Commun. Math. Phys. 135, 547–580 (1991)

Communications in Mathematical Physics
© Springer-Verlag 1991

BRST Cohomology and Highest Weight Vectors. I

Bong H. Lian[1] **and Gregg J. Zuckerman**[2]*

[1] Center for Theoretical Physics, Yale University, New Haven, CT 06520, USA
[2] Department of Mathematics, Yale University, New Haven, CT 06520, USA

Received April 18, 1990

Abstract. We initiate a program to study certain recent problems in non-compact coset CFT by the BRST approach. We derive a reduction formula for the BRST cohomology by making use of a twisting by highest weight modules. As illustrations, we apply the formula to the bosonic string model and a rank one non-compact coset model [DPL]. Our formula provides a completely new approach to non-compact coset construction.

0. Introduction

In recent years, much effort has been focused on studying aspects of conformal field theory models, 2D gravity, string theories and their mutual relations. These theories are often accompanied by rich algebraic structures from which many physical quantities (correlations, critical exponents, string susceptibilities, etc.) can be drawn. A single algebraic structure playing one role in a given model can often play entirely different roles in others. For example, the Virasoro algebra is the constraint algebra in string theory, but becomes a symmetry algebra in CFT. And yet, it is part of a "hidden symmetry algebra" (via the energy momentum tensor) in any theory with a current algebra structure. Table 1 gives a list of problems incorporating the above three roles of algebraic structures. Although the problems have quite different origins, each of them involves solving a system of first class constraints. This point of view therefore suggests to us a unified method to study these problems – the BRST approach.

In this approach, one starts with a quantum state space $\mathscr{H}$ which carries a representation of (usually) a large "hidden" symmetry algebra, $\mathscr{G}$, of the problem. Due to the presence of constraints, one introduces in a natural way some auxiliary degrees of freedom – the ghost states, $\mathscr{H}^{gh}$. Then the constraints, which form a subalgebra of $\mathscr{G}$, can be imposed simultaneously on the enlarged space $\mathscr{H} \otimes \mathscr{H}^{gh}$

* Partially supported by NSF Grant DMS-8703581

Table 1.

	Phy. Problem [ref]	Hidden Symmetry	Constraints	Symmetry
1.	(Super) strings [GSW]	(super) Heisenberg	Vir(Vir$_\varepsilon$)	Wigner little group
2.	Non-compact coset models [DPL, Ba, KS]	$\widehat{su}(p,q)$	$\hat{u}(1) \oplus \widehat{su}(p) \oplus \widehat{su}(q)$	$N=2$ superconformal algebra
3.	2D gravity [KPZ, HPV]	Vir $\ltimes \widehat{sl}(2,\mathbf{R})$	Vir $\times \hat{n}_-$	—
4.	Quantum Hamiltonian reduction. [BO, FF]	Vir $\ltimes \widehat{sl}(2,\mathbf{R})$	$\hat{n}_-$	Vir
5.	”	Vir $\ltimes \widehat{sl}(n,\mathbf{R})$	$\hat{n}_-$	W_n-algebra

via a square zero BRST operator Q. The resulting quotient space $\operatorname{Ker} Q/\operatorname{Im} Q$ is the space of physical states. This space can carry symmetry of its own, namely, any algebra represented on $\mathscr{H} \otimes \mathscr{H}^{\mathrm{gh}}$ that is BRST-invariant becomes a symmetry algebra of the quotient space itself.

Our goal in this paper is to initiate a program to study problems of the kinds in Table 1 via our unified approach, and to explain the scope of our method. Our results will be reported in a series of investigations. Although the BRST approach is a common theme in physics, its associated mathematical tools – the theory of semi-infinite cohomology – is less well-known. One of our purposes of this paper is to develop and to illustrate the tools suitable for our problems. To illustrate the main ideas, we will focus first on problems of type 2 in Table 1. We will show in detail (Chap. 3), an application to the case $su(1,1)$ [DPL] by a completely new approach to that problem. Our on-going work is to generalize to higher-rank algebras. This work will be reported elsewhere.

Let $\mathscr{G}$ be a graded (super) Lie algebra. The theory of semi-infinite cohomology is the study of the functor (or its relative versions) $W \rightsquigarrow H^*_\infty(\mathscr{G}, W)$ (Chap. 1) from a suitable category $\mathscr{O}$ of $\mathscr{G}$-modules to the category of graded vector spaces. Certain foundational results for non-super Lie algebras have been established in [Fe, FGZ]. In this paper, we expand the results in these references and study the above functor for general Lie superalgebras for the first time. (See however [Fe].) Analogues of the results in [Fe, FGZ, Zu, LZ1] will be proved. Our primary goal, however, is to initiate a program to study a twisted version of the above functor, namely, the composite functor:

$$W \rightsquigarrow W \otimes_{\mathbf{C}} V \rightsquigarrow H^*_\infty(\mathscr{G}, W \otimes V).$$

We will begin the program by studying semi-infinite cohomology in the case when $\mathscr{G}$ is graded by an abelian subalgebra $\mathscr{H}$, V is a highest weight module, and W is $\mathscr{H}$-diagonalizable. The reasons for our assumptions are

(1) they hold in one of the most physically relevant cases [DPL, Ba, KS];

(2) they have several technical advantages over others because the highest weight modules, V, are relatively well-studied. Thus to initiate our program, this is a good place to start. Note however that W need not be a highest weight module.

This paper will be the first in a series of our investigations of the above twisted semi-cohomology functor. We note here that E. Frenkel and B. Feigin have recently studied semi-infinite cohomology in connection with flag manifolds and representations of affine Lie algebras [FF]. The cohomology with coefficients in a tensor product module was briefly alluded to in [Fe, FGZ].

We now summarize the organization of this paper. The readers who wish to defer the technical details may skip Chaps. 1 and 2, because applications in Chap. 3 illustrate most of the main ideas, (particularly Theorem 3.5) in a very concrete setting. We hope that the list of symbols given in Chap. 1 will help those readers identify the notations in Chap. 3 without going through all of Chaps. 1 and 2. Theorem 3.13 will be the main result in Chap. 3. It illustrates how unitarity of the"$so(2,1)/u(1)$ coset module" follows from our general BRST approach. The idea is to use the reduction formula (Theorem 2.13) to identify the coset state space with the BRST cohomology. The unitarity of the latter can be studied using general principles in cohomology theory. Part of the proof also relies on Wallach's method of deforming a module in the parameter space of modules (Remark 3.15).

In Chap. 1, we introduce the notations and construct the BRST (semi-infinite) complex and its relative subcomplexes. The readers who are already familiar with the language of semi-infinite cohomology but who are interested in the general results may want to use Chap. 1 as a notational guide and begin on Chap. 2. Here we begin by introducing the notion of a reduced complex. (For the experts, this is essentially the associated graded complex with respect to the filtration in Proposition 2.12.) This new complex will play an important role in deriving the reduction formula, our fundamental result (Theorem 2.13). Following from it is Corollary 2.19, a strong version of the Vanishing Theorem. Corollary 2.30 gives a sufficient condition on the weight λ for the weight space $W[\lambda]$ (of a $\mathscr{G}$-module W) to decompose into the space of highest weight vectors and a canonical complement.

1. Definitions and Notations

Notation 1. Let Γ be an abelian group. Let M be a complex Γ-graded vector space, $M=\bigoplus_{\alpha\in\Gamma} M_\alpha$ with $\dim M_\alpha < +\infty$ for all α. The following notations apply:

(a) $M' = \bigoplus_{\alpha\in\Gamma} M'_\alpha$: the restricted dual of M.
(b) $\bigoplus_{\alpha\in\Gamma} M^*_\alpha$: the restricted antidual of M.
(c) If $\{e^i_\alpha\}_{i=1}^{\dim M_\alpha}$ is a basis of M_α, $\{e^{i'}_\alpha\}_{i=1}^{\dim M_\alpha}$ is its dual.
(d) $\wedge(M)$ or $\mathrm{Etr}(M)$: the exterior algebra of M.
(e) $\vee(M)$ or $\mathrm{Sym}(M)$: the symmetric algebra of M.
(f) If M is $\mathbf{Z}_2$-graded, $M^{(0)}$, $(M^{(1)})$ is the even (odd) subspace of M.
(g) $\langle x', y\rangle = x'(y)$ for $x'\in M'$, $y\in M$ defines the natural bilinear pairing between M' and M.

Notation 2. The following is a list of symbols frequently used throughout this article:

(a) $\mathscr{G}$: a complex Lie superalgebra

(b) $\mathscr{H}_0$: an ad-diagonalizable abelian subalgebra of $\mathscr{G}$ with $\dim \mathscr{H}_0 = l+1$
(c) $\mathscr{G}_\alpha$: an eigenspace of $\mathscr{H}_0$ (see A1 below).
(d) Δ_0, $\Delta_\pm$: the roots of $\mathscr{H}_0$ (see A1, A2) in $\mathscr{G}$.
(e) $\tilde{\Gamma} \subset \mathscr{G}'_0$: an abelian group of rank $n+1 = \dim \mathscr{G}_0$ (see paragraph after A5)
(f) $\Gamma = \tilde{\Gamma} \cap \mathscr{H}'_0$: a subgroup of rank $l+1$ (see A2)
(g) $\sigma: \mathscr{G} \to \mathscr{G}$: an involution (see A3)
(h) $|\cdot|$: a height function (see paragraph after A5)
(i) $I^{(s)}_\alpha$: (see Remark 1.1(ii))
(j) $\mathrm{Spec}_W \mathscr{G}_0$: (see Remark 1.1(iii))
(k) deg: (see Remark 1.1 (v))
(l) $\Omega_\infty \mathscr{G}, \varepsilon(\cdot), \imath(\cdot)$: (see paragraph after Remark 1.4)
(m) $:\varepsilon(e^{j'}_\alpha)\imath(e^i_\alpha):$ normal ordering (see Definition (1.6))
(n) $\bar{\beta}$: (see Proposition 1.8(ii))
(o) f deg: (see Remark 2.8(i))
(p) $W[\alpha]$, $W[\alpha, \mathscr{H}'_0]$: weight spaces (see Remark 1.1(iii))
(q) $M(\Lambda)$, $L(\Lambda)$: Verma module and its irreducible quotient.
(r) rad M: (see Definition 1.14(ii))
(s) $W^{\mathscr{G}_+}$: $\mathscr{G}_+$-invariant elements of W.
(t) $[,]$: graded commutator (see Remark 1.7(iii)).
(u) ch_q, sign_q: Definition 1.14.

Let $\mathscr{G}$ be a Lie superalgebra with an ad-diagonalizable nontrivial abelian subalgebra $\mathscr{H}_0 \subseteq \mathscr{G}^{(0)}$. We make the following assumptions on $\mathscr{G}$:

A1: Let $\mathscr{G}_\alpha = \{x \in \mathscr{G}: [h, x] = \langle \alpha, h \rangle x \ \forall h \in \mathscr{H}_0\}$ for $\alpha \in \mathscr{H}'_0$, and let $\Delta_0 = \{\alpha \in \mathscr{H}'_0 | \mathscr{G}_\alpha \neq 0\}$. There is an abelian group $\Gamma \subseteq \mathscr{H}'_0$ of rank $l+1 = \dim \mathscr{H}_0 < +\infty$, such that $\Delta_0 \subseteq \Gamma$. For all $\alpha \in \Delta_0$, $\dim \mathscr{G}_\alpha < \infty$.

