_{G_L, \gamma^{-3}\Gamma, h_L}$ is sufficiently small, then

$$\|S(K)\|_{G,\Gamma,h} \leq \mathscr{O}(1)L^d \|K\|_{G_L,\gamma^{-3}\Gamma,h_L} \tag{103}$$

$$|\mathscr{S}(K)|_{\Gamma,h} \leq \mathscr{O}(1)L^d |K|_{\gamma^{-3}\Gamma,h_L}. \tag{104}$$

Proof. The bound on the kernels is the special case where the large field regulator G is concentrated at $\phi = 0$. We only prove the first bound. We give the proof only for the case of translation invariant K. We rewrite (42) as

$$\mathscr{S}(K)(Z,\psi) = \sum_N 1/N! \sum_{(X_1,\ldots,X_N)} (e^{-V})_{L^{-1}}(Z \backslash L^{-1}X, \psi)) \prod_i K(X_i, \psi_{L^{-1}}), \tag{105}$$

where the X_i are disjoint but the L-closures $\bar{X}_i^L$ overlap and fill LZ. Using

$$G(Z,\phi)^{-1} = G(Z \setminus L^{-1}X, \phi)^{-1} \prod_i G_L(X_i, \phi_{L^{-1}})^{-1} \tag{106}$$

we obtain by (64)

$$\|\mathscr{S}(K)(Z)\|_{G,h} \leq \sum_N 1/N! \sum_{(X_1,\ldots,X_N)} \|(e^{-V})_{L^{-1}}(Z \setminus L^{-1}X)\|_{G,h} \prod_i \|K(X_i)\|_{G_L,h_L}.$$

By (64) and the small V hypothesis,

$$\|(e^{-V})_{L^{-1}}(Z \setminus L^{-1}X)\|_{G,h} \leq \prod_{\Delta \subset Z} \|(e^{-V})_{L^{-1}}(\Delta \setminus L^{-1}X)\|_{G(\Delta\setminus L^{-1}X),h} \leq \gamma(Z). \tag{107}$$

Now multiply by $\Gamma(Z)$. By the connectedness we have $(\gamma\Gamma)(Z) \leq \prod_i (\gamma\Gamma)(L^{-1}\bar{X}_i^L)$. Furthermore we have the bound (58) for some constant $\mathscr{O}(1)$:

$$(\gamma\Gamma)(L^{-1}\bar{X}^L) \leq \mathscr{O}(1)(\gamma^{-3}\Gamma)(X).$$

Summing over Z with a pin and using a spanning tree argument[2] we obtain

$$\|\mathscr{S}(K)\|_{G,\Gamma,h} \leq \sum_{N=1}^{\infty} \mathscr{O}(1)^{N-1}(L^d\|K\|_{G_L,\gamma^{-3}\Gamma,h_L})^N.$$

This gives the result.

□

We can replace the h_L by h in the right hand side of the above theorem, because the norm on the right hand side becomes larger when h_L is increased to h. If we also know that low dimensional derivatives vanish at zero, we can gain some critical factors of L^{-1} when we make this replacement, at least for small sets. Our next goal is to see how this is accomplished.

A key role is played by an estimate dominating functionals K with derivatives satisfying $K_n(X, \phi = 0) = 0$ for $\dim n < p$ by a norm involving only derivatives with $\dim n \geq p$ for all ϕ (not just $\phi = 0$). This originally appeared as Lemma 4.3 in [BY90] for functionals depending only on $\partial\phi$. The proof involved using a Sobolev inequality to dominate fields $\partial\phi$ by $G(0, \epsilon)$ and does not work for plain fields ϕ. We have a modification using the factors e^{-V} to dominate the ϕ's. The details follow.

Lemma 6. *There is a constant D such that the sup norm on a small set X satisfies*

$$\|\phi\|_{\infty,X} \leq DL\|\partial\phi\|_{\infty,\bar{X}^L} + DL^{-d/2}\|\phi\|_{2,\bar{X}^L\setminus X}$$

where $\|\phi\|_{2,\bar{X}^L\setminus X}$ is the L_2 norm on $\bar{X}^L \setminus X$.

Proof. Let $Y = \bar{X}^L \setminus X$. Note that Y is not empty for a small set X. For $x \in X$ we have

$$\phi(x) = -|Y|^{-1}\int_Y \left(\int_x^y d\phi\right) dy + |Y|^{-1}\int_Y \phi(y)dy.$$

The first term is bounded using $|x - y| \leq \mathscr{O}(L)$. The second term is bounded by $|Y|^{-1/2}\|\phi\|_{2,Y}$ by the Schwarz inequality and since $|Y| = \mathscr{O}(L^d)$ the result follows.

□

We define, for p a nonnegative integer,

$$\bar{G}(\epsilon_0, \epsilon_1, X) = G_L(\epsilon_0 = 0, \epsilon_1, \bar{X}^L)\; G_L(\epsilon_0, \epsilon_1 = 0, \bar{X}^L \setminus X), \tag{108}$$

$$\dim(n) = n_0\frac{d-2}{2} + n_1\frac{d}{2}, \tag{109}$$

$$\|K\|_{G,\Gamma,h,\dim\geq p} = \sum_{n:\dim(n)\geq p} (h^n/n!)\; \|K_n\|_{G,\Gamma}. \tag{110}$$

[2] described in the proof of Lemma 5.1 of [BY90]

Lemma 7. *Suppose K is supported on small sets and $K_n(X, \phi = 0) = 0$ for n with $\dim n < p$. Let $a = \min\{\epsilon_0 h_0^2,\ \epsilon_1 h_0^2,\ \epsilon_1 h_1^2\} > 0$. Then there exists $c(p, a)$ such that for all Γ,*

$$\|K\|_{\bar{G}(\epsilon_0,\epsilon_1),\Gamma,h_L} \leq c(p, a)\|K\|_{G_L(0,\epsilon_1),\Gamma,h_L,\dim\geq p}.$$

Proof. Let $F \in \mathscr{C}(\bar{X}' \times \ldots \times \bar{X}')$ be a test function for the derivative $K_n(X, \psi)$ and $\underline{\Delta} = (\Delta_1, \ldots, \Delta_N)$. By the fundamental theorem of calculus, if $\dim(n) < p$,

$$K_n(X, \psi; F1_{\underline{\Delta}}) = \sum_{m\geq n, |m|=|n|+1} \int_0^1 dt \sum_{\Delta} K_m(X, t\psi; F1_{\underline{\Delta}} \otimes \psi 1_{\Delta}). \tag{111}$$

We evaluate this at $\psi = \psi_\phi$. We also have, by a Sobolev inequality, that

$$\begin{aligned}
|\partial\phi|_{\infty,X} &\leq |\partial\phi|_{\infty,\bar{X}^L} \\
&= L^{-d/2}|\partial\phi_L|_{\infty,L^{-1}\bar{X}^L} \\
&\leq L^{-d/2}\,\mathscr{O}(1)\frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{\epsilon_1}}G^{1-t^2}(0, \epsilon_1, L^{-1}\bar{X}^L, \phi_L) \\
&\leq L^{-d/2}\mathscr{O}(1)\frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{\epsilon_1}}\bar{G}^{1-t^2}(X, \phi)
\end{aligned}$$

and by Lemma 6, $G_L(\epsilon_0, 0) = G(L^{-2}\epsilon_0, 0)$ and a Sobolev inequality

$$\begin{aligned}
|\phi|_{\infty,X} &\leq DL|\partial\phi|_{\infty,\bar{X}^L} + DL^{-d/2}|\phi|_{2,\bar{X}^L\setminus X} \\
&\leq \mathscr{O}(1)\frac{1}{\sqrt{1-t^2}}\left[\frac{L^{1-d/2}}{\sqrt{\epsilon_1}} + \frac{L^{1-d/2}}{\sqrt{\epsilon_0}}\right]\bar{G}^{1-t^2}(X, \phi).
\end{aligned}$$