A2: There is a set of generators $\{\alpha_i\}_{i=0}^l$ of Γ such that $\alpha \in \Delta_0 \Rightarrow \alpha = 0$ or $\alpha \in \Gamma_+$, where $\Gamma_\pm = \sum_{i=0}^{l} \mathbf{Z}_\pm \alpha_i$. Set $\Delta_\pm = \Delta_0 \cap \Gamma_\pm$, $\mathscr{G}^{(\kappa)}_\pm = \sum_{\alpha \in \Delta_\pm} \mathscr{G}^{(\kappa)}_\alpha$.
A3: There is an antilinear involution σ, such that $\sigma(\mathscr{G}_\alpha) = \mathscr{G}_{-\alpha}$. Denote also the induced involution on $\mathscr{G}'$ by σ. Note that for $x' \in \mathscr{G}'$, $y \in \mathscr{G}$, $\langle x', y \rangle = \overline{\langle \sigma(x'), \sigma(y) \rangle}$.
A4: $\mathscr{G}_0 = \mathscr{H}_0 \oplus \mathrm{cent.}(\mathscr{G})$.
A5: We will consider only the case when $\mathscr{G}^{(1)}_0 = 0$. (Neveu–Schwarz type algebra). The case $\mathscr{G}^{(1)}_0 \neq 0$ (Ramond type) will be dealt with elsewhere.

We embed the abelian group Γ in $\tilde{\Gamma} = \sum_{i=1}^{n} \mathbf{Z}\alpha_i \subseteq \mathscr{G}'_0$, where $\{\alpha_i\}_{i=l+1}^n$ is a basis of $\mathrm{cent.}(\mathscr{G}_0)'$. We define an integer grading $|\cdot|$ on $\tilde{\Gamma}$ by $|\Sigma n_i \alpha_i| = \Sigma n_i$. We introduce a partial ordering on $\mathscr{H}'_0$ by setting $\lambda \geqslant \mu$ if $\lambda - \mu \in \Gamma_+$.

Remark 1.1.

(i) The above class of $\mathscr{G}$ includes all (super) Kac-Moody [K1, K2], (super) Virasoro, Vir and $u(\hat{1})$ algebras (see Eq. (3.3)).
(ii) We will write $\Delta^{(\kappa)}_\pm = \{\alpha \in \Delta_\pm | \mathscr{G}^{(\kappa)}_\alpha \neq 0\}$; $I^{(\kappa)}_\alpha = \{1, \ldots, \dim \mathscr{G}^{(\kappa)}_\alpha\}$ for $\alpha \in \Delta^{(\kappa)}_\pm$, and denote a $\mathbf{Z}_2$-graded basis of $\mathscr{G}^{(\kappa)}_\alpha$ by $\{e^i_\alpha\}_{i \in I^{(1)}_\alpha}$ for $\alpha \in \Delta^{(0)}_0$ and $\{f^i_\alpha\}_{i \in I^{(1)}_\alpha}$ for $\alpha \in \Delta^{(1)}_0$.
(iii) Let W be any $\mathscr{G}'_0$-graded vector space – i.e.

$$W = \sum_{\lambda \in \mathscr{G}'_0} W[\lambda] \quad \text{(direct sum).}$$

$W[\lambda]$ is called a $\mathscr{G}'_0$-weight space of weight λ. Let $\mathrm{Spec}_W \mathscr{G}_0 = \{\lambda | W[\lambda] \neq 0\}$. We note that when W is $\mathscr{G}'_0$-graded, it is also $\mathscr{H}'$-graded, for any subspace $\mathscr{H} \subset \mathscr{G}_0$. That is, for $\mu \in \mathscr{H}'$, if we let $W[\mu, \mathscr{H}'] = \sum_{\lambda|_{\mathscr{H}} = \mu,\, \lambda \in \mathrm{Spec}_W \mathscr{G}_0} W[\lambda]$, then $W = \sum_{\mu \in \mathscr{H}'} W[\mu, \mathscr{H}']$ is a direct sum. The set of $\mathscr{H}'$-weights is denoted by $\mathrm{Spec}_W \mathscr{H}$. Of particular importance in our discussion will be $\mathrm{Spec}_W \mathscr{H}_0$, where $\mathscr{H}_0$ is an abelian subalgebra of $\mathscr{G}$. We will sometimes write the $\mathscr{H}'_0$-weight spaces as $W[\mu]$ rather than $W[\mu, \mathscr{H}'_0]$.

(iv) Now let W be a $\mathscr{H}'_0$-graded vector space. We call W *$\mathscr{H}_0$-bounded above*, if there are $\lambda_1, \ldots, \lambda_\kappa \in \mathscr{H}'_0$ such that $\mathrm{Spec}_W \mathscr{H}_0 \subseteq \bigcup_{i=1}^{\kappa} (\lambda_i + \Gamma_-)$. We have a similar definition for $\mathscr{G}'_0$-graded spaces where Γ_- is replaced by $\tilde{\Gamma}_-$. It is easy to check that if W is $\mathscr{G}_0$-bounded above, then it is $\mathscr{H}_0$-bounded above. Note that the converse need not be true even though $\mathscr{G}_0 = \mathscr{H}_0 \oplus \mathrm{Cent}(\mathscr{G})$. This is because $\mathrm{Cent}(\mathscr{G})$ need not act by constants in W.

(v) When W is $\mathscr{H}_0$-bounded above, we define a $\mathbf{Z}$-grading deg on W as follows: Fix a minimal set of $\lambda_1 \cdots \lambda_m \in \mathscr{H}'_0$ such that $\mathrm{Spec}_W \mathscr{H}_0 \subseteq \bigcup_{i=1}^{m} (\lambda_i + \Gamma)$. For $v \in W[\alpha]$, $\alpha \in \mathrm{Spec}_W \mathscr{H}_0$, set $\deg v = |\alpha - \lambda_i| \in \mathbf{Z}$, where $\alpha \in \lambda_i + \Gamma$; deg is well-defined because the sets $(\lambda_i + \Gamma)$ are mutually disjoint. We will write $\deg W = \{n \in \mathbf{Z} | W_n \neq 0\}$, where $W_n = \{v \in W | \deg v = n\}$. We note that because W is $\mathscr{H}'_0$-bounded above, $\deg W$ is also bounded above as an integer subset.

(vi) The above remarks (iii), (iv), (v) apply to any $\mathscr{G}_0$-diagonalizable $\mathscr{G}$-module $M = \sum_{\lambda \in \mathscr{G}'_0} M[\lambda]$, where $\mathscr{G}_0$ acts by a weight $\lambda \in \mathscr{G}'_0$ on $M[\lambda]$.

Definition 1.2. *$\mathcal{O}$ denotes the category of all $\mathscr{G}_0$-diagonalizable $\mathbf{Z}_2$-graded $\mathscr{G}$-modules such that for $V \in \mathcal{O}$.*

$\mathcal{O}1$. the $\mathbf{Z}_2$-grading of V is consistent with that of $\mathscr{G}$;

$\mathcal{O}2$. V is $\mathscr{H}_0$-bounded above. Thus V is also bounded above as $\mathbf{Z}$-graded space;

$\mathcal{O}3$. for $x \in \mathscr{G}_\alpha$, $\deg x = |\alpha|$, i.e. $x(V_n) \subset V_{n+|\alpha|}$.

Definition 1.3. *Let $\mathcal{O}_0$ be the subcategory of $\mathcal{O}$ such that*

$\mathcal{O}4$. Each $V \in \mathcal{O}_0$ is $\mathscr{G}_0$-bounded above.

Remark 1.4. $\mathcal{O}$ and $\mathcal{O}_0$ are closed under tensor products, direct sums and quotients. Furthermore, if V is in $\mathcal{O}$ or $\mathcal{O}_0$ so are its submodules.

We now define a special $\mathscr{G}$-module $\Omega_\infty = \Omega_\infty \mathscr{G}$ in $\mathcal{O}_0$. Let $\mathscr{B} = \mathscr{G}_+ + \mathscr{G}_0$ (Borel-subalgebra). Let $\wedge_\infty = \wedge_\infty \mathscr{G}^{(0)}$ be the $\mathbf{Z}_2 \oplus \mathbf{Z}$-graded vector space: $\wedge_\infty = \wedge(\mathscr{B}^{(0)'} \oplus \mathscr{G}^{(0)}_-)$. It has a canonical basis of the form

$$\omega = e^{i'_1}_{\alpha_1} \wedge \cdots \wedge e^{i'_n}_{\alpha_n} \wedge e^{j_1}_{\beta_1} \wedge \cdots \wedge e^{j_m}_{\beta_m};$$

ω is even (odd) if $n + m \in 2\mathbf{Z} (\in 2\mathbf{Z} + 1)$ and $\deg \omega = -\sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{m} |\beta_i|$. Let $\vee_\infty = \vee_\infty \mathscr{G}^{(1)} = \vee(\mathscr{B}^{(1)'} \oplus \mathscr{G}^{(1)}_-)$. It is a $\mathbf{Z}_2$-even, $\mathbf{Z}$-graded vector space. It has a canonical basis of the form

$$\omega = f^{i'_1}_{\alpha_1} \vee \cdots \vee f^{i'_n}_{\alpha_n} \vee f^{j_1}_{\beta_1} \vee \cdots \vee f^{j_m}_{\beta_m},$$

and $\deg \omega = -\sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{m} |\beta_i|$. We note that the $\mathbf{Z}$-gradings in both $\wedge_\infty$ and

$\vee_\infty$ are $\mathbf{Z}$-valued. Let $\Omega_\infty = \Omega_\infty \mathscr{G} = \wedge_\infty \mathscr{G}^{(0)} \otimes \vee_\infty \mathscr{G}^{(1)}$. Ω_∞ has an obvious $\mathbf{Z}_2 \oplus \mathbf{Z}$-graded structure. We note that $\wedge_\infty \mathscr{G}^{(0)}$ is the space of semi-infinite forms introduced in [Fe] and [FGZ]. We now define a $\mathscr{G}$-action on $\Omega_\infty \mathscr{G}$. First we extend the pairing $\langle , \rangle$: $\mathscr{G}' \times \mathscr{G} \to \mathbf{C}$ to $\langle , \rangle$: $(\mathscr{G}' \oplus \mathscr{G}) \times (\mathscr{G}' \oplus \mathscr{G}) \to \mathbf{C}$ by letting $\langle \mathscr{G}', \mathscr{G}' \rangle = 0 = \langle \mathscr{G}, \mathscr{G} \rangle$ and $\langle a, x \rangle = \langle x, a \rangle$ for $a \in \mathscr{G}$, $x \in \mathscr{G}'$. Let

$$\omega = (x_1 \wedge \cdots \wedge x_n) \otimes (y_1 \vee \cdots \vee y_m) \in \Omega_\infty \mathscr{G},$$
$$u \in \mathscr{B}^{(0)\prime}, \quad x \in \mathscr{G}_-^{(0)\prime}, \quad y \in \mathscr{B}^{(1)\prime}, \quad z \in \mathscr{G}_-^{(1)\prime},$$
$$a \in \mathscr{B}^{(0)}, \quad b \in \mathscr{G}_-^{(0)}, \quad c \in \mathscr{B}^{(1)}, \quad d \in \mathscr{G}_-^{(1)}.$$

Define

$$\varepsilon(u) \cdot \omega = u \wedge x_1 \wedge \cdots \wedge x_n \otimes y_1 \vee \cdots \vee y_m,$$

$$\varepsilon(x) \cdot \omega = \sum_{k=1}^{n} (-1)^{k-1} \langle x, x_k \rangle x_1 \wedge \cdots \hat{x}_k \cdots \wedge x_n \otimes y_1 \vee \cdots \vee y_m,$$

$$\varepsilon(y) \cdot \omega = x_1 \wedge \cdots \wedge x_n \otimes y \vee y_1 \vee \cdots \vee y_m,$$

$$\varepsilon(z) \cdot \omega = \sum_{k=1}^{m} \langle z, y_k \rangle x_1 \wedge \cdots \wedge x_n \otimes y_1 \vee \cdots \hat{y}_k \cdots \vee y_m,$$

$$\iota(a) \cdot \omega = \sum_{k=1}^{n} (-1)^{k-1} \langle a, x_k \rangle x_1 \wedge \cdots \hat{x}_k \cdots \wedge x_n \otimes y_1 \vee \cdots \vee y_m,$$

$$\iota(b) \cdot \omega = b \wedge x_1 \wedge \cdots \wedge x_n \otimes y_1 \vee \cdots \vee y_m,$$

$$\iota(c) \cdot \omega = -\sum_{k=1}^{m} \langle c, y_k \rangle x_1 \wedge \cdots \wedge x_n \otimes y_1 \vee \cdots \hat{y}_k \cdots \vee y_m,$$

$$\iota(d) \cdot \omega = x_1 \wedge \cdots \wedge x_n \otimes d \vee y_1 \vee \cdots \vee y_m.$$

The (anti)-commutation relations and gradings of these linear operators are summarized as

Proposition 1.5. *For* $x \in \mathscr{G}_\alpha^{(0)\prime}$, $y \in \mathscr{G}_\alpha^{(1)\prime}$, $a \in \mathscr{G}_\beta^{(0)}, b \in \mathscr{G}_\beta^{(1)}$,
(i) $\{\varepsilon(x), \iota(a)\} = \langle x, a \rangle$,
(ii) $[\varepsilon(y), \iota(b)] = \langle y, b \rangle$,
(iii) *All other (anti)-commutations are zero,*
(iv) $\varepsilon(x)$, $\iota(a)(\varepsilon(y), \iota(b))$ *are* $\mathbf{Z}_2$*-odd* ($\mathbf{Z}_2$*-even*).
(v) $\deg \varepsilon(x) = \deg \varepsilon(y) = -|\alpha|$,
$\deg \iota(a) = \deg \iota(b) = |\beta|$.