By these two bounds and $\int_0^1 dt\, \frac{1}{\sqrt{1-t^2}} < \infty$ we obtain

$$\|K_n\|_{\bar{G},\Gamma} \leq \mathscr{O}(1) \sum_{m\geq n, |m|=|n|+1} \left(\frac{L^{1-d/2}}{\sqrt{\epsilon_1}} + \frac{L^{1-d/2}}{\sqrt{\epsilon_0}}\right)^{m_0-n_0} \tag{112}$$

$$\left(\frac{L^{-d/2}}{\sqrt{\epsilon_1}}\right)^{m_1-n_1} \|K_m\|_{\bar{G},\Gamma}. \tag{113}$$

We iterate this equation starting with n with $\dim n < p$ and obtain

$$\|K_n\|_{\bar{G},\Gamma} \leq C(p) \sum_{m\geq n:\dim m=p} \left(\frac{L^{1-d/2}}{\sqrt{\epsilon_1}} + \frac{L^{1-d/2}}{\sqrt{\epsilon_0}}\right)^{m_0-n_0}, \tag{114}$$

$$\left(\frac{L^{-d/2}}{\sqrt{\epsilon_1}}\right)^{m_1-n_1} \|K_m\|_{\bar{G},\Gamma}. \tag{115}$$

Recall that $h_L = (L^{1-d/2}h_0, L^{-d/2}h_1)$. We multiply both sides by $\frac{h_L^n}{n!}$ and sum over n with $\dim n < p$ to obtain

$$\|K\|_{\bar{G},\Gamma,h_L,\dim(n)<p} \leq C(p) \sum_{m\geq n:\dim m=p} \left(\frac{1}{h_0\sqrt{\epsilon_1}}+\frac{1}{h_0\sqrt{\epsilon_0}}\right)^{m_0-n_0} \cdot \left(\frac{1}{h_1\sqrt{\epsilon_1}}\right)^{m_1-n_1} \|K\|_{\bar{G},\Gamma,h_L,\dim=p}.$$

We complete the proof of the lemma by using the hypotheses on ϵ and h to bound the sum by $c(p,a)$ and then add $\|K\|_{\bar{G},\Gamma,h_L,\dim\geq p}$ to both sides.

□

Now consider the linearization $\mathscr{S}_1$ of $\mathscr{S}$ which is given by

$$\mathscr{S}_1(K)(Z) = \sum_{X:\bar{X}^L=LZ} K(X,\psi_{L^{-1}})\,(e^{-V})_{L^{-1}}(Z\setminus L^{-1}X,\psi)).$$

Theorem 7. *Let V be translation invariant and let $G=G(0,\epsilon_1), g=G(\epsilon_0,0)$. Suppose g and h are such that*

$$\|(e^{-V})_{L^{-1}}(X)\|_{g^{-1},h} \leq 2$$

for all L^{-1}-polymers X contained in a unit block Δ.

1. *If $K(X)$ is supported on large sets then*

$$\|\mathscr{S}_1(K)\|_{G,\Gamma,h} \leq \mathscr{O}(1)L^{-1}\|K\|_{G_L,\gamma^{-3}\Gamma,h_L}.$$

2. *Let $K(X)$ be supported on small sets with $K_n(X,\phi=0)=0$ for $\dim n<p$. Let $a=\min\{\epsilon_0h_0^2,\ \epsilon_1h_0^2,\ \epsilon_1h_1^2\}>0$ (as in the last lemma) then there exists $C(a,p)$ such that*

$$\|\mathscr{S}_1(K)\|_{G,\Gamma,h} \leq C(a,p)L^d\|K\|_{G_L,\gamma^{-3}\Gamma,h_L,\dim\geq p}.$$

Remarks. The same estimates hold for the kernels. Notice that in the small set estimate a factor of L^{-p} is gained if the norm on the right hand side with h_L is bounded in terms of the norm with h.

Proof. (1) We give the proof assuming translation invariance of K. Proceeding as in the proof of Theorem 6 we obtain

$$\|\mathscr{S}_1(K)(Z)\|_{G,h} \leq \gamma(Z) \sum_{X:\bar{X}^L=LZ} \|K(X)\|_{G_L,h_L}$$

which leads to

$$\|\mathscr{S}_1(K)\|_{G,\Gamma,h} \leq L^d \sum_{X\supset\Delta_0} (\gamma\Gamma)(L^{-1}\bar{X}^L)\|K(X)\|_{G_L,h_L}.$$

But for X large we have the bound $(\gamma\Gamma)(L^{-1}\bar{X}^L) \leq \mathscr{O}(1)L^{-d-1}(\gamma^{-3}\Gamma)(X)$ which gives the result.

(2) Let $\bar{G}=\bar{G}(\epsilon_0,\epsilon_1)$. This time we use

$$\begin{aligned} G(Z,\phi) &\equiv G_L(LZ,\phi_{L^{-1}}) \\ &= G_L(\bar{X}^L,\phi_{L^{-1}})g_L(\bar{X}^L\setminus X,\phi_{L^{-1}})g^{-1}(Z\setminus L^{-1}X,\phi) \\ &\equiv \bar{G}(X,\phi_{L^{-1}})g^{-1}(Z\setminus L^{-1}X,\phi) \end{aligned}$$

and obtain

$$\|\mathscr{S}_1(K)(Z)\|_{G,h} \leq \sum_{X:\bar{X}^L=LZ} \|K(X)\|_{\bar{G},h_L} \|(e^{-V})_{L^{-1}}(Z \setminus L^{-1}X)\|_{g^{-1},h}$$
$$\leq \gamma(Z) \sum_{X:\bar{X}^L=LZ} \|K(X)\|_{\bar{G},h_L}$$

so that

$$\|\mathscr{S}_1(K)\|_{G,\Gamma,h} \leq L^d \sum_{X\supset\Delta_0} (\gamma\Gamma)(L^{-1}\bar{X}^L)\|K(X)\|_{\bar{G},h_L}.$$

Now use the previous lemma and the bound $(\gamma\Gamma)(L^{-1}\bar{X}^L) \leq \mathscr{O}(1)(\gamma^{-3}\Gamma)(X)$ to complete the proof.

□

Remark. In [BDH93] we have given stronger versions of these theorems that allow a larger class of polymers, preparing the way for problems such as infrared ϕ_4^4.

7. Main Theorem

Now we return to the ϕ_3^4 model and study the renormalization group flow using the machinery we have been developing. The starting point is the density

$$\begin{aligned} Z^N &= e^{-V^N(\Lambda_N)} \\ &= \mathscr{E}xp(\Box e^{-V^N})(\Lambda_N) \\ V^N(X) &= \lambda_N V_4(X; v_N) + \mu_N^N V_2(X, v_N). \end{aligned} \tag{116}$$

Here $\lambda_N = L^{-N}\lambda$ and we make a basic mass renormalization by choosing

$$\mu_N^N = 48\lambda_N^2 \int_{\Lambda_N} v_N(x-y)^3 dy, \tag{117}$$

following the suggestion of second order perturbation theory as in Sect. 3. We do not renormalize the energy.

After $N-i$ renormalization group transformations we have a density Z_i^N on Λ_i. We will find constants Ω_i^N, μ_i^N and polymer activities K_i^N such that

$$\begin{aligned} Z_i^N &= e^{\Omega_i^N|\Lambda_i|}\mathscr{E}xp(A_i^N)(\Lambda_i) \\ &= e^{\Omega_i^N|\Lambda_i|}\mathscr{E}xp(\Box e^{-V_i^N} + K_i^N)(\Lambda_i) \\ V_i^N(X) &= \lambda_i V_4(X, v_i) + \mu_i^N V_2(X, v_i). \end{aligned} \tag{118}$$

(Hereafter the superscript N is suppressed.)

To write Z_i in this form we assume it has been done for i and derive the form for $i-1$. For the fluctuation step we have

$$\begin{aligned} \mu_C * Z_i &= e^{\Omega_i|\Lambda_i|}\mathscr{E}xp(A^\#) \\ &= e^{\Omega_i|\Lambda_i|}\mathscr{E}xp(\Box e^{-V^\#} + K^\#) \\ V^\#(X) &= \lambda_i V_4(X, v^\#) + \mu_i V_2(X, v^\#). \end{aligned} \tag{119}$$

Here $A^{\#} = \bar{\mathscr{F}}(A_i)$ and we have defined

$$K^{\#} = \mathscr{F}(K_i) = A^{\#} - \Box e^{-V^{\#}}. \tag{120}$$

The Wick ordering $v^{\#} = v_i - C$ now matches the free measure.

Next we extract $F = F_0 + F_1$ where

$$\begin{aligned} F_0(Y) &= \alpha_0(Y), & (121)\\ F_1(Y) &= \alpha_1(Y)V_2(Y, v^{\#}). & (122)\end{aligned}$$

The parameters $\alpha_0(Y), \alpha_1(Y)$ are still to be specified, but they will be invariant under lattice symmetries. Then we find

$$\begin{aligned} \mu_C * Z_i &= e^{(\Omega_i + \delta\Omega_i)|\Lambda_i|}\mathscr{E}xp(\Box e^{-V^*} + K^*) & (123)\\ V^*(X) &= \lambda_i V_4(X, v^{\#}) + (\mu_i + \delta\mu_i)V_2(X, v^{\#}) \end{aligned}$$

where

$$\begin{aligned} K^* &= \mathscr{E}(K^{\#}, F), & (124)\\ \delta\Omega_i &= \sum_{Y \supset \Delta} \alpha_0(Y), & (125)\\ \delta\mu_i &= -\sum_{Y \supset \Delta} \alpha_1(Y). & (126)\end{aligned}$$

Finally we scale to obtain Z_{i-1} which has the claimed form if we define

$$\begin{aligned} K_{i-1} &= \mathscr{S}(K^*) & (127)\\ \lambda_{i-1} &= L\lambda_i \\ \mu_{i-1} &= L^2[\mu_i + \delta\mu_i] \\ \Omega_{i-1} &= L^3[\Omega_i + \delta\Omega_i]. \end{aligned}$$

Note that $K_{i-1} = (\mathscr{S}\mathscr{E}\mathscr{F})(K_i)$ and that our notation can be summarized by

$$K_i \xrightarrow{\mathscr{F}} K^{\#} \xrightarrow{\mathscr{E}} K^* \xrightarrow{\mathscr{S}} K_{i-1}.$$

We shall write $K_i = Q_i \exp(-V_i) + R_i$ where Q_i includes the terms which are second order in λ_i and R_i is the remainder. We track the flow of the Q_i as in perturbation theory (Chapter 3), now including the $\partial\phi$ terms and constants and give general bounds on the remainder.

We introduce quantities $Q^{2n}(v, w; X, \phi)$ for $0 \le n \le 3$ by setting

$$Q^{2n}(v, w; \Delta \cup \Delta', \phi) = \int_{\Delta \times \Delta' \cup \Delta' \times \Delta} :\phi(x)^n:_v \, w(x-y) :\phi(y)^n:_v \, dx dy \tag{128}$$

if $|X| = 1, 2$ and defining $Q_{2n}(v, w; X, \phi) = 0$ if $|X| \ge 3$. Then the following formula defines Q_i:

$$\begin{aligned} Q_i(X, \phi) = {} & \lambda_i^2[8Q^6(v_i, w_i; X, \phi) + 36Q^4(v_i, (w_i)^2; X, \phi) + & (129)\\ & 48Q^2((w_i)^3\chi_L^c; X, \phi) + 12Q^0((w_i)^4\chi_L^c; X)] + Q_i'(X, \phi). \end{aligned}$$

Here $\chi(x,y) = 1$ if $\Delta_x \cup \Delta_y$ is a small set (Δ_x = the unit block containing x), and is 0 otherwise. Also $\chi^c = 1 - \chi$ and $\chi_L(x,y) = \chi(Lx, Ly)$. The last term $Q_i'(X,\phi)$ is supported on small sets X with $|X| \leq 2$ and has the form

$$Q_i'(X,\phi) = \lambda_i^2 \int_{X\times X} \phi(x) q_i^\mu(X;x,y)(\partial_\mu\phi)(y)dxdy \tag{130}$$

where the kernel q_i is a function (not just a measure) to be specified further.

All the functionals K_i, V_i, Q_i, R_i are to be regarded as functions of $\psi \in C(\Lambda_i')$. For V, Q one replaces $\phi(x)$ by $\psi(\emptyset, x)$ and $(\partial_\mu\phi)(x)$ by $\psi(\mu, x)$. As functions of ψ these functionals have norms of the form discussed in Sect. 5.

We now make specific choices for the norms

$$\|\cdot\|_i \equiv \|\cdot\|_{G_i,\Gamma,h_i}, \tag{131}$$

$$|\cdot| \equiv |\cdot|_{\Gamma,1}. \tag{132}$$

We take

$$G_i \equiv G(0,\kappa_i) \tag{133}$$

where G is given by (53) and $\kappa_i = \lambda_i^{1/2}$. The large set regulator Γ is as defined in Sect. 5. Finally

$$h_i = (h_{i0}, h_{i1}) = (\delta\lambda_i^{-1/4}, \delta\lambda_i^{-1/4}) \tag{134}$$

for some constant δ. This is the largest choice of h consistent with Theorem 1.

As a reference point for the mass we take a local version of second order perturbation theory:

$$\hat{\mu}_i = L^{2(N-i)}\hat{\mu}_N - 48\lambda_i^2 \int_{\Delta\times\Lambda_i} (w_i(x-y))^3 \chi_L(x,y)dxdy. \tag{135}$$

As noted previously this is bounded uniformly in N. For the change in the second order mass we use

$$\delta\hat{\mu}_i = -48\lambda_i^2 \int_{\Delta\times\Lambda_i} (w^\#(x-y)^3\chi(x,y) - w_i(x-y)^3\chi_L(x,y))dxdy \tag{136}$$

where we recall that $w^\# = w_i + C$. We still have $\hat{\mu}_{i-1} = L^2(\hat{\mu}_i + \delta\hat{\mu}_i)$.

All the results to follow are obtained under the following hypotheses. Fix $0 < \epsilon < 1/2$. We assume that δ is sufficiently small, that L is sufficiently large (depending on δ), and that λ is sufficiently small (depending on δ, L). Constants that may depend on δ are denoted by $\mathcal{O}(1)$ and constants that may depend on L are denoted by the letter C whose value may vary from line to line. A constant of this type whose value does not vary from line to line is denoted by C_1, C_2, etc.

Now we are ready to state the main theorem which gives bounds on the polymer activities K_i and the effective masses μ_i uniform in N. (Since we have not renormalized the energy we do not get good bounds on Ω_i.)

Theorem 8. *Under the above hypotheses there is a choice of $\alpha_0(Y), \alpha_1(Y)$ and a constant C_1 so that for all i, N with $0 \leq i \leq N$ the polymer activities have the form*

$$K_i = Q_i \exp(-V_i) + R_i$$

where $|Q_i'|_{\gamma\Gamma,1} \leq C_1\lambda_i^2$ and

$$\begin{aligned} \|R_i\|_i &\leq \lambda_i^{1-\epsilon} \\ |R_i| &\leq \lambda_i^{3-\epsilon} \\ |\mu_i - \hat{\mu}_i| &\leq \lambda_i^{3-\epsilon}. \end{aligned}$$

The bounds on R_i are smaller than the following bounds on $Q_i e^{-V_i}$ so the R_i really are remainder terms.