Definition 1.6. *For* $x \in \mathscr{G}$ *and some fixed* $\bar{\beta} \in \mathscr{G}'_0$, *define the linear operator*

$$\begin{aligned} \eta(x) = & \sum_{\alpha \in \Delta_0} \sum_{i \in I_\alpha^{(0)}} :\varepsilon(e_\alpha^{i'}) \iota([e_\alpha^i, x]): \\ & - \sum_{\alpha \in \Delta_0} \sum_{i \in I_\alpha^{(1)}} :\varepsilon(f_\alpha^{i'}) \iota([f_\alpha^i, x]): \\ & + \langle \bar{\beta}, x \rangle, \end{aligned}$$

where

$$
\begin{aligned}
:\varepsilon(e_\beta^{j'})\iota(e_\alpha^i):&= -\iota(e_\alpha^i)\varepsilon(e_\beta^{j'}) && \text{if } \beta<0\\
&= \varepsilon(e_\beta^{j'})\iota(e_\alpha^i) && \text{if } \beta>0\\
&= \tfrac{1}{2}(\varepsilon(e_\beta^{j'})\iota(e_\alpha^i) - \iota(e_\alpha^i)\varepsilon(e_\beta^{j'})) && \text{if } \beta=0
\end{aligned}
$$

and

$$
\begin{aligned}
:\varepsilon(f_\beta^{j'})\iota(f_\alpha^i):&= \iota(f_\alpha^i)\varepsilon(f_\beta^{j'}) && \text{if } \beta<0\\
&= \varepsilon(f_\beta^{j'})\iota(f_\alpha^i) && \text{if } \beta>0.
\end{aligned}
$$

Note that because $\mathscr{G}_0^{(1)}=0$, we need not consider the case $\beta=0$.

Remark 1.7.
(i) In view of the (anti)-commutation relations in Proposition 1.5, the above normal ordering is well-defined. In fact, it is necessary in the expression of $\eta(x)$ only when $x\in\mathscr{G}_0$.
(ii) $\eta(x)$ is well-defined, i.e. only finitely many terms act on each $\omega\in\Omega_\infty$. The normal ordering ensures that $:\varepsilon(e_\alpha^{i'})\iota([e_\alpha^i,x]):\omega=0=\ :\varepsilon(f_\alpha^{i'})\iota([f_\alpha^i,x]):\omega$ for all but finitely many $\alpha\in\Delta_0$.
(iii) $[,]$ here really denotes the Lie super bracket of $\mathscr{G}$. We will use the same notation for graded commutators.

Proposition 1.8. [cf. FGZ]:
(i) *There exists a two-cocycle γ (depending on the choice of $\bar\beta$) such that*

$$[\eta(x),\eta(y)]=\eta([x,y])+\gamma(x,y)$$

for $x,y\in\mathscr{G}$, and $\gamma(\mathscr{G}_\alpha,\mathscr{G}_\beta)=0$ for $\alpha+\beta\neq 0$.
(ii) *if γ is a coboundary, then there exists a choice of $\bar\beta$ such that $\gamma=0$. In this case $(\Omega_\infty\mathscr{G},\eta)$ is a $\mathscr{G}$-module in $\mathscr{O}_0$.*

Proof.
(i) It follows from a long but straightforward computation that the operator $[\eta(x),\eta(y)]-\eta([x,y])$ is a multiple of $\mathrm{Id}_{\Omega_\infty}$. Denote it as $\gamma(x,y)\mathrm{Id}_{\Omega_\infty}$. The cocycle properties of γ follow from its definition. One can also check that $\gamma(\mathscr{G}_\alpha,\mathscr{G}_\beta)=0$ for $\alpha+\beta\neq 0$.
(ii) Suppose for some fixed $\bar\beta=\lambda,\gamma$ is a coboundary, say $\gamma(x,y)=\zeta_\lambda([x,y])$ for some linear map $\zeta_\lambda\in\mathscr{G}'$. Part (i) implies that $\zeta_\lambda\in\mathscr{G}_0'$. Denote by η_λ the η in Definition 1.6 corresponding to $\bar\beta=\lambda$. Let $\eta(x)=\eta_\lambda(x)+\langle\zeta_\lambda,x\rangle$ for $x\in\mathscr{G}$. Then one has $[\eta(x),\eta(y)]=\eta([x,y])$. Thus if we now choose $\bar\beta=\lambda+\zeta_\lambda$, then the corresponding γ is zero. It follows that (Ω_∞,η) is a representation of $\mathscr{G}$. Proposition 1.5(iv), (v) imply that $\deg\eta(x)=|\alpha|$ for $x\in\mathscr{G}_\alpha$ and that $\eta(x)$ has a consistent Z_2-grading. Note also that $\mathrm{Spec}_{\Omega_\infty}\mathscr{G}_0\subseteq\bar\beta+\Gamma_-$. Thus (Ω_∞,η) defines a module in $\mathscr{O}_0$. ■

Remark 1.9.
(i) In latter applications, we can explicitly determine $\bar\beta$. For example, recall that the second cohomology $H^2(\mathscr{G},\mathbf{C})=0$ for an affine Kac-Moody $\mathscr{G}[\mathbf{F}]$. Let $e_i, f_i, \alpha_i^\vee$, $i=0,\dots,l$, be a set of Chevelley generators of $\mathscr{G}$, where $\langle\alpha_i,\alpha_j^\vee\rangle=a_{ij}$ is a Cartan matrix of $\mathscr{G}$. Then it follows from direct computations that $\gamma=0$ provided that

$$\langle\bar\beta,[f_i,e_i]\rangle=\langle e_i',[e_i,[e_i,f_i]]\rangle.$$

This gives $\langle \bar{\beta}, \alpha_i^\vee \rangle = a_{ii}$ for $i = 0, \ldots, l$. Thus it is enough to choose $\bar{\beta} = 2\rho$, where $\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii}$.
From now on, we will assume that $\bar{\beta} \in \mathscr{G}'_0$, can be chosen so that $\gamma = 0$.
(ii) Recall that [FGZ] $\wedge_\infty \mathscr{G}^{(0)}$ has a unique hermitian form such that $\langle e_0^{1'} \wedge \cdots \wedge e_0^{n'}, 1 \rangle = 1, \varepsilon(x)^\dagger = -\varepsilon(\sigma(x))$ and $\iota(a)^\dagger = -\iota(\sigma(a))$ for $x \in \mathscr{G}^{(0)'}$, $a \in \mathscr{G}^{(0)}$, where $\{e_0^{1'}, \ldots, e_0^{n'}\}$ is a fixed basis of $\mathscr{G}_0^{(0)'}$. Similarly, there is a unique hermitian form on $\vee_\infty \mathscr{G}^{(1)}$ such that $\langle 1, 1 \rangle = 1, \varepsilon(x)^\dagger = -\sqrt{-1}\varepsilon(\sigma(x))$ and $\iota(a)^\dagger = -\sqrt{-1}\iota(\sigma(a))$ for $x \in \mathscr{G}^{(1)'}$, $a \in \mathscr{G}^{(1)}$. The space $\Omega_\infty = \wedge_\infty \otimes \vee_\infty$ is then given the tensor product of the two forms. One can check that if $\gamma = 0$ (Proposition 1.8(ii)) for some choice $\bar{\beta} = \lambda$, then $\gamma = 0$ for the new choice $\bar{\beta} = \frac{1}{2}(\sigma(\lambda) - \lambda)$. We will always use the latter so that $\sigma(\bar{\beta}) = -\bar{\beta}$. It then follows that (Definition 1.6) $\eta(x)^\dagger = -(-\sqrt{-1})^s \eta(\sigma(x))$ for $x \in \mathscr{G}^{(s)}$, $s = 0, 1$.

Definition 1.10. (cf. [Fe], [FGZ]) *For $(V, \pi) \in \mathcal{O}$, let $C_\infty(\mathscr{G}, V) = V \otimes \Omega_\infty \mathscr{G}$ and define the linear operator d on $C_\infty(\mathscr{G}, V)$ by*

$$\begin{aligned} d = & \sum_{\alpha \in \Delta_0} \sum_{i \in I_\alpha^{(0)}} \pi(e_\alpha^i) \varepsilon(e_\alpha^{i'}) \\ & + \sum_{\alpha \in \Delta_0} \sum_{i \in I_\alpha^{(1)}} \pi(f_\alpha^i) \varepsilon(f_\alpha^{i'}) \\ & - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_0} \sum_{i \in I_\alpha^{(0)} j \in I_\beta^{(0)}} :\iota([e_\alpha^i, e_\beta^j]) \varepsilon(e_\alpha^{i'}) \varepsilon(e_\beta^{j'}): \\ & + \sum_{\alpha, \beta \in \Delta_0} \sum_{i \in I_\alpha^{(0)} j \in I_\beta^{(1)}} :\iota([e_\alpha^i, f_\beta^j]) \varepsilon(e_\alpha^{i'}) \varepsilon(f_\beta^{j'}): \\ & - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_0} \sum_{i \in I_\alpha^{(1)} j \in I_\beta^{(1)}} \iota([f_\alpha^i, f_\beta^j]) \varepsilon(f_\alpha^{i'}) \varepsilon(f_\beta^{j'}) \\ & + \varepsilon(\bar{\beta}). \end{aligned}$$

Again, the normal ordering ensures that at most finitely many terms in each sum act on a given $\omega \in C_\infty(\mathscr{G}, V)$. The last sum needs no normal ordering because the operators in the term commute.

Proposition 1.11.
(i) $d^2 = 0$.
(ii) *If V has a sesquilinear form $\langle , \rangle_V$ such that $\langle \pi(a) \cdot, \cdot \rangle_V = -(-\sqrt{-1})^s \langle \cdot, \pi(\sigma(a)) \cdot \rangle_V$, for $a \in \mathscr{G}^{(s)}$ and $\langle V^{(0)}, V^{(1)} \rangle_V = 0$, then with respect to $\langle , \rangle_{C_\infty} \equiv \langle , \rangle_V \otimes \langle , \rangle_{\Omega_\infty}$, we have $\langle d \cdot, \cdot \rangle_{C_\infty} = \langle \cdot, d \cdot \rangle_{C_\infty}$.*

Proof.
(i) Since the detailed computation is quite long, we will only sketch how it is done. By breaking up and regrouping the terms, one can rewrite d in the form:

$$d = d_0 + \sum_{i \in I_\alpha^{(0)}} (\pi(e_0^i) + \eta(e_0^i)) \varepsilon(e_0^{i'}) + \psi,$$

where

$$\psi = -\sum_{\alpha \in \Delta_+} \left[\sum_{i, j \in I_\alpha^{(0)}} \iota([e_\alpha^i, e_{-\alpha}^j]) \varepsilon(e_\alpha^{i'}) \varepsilon(e_{-\alpha}^{j'}) + \sum_{i, j \in I_\alpha^{(1)}} \iota([f_\alpha^i, f_{-\alpha}^j]) \varepsilon(f_\alpha^{i'}) \varepsilon(f_{-\alpha}^{j'}) \right],$$

where d_0 has the same form as d with $\sum_{\alpha\in\Delta_0}$, $\sum_{\alpha,\beta\,\Delta_0}$ replaced by $\sum_{\alpha\in\Delta_\pm}$, $\sum_{\substack{\alpha,\beta\in\Delta_\pm\\ \alpha+\beta\neq 0}}$ respectively, and the normal ordering removed. One can now compute $\{d,d\}$ using the above decomposition of d and show that the terms in this anticommutators cancel exactly.
(ii) Let $u,v\in V^{(s)}$, and fix $s\in\{0,1\}$. Let $w,t\in\Omega_\infty$, $x\in\mathscr{G}^{(0)'}$: Then we have

$$\begin{aligned}\langle\varepsilon(x)u\otimes w, v\otimes t\rangle_{C_\infty} &= (-1)^s\langle u,v\rangle_V\cdot\langle\varepsilon(x)w,t\rangle_{\Omega_\infty}\\ &= -(-1)^s\langle u,v\rangle_V\cdot\langle w,\varepsilon(\sigma(x))t\rangle_{\Omega_\infty}\\ &= -\langle u\otimes w,\varepsilon(\sigma(x))v\otimes t\rangle_{C_\infty}.\end{aligned}$$

More generally, for $x\in\mathscr{G}^{(s)'}$, $a\in\mathscr{G}^{(s)}$ we have the hermiticity properties:

$$\langle\varepsilon(x)\cdot,\cdot\rangle = \langle\cdot,-(\sqrt{-1})^s\varepsilon(\sigma(x))\rangle, \tag{1}$$

$$\langle\iota(a)\cdot,\cdot\rangle = \langle\cdot,-(\sqrt{-1})^s\iota(\sigma(a))\rangle, \tag{2}$$

$$\langle\pi(a)\cdot,\cdot\rangle = \langle\cdot,-(-\sqrt{-1})^s\pi(\sigma(a))\rangle, \tag{3}$$

$$\langle\eta(a)\cdot,\cdot\rangle = \langle\cdot,-(-\sqrt{-1})^s\eta(\sigma(a))\rangle. \tag{4}$$

Using (1) through (4), and the expression for d in part (i), we get $\langle d\cdot,\cdot\rangle = \langle\cdot,d\cdot\rangle$. ■

Definition 1.12.
(i) *For* $(V,\pi)\in\mathcal{O}$, *let* $\theta(x)=\pi(x)+\eta(x)$ *for* $x\in\mathscr{G}$. *Note that* $(C_\infty(\mathscr{G},V),\theta)\in\mathcal{O}$.
(ii) *Let* $U\in\mathrm{End}_{\mathbf{C}}\,C_\infty(\mathscr{G},V)$ *be*

$$U=\sum_{\alpha\in\Delta_0}\sum_{i\in I_\alpha^{(0)}}:\varepsilon(e_\alpha^{i'})\iota(e_\alpha^i): - \sum_{\alpha\in\Delta_0}\sum_{i\in I_\alpha^{(1)}}:\varepsilon(f_\alpha^{i'})\iota(f_\alpha^i): + \tfrac{1}{2}\dim\mathscr{G}_0.$$

U is called the ghost number operator [GSW].

Proposition 1.13. *For* $a\in\mathscr{G}$, $x\in\mathscr{G}'$,
(i) $[d,\iota(a)]=\theta(a)$.
(ii) $[d,\theta(a)]=0$.
(iii) $[U,\iota(a)]=-\iota(a)$,
$[U,\varepsilon(x)]=\varepsilon(x)$.
(iv) $[U,d]=d$.
(v) $[U,\theta(a)]=0$.
(vi) *U is diagonalizable with* $\mathrm{Spec}_{C_\infty}\,U=\mathbf{Z}$.
Then n^{th} *eigenspace has the form* $C_\infty^n(\mathscr{G},V)=V\otimes\Omega_\infty^n\mathscr{G}$, *where*

$$\begin{aligned}\Omega_\infty^n = \mathrm{Span}\{\omega = e_{\alpha_1}^{i_1'}\wedge\cdots\wedge e_{\alpha_p}^{i_p'}\wedge e_{\beta_1}^{j_1}\wedge\cdots\wedge e_{\beta_q}^{j_q}\\ \otimes f_{\gamma_1}^{k_1'}\vee\cdots\vee f_{\gamma_r}^{k_r'}\vee f_{\delta_1}^{l_1}\vee\cdots\vee f_{\delta_s}^{l_s}\,|\,p-q+r-s=n\}.\end{aligned}$$

(See definition of basis vectors after Remark 1.4.)

Proof.
(i) can be obtained using the expression of d in the proof of Proposition 1.11(i).
(ii) follows from (i).
(iii) Break up the sum in U using the definition of the normal ordering. Then $[U,\iota(a)]=-\iota(a)$, $[U,\varepsilon(x)]=\varepsilon(x)$ can be seen by inspection.
(iv) is a direct consequence of (iii).
(v) is obtained from (i), (iii) and (iv).

(vi) Recall that the vectors of the form ω above constitute a canonical basis of Ω_∞. By (iii), it follows that the vectors of the form $v \otimes \omega, v \in V$ are eigenvectors of U. Thus $V \otimes \Omega^n_\infty$ is the eigenspace of U with eigenvalue $n = p - q + r - s$. ■

Propositions 1.11 and 1.13(iv), (vi) imply that $(C^*_\infty(\mathscr{G}, V), d)$ is a cochain complex. Its cohomology groups $H^*_\infty(\mathscr{G}, V)$ are called the *semi-infinite cohomology* of $\mathscr{G}$ with coefficients in V. This theory was introduced by Feigin [Fe] and further developed by Frenkel, Garland and Zuckerman [FGZ) for general **Z**-graded Lie algebras. Several important special cases were also studied in the past [LZ, Zu and references therein].

Definition 1.14.
(i) *if* V *is a* $\mathscr{G}_0'$*-graded vector space with* $\dim V^\alpha < +\infty$, *we write* $ch_q V = \sum_{\alpha \in \mathscr{G}_0'} \dim V^{-\alpha} q^\alpha$.
(ii) *If, furthermore,* V *has a sesquilinear form with* $\langle V^\alpha, V^\beta \rangle = 0$ *for* $\alpha \neq \beta$, *then we write*

$$\mathrm{sign}_q V = \sum_{\alpha \in \mathscr{G}_0'} \mathrm{sign}\, V^{-\alpha} q^\alpha,$$

where sign $V^{-\alpha}$ *is the difference between the number of positive signs and the number of negative signs in the diagonalized form* $\langle , \rangle|_{V^\alpha \times V^\alpha}$. *Let* rad $V = \{v \in V | \langle v, V \rangle = 0\}$. *If* rad$V = 0$, *we call* V *a hermitian vector space.*
(iii) *If* $(V, \pi) \in \mathcal{O}$ *has a non-degenerate hermitian form* $\langle , \rangle_V$ *such that* $\pi(a)^\dagger = -(-\sqrt{-1})^s \pi(\sigma(a))$ *for* $a \in \mathscr{G}^{(s)}$, *then* V *is called a hermitian* $\mathscr{G}$*-module.*

Proposition 1.15.

(i) $$\sum_{m \in \mathbf{Z}} ch_q \Omega^m_\infty \mathscr{G} = 2^{\dim \mathscr{G}_0} q^{-\bar{\beta}} \prod_{\alpha \in \Delta_+} (1 + q^\alpha)^{2 \dim \mathscr{G}^{(0)}_\alpha} \cdot (1 - q^\alpha)^{-2 \dim \mathscr{G}^{(1)}_\alpha}.$$

(ii) $$\sum_{m \in \mathbf{Z}} (-1)^m ch_q \Omega^m_\infty \mathscr{G} = 0.$$

Both are results of straightforward computations.

We note that Propositions 1.13 (i) and (ii) imply that for any subalgebra $\mathscr{A} \subset \mathscr{G}$, there is a relative subcomplex given by

$$C^*_\infty(\mathscr{G}, \mathscr{A}; V) = \{\omega \in C^*_\infty(\mathscr{G}, V) \colon \iota(a)\omega = 0 = \theta(a)\omega, \text{ for all } a \in \mathscr{A}\};\ d_0 = d|_{C^*_\infty(\mathscr{G}, \mathscr{A}; V)}.$$

Of particular interest is $\mathscr{A} = \mathscr{G}_0$. We will study this case with coefficients in $V = V_1 \otimes V_2$. Our main focus will be for V_2 belonging to the Verma modules of $\mathscr{G}$, their quotients or their submodules.

Definition 1.16. *The cohomology groups of the subcomplex* $(C^*_\infty(\mathscr{G}, \mathscr{G}_0; V), d_0)$ *are denoted by* $H^*_\infty(\mathscr{G}, \mathscr{G}_0; V)$.

Remark 1.17.
(i) In the course of the following discussion, we will encounter several different (co-)chain complexes and their (co-)homology groups. Given a (co-)chain complex (C_*, d) (or (C^*, d)) we will generically denote its (co-)homology groups by $H_*(C, d)$

(or $H^*(C,d)$). When the context is clear, we may drop the d. We will also reserve the notation such as that in Definition 1.16 for special cases.
(ii) Note that d_0 in Definition 1.16 is given explicitly in the proof of Proposition 1.11.

2. Relative Semi-Infinite Cohomology with Coefficients in $V_1 \otimes V_2$

We note here that the relative cohomology with a tensor product module as coefficients was briefly alluded to in [Fe, FGZ]. The case of $\mathscr{G} = \mathrm{Vir}$ was also studied by one of us [Zu].

The main result of this section is the "reduction formula" (Theorem 2.13). This formula establishes a connection between the classical and the semi-infinite cohomology. First we will define a "reduced complex." This will be the bridge between the above two objects. We will then show a few important consequences of the reduction formula. In particular, we will prove a strong version of the Vanishing Theorem (Corollary 2.26) and derive a necessary and sufficient condition for unitarity of the cohomology group. We will also obtain a "decomposition formula" for modules in $\mathcal{O}$. Some applications of these consequences will be discussed in Sect. 3.

2.1. The Reduced Complex. In this section, we will always asume that V_+, V_- are $\mathscr{G}$-modules in $\mathcal{O}$. We first state a theorem.

Theorem 2.1. *Let $\mathscr{L} = \mathscr{L}^{(0)} \oplus \mathscr{L}^{(1)}$ be a Lie superalgebra and $U\mathscr{L}$ be its ($\mathbf{Z}_2$-graded) universal enveloping algebra. Let $C_n = \sum_{n=p+q} U\mathscr{L} \otimes \wedge^p \mathscr{L}^{(0)} \otimes \vee^q \mathscr{L}^{(1)}$. Define $d: C_n \to C_{n-1}$ by*

$$\begin{aligned}
&d(u \otimes x_1 \wedge \cdots \wedge x_p \otimes y_1 \vee \cdots \vee y_q)\\
&= (-1)^u \sum_{1 \le i \le p} (-1)^{i+1} x_i \cdot u \otimes x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_p \otimes y_1 \vee \cdots \vee y_q\\
&\quad + \sum_{1 \le i \le q} y_i \cdot u \otimes x_1 \wedge \cdots \wedge x_p \otimes y_1 \vee \cdots \hat{y}_i \cdots \vee y_q\\
&\quad - (-1)^u \cdot \sum_{1 \le i < j \le p} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_j \otimes y_1 \vee \cdots \vee y_q\\
&\quad + (-1)^u \sum_{\substack{1 \le i \le p \\ 1 \le j \le q}} (-1)^i u \otimes x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_p \otimes [x_i, y_i] \vee \cdots \hat{y}_j \cdots \vee y_q\\
&\quad - (-1)^u \sum_{1 \le i < j \le q} u \otimes [y_i, y_j] \wedge x_1 \wedge \cdots \wedge x_p \otimes y_1 \vee \cdots \hat{y}_i \cdots \hat{y}_j \cdots \vee y_q,
\end{aligned}$$

where $(-1)^u$ is the $\mathbf{Z}_2$-grading of $u \in U\mathscr{L}$, $x_1 \wedge \cdots \wedge x_p \in \wedge^p \mathscr{L}^{(0)}$, $y_1 \vee \cdots \vee y_q \in \vee^q \mathscr{L}^{(1)}$. Let $\varepsilon: C_0 \to \mathbf{C}$ be the augmentation map. Then the sequence

$$C: \cdots C_n \xrightarrow{d} C_{n-1} \cdots \to C_0 \xrightarrow{\varepsilon} \mathbf{C} \to 0$$

is an $\mathscr{L}$-free resolution of the trivial $\mathscr{L}$-module $\mathbf{C}$. In fact, there is a contracting homotopy for the complex (C_*, d).