Lemma 8. *There is a constant C_2 such that for all i, N with* $0 \leq i \leq N$

$$\begin{aligned} \|Q_i e^{-V_i}\|_i &\leq C_2 \lambda_i^{1/2} \leq \lambda_i^{1/2-\epsilon} \\ |Q_i e^{-V_i}| &\leq C_2 \lambda_i^2 \leq \lambda_i^{2-\epsilon}. \end{aligned}$$

Proof. Since $\kappa_i \lambda_i^{-1/2} = 1 > a$, Corollary 1 implies

$$\|Q_i e^{-V_i}\|_i \leq \mathcal{O}(1) |Q_i|_{\gamma\Gamma, h_i}.$$

To estimate $|Q_i|_{\gamma\Gamma,h_i}$ we note that

$$\begin{aligned} \int_{\Delta\times\Delta'} |w_i^p(x,y)| dx dy &\leq C \exp[-\alpha d(\Delta,\Delta')], \qquad \text{for } p = 1,2 \\ \int_{\Delta\times\Delta'} |w_i^p(x,y)| \chi_L^c(x,y) dx dy &\leq C \exp[-\alpha d(\Delta,\Delta')], \qquad \text{for } p = 3,4 \end{aligned}$$

for some C and $0 < \alpha < 1$, both depending on L. Note that in the second bound the characteristic function enforces that $|x-y| \geq 1/L$: this is needed since $w_i(x,y)$ has the singularity $\mathcal{O}(|x-y|^{-1})$ as $|x-y| \to 0$. Using also

$$\sum_{\Delta'} (\gamma\Gamma)(\Delta,\Delta') e^{-\alpha d(\Delta,\Delta')} \leq C$$

we find that the first four terms in Q_i have norms bounded by $C\lambda_i^2 h_i^6$. For the last term we use $|Q_i'|_{\gamma\Gamma,h_i} = h_i^2 |Q_i'|_{\gamma\Gamma,1} \leq C_1 \lambda_i^2 h_i^2$. Thus we have

$$|Q_i|_{\gamma\Gamma,h_i} \leq C\lambda_i^2 h_i^6 \leq C_2 \lambda_i^{1/2}.$$

Similarly,

$$|Q_i e^{-V_i}|_{\Gamma,1} \leq |Q_i|_{\gamma\Gamma,1} \leq C_2 \lambda_i^2.$$

□

The proof of Theorem 8 is by induction on i working down from $i = N$. Clearly the result is true for $i = N$, since $K_N = 0$. The proof of the inductive step $i \to i-1$ is broken up into three lemmas, each analyzing a piece of the transformation $K_{i-1} = \mathcal{S}\mathcal{E}\mathcal{F} K_i$.

To control the fluctuation step we introduce a norm $\|\cdot\|_\#$ with regulators:

$$\begin{aligned} G^\#(X,\phi) &= G(0,\kappa_{i-1}; L^{-1}X, \phi_L), \\ \Gamma^\#(X) &= \gamma(X)^{-1}\Gamma(X), \\ h^\# &= (1/2)h_i \end{aligned} \tag{137}$$

and also the norm $|\cdot|_\# = |\cdot|_{\Gamma^\#,1/2}$.

Lemma 9. *$K^\#$ has the form*

$$K^\# = Q^\# \exp(-V^\#) + R^\#$$

where

$$\begin{aligned} Q^\# \;&=\; \lambda_i^2\Big[8Q^6(v^\#, w^\#) + 36Q^4(v^\#, (w^\#)^2) + 48Q^2((w^\#)^3 - (w_i)^3\chi_L) \\ &\quad +12Q^0((w^\#)^4 - (4(w_i)^3C + (w_i)^4)\chi_L)\Big] + Q_i' + \Delta_C Q_i' \end{aligned}$$

and where

$$\begin{aligned} \|R^\#\|_\# \;&\le\; \mathscr{O}(1)\lambda_i^{1-\epsilon} \\ |R^\#|_\# \;&\le\; \mathscr{O}(1)\lambda_i^{3-\epsilon}. \end{aligned}$$

The extraction step is controlled using the norm $\|\cdot\|_*$ defined with the regulators $G^* = G^\#$, $\Gamma^*(X) = \gamma(X)^{-3}\Gamma(X)$, and $h^* = h^\#$. We also define $|\cdot|_* = |\cdot|_{\Gamma^*,1/2}$.

Lemma 10. *K^* has the form*

$$K^* = Q^* \exp(-V^*) + R^* + S^*$$

with

$$\begin{aligned} Q^* \;&=\; \lambda_i^2\Big[8Q^6(v^\#, w^\#) + 36Q^4(v^\#, (w^\#)^2) \\ &\quad +48Q^2((w^\#)^3\chi^c) + 12Q^0((w^\#)^4\chi^c)\Big] + Q_i''. \end{aligned}$$

Here Q_i'' has the form (130) and satisfies $|Q_i''|_{\gamma\Gamma,1} \le \mathscr{O}(1)C_1\lambda_i^2$. Also $R_n^*(X, \phi = 0) = 0$ for X small and $\dim n = n_0/2 + 3n_1/2 < 2$ *and*

$$\begin{aligned} \|R^*\|_* \;&\le\; \mathscr{O}(1)\lambda_i^{1-\epsilon} \\ |R^*|_* \;&\le\; \mathscr{O}(1)\lambda_i^{3-\epsilon} \\ \|S^*\|_* \;&\le\; \mathscr{O}(1)\lambda_i^{3/2-\epsilon} \\ |S^*|_* \;&\le\; \mathscr{O}(1)\lambda_i^{7/2-\epsilon} \\ |\delta\mu_i - \delta\hat{\mu}_i| \;&\le\; \mathscr{O}(1)\lambda_i^{3-\epsilon}. \end{aligned}$$

Finally the proof of the theorem is completed by the scaling step:

Lemma 11. *$K_{i-1} = \mathscr{S}(K^*)$, Q_{i-1}' and $\mu_{i-1} = L^2(\mu_i + \delta\mu_i)$ satisfy the conditions of Theorem 8.*

Now we prove each of these lemmas.

Proof of lemma 9. The proof relies on Theorems 2,3 and 4. These will apply once we have checked that the homotopy hypothesis 78 can be satisfied since $\|C_i\|_\theta \le C$ for all i (even for $i = 1$) which follows from standard bounds on such covariances. In the Appendix to this section we show that the homotopy hypothesis is satisfied.

Let $B(t) = (\Box + Q(t))e^{-V(t)}$ where

$$\begin{aligned} V(t,X) &= \lambda_i V_4(X,v_t) + \mu_i V_2(X,v_t) \\ Q(t) &= \lambda^2\Big[8Q^6(v_t,w_t) + 36Q^4(v_t,w_t^2) + 48Q^2(v_t,w_t^3 - w_i^3\chi_L) \\ &\quad +12Q^0(w_t^4 - (4tw_i^3C + w_i^4)\chi_L) + Q' + t\Delta_C Q'\Big] \end{aligned}$$

with $v_t = v_i - tC$ and $w_t = w_i + tC$. Then $B(t)$ interpolates between $B(0) = (\Box + Q_i)e^{-V_i}$ and $B(1) = (\Box + Q^\#)e^{-V^\#}$. It is also an approximate solution of the fluctuation equation (35). Indeed we will show that the discrepency

$$E(t) = \frac{\partial B}{\partial t} - \Delta_C B - \frac{1}{2}C\left(\frac{\partial B}{\partial \psi} \,\circ\, \frac{\partial B}{\partial \psi}\right)$$

satisfies

$$\|E(t)\|_i \leq C\lambda_i, \tag{138}$$

$$|E(t)| \leq C\lambda_i^3. \tag{139}$$

Recalling that $A(t) = \bar{\mathscr{F}}_t(A_i)$ is the exact flow of the fluctuation equation, and that $A^\# = A(1)$, it follows from Theorem 3 that

$$\begin{aligned} R^\# &\equiv A(1) - B(1) \\ &= \int_0^1 \bar{\mathscr{F}}_1(B_i + tR_i; R_i)dt - \int_0^1 (\bar{\mathscr{F}}_{1-t})_1(B(t); E(t))dt. \end{aligned}$$

Now the proof of Lemma 9 follows from Theorem 4 since

$$\|R^\#\|_\# \leq \mathscr{O}(1)\|R_i\|_i + \mathscr{O}(1)\sup_t \|E(t)\|_i \leq \mathscr{O}(1)\lambda_i^{1-\epsilon}. \tag{140}$$

Similarly using Theorem 4 we get $|R^\#|_\# \leq \mathscr{O}(1)\lambda_i^{3-\epsilon}$.

We prove (138) and (139) by first defining

$$J(X) = \begin{cases} \frac{1}{2}\int_{\Delta_1\times\Delta_2\cup\Delta_2\times\Delta_1} C(x-y)\frac{\partial V(t)}{\partial\phi(x)}\frac{\partial V(t)}{\partial\phi(y)} & \text{if } X = \Delta_1 \cup \Delta_2 \\ 0 & \text{otherwise,} \end{cases} \tag{141}$$

and then writing

$$\begin{aligned} E(t) &= \left(\frac{\partial}{\partial t} - \Delta_C + J\right)\Box\, e^{-V(t)} && \text{call this } I, \\ &\quad +Q(t)\cdot\left(\frac{\partial}{\partial t} - \Delta_C\right)e^{-V(t)} && \text{call this } II, \\ &\quad -C\left(\frac{\partial Q(t)}{\partial\phi}, \frac{\partial e^{-V(t)}}{\partial\phi}\right) && \text{call this } III, \\ &\quad +\left\{(\frac{\partial}{\partial t} - \Delta_C)Q(t) - J\Box\right\}e^{-V(t)} && \text{call this } IV, \\ &\quad -(J - J\Box)e^{-V(t)} && \text{call this } V, \\ &\quad -C\left(\frac{\partial\,\Box\, e^{-V(t)}}{\partial\phi} \,\circ\, \frac{\partial Q(t)e^{-V(t)}}{\partial\phi}\right) && \\ &\quad -\frac{1}{2}C\left(\frac{\partial Q(t)e^{-V(t)}}{\partial\phi} \,\circ\, \frac{\partial Q(t)e^{-V(t)}}{\partial\phi}\right) && \text{call this } VI. \end{aligned}$$

(The first four terms come from $(\frac{\partial}{\partial t} - \Delta_C)B$ and the last two from $-\frac{1}{2}C(\frac{\partial B}{\partial \phi} \circ \frac{\partial B}{\partial \phi})$.)

We now proceed to estimate each term

1. I vanishes because $(\frac{\partial}{\partial t} - \Delta_C)V(t) = 0$ by the definition of Wick ordering.
2. II has the form $QPe^{-V(t)}$ for some polynomial P. The polynomial has terms labeled by two vertex tree diagrams. (Again we use $(\frac{\partial}{\partial t} - \Delta_C)V(t) = 0$ to suppress the single vertex term.) Each vertex either comes from a mass counterterm and is $\mathcal{O}(\mu_i) = \mathcal{O}(\lambda_i^2)$ or from the interaction and is $\mathcal{O}(\lambda_i)$.
   By Corollary 1 and a variation of Lemma 8 we estimate this term by
$$\begin{aligned} \|QPe^{-V(t)}\|_i &\leq |QP|_{\gamma\Gamma,h_i} \\ &\leq |Q|_{\gamma\Gamma,h_i}|P|_{\gamma\Gamma,h_i} \\ &\leq C\lambda_i. \end{aligned} \tag{142}$$
   Here the bound on P can be patterned on the bound on Q. Similarly
$$|QPe^{-V(t)}| \leq C\lambda_i^4. \tag{143}$$
3. III has the form $Pe^{-V(t)}$ where P is a polynomial with $\mathcal{O}(\lambda_i^3)$ coefficients (or smaller). It has terms labeled by three vertex tree diagrams localized in at most two squares. More precisely it is a tree provided we regard the terms from Q as having single propagators $w_i, w_i^2, w_i^3\chi_L^c$ or q_i. Since there are at most 8 fields in any term, it is straightforward to bound the norm by $C\lambda_i^3 h_i^8 \leq C\lambda_i$ and the kernel norm by $C\lambda_i^3$. In making this estimate for the terms involving q_i one can use the fact that it is supported on small sets X and that for $\Delta, \Delta' \subset X$
$$\int_{\Delta\times\Delta'} |q_i^\mu(X;x,y)|dxdy \leq \lambda_i^{-2}|Q_i'|_{\gamma\Gamma,1} \leq C_1.$$
4. $IV + V = Pe^{-V(t)}$ where P is a two vertex tree diagram of the form $\mathcal{O}(\lambda_i\mu_i) + \mathcal{O}(\mu_i^2)$, i.e. at least one vertex is from the mass counterterm. To see this we compute
$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta_C)Q(t) &= \lambda_i^2\Big[8Q^6(v_t,C) - (8)(9)Q^4(v_t,w_tC) \\ &\quad +(36)(2)Q^4(v_t,w_tC) - (36)(4)Q^2(v_t,w_t^2C) \\ &\quad +(48)(3)Q^2(v_t,w_t^2C) - 48Q^0(w_t^3C - w_i^3C\chi_L) \\ &\quad +12Q^0(4w_t^3C - 4w_i^3C\chi_L)\Big] \\ &= 8\lambda_i^2Q^6(v_t,C) \end{aligned} \tag{144}$$
   and
$$J = 8\lambda_i^2Q^6(v_t,C) + \mathcal{O}(\lambda_i\mu_i) + \mathcal{O}(\mu_i^2). \tag{145}$$
   These remainder terms have norms bounded by $C\lambda_i^3h_i^4 \leq C\lambda_i^2$ and kernel norms bounded by $C\lambda_i^3$.

5. VI can also be written in the form $Pe^{-V(t)}$ and treated similarly, although the details are a bit more involved. For example, one contribution to P evaluated on $X = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ is

$$1/2Q(\Delta_1 \cup \Delta_2)Q(\Delta_3 \cup \Delta_4)\ C\left(\frac{\partial V(\Delta_1)}{\partial\phi}, \frac{\partial V(\Delta_3)}{\partial\phi}\right).$$

This is a product of three tree diagrams. Each diagram has $\mathcal{O}(\lambda_i^2)$ coefficients and at most 6 fields for an $C\lambda_i^{1/2}$ bound. Overall the term is bounded by $C\lambda_i^{3/2}$. Note that the localizations overlap in such a way that in the sum over X there is always adequate decay to cancel the growth of $\gamma\Gamma(X)$: do the sums in the order Δ_4, Δ_3, Δ_2.

□

Proof of lemma 10. We write

$$K^* = \mathcal{E}(K^\#, F) = \mathcal{E}_1(K^\#, F) + \mathcal{E}_{\geq 2}(K^\#, F)$$

where $\mathcal{E}_1$ is the linearization of the extraction step and $\mathcal{E}_{\geq 2}$ is the remainder. The first term is

$$\mathcal{E}_1(K^\#, F) = K^\# - Fe^{-V^\#}$$

which, if we write $F = F_Q + F_R$, can be expressed as

$$\mathcal{E}_1(K^\#, F) = (Q^\# - F_Q)e^{-V^\#} + (R^\# - F_R e^{-V^\#}).$$

Now K^* can be written in the form $Q^* e^{-V^*} + R^* + S^*$ if we define

$$Q^* = Q^\# - F_Q, \tag{146}$$

$$R^* = (R^\# - F_R e^{-V^\#}), \tag{147}$$

$$S^* = \mathcal{E}_{\geq 2}(K^\#, F) + Q^*(e^{-V^\#} - e^{-V^*}). \tag{148}$$

We choose F_Q and F_R to cancel the local low order terms in ϕ in $Q^\#$ and $R^\#$ for small sets. Let $X = \Delta \cup \Delta'$ be a small set. We define $F_Q(X)$ by taking the constant terms in $Q^\#$, and also inserting the identity

$$\phi(x)\phi(y) = \frac{1}{|X|}\int_X dz\left[(\phi(z))^2 + \int_0^1 \frac{d}{ds}\phi(\gamma_{xz}(s))\phi(\gamma_{yz}(s))\right] \tag{149}$$

into the Q^2 term in $Q^\#$ and retaining the first term. Here $\gamma_{xz}(s)$ is some standard choice of path in X from z to x. This can be done in a way that is invariant under lattice symmetries (see [BK93] for a detailed discussion). The complete definition of F_Q is then

$$F_Q(X) = 12\lambda_i^2 Q^0(X, (w^\#)^4 - (4w_i^3 C - w_i^4)\chi_L) + \Delta_C Q_i' \tag{150}$$
$$+48\lambda_i^2\left[\frac{1}{|X|}\int_X dz(\phi(z))^2\right]\int_{\Delta\times\Delta'\cup\Delta'\times\Delta}\left[(w^\#)^3 - w_i^3\chi_L(x,y)\right]dxdy$$

for $X = \Delta \cup \Delta'$ and small and $F_Q(X) = 0$ otherwise.