This is a generalization of a theorem in the classical theory of Lie algebra homology (a good exposition is given in [HS]; consult also [Le]). The proof of this case of a Lie superalgebra requires only a slight modification of the ordinary case. We note that the theorem holds if we replace $U\mathscr{L}$ by any $\mathscr{L}$-free module.

Definition 2.2. *Let $(V_+,\pi_+)(V_-,\pi_-)$ be $\mathscr{G}$-modules in $\mathcal{O}$. Let $C^*_{\text{red}} = C^*_{\text{red}}(\mathscr{G}_1\mathscr{G}_0; V_+\otimes V_-) = V_+\otimes V_-\otimes\Omega^*_\infty(\mathscr{G}/\mathscr{G}_0)$ and $\tilde{d}_0$ be a linear operator on C^*_{red} defined by $\tilde{d}_0 = d_+ + d_-$, where*

$$d_\pm = \sum_{\alpha\in\Delta_\pm}\sum_{i\in I^{(0)}_\alpha} \varepsilon(e^{i'}_\alpha)\pi_\pm(e^i_\alpha) + \sum_{\alpha\in\Delta_\pm}\sum_{i\in I^{(1)}_\alpha} \varepsilon(f^{i'}_\alpha)\pi_\pm(f^i_\alpha)$$
$$-\frac{1}{2}\sum_{\alpha,\beta\in\Delta_\pm}\sum_{i\in I^{(0)}_\alpha j\in I^{(0)}_\beta} \varepsilon(e^{i'}_\alpha)\varepsilon(e^{j'}_\beta)\iota([e^i_\alpha, e^j_\beta])$$
$$+\sum_{\alpha,\beta\in\Delta_\pm}\sum_{i\in I^{(0)}_\alpha j\in I^{(1)}_\beta} \varepsilon(e^{i'}_\alpha)\varepsilon(f^{j'}_\beta)\iota([e^i_\alpha, f^j_\beta])$$
$$-\frac{1}{2}\sum_{\alpha,\beta\in\Delta_\pm}\sum_{i\in I^{(1)}_\alpha j\in I^{(1)}_\beta} \varepsilon(f^{i'}_\alpha)\varepsilon(f^{j'}_\beta)\iota([f^i_\alpha, f^j_\beta]).$$

A direct computation gives

Proposition 2.3. *$d^2_+ = d^2_- = d_+d_- + d_-d_+ = 0$. Hence $\tilde{d}^2_0 = 0$.*

Remark 2.4.
(i) Note that $\tilde{d}_0$ is really a reduced form of d_0 (see Remark 1.17(ii)). Normal ordering is not required here because the terms in each sum in $\tilde{d}_0$ (anti-) commute. From the expressions of $d_\pm$, it is obvious that $[U,d_\pm] = d_\pm$. Thus (C^*_{red}, d) is a cochain complex, which will be called the reduced complex.
(ii) Observe that there is a canonical isomorphism of cochain complexes

$$(C^*_{\text{red}}, \tilde{d}_0) \cong (C^*(\mathscr{G}_+, V_+), d_+)\otimes(C^*(\mathscr{G}_-, V_-), d_-),$$

where

$$C(\mathscr{G}_\pm, V_\pm) = V_\pm\otimes\Omega_\infty\mathscr{G}_\pm,$$
$$\Omega_\infty\mathscr{G}_+ = \wedge\mathscr{G}^{(0)\prime}_+\otimes\vee\mathscr{G}^{(1)\prime}_+,$$
$$\Omega_\infty\mathscr{G}_- = \wedge\mathscr{G}^{(0)}_-\otimes\vee\mathscr{G}^{(1)}_-.$$

By convention, we set the weight of $1\otimes1\in\Omega_\infty\mathscr{G}_-$ to be $\bar{\beta}$, and that of $1\otimes1\in\Omega_\infty\mathscr{G}_+$ to be 0. Then the isomorphism above, which maps $v_+\otimes v_-\otimes1\otimes1\in V_+\otimes V_-\otimes\wedge_\infty\mathscr{G}^{(0)}\otimes\vee_\infty\mathscr{G}^{(1)}$ to $(v_+\otimes1\otimes1)\otimes(v_-\otimes1\otimes1)\in C(\mathscr{G}_+, V_+)\otimes C(\mathscr{G}_-, V_-)$, is weight preserving.

(iii) We note also that the cohomology of $(C^*(\mathscr{G}_-, V_-), d_-)$ (cf. Theorem 2.1 and Definition 2.2) is precisely the classical (Cartan–Chevalley–Eilenberg) homology of the subalgebra $\mathscr{G}_-$ with coefficients in V_-. We will denote the cohomology of $(C^*(\mathscr{G}_\pm, V_\pm), d_\pm)$ by $H^*(\mathscr{G}_\pm, V_\pm)$. When passing to cohomology the isomorphism of complexes above gives

Proposition 2.5. (*Künneth formula*). *There is a canonical isomorphism of $\mathscr{G}'_0$-graded vector spaces:*

$$H^n(C_{\text{red}}, \tilde{d}_0) \cong \sum_{n=b-a} H^b(\mathscr{G}_+, V_+)\otimes H^{-a}(\mathscr{G}_-, V_-)$$

(*see* [HS], *Chap.* 5).

Proof. Recall that the isomorphism in Remark 2.4(ii) preserves the $\mathscr{G}'_0$-weights and that in Definition 2.2, $d_\pm$, $\tilde{d}_0$ all carry weight 0. Thus $H^*(C_{\text{red}}, \tilde{d}_0)$, $H^*(\mathscr{G}_\pm, V_\pm)$ are all $\mathscr{G}'_0$-graded spaces and the induced isomorphism on the cohomology groups is weight preserving. ■

Remark 2.6. Besides the natural **Z**-graded structure (i.e. ghost number) that the space C_{red} has as a complex, there are two other important graded structures. The first one is the $\mathscr{G}'_0$-graded structure inherited from its factor spaces $V_\pm, \Omega_\infty\mathscr{G}_\pm$, i.e.

$$C_{\text{red}} = V_+ \otimes V_- \otimes \Omega_\infty \mathscr{G}_+ \otimes \Omega_\infty \mathscr{G}_-.$$

Thus we can write as a direct sum

$$C_{\text{red}}[\lambda] = \sum_{\lambda=\alpha+\beta+\gamma+\delta} V_+[\alpha] \otimes V_-[\beta] \otimes \Omega_\infty \mathscr{G}_+[\gamma] \otimes \Omega_\infty \mathscr{G}_-[\delta].$$

Since $\tilde{d}_0$ carries $\mathscr{G}'_0$-weight 0, each $(C_{\text{red}}[\lambda], \tilde{d}_0)$ is itself a complex. This gives

Lemma 2.7. *For each* $\lambda \in \text{Spec}_{C_{\text{red}}}\mathscr{G}_0$, *there is a canonical isomorphism*

$$H^*(C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-), \tilde{d}_0)[\lambda] \cong H^*(C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-)[\lambda], \tilde{d}_0).$$

Remark 2.8.
(i) The second graded structure, $f\deg$, on C_{red} is canonically induced by the isomorphism $C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-) \cong C(\mathscr{G}_+, V_+) \otimes C(\mathscr{G}_-, V_-)$ as follows (cf. [FGZ]). It is enough to define $f\deg$ on $C(\mathscr{G}_+, V_+)$, $C(\mathscr{G}_-, V_-)$ separately. $V_\pm, \Omega_\infty\mathscr{G}_\pm$ are by definition $\mathscr{G}'_0$-graded vector spaces. Hence they are also $\mathscr{H}'_0$-graded (Remark 1.1 (iii)). This means that they have the **Z**-graded structure deg, defined in Remark 1.1(v). Extend deg to $C(\mathscr{G}_\pm, V_\pm) = V_\pm \otimes \Omega_\infty \mathscr{G}_\pm$. Let $\omega_\pm \in C(\mathscr{G}_\pm, V_\pm)$ be homogeneous elements. Define

$$f\deg \omega_\pm = \mp \deg \omega_\pm,$$
$$f\deg \omega_+ \otimes \omega_- = f\deg \omega_+ + f\deg \omega_-.$$

We note that $f\deg$ on $C_{\text{red}} = C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-)$ is compatible with the $\mathscr{G}'_0$-grading. We will write, for $q \in \mathbf{Z}$, $\lambda \in \mathscr{G}'_0$,

$$C_{\text{red}}[\lambda]^q = \{\omega \in C_{\text{red}}[\lambda] : f\deg \omega = q\}.$$

(ii) The grading $f\deg$ plays two important roles. The first one is given by

Proposition 2.9.
(i) *For each* $\mu \in \text{Spec}_{C_{\text{red}}}\mathscr{G}_0$, *there is a* **Z**-*graded structure* $f\deg$, *naturally induced on* $H^*(C_{\text{red}}[\mu], \tilde{d}_0)$ *and* $H^*(C_{\text{red}}, \tilde{d}_0)[\mu]$, *such that*

$$H^*(C_{\text{red}}[\mu], \tilde{d}_0)^q \cong H^*(C_{\text{red}}[\mu]^q, \tilde{d}_0)$$
$$\cong H^*(C_{\text{red}}, \tilde{d}_0)[\mu]^q.$$

(ii) *One has similar statements for the complexes* $(C^*(\mathscr{G}_\pm, V_\pm), d_\pm)$.