Notice that this has the form

$$F_Q(X) = \alpha_{0Q}(X)|X| + \alpha_{1Q}(X)V_2(X, v^\#)$$

where

$$\alpha_{1Q}(X) = \frac{48\lambda_i^2}{|X|}\int_{\Delta\times\Delta'\cup\Delta'\times\Delta} [(w^\#)^3\chi - w_i^3\chi_L]$$

and α_0 satisfies $|\alpha_0(X)| \leq C\lambda_i^2$. The Q extractions lead to the perturbative change in the mass (cf. Eq. (136))

$$-\sum_{X\supset\Delta} \alpha_{1Q}(X) = -48\lambda_i^2 \int_{\Delta\times\Lambda_i} [(w^\#)^3\chi - w_i^3\chi_L] \equiv \delta\hat{\mu}_i. \tag{151}$$

The definition of $Q^* = Q^\# - F_Q$ agrees with the definition in the lemma provided we define

$$Q_i'' = Q_i' + \delta Q \tag{152}$$

where for $X = \Delta \cup \Delta'$ and small

$$\begin{aligned} \delta Q(X) &= 48\lambda_i^2 \int_{\Delta\times\Delta'\cup\Delta'\times\Delta} dx\, dy\, [(w^\#)^3 - w_i^3\chi_L](x, y) \\ &\quad \left[\frac{1}{|X|}\int_X dz \int_0^1 \frac{d}{ds}\phi(\gamma_{xz}(s))\phi(\gamma_{yz}(s))ds\right]. \end{aligned}$$

We redefine it as a functional of ψ by carrying out the s derivatives and replacing $(\partial_\mu\phi)(x) \equiv \partial_\mu\psi(x, \emptyset)$ by $\psi(x, \mu)$. Then δQ has the required form.

Taking into account that the leading singularities in $(w^\#)^3$ and w_i^3 cancel so that $(w^\#)^3 - w_i^3$ is locally integrable uniformly in i and N, we can bound δQ using the techniques in the proof of Lemma 8 and obtain $|\delta Q|_{\gamma\Gamma,1} \leq C_1\lambda_i^2$ provided C_1 is sufficiently large; this fixes C_1. Then $|Q''|_{\gamma\Gamma,1} \leq \mathcal{O}(1)C_1\lambda_i^2$

For the choice of F_R we proceed similarly. Let $R^{\#,\leq 2}$ be the expansion of $R^\#$ to second order in ϕ on small sets and zero on large sets. In the quadratic piece we localize the ϕ dependence using (149). This means we replace $R^{\#,\leq 2}$ by $R^{\#,\leq 2} - \delta R$ where

$$\delta R = 1/2 \int R_{2,0}^\#(X, 0; x, y)[\frac{1}{|X|}\int_X dz \int_0^1 \frac{d}{ds}\phi(\gamma_{xz}(s))\phi(\gamma_{yz}(s))ds]dxdy.$$

Then we choose F_R so that $(F_R e^{-V^\#})^{\leq 2} = R^{\#,\leq 2} - \delta R$. We find we should take (for small sets only):

$$F_R(X) = \alpha_{0R}(X)|X| + \alpha_{1R}(X)V_2(X, v^\#)$$

where α_{0R}, α_{1R} are the solutions of the equations

$$\begin{aligned} (\alpha_{0R}(X) - \alpha_{1R}(X)v^\#(0))e^{-b} &= R^\#(X, 0) \\ (\alpha_{1R}(X) - \alpha_{0R}(X)a)e^{-b} &= (2|X|)^{-1}\int R_{2,0}^\#(X, 0; x, y)dxdy. \end{aligned} \tag{153}$$

Here a, b are the coefficients of the quadratic and constant terms in $V^\#$:

$$\begin{aligned} a &= -6\lambda_i v^\#(0) + \mu_i \\ b &= 3\lambda_i(v^\#(0))^2 - \mu_i v^\#(0). \end{aligned}$$

The contribution to the change in mass is by definition $\delta\mu_i - \delta\hat{\mu}_i$ and is given by

$$\delta\mu_i - \delta\hat{\mu}_i = -\sum_{X\supset\Delta} \alpha_{1R}(X).$$

Now taking into account that $v^{\#}(0)$ is bounded we have that a, b are $\mathcal{O}(\lambda_i)$, and so one can show

$$|\delta\mu_i - \delta\hat{\mu}_i| \le \sum_{X\supset\Delta} |\alpha_{1R}(X)| \le \mathcal{O}(1)|R^{\#}|_{\#} \le \mathcal{O}(1)\lambda_i^{3-\epsilon}$$

which gives the bound on $\delta\mu_i - \delta\hat{\mu}_i$.

We write

$$R^* = (R^{\#} - F_R e^{-V^{\#}}) = [R^{\#} - R^{\#,\le 2}] + \delta R + [(F_R e^{-V})^{\le 2} - F_R e^{-V}]. \qquad (154)$$

Note that in the term δR we can again replace $\partial_\mu\phi$ by $\psi(x,\mu)$ and then $R_n^*(X,\phi = 0) = 0$ for X small and $\dim(n) < 2$ as claimed. To bound R^* note that the terms $\|\delta R\|_{\#}$ and $\|R^{\#} - R^{\#,\le 2}\|_{\#}$ are bounded by $\mathcal{O}(1)\|R^{\#}\|_{\#} \le \mathcal{O}(1)\lambda_i^{1-\epsilon}$. Since $\alpha_R = \mathcal{O}(\lambda_i^{3-\epsilon})$ we also have $\|F_R e^{-V} - (F_R e^{-V})^{\le 2}\|_{\#} \le C\lambda_i^{5/2-\epsilon} \le \mathcal{O}(1)\lambda_i^{1-\epsilon}$. Thus $\|R^*\|_* \le \|R^*\|_{\#} \le \mathcal{O}(1)\lambda_i^{1-\epsilon}$. Similarly the kernel norm satisfies $|R^*|_* \le \mathcal{O}(1)\lambda_i^{3-\epsilon}$.

Finally consider S^*. For the bound on $\mathcal{E}_{\ge 2}(K^{\#}, F)$ we shall apply Corollary 2. The hypothesis (98) is verified by using Theorem 1 with a value $r = \mathcal{O}(1)\lambda^{1/2}$. With $\|\alpha\| = \|\alpha_0\| + r^{-1}\|\alpha_1\|$ we find that

$$\begin{aligned} \|\mathcal{E}_{\ge 2}(K^{\#}, F)\|_* &\le \mathcal{O}(1)\|\alpha\|_{\Gamma^{\#}}\|K^{\#}\|_{\#} \\ |\mathcal{E}_{\ge 2}(K^{\#}, F)|_* &\le \mathcal{O}(1)\|\alpha\|_{\Gamma^{\#}}\, |K^{\#}|_{\#}. \end{aligned}$$

Since $\|K^{\#}\|_{\#} \le \mathcal{O}(1)\lambda_i^{1/2-\epsilon}$, $|K^{\#}|_{\#} \le \mathcal{O}(1)\lambda_i^{2-\epsilon}$, and $\|\alpha\|_{\Gamma^{\#}} \le C\lambda_i^{3/2}$ we obtain the bounds $\mathcal{O}(1)\lambda_i^{3/2-\epsilon}$ and $\mathcal{O}(1)\lambda_i^{7/2-\epsilon}$. Finally $Q^*(e^{-V^{\#}} - e^{-V^*})$ supports a similar bound (details are left to the reader) and we have the required $\|S^*\|_* \le \mathcal{O}(1)\lambda_i^{3/2-\epsilon}$ and $|S^*|_* \le \mathcal{O}(1)\lambda_i^{7/2-\epsilon}$. This completes the proof of Lemma 10.