Proof. (i) The first isomorphism follows trivially from the observation that $\tilde{d}_0$ preserves $f\deg$ on each $C_{\text{red}}[\mu]$. The second isomorphism follows from Lemma 2.7 and the fact that the $f\deg$ grading and $\mathscr{G}'_0$-grading are compatible.
(ii) The same argument applies to $(C(\mathscr{G}_\pm, V_\pm), d_\pm)$. ■

Remark 2.10. We note that by definition

$$C^*_\infty(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-) = C^*_{\text{red}}[0].$$

(See remarks preceding Definition 1.16.) Thus f deg defines a grading on C^*_∞ as well. It turns out that it defines a filtration on the complex (C^*_∞, d_0) such that $(C^*_{\text{red}}[0], \tilde{d}_0)$ is the associated graded complex of (C^*_∞, d_0). This is the key role played by f deg. Hence we let, for $q \in \mathbf{Z}$,

$$D^q = \{\omega \in C_{\text{red}}[0] = C_\infty(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-) \\ : f \deg \omega = q\} = C_{\text{red}}[0]^q \tag{2.1}$$

$$B^q = \sum_{p \geqq q} D^p = D^q + B^{q+1}. \tag{2.2}$$

Lemma 2.11. *Let* $C_{\text{red}}[0, \mathscr{H}'_0]$ *be the zeroth* $\mathscr{H}'_0$*-weight space of* $C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-)$*. Then the set of* f deg *values on* $C_{\text{red}}[0, \mathscr{H}'_0]$, $f \deg C_{\text{red}}[0, \mathscr{H}'_0]$, *is bounded above and below.*

Proof. By definition,

$$C_{\text{red}}[0, \mathscr{H}'_0] = \sum_{\alpha+\beta+\gamma=0} V_+[\alpha, \mathscr{H}'_0] \otimes V_-[\beta, \mathscr{H}'_0] \otimes \Omega_\infty[\gamma, \mathscr{H}'_0], \tag{1}$$

where α, β, γ ranges over $\text{Spec}_{V_+} \mathscr{H}'_0, \text{Spec}_{V_-} \mathscr{H}'_0$, $\text{Spec}_{\Omega_\infty} \mathscr{H}'_0$ respectively. Since $V_\pm \in \mathcal{O}$, the set det $V_\pm$ are bounded above by some $N > 0$ (Definition 1.2). Recall that (Remark 1.1 (v)) to define deg on $V_\pm$, we fixed some minimal finite subsets $A, B \subseteq \mathscr{H}'_0$ such that

$$\text{Spec}_{V_+} \mathscr{H}_0 \subseteq \bigcup_{\mu \in A} (\mu + \Gamma),$$
$$\text{Spec}_{V_-} \mathscr{H}_0 \subseteq \bigcup_{\nu \in B} (\nu + \Gamma).$$

Recall also that $\text{Spec}_{\Omega_\infty} \mathscr{H}_0 \subseteq \bar{\beta} + \Gamma_-$. Thus for every α, β, γ in the range of summation in Eq 1. there are unique $\mu(\alpha) \in A, \lambda_1(\alpha) \in \Gamma, \nu(\beta) \in B, \lambda_2(\beta) \in \Gamma, \lambda_3(\gamma) \in \Gamma_-$, such that (see Remark 1.1 (v))

$$\alpha = \mu(\alpha) + \lambda_1(\alpha),$$
$$\beta = \nu(\beta) + \lambda_2(\beta),$$
$$\gamma = \bar{\beta} + \lambda_3(\gamma).$$

By definition (see Remark 1.1(v)), for v_+, v_-, u in $V_+[\alpha, \mathscr{H}'_0], V_-[\beta, \mathscr{H}'_0], \Omega_\infty[\gamma, \mathscr{H}'_0]$ respectively, we have

$$\deg v_+ = |\lambda_1(\alpha)|,$$
$$\deg v_- = |\lambda_2(\beta)|,$$
$$\deg u = |\lambda_3(\gamma)|.$$

In Eq. (1), $\alpha + \beta + \gamma = 0$ implies that

$$\mu(\alpha) + \nu(\beta) + \bar{\beta} = -(\lambda_1(\alpha) + \lambda_2(\beta) + \lambda_3(\gamma)).$$

Since the left-hand side varies over a (finite) subset of $A + B + \bar{\beta}$, we have a uniform bound $K > 0$ such that

$$||\lambda_1(\alpha)| + |\lambda_2(\beta)| + |\lambda_3(\gamma)|| < K \tag{2}$$

for all α, β, γ in $\mathrm{Spec}_{V_+} \mathscr{H}_0$ $\mathrm{Spec}_{V_-} \mathscr{H}_0$, $\mathrm{Spec}_{\Omega_\infty} \mathscr{H}_0$, respectively, with $\alpha + \beta + \gamma = 0$. (Reminder: for $\lambda \in \Gamma$, $\lambda = \sum_{i=0}^{l} n_i \alpha_i \Rightarrow |\lambda| \stackrel{\text{def}}{=} \sum n_i$.) Now let $\omega = v_+ \otimes v_- \otimes x \wedge a \otimes y \vee b \in C_{\text{red}}[0, \mathscr{H}'_0]$ be a homogeneous element, with $v_+ \in V_+[\alpha, \mathscr{H}']$, $v_- \in V_-[\beta, \mathscr{H}'_0]$, $x \wedge a \otimes y \vee b \in \Omega_\infty[\gamma, \mathscr{H}'_0]$, $\alpha + \beta + \gamma = 0$. Then

$$f \deg \omega = -\deg v_+ + \deg v_- - \deg x + \deg a - \deg y + \deg b. \tag{3}$$

Recall that $\deg x$, $\deg a$, $\deg y$, $\deg b$ are non-positive (see paragraph after Remark 1.4), and that $\deg v_\pm \leqq N$ as mentioned earlier. Therefore,

$$\begin{aligned} \deg v_+ &= |\lambda_1(\alpha)| < N, \\ \deg v_- &= |\lambda_2(\beta)| < N, \\ \deg x \wedge a \otimes y \vee b &= |\lambda_3(\gamma)| \leqq 0. \end{aligned}$$

Thus (2) and (3) imply that

$$\begin{aligned} f \deg \omega &\leqq -\deg v_+ + \deg v_- - \deg x - \deg a - \deg y - \deg b \\ &= -|\lambda_1(\alpha)| + |\lambda_2(\beta)| - |\lambda_3(\gamma)| \\ &= -|\lambda_1(\alpha)| - |\lambda_2(\beta)| - |\lambda_3(\gamma)| + 2|\lambda_2(\beta)| \\ &< K + 2N \end{aligned}$$

Similarly

$$\begin{aligned} f \deg \omega &\geqq -\deg v_+ + \deg v_- + \deg x \\ &\quad + \deg a + \deg y + \deg b \\ &= -|\lambda_1(\alpha)| + |\lambda_2(\beta)| + |\lambda_3(\gamma)| \\ &> -K - 2N. \end{aligned}$$

Hence $|f \deg \omega| \leqq K + 2N$ for all homogeneous elements $\omega \in C_{\text{red}}[0, \mathscr{H}'_0]$. ■

We are now ready to prove (Eq. (2.1), (2.2))

Proposition 2.12. *$\{B^q\}_{q \in \mathbb{Z}}$ is a finite filtration of the complex $(C^*_\infty(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-), d_0)$ – i.e. $B^{q+1} \subseteq B^q$, $d_0 B^q \subseteq B^q$ and there exists q_0, q_1, such that*

$$B^q = C_\infty(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-) \quad \text{for} \quad q \leqq q_0,$$
$$B^q = 0 \qquad \text{for} \quad q \geqq q_1.$$

Proof. $B^{q+1} \subseteq B^q$ is obvious.

Using the proof of Proposition 1.11 and combining it with Definition 2.2 we can write d_0 as

$$d_0 = d_+ + d_- + \delta_0, \tag{1}$$

where

$$\begin{aligned} \delta_0 = &\sum_{\alpha \in \Delta_+} \sum_{i \in I_\alpha^{(0)}} \pi_-(e_\alpha^i) \varepsilon(e_\alpha^{i'}) + \sum_{\alpha \in \Delta_+} \sum_{i \in I_\alpha^{(1)}} \pi_-(f_\alpha^i) \quad \varepsilon(f_\alpha^{i'}) \\ &+ (\Delta_+, \pi_- \quad \text{replaced by } \Delta_-, \pi_+) + \sum_{\substack{\alpha \in \Delta_- \beta \in \Delta_+ \\ \alpha + \beta \neq 0}} \Bigg\{ - \sum_{i \in I_\alpha^{(0)} j \in I_\beta^{(0)}} \iota([e_\alpha^i, e_\beta^j]) \varepsilon(e_\alpha^{i'}) \varepsilon(e_\beta^{j'}) \\ &+ \sum_{i \in I_\alpha^{(0)} j \in I_\beta^{(1)}} \iota([e_\alpha^i, f_\beta^j]) \varepsilon(e_\alpha^{i'}) \varepsilon(f_\beta^{j'}) + \sum_{i \in I_\beta^{(0)} j \in I_\alpha^{(1)}} \iota([e_\beta^i, f_\alpha^j]) \varepsilon(e_\beta^{i'}) \varepsilon(e_\alpha^{j'}) \\ &- \sum_{i \in I_\alpha^{(1)} j \in I_\beta^{(1)}} \iota([f_\alpha^i, f_\beta^j]) \varepsilon(f_\alpha^{i'}) \varepsilon(f_\beta^{j'}) \Bigg\}. \end{aligned} \tag{2}$$

It is easy to check that $d_{\pm}B^q \subseteq B^q$ and $\delta_0 B^q \subseteq B^{q+1}$. Thus $d_0 B^q \subseteq B^q$. To prove finiteness of $\{B^q\}_{q\in\mathbf{Z}}$, it is enough to show that the subset of integers, $f\deg C_\infty(\mathscr{G},\mathscr{G}_0; V_+ \otimes V_-)$ is bounded from above and below. By Remark 2.10 and Remark 1.1 (iii) we have

$$C_\infty(\mathscr{G},\mathscr{G}_0; V_+\otimes V_-) = C_{\text{red}}[0] \subseteq C_{\text{red}}[0, \mathscr{H}'_0].$$

Thus

$$f\deg(C_\infty) \subseteq f\deg(C_{\text{red}}[0, \mathscr{H}'_0]).$$

Now, the boundedness of $f\deg(C_\infty)$ follows from Lemma 2.11. ■

2.2. *Main results.* Having defined the reduced complex, we are now ready to use its properties to prove our main result. The proof will be done via several lemmas. In this section we will always assume that $V_- = M(\Lambda)$ is the Verma module of $\mathscr{G}$ with highest weight $\Lambda\in\mathscr{G}'_0$, and $V_+ = W$ is any $\mathscr{G}$-module in $\mathscr{O}$. We will sometimes abbreviate $C^*_{\text{red}}(\mathscr{G},\mathscr{G}_0; W\otimes M(\Lambda))$ as C^*_{red}. B^q, D^q in Eqs (2.1), (2.2) are now spaces defined in terms of $M(\Lambda)$ and W.

Theorem 2.13. (*Reduction formula*). *There is a canonical isomorphism* $H^*_\infty(\mathscr{G},\mathscr{G}_0; W\otimes M(-\bar{\beta}-\lambda)) \cong H^*(\mathscr{G}_+, W)[\lambda]$ *for each* $\lambda\in\mathscr{G}'_0$. (*See Remark 2.4*(iii).)

Lemma 2.14. *For each* $\Lambda\in\mathscr{G}'_0$, *there is a canonical isomorphism* $H^*(C_{\text{red}}(\mathscr{G},\mathscr{G}_0; W\otimes M(\Lambda))) \cong H^*(\mathscr{G}_+, W)$ *such that* $H^*(C_{\text{red}})[\mu] \to H^*(\mathscr{G}_+, W)[\mu-\Lambda-\bar{\beta}]$ *for all weights* $\mu\in\mathscr{G}'_0$.

Proof. By Theorem 2.1,

$$\begin{aligned} H^n(\mathscr{G}_-, M(\Lambda)) &= 0 && \text{for} \quad n\neq 0 \\ &\cong \mathbf{C}v_\Lambda \otimes 1\otimes 1 && \text{for} \quad n=0, \end{aligned}$$

where v_Λ is a highest weight vector of $M(\Lambda)$ having weight Λ, and $1\otimes 1 \in \wedge\mathscr{G}^{(0)}_- \otimes \vee\mathscr{G}^{(1)}_-$ has weight $\bar{\beta}$. Composing the isomorphism of Proposition 2.4 and the above, we have

$$\begin{aligned} H^*(C_{\text{red}})[\mu] &\cong [H^*(\mathscr{G}_+, W)\otimes(\mathbf{C}v_\Lambda\otimes 1\otimes 1)][\mu] \\ &\cong H^*(\mathscr{G}_+, W)[\mu-\Lambda-\bar{\beta}]. \quad ■ \end{aligned}$$

Lemma 2.15. *For each* $\lambda\in\mathscr{G}'_0$, *there is at most* $q\in\mathbf{Z}$ *such that* $H^*(\mathscr{G}_+, W)[\lambda]^q \cong 0$. (*See notation in Proposition 2.9.*)

Proof. We note that

$$\text{Spec}_{H^*(\mathscr{G}_+, W)}\mathscr{H}_0 \subseteq \text{Spec}_W \mathscr{H}_0 + \text{Spec}_{\wedge\mathscr{G}^{(0)\prime}_+}\mathscr{H}_0 + \text{Spec}_{\wedge\mathscr{G}^{(1)\prime}_+}\mathscr{H}_0. \tag{1}$$

Let $\lambda\in\text{Spec}_{H^*(\mathscr{G}_+, W)}\mathscr{G}_0$. Then $\lambda|_{\mathscr{H}_0}\in\text{Spec}_{H^*(\mathscr{G}_+, W)}\mathscr{H}_0$. By definition, $\text{Spec}_W\mathscr{H}_0 \subseteq \bigcup_{j=1}^m \lambda_j + \Gamma$, for some λ_j's$\in\mathscr{H}'_0$, where the $\lambda_j+\Gamma$ are mutually disjoint. Also $\text{Spec}_{\wedge\mathscr{G}^{(0)\prime}_+}\mathscr{H}_0 \subseteq \Gamma_-$, $\text{Spec}_{\vee\mathscr{G}^{(1)\prime}_+}\mathscr{H}_0 \subseteq \Gamma_-$. Thus by (1), for fixed λ, there are $i\leq m$ and $\alpha,\beta,\gamma\in\Gamma$ such that

$$\lambda|_{\mathscr{H}_0} = \lambda_i + \alpha + \beta + \gamma. \tag{2}$$

Because the $\lambda_j+\Gamma$ are mutually disjoint, such an i is unique.

Now if $\psi\in H^*(\mathscr{G}_+, W)[\lambda]^q$ and $\psi\neq 0$, then ψ is represented by some cocycle

$\omega \in C(\mathscr{G}_+, W)[\lambda]$ of the form $\omega = \sum_{ljk} v^l \otimes x^j \otimes y^k$ with $f\deg\omega = -\deg v^i - \deg x^j - \deg y^k = q$. But the middle sum here is precisely the integer $-|\lambda_{|\mathscr{H}_0} - \lambda_i|$. Thus we have $q = -|\lambda|_{\mathscr{H}_0} - \lambda_i|$. Since λ_i is unique and λ is fixed, q is also fixed. ∎

Lemma 2.16. *For fixed* $\lambda \in \mathscr{G}_0'$, *let* $D^q = C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda))[0]^q$ (cf. Eq. (2.1)) *for* $q \in \mathbf{Z}$. *Then* $H^*(D^q, \tilde{d}_0) \cong H^*(\mathscr{G}_+, W)[\lambda]^q$. *Thus there is at most one* $q \in \mathbf{Z}$ *such that* $H^*(D^q, \tilde{d}_0) \neq 0$.

Proof. Lemma 2.14 for $\Lambda = -\bar{\beta} - \lambda$, gives

$$H^*(C_{\text{red}}(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda)), \tilde{d}_0)[0]^q \cong H^*(\mathscr{G}_+, W)[\lambda]^q.$$

By Proposition 2.9(ii), the left-hand side is just $H^*(D^q, \tilde{d}_0)$. By Lemma 2.15 the right-hand side is non-zero for at most one $q \in \mathbf{Z}$. ∎

We are now ready to prove Theorem 2.13.

Proof. (Theorem 2.13). As in Eqs. (2.1), (2.2), we let

$$D^q = \{\omega \in C_{\text{red}}[0] = C_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda)) \mid f\deg\omega = q\},$$

$$B^q = \sum_{p \geqq q} D^p = D^q + B^{q+1}.$$

Then by Proposition 2.12, for $V_- = M(-\bar{\beta} - \lambda)$, $V_+ = W$, we have a long exact sequence

$$\cdots \to H^{n-1}(D^q, \tilde{d}_0) \to H^n(B^{q+1}, d_0) \to H^n(B^q, d_0) \to H^n(D^q, \tilde{d}_0) \to \cdots. \tag{1}$$

For $q \in \mathbf{Z}$ such that $H^*(D^q, \tilde{d}_0) = 0$, we have

$$H^*(B^{q+1}, d_0) \cong H^*(B^q, d_0). \tag{2}$$

If $H^*(D^q, \tilde{d}_0) = 0$ for all q, then we set $p = 0$. Otherwise, there is a unique p such that $H^*(D^p, \tilde{d}_0) \neq 0$, by Lemma 2.16. Thus in both cases, Eq. (2) holds for all $q \neq p$. By finiteness of the filtration $\{B^q\}_{q \in \mathbf{Z}}$, we have

$$H^*(B^q, d_0) = 0 \qquad \text{for} \quad q > p, \tag{3}$$

$$H^*(D^p, \tilde{d}_0) \cong H^*(B^q, d_0) = H^*_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda)) \quad \text{for} \quad q \leqq p. \tag{4}$$

Now the reduction formula follows from Lemma 2.16 and Eq. (4). ∎

Corollary 2.17.

$$H^0_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda)) \cong W^{\mathscr{G}_+}[\lambda].$$

Proof. A simple calculation shows that $H^0(\mathscr{G}_+, W) \cong W^{\mathscr{G}_+}$, the space of $\mathscr{G}_+$-invariant vectors in W. ∎

2.3. Important Consequences. We now proceed to an important consequence of the reduction formula (for the case of a Hermitian $\mathscr{G}$-module W) which relates $W^{\mathscr{G}_+}[\lambda]/\text{rad}$ with a zeroth relative cohomology group (Theorem 2.25). In some special cases, the space $W^{\mathscr{G}_+}[\lambda]/\text{rad}$ is known as the "physical Hilbert space" (see Sect. 3 for examples.) From now on we will only consider weights $\lambda \in \mathscr{G}_0'$ that satisfy the reality condition $\sigma\lambda = -\lambda$.

Lemma 2.18. *Let* (B^*, d) *be a subcomplex of a cochain complex* (A^*, d) *such that*

$H^n(C)=0$ for $C=A,B$ or $A/B, n\neq 0$. Then there is an isomorphism $H^0(A/B)\cong H^0(A)/i^*H^0(B)$, where i^* is induced by i: $B\hookrightarrow A$. If, furthermore, A has a sesquilinear form such that $(d\cdot,\cdot)=(\cdot,d\cdot)$ and $BC\subseteqq \operatorname{rad} A$, then the isomorphism is an isometry.

Proof. The short exact sequence

$$0\to B\xrightarrow{i} A\xrightarrow{p} A/B\to 0,$$

where p is the canonical map, gives a long exact sequence. But since this sequence collapses we get the desired isomorphism. The second part is a trivial exercise. ∎

Remark 2.19.
(i) To apply the lemma, we define a hermitian form on $\Omega_\infty=\Omega_\infty(\mathscr{G}/\mathscr{G}_0)$. Recall that $\Omega_\infty=\wedge(\mathscr{G}_+^{(0)'}\oplus\mathscr{G}_-^{(0)})\otimes\vee(\mathscr{G}_+^{(1)'}\oplus\mathscr{G}_-^{(1)})$. A hermitian form is uniquely defined by letting $\langle 1\otimes 1, 1\otimes 1\rangle_{\Omega_\infty}=1, \varepsilon(x)^\dagger=-(\sqrt{-1})^s\varepsilon(\sigma(x))$ and $\iota(a)^\dagger=-(\sqrt{-1})^s\iota(\sigma(a))$ for $x\in\mathscr{G}^{(s)'}$, $a\in\mathscr{G}^{(s)}$. We note that $\langle,\rangle|_{\Omega_\infty^n[\alpha]\otimes\Omega_\infty^m[\beta]}$ is zero when $\alpha\neq\beta$ or $n+m\neq 0$, and non-degenerate otherwise.
(ii) When $(V,\pi)\in\mathscr{O}$ is given a sesquilinear form $\langle,\rangle_V$, we will always assume that

$$\langle\pi(a)\cdot,\cdot\rangle=\langle\cdot,-(-\sqrt{-1})^s\pi(\sigma(a))\rangle,$$

for $a\in\mathscr{G}^{(s)}$, and that $\langle V^{(0)},V^{(1)}\rangle=0$. Then it is easy to check that the relative subcomplex $(C_\infty^*(\mathscr{G},\mathscr{G}_0;V),d_0)$ has a form $\langle,\rangle_{C_\infty}$ given by restricting $\langle,\rangle_V\otimes\langle,\rangle_{\Omega_\infty}$ to the subspace $C_\infty^*(\mathscr{G},\mathscr{G}_0;V)\subseteqq V\otimes\Omega_\infty^*(\mathscr{G}/\mathscr{G}_0)$. Furthermore, $\langle d_0\cdot,\cdot\rangle_{C_\infty}=\langle\cdot,d_0\cdot\rangle_{C_\infty}$ (cf. Proposition 1.11(ii)). Note also that if $\langle,\rangle_V$ is non-degenerate, so are $\langle,\rangle|_{C_\infty^n\times C_\infty^{-n}}$ for all n.

(iii) We will need the following observation later: $U\subseteqq\operatorname{rad}V$ then $C_\infty(\mathscr{G},\mathscr{G}_0;U)\subseteqq\operatorname{rad}C_\infty(\mathscr{G},\mathscr{G}_0;V)$ as complexes. The same is true when "$\subseteqq$" is replaced by "$=$".

Corollary 2.20. *Suppose $(V,\pi)\in\mathscr{O}$ has a sesquilinear form and $U\subset\operatorname{rad}V$ is a submodule. If $H_\infty^n(\mathscr{G},\mathscr{G}_0;K)=0$ for $K=V,U$ or V/U $n\neq 0$, then there is an isometry*

$$H_\infty^0(\mathscr{G},\mathscr{G}_0;V/U)\cong H_\infty^0(\mathscr{G},\mathscr{G}_0;V)/i^*H_\infty^0(\mathscr{G},\mathscr{G}_0;U).$$

Proof. It follows from Lemma 2.18 and the observation made in Remark 2.13(iii). ∎

Lemma2.21. (cf. [FGZ, *Theorem 1.12*]) *If $V_+,V_-\in\mathscr{O}$ with V_+ hermitian and $\mathscr{G}_-$-free and* $\dim V_+[\alpha]<+\infty\,\forall\alpha\in\mathscr{G}_0'$, *then $H_\infty^n(\mathscr{G},\mathscr{G}_0;V_+\otimes V_-)=0$ for $n\neq 0$.*

Proof. Let D^q, B^q be defined by Eqs. (2.1), (2.2). By Proposition 2.5 and Lemma 2.7 we have, for $\lambda\in\mathscr{G}_0'$,

$$\sum_{q\in\mathbf{Z}}H^n(C_{\text{red}}[\lambda]^q,\tilde{d}_0)\cong\sum_{n=b-a}[H^b(\mathscr{G}_+,V_+)\otimes H^{-a}(\mathscr{G}_-,V_-)][\lambda]. \quad (1)$$

Note that for $C_{\text{red}}[0]^q=D^q$ (Eq. (2.1)). By hypothesis, $V_+^*\cong V_+$ is $\mathscr{G}_-$-free. Thus applying Theorem 2.1 to $C_n=C^{-n}(\mathscr{G}_-,V_+^*), n\geqq 0$, we have a contracting homotopy $\Sigma_n: C_n\to C_{n+1}$ such that $d_-\Sigma_n+\Sigma_{n-1}d_-=1$ for $n>0$. Using the natural hermitian pairing between $C^{-n}(\mathscr{G}_-,V_+^*)$ and $C^n=C^n(\mathscr{G}_+,V_+)$ for each n, we can define a homotopy $\sigma_n: C^n\to C^{n-1}$, such that $d_+\sigma_n+\sigma_{n+1}d_+=1$ for $n>0$. Thus $H^n(\mathscr{G}_+,V_+)=0$ for $n>0$. Equation (1) now implies that $H^n(D^q,\tilde{d}_0)=0$ for $q\in\mathbf{Z}, n>0$. From the long exact sequence

$$\begin{aligned}\cdots \to H^{n-1}(D^q, \tilde{d}_0) \to H^n(B^{q+1}, d_0)\\ \to H^n(B^q, d_0) \quad \to H^n(D^q, \tilde{d}_0) \to \cdots\end{aligned} \tag{2}$$

we conclude that for $n > 0$, there is a diagram

$$H^n(B^{q+1}) \to H^n(B^q) \to 0 \tag{3}$$

for each q. By finiteness of $\{B^q\}_{q\in\mathbf{Z}}$, it follows that $H^n_\infty(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-) = 0$ for $n > 0$.