□

Proof of lemma 11. We have

$$K_{i-1} = \mathcal{S}(K^*) = \mathcal{S}_1(K^*) + \mathcal{S}_{\ge 2}(K^*)$$

where $\mathcal{S}_1$ is the linearization of $\mathcal{S}$ and $\mathcal{S}_{\ge 2}$ is the remainder and further

$$\mathcal{S}_1(K^*) = \mathcal{S}_1(Q^* e^{-V^*}) + \mathcal{S}_1(R^*) + \mathcal{S}_1(S^*).$$

Let $\mathcal{S}_1^0$ be the linearized scaling operator with $V = 0$, that is

$$\mathcal{S}_1^0(K)(Z,\psi) = \sum_{X:\bar{X}^L = LZ} K(X, \psi_{L^{-1}}).$$

Then the first term is computed as

$$\mathcal{S}_1(Q^* e^{-V^*}) = \mathcal{S}_1^0(Q^*)e^{-V_{i-1}} = Q_{i-1}e^{-V_{i-1}}$$

provided we define

$$Q'_{i-1} = \mathscr{S}_1^0(Q'').$$

The remainder is then given by

$$R_{i-1} = \mathscr{S}_{\geq 2}(K^*) + \mathscr{S}_1(S^*) + \mathscr{S}_1(R^*). \tag{155}$$

To estimate the remainder we use Theorem 6 and Theorem 7 to bound each term separately. Thus we need that

$$\|(e^{-V^*})_{L^{-1}}(X)\|_{g,h_{i-1}} = \|e^{-V_{i-1}}(X)\|_{g,h_{i-1}} < 2$$

holds for any L^{-1} scale polymer $X \subset \Delta$, both for $g = 1$ (and hence for G_{i-1}) and for $g = G(-\epsilon_0, 0)$ with $\epsilon_0 = a\lambda_{i-1}^{-1/2}$ and $a = \mathscr{O}(1)$ (but small). This follows by Theorem 1.

The first two terms of (155) are higher order in λ_i than we need, and we use this extra smallness to cancel any growth factor. By Theorem 6 and Lemma 10 we have that $s \to \mathscr{S}(sK^*)$ is analytic in say $|s| \leq \lambda_i^{-1/2+\epsilon/2}$ and is bounded there by

$$\|\mathscr{S}(sK^*)\|_{i-1} \leq \mathscr{O}(1)L^3\|sK^*\|_* \leq \mathscr{O}(1)L^3C\lambda_i^{\epsilon/2}.$$

Since

$$\mathscr{S}_{\geq 2}(K^*) = \frac{1}{2\pi i}\oint \frac{\mathscr{S}(sK^*)}{s^2(s-1)}ds$$

around the circle $|s| = \lambda_i^{-1/2+\epsilon/2}$ we get the bound

$$\|\mathscr{S}_{\geq 2}(K^*)\|_{i-1} \leq \mathscr{O}(1)L^3C\lambda_i^{1-\epsilon/2} \leq \mathscr{O}(1)\lambda_i^{1-\epsilon}.$$

Similarly we have

$$\|\mathscr{S}_1(S^*)\|_{i-1} \leq \mathscr{O}(1)L^3\lambda_i^{3/2-\epsilon} \leq \mathscr{O}(1)\lambda_i^{1-\epsilon}.$$

For the last term we need the more delicate estimate given by Theorem 7. Separating large and small set contributions gives

$$\|\mathscr{S}_1(R^*)\|_{i-1} \leq \mathscr{O}(1)L^{-1}\|R^*\|_* + \mathscr{O}(1)L^3\|R^*\|_{G^*,\Gamma^*,(h_{i-1})_L,dim\geq 2}$$

where

$$(h_{i-1})_L = (L^{-3/4}\delta\lambda_i^{-1/4}, L^{-7/4}\delta\lambda_i^{-1/4}).$$

The theorem is applicable since

$$\min\{\epsilon_0 h_{i-1}^2, \kappa_{i-1}h_{i-1}^2\} = \min\{a, 1\} = a = \mathscr{O}(1) > 0.$$

Now we can extract some powers of L^{-1} in the second term in passing from $(h_{i-1})_L$ to $h^* = (\frac{1}{2}\delta\lambda_i^{-1/4}, \frac{1}{2}\delta\lambda_i^{-1/4})$. The worst term with dim ≥ 2 is a $\phi\ \partial\phi$ term which gives $L^{-5/2}$. Then we may continue with

$$\|\mathscr{S}_1(R^*)\|_{i-1} \leq \mathscr{O}(1)L^{1/2}\lambda_i^{1-\epsilon}.$$

Combining the above bounds gives the required result $\|R_{i-1}\|_{i-1} \leq \mathscr{O}(1)L^{1/2}\lambda_i^{1-\epsilon} \leq \lambda_{i-1}^{1-\epsilon}$, where we use $\lambda_i = L^{-1}\lambda_{i-1}$.

Now for the bound on the kernels of R_{i-1}. Again for the first two terms of (155) we have a higher power of λ_i than we need:

$$\begin{aligned}|\mathscr{S}_{\geq 2}(K^*)| &\leq \mathscr{O}(1)L^3\lambda_i^{4-\epsilon}\\ &\leq \mathscr{O}(1)\lambda_i^{3-\epsilon}\\ |\mathscr{S}_1(S^*)| &\leq \mathscr{O}(1)L^3\lambda_i^{7/2-\epsilon}\\ &\leq \mathscr{O}(1)\lambda_i^{3-\epsilon}.\end{aligned}$$

And for the last term we again use Theorem 7 to obtain:

$$\begin{aligned}|\mathscr{S}_1(R^*)| &\leq \mathscr{O}(1)L^{-1}|R^*|_* + \mathscr{O}(1)L^3|R^*|_{\Gamma^*,(1)_L,dim\geq 2}\\ &\leq \mathscr{O}(1)L\lambda_i^{3-\epsilon}.\end{aligned}$$

Combining the above we get the required $|R_{i-1}|_\Gamma \leq \mathscr{O}(1)L\lambda_i^{3-\epsilon} \leq \lambda_{i-1}^{3-\epsilon}$.

The previous bound is also a model for the bound on $Q'_{i-1} = \mathscr{S}_1^0(Q'')$. Since $(Q'')_n(X,0) = 0$ for $\dim(n) < 2$ we have as above

$$|Q'_{i-1}|_{\gamma\Gamma,1} \leq \mathscr{O}(1)L|Q''|_{\gamma\Gamma,1} \leq \mathscr{O}(1)C_1L\lambda_i^2 \leq C_1\lambda_{i-1}^2.$$

Finally we note that the mass term has the correct behavior:

$$\begin{aligned}|\mu_{i-1} - \hat{\mu}_{i-1}| &\leq L^2|\mu_i - \hat{\mu}_i| + L^2|\delta\mu_i - \delta\hat{\mu}_i|\\ &\leq \mathscr{O}(1)L^2\lambda_i^{3-\epsilon}\\ &\leq \lambda_{i-1}^{3-\epsilon}.\end{aligned}$$

This completes the proof of Lemma 11 and the main theorem.

□

7.1. Appendix: The Homotopy Hypothesis. We have defined $G_i \equiv G(0,\kappa_i)$ and $G^{\#}(X,\phi) \equiv G(0,\kappa_{i-1};L^{-1}X,\phi_L)$. Recall that $\kappa_i = \sqrt{\lambda_i}$. To apply Theorems 2,3 and 4 let $G_i(t)$ be the geometric interpolation

$$G_i(t) = G_i^{1-t}(\gamma G^{\#})^t, \tag{156}$$

where $t \in [0,1]$ and $\gamma = \gamma(X) \equiv 2^{|X|}$.

Lemma 12. *Given $L \geq 2$, there exists $\lambda_0 > 0$ such that $\forall\lambda \in [0,\lambda_0]$, $\forall i \geq 0$ and $\forall s < t \in [0,1]$*

$$\mu_{(t-s)C} * G_i(s,X) \leq G_i(t,X). \tag{157}$$

Proof. Let $U(s,\phi) := \log G(s,\phi)$. It is enough to prove that

$$\frac{\partial U}{\partial s} - \Delta_C U - \frac{1}{2}C(\frac{\partial U}{\partial\phi},\frac{\partial U}{\partial\phi}) \geq 0 \tag{158}$$

because of the implications

$$\begin{aligned}\frac{\partial U}{\partial s} - \Delta_C U - \frac{1}{2}C(\frac{\partial U}{\partial\phi},\frac{\partial U}{\partial\phi}) \geq 0 &\Rightarrow \mu_{(t-s)C} * \left(\frac{\partial G(s)}{\partial s} - \Delta_C G(s)\right) \geq 0\\ &\Rightarrow \frac{\partial}{\partial s}\mu_{(t-s)C} * G(s,X) \geq 0 \text{ for } s \in (0,t)\\ &\Rightarrow \mu_{(t-s)C} * G(s,X) \leq G(t,X).\end{aligned}$$

From the definitions

$$U = t\log(2)|X| + \sqrt{\lambda_i} \sum_{1\leq|\alpha|\leq s} \int_X |\partial^\alpha\phi|^2 \cdot \left((L^{2|\alpha|-3/2}-1)t+1\right) \tag{159}$$

from which we verify (158): for example, if we choose λ_0 small so that

$$\sqrt{\lambda_0} \sup_{x,y} |\partial_x^\alpha \partial_y^\beta C(x-y)| \tag{160}$$

is small for $1 \leq |\alpha|, |\beta| \leq s$ then the ϕ independent term in $\partial U/\partial t$ dominates $\Delta_C U$ for all i. To dominate $C(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi})$, we use

$$|C(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi})| \leq \|C\|\lambda_i \sum_{1\leq|\alpha|\leq s} (L^{2|\alpha|-3/2})^2 \int_X |\partial^\alpha\phi|^2 \tag{161}$$

which is smaller than the ϕ dependent terms in $\partial U/\partial t$ when $\sqrt{\lambda_0}\|C\|$ is sufficiently small. Here $\|C\|$ is an L_2 norm in $x-y$ on the (matrix-valued) kernels $|\partial_x^\alpha \partial_y^\beta C(x-y)|$, $1 \leq |\alpha|, |\beta| \leq s$.

□

8. The Generating Functional and Correlations

Now we consider the generating functional $S^N(\rho)$. From (22) and (118) this can be written

$$\begin{aligned} S^N(\rho) &= \exp(-1/2(\rho, \tilde{v}_N\rho) + \Omega_0^N)\bar{S}^N(\rho), \\ \bar{S}^N(\rho) &= (\mathscr{E}xpA_0^N)(\Delta, \phi = i\tilde{v}_N\rho) = A_0^N(\Delta, i\tilde{v}_N\rho). \end{aligned} \tag{162}$$

The truncated correlation functions are the functional derivatives of $\log S(\rho)$ with respect to ρ. For the two point function one has

$$\begin{aligned} \langle\phi(\rho_1)\phi(\rho_2)\rangle^T &= (-i)^2 \frac{\partial^2}{\partial s_1 \partial s_2} \log S(s_1\rho_1 + s_2\rho_2)\Big|_{s=0} \\ &= (\rho_1, \tilde{v}_N\rho_2) + \frac{\partial^2}{\partial s_1 \partial s_2} \log \bar{S}^N(s_1\rho_1 + s_2\rho_2)\Big|_{s=0} \end{aligned}$$

and for the truncated n-point function

$$\langle\phi(\rho_1), \ldots, \phi(\rho_n)\rangle^T = (-i)^n \frac{\partial^n}{\partial s_1 \ldots \partial s_n} \log \bar{S}(s_1\rho_1 + \cdots + s_n\rho_n))\Big|_{s=0}.$$

Now we can give a bound on the correlation functions that is uniform in N.

Theorem 9. *Suppose the hypotheses of Theorem 8 hold.*

1. *For any $p > 3$ there is a constant R so that $\bar{S}(\rho)$ is analytic in the ball $\|\rho\|_p \leq R^{-1}\lambda^{-1/4}$ in $L^p(\Delta)$ and satisfies there*

$$|\bar{S}(\rho) - 1| \leq 1/2.$$

2. *For* $\rho_i \in L^p(\Delta)$

$$\begin{aligned} |\langle\phi(\rho_1)\phi(\rho_2)\rangle - (\rho_1, \tilde{v}_N\rho_2)| &\leq 2^2R^2\lambda^{1/2}\|\rho_1\|_p\|\rho_2\|_p \\ |\langle\phi(\rho_1), ..., \phi(\rho_n)\rangle^T| &\leq n^nR^n\lambda^{n/4}\prod_{i=1}^{n}\|\rho_i\|_p. \end{aligned}$$

Proof. (1.) We start with the analyticity properties of

$$A_0^N(\Delta, \phi) = e^{-V_0^N(\Delta,\phi)} + K_0^N(\Delta, \phi).$$

By the main theorem $K_0^N(\Delta, \psi)$ is analytic in $\|\psi_0\|_\infty < h_0, \|\psi_1\|_\infty < h_0$ and if we make a Taylor expansion around $\psi = 0$ we find that

$$|K_0^N(\Delta, \psi)| \leq \Gamma(\Delta)(1 - \frac{\|\psi_0\|_\infty}{h_0})^{-1}(1 - \frac{\|\psi_1\|_\infty}{h_0})^{-1}\|K_0^N\|_{G_0,\Gamma,h_0}.$$

This gives analyticity and a bound for $K_0^N(\Delta, \phi) = K_0^N(\Delta, \psi_\phi)$. If we also use $\|K_0^N\|_{G,\Gamma,h_0}$
$\leq 2\lambda^{1/2-\epsilon}$ and take $\|\phi\|_\infty \leq h_0/2$and $\|\partial\phi\|_\infty \leq h_0/2$ we find

$$|K_0^N(\Delta, \phi)| \leq 1/4.$$

With the same restrictions on ϕ we have $|V_0^N(\Delta, \phi)| \leq \mathcal{O}(1)\lambda h_0^4 = \mathcal{O}(1)\delta$. Thus taking δ smaller if necessary we have

$$|e^{V_0^N(\Delta,\phi)} - 1| \leq 1/4.$$

Thus if $\|\phi\|_\infty$ and $\|\partial\phi\|_\infty$ are less than $h_0/2 = \delta\lambda^{-1/4}/2$ we have that $A_0^N(\Delta, \phi)$ is analytic and satisfies

$$|A_0^N(\Delta, \phi) - 1| \leq 1/2.$$

Now specialize to $\bar{S}(\rho) = A_0^N(\Delta, i\tilde{v}_N\rho)$. Since $\tilde{v}_N(x)$ and $\partial\tilde{v}_N(x)$ have the singularities $\mathcal{O}(|x|^{-1})$ and $\mathcal{O}(|x|^{-2})$, the best we can say about both of them is that they are in $L^q(\Delta)$ for $q < 3/2$. For $p > 3$ take $q < 3/2$ so $1/q + 1/p = 1$. Let $R_q = \max(\|\tilde{v}_N\|_q, \|\partial\tilde{v}_N\|_q)$. By Young's inequality we have

$$\begin{aligned} \|\tilde{v}_N * \rho\|_\infty &\leq R_q\|\rho\|_p \\ \|\partial\tilde{v}_N * \rho\|_\infty &\leq R_q\|\rho\|_p. \end{aligned}$$

Thus if $\|\rho\|_p \leq R^{-1}\lambda^{-1/4}$ with $R = 2R_q/\delta$ these quantities are bounded by $\delta\lambda^{-1/4}/2$ and so $\bar{S}(\rho)$ is analytic and satisfies $|\bar{S}(\rho) - 1| \leq 1/2$.

(2.) By part (1.) for $\|\rho\|_p \leq R^{-1}\lambda^{-1/4}$ we have that $\log\bar{S}(\rho)$ is analytic and satisfies $|\log\bar{S}(\rho)| \leq 1$. It follows that for $\|\rho_i\|_p \leq R^{-1}\lambda^{-1/4}n^{-1}$ the function $\log\bar{S}(s_1\rho_1 + \cdots + s_n\rho_n)$ is analytic in $|s_i| < 1$ and is also bounded by 1. By Cauchy bounds the derivatives satisfy

$$\left|\left[\frac{\partial^n}{\partial s_1 \ldots \partial s_n}\log\bar{S}(s_1\rho_1 + \cdots + s_n\rho_n)\right]_{s=0}\right| \leq 1.$$

This gives the bounds of the theorem with the restriction on the ρ_i. The general case of $\rho_i \in L^p(\Delta)$ follows by linearity.

Remarks. The fact that the test functions can be in L^p for any $p > 3$ is a limitation on how singular the truncated correlation functions can be at coinciding points. The result is probably not optimal and one could try for a lower value of p and hence more regularity. The best one could hope for would be $p > 12/11$, for example this is needed so that $V_0^N(\Delta, i\tilde{v}_N\rho)$ is well defined. In any case to do better one would have to get better regularity for the derivatives of the polymer activities K_i^N, possibly by using a stronger norm.

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