We now interchange the roles of $V_\pm$ and define a similar filtration $\{B'^q\}_{q\in\mathbf{Z}}$, $D'^q = B'^q/B'^{q+1}$ and the induced differential $\tilde{d}'_0$ on D'^q. We obtain

$$\sum_{q\in\mathbf{Z}} H^n(D'^q, \tilde{d}'_0) \cong \sum_{n=b-a} [H^b(\mathscr{G}_+, V_-) \otimes H^{-a}(\mathscr{G}_-, V_+)][0]. \tag{4}$$

The fact that V_+ is $\mathscr{G}_-$-free implies that $H^{-a}(\mathscr{G}_-, V_+) = 0$ for all $a > 0$. As before, using the finiteness of the filtration, we conclude that $H^n_\infty(\mathscr{G}, \mathscr{G}_0; V_+ \otimes V_-) = 0$ for $n < 0$. ■

Lemma 2.22. *Suppose $W \in \mathcal{O}$ is hermitian, $\mathscr{G}_-$-free and* $\dim W[\alpha] < \infty\ \forall \alpha \in \mathscr{G}'_0$. *Then for each $\lambda \in \mathscr{G}'_0$, there is an isometry*

$$\begin{aligned} H^n_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda)) &= 0 \qquad \textit{for} \quad n \neq 0, \\ &\cong W^{\mathscr{G}_+}[\lambda] \quad \textit{for} \quad n = 0. \end{aligned}$$

Proof. The vanishing part follows from Lemma 2.21. One can check that the composed isomorphism given by the proof of Corollary 2.17 and Theorem 2.13,

$$W^{\mathscr{G}_+}[\lambda] \xrightarrow{\sim} H^0(\mathscr{G}_+, W)[\lambda] \xrightarrow{\sim} H^0_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(-\bar{\beta} - \lambda)), \tag{1}$$

is given by

$$\omega \mapsto [\omega \otimes 1] \mapsto [\omega \otimes v_{-\bar{\beta}-\lambda} \otimes 1 \otimes 1], \tag{2}$$

where [ ] denotes the cohomology class. This is obviously an isometry. ■

Lemma 2.23. (*Poincaré duality*). *Suppose (C^*, d) is a cochain complex with* $\dim C^* < +\infty$ *and a nondegenerate hermitian pairing $\langle , \rangle$: $C^n \times C^{-n} \to \mathbf{C}$ for each n, such that $d^\dagger = d$. Then there is an induced nondegenerate hermitian form $\langle , \rangle$: $H^n \times H^{-n} \to \mathbf{C}$ for each n.*

The reader may refer to [LZ, Lemma 3.1] for a proof.

Lemma 2.24. *Let $W \in \mathcal{O}, \Lambda \in \mathscr{G}'_0$, If* $\dim W[\alpha] < +\infty$ *for all* $\alpha \in \mathrm{Spec}_W \mathscr{G}_0$, *then the vector space $C_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(\Lambda))$ is finite dimensional.*

Proof. In Remark 2.10, we noted that

$$C_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(\Lambda)) = C_{\mathrm{red}}(\mathscr{G}, \mathscr{G}_0; W \otimes M(\Lambda))[0] \equiv C_{\mathrm{red}}[0]. \tag{1}$$

By definition $C_{\mathrm{red}} = W \otimes M(\Lambda) \otimes \Omega_\infty$, where $\Omega_\infty = \Omega_\infty(\mathscr{G}/\mathscr{G}_0)$ (Definition 2.2). Thus

$$C_{\mathrm{red}}[0] = \sum_{\alpha+\beta+\gamma=0} W[\alpha] \otimes M(\Lambda)[\beta] \otimes \Omega_\infty[\gamma]. \tag{2}$$

Here α, β, γ ranges over the subsets $\mathrm{Spec}_W \mathscr{G}_0, \mathrm{Spec}_{M(\Lambda)} \mathscr{G}_0, \mathrm{Spec}_{\Omega_\infty} \mathscr{G}_0$, of $\mathscr{G}'_0$

respectively. Because $W\in\mathcal{O}$ (instead of in $\mathcal{O}_0$) we have little control over the set $\mathrm{Spec}_W\mathcal{G}_0$. Fortunately, $\mathrm{Spec}_{M(\Lambda)}\mathcal{G}_0$ and $\mathrm{Spec}_{\Omega_\infty}\mathcal{G}_0$ are well under control.

As a $\mathcal{G}'_0$-graded space, $M(\Lambda)$ is isomorphic to $U(\mathcal{G}_-)\otimes \mathbf{C}v_\Lambda$, where v_Λ is a highest weight vector. Thus

$$\mathrm{Spec}_{M(\Lambda)}\mathcal{G}_0 \subseteq \Lambda + \Gamma_-. \tag{3a}$$

(Note: If the reader wonders why it is $\Lambda+\Gamma_-$ and not $\Lambda+\tilde{\Gamma}_-$ on the right-hand side, recall that $\mathcal{G}$, which satisfies conditions A1-A5 in Sect. 1, is graded by the sublattice $\Gamma=\sum_{i=0}^{\ell}\mathbf{Z}\alpha_i\subseteq\tilde{\Gamma}$, not $\tilde{\Gamma}=\sum_{i=0}^{n}\mathbf{Z}\alpha_i$ itself. This is important because the following finiteness argument depends crucially on this fact.) Similarly,

$$\mathrm{Spec}_{\Omega_\infty}\mathcal{G}_0 \subseteq \bar{\beta} + \Gamma_-. \tag{3b}$$

By construction, each of the $\mathcal{G}'_0$-weight spaces $M(\Lambda)[\beta], \Omega_\infty[\gamma]$ is finite dimensional. Using also the assumption that $\dim W[\alpha] < +\infty$, we see that each summand in Eq. (2) is finite dimensional.

Therefore, it is enough to show that the summation in Eq. (2) admits only finitely many pairs $(\beta,\gamma)\in\mathrm{Spec}_{M(\Lambda)}\mathcal{G}_0\times\mathrm{Spec}_{\Omega_\infty}\cdot\mathcal{G}_0$, for then there are only finitely many admissible $\alpha=-\beta-\gamma\in\mathrm{Spec}_W\mathcal{G}_0$. By Remark 1.1 (iii), the right-hand side of (2) is a subspace of

$$\sum_{\mu+\nu+\rho=0} W[\mu,\mathcal{H}'_0]\otimes M(\Lambda)[\nu,\mathcal{H}'_0]\otimes\Omega_\infty[\rho,\mathcal{H}'_0], \tag{4}$$

where μ,ν,ρ range over

$$\begin{aligned}
&\mu\in\mathrm{Spec}_W\mathcal{H}_0\subseteq\bigcup_{\mu\in A}(\lambda+\Gamma_-)\subseteq\mathcal{H}'_0,\\
&\nu\in\mathrm{Spec}_{M(\Lambda)}\mathcal{H}_0\subseteq\Lambda|_{\mathcal{H}_0}+\Gamma_-\subseteq\mathcal{H}'_0,\\
&\rho\in\mathrm{Spec}_{\Omega_\infty}\mathcal{H}_0\subseteq\bar{\beta}|_{\mathcal{H}_0}+\Gamma_-\subseteq\mathcal{H}'_0,
\end{aligned}\tag{5}$$

A being a finite subset of $\mathcal{H}'_0$. Equations (5) and $\mu+\nu+\rho=0$ imply that there are only finitely many triples (μ,ν,ρ) admissible in Eq. (4).

Comparing Eqs. (3a), (3b) and Eqs. (5), we see that elements of $\mathrm{Spec}_{M(\Lambda)}\mathcal{H}_0$ and $\mathrm{Spec}_{M(\Lambda)}\mathcal{G}_0$ are in one-to-one correspondence. Similarly for $\mathrm{Spec}_{\Omega_\infty}\mathcal{H}_0$ and $\mathrm{Spec}_{\Omega_\infty}\mathcal{G}_0$. This means that there can be only finitely many $(\beta,\gamma)\in\mathrm{Spec}_{M(\Lambda)}\mathcal{G}_0\times\mathrm{Spec}_{\Omega_\infty}\mathcal{G}_0$ admissible in Eq. (2). This completes the proof of the lemma. ■

Theorem 2.25. *Let* $W\in\mathcal{O}$ *be hermitian,* $\mathcal{G}$*-free and* $\dim W[\alpha]<+\infty$ *for all* $\alpha\in\mathrm{Spec}_W\mathcal{G}_0$. *For each* $\lambda\in\mathcal{G}'_0$, *there is an isometry*

$$\begin{aligned}
H^n_\infty(\mathcal{G},\mathcal{G}_0;W\otimes L(-\bar{\beta}-\lambda)) &= 0 \quad \text{for} \quad n\neq 0\\
&\cong W^{\mathcal{G}_+}[\lambda]/\mathrm{rad} \quad \text{for} \quad n=0,
\end{aligned}$$

where $L(-\bar{\beta}-\lambda)$ *is the irreducible quotient of* $M(-\bar{\beta}-\lambda)$.

Proof. First apply Lemma 2.21 to $V_+=W$ and $V_-=M$, $\mathrm{rad}\,M$, $M/\mathrm{rad}\,M$, where $M=M(-\bar{\beta}-\lambda)$. Then Corollary 2.20 implies that

$$H^0_\infty(\mathcal{G},\mathcal{G}_0;W\otimes(M/\mathrm{rad}\,M))\cong H^0_\infty(\mathcal{G},\mathcal{G}_0;W\otimes M)/i^*H^0_\infty(\mathcal{G},\mathcal{G}_0;W\otimes\mathrm{rad}\,M) \tag{1}$$

is an isometry. Now $W\otimes(M/\mathrm{rad}\,M)$ is hermitian. Thus by Remark 2.19 (ii), there is a hermitian pairing between C^n_∞ and C^{-n}_∞, where $C^*_\infty = C^*_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes (M/\mathrm{rad}\, M))$. By Lemma 2.24 $C^*_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M(\Lambda))$ is finite dimensional. Thus Lemma 2.23 implies that the left-hand side of Eq. (1) is hermitian. It follows from Eq. (1) that

$$i^* H^0_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes \mathrm{rad}\, M) = \mathrm{rad}\, H^0_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes M). \tag{2}$$

Now by Lemma 2.22, Eqs. (1), (2) give the desired result. ■

Corollary 2.26. (*Vanishing Theorem*). *For W as in Theorem 2.25,*

$$\begin{aligned} H^n_\infty(\mathscr{G}, \mathscr{G}_0; W) &\cong W^{\mathscr{G}^+}[-\bar{\beta}]/\mathrm{rad} \quad \textit{for} \quad n = 0 \\ &= 0 \qquad\qquad\qquad\quad \textit{for} \quad n = 0. \end{aligned}$$

Proof. $L(0) = \mathbf{C}$. ■

Corollary 2.27. *For W as in Theorem 2.25 and* $\lambda \in \mathscr{G}'_0$,

(i)

$$\dim(W^{\mathscr{G}^+}[\lambda]/\mathrm{rad}) = \left[ch_q W \cdot ch_q L(-\bar{\beta} - \lambda) q^{-\bar{\beta}} \prod_{\alpha \in \Delta_+} (1 - q^\alpha)^{2 \dim \mathscr{G}^{(0)}_\alpha} (1 + q^\alpha)^{-2 \dim \mathscr{G}^{(1)}_\alpha} \right]_{q^0},$$

(ii)

$$\begin{aligned} &\mathrm{sign}(W^{\mathscr{G}^+}[\lambda]/\mathrm{rad}) \\ &= \left[\mathrm{sign}_q W \cdot \mathrm{sign}_q L(-\bar{\beta} - \lambda) q^{-\bar{\beta}} \prod_{\alpha \in \Delta_+} [(1 - q^\alpha)(1 + q^\alpha)]^{\dim \mathscr{G}^{(0)}_\alpha - \dim \mathscr{G}^{(1)}_\alpha} \right]_{q^0}. \end{aligned}$$

Here $[\cdots]_{q^0}$ = *constant term of* $[\cdots]$.

Proof.

(i) By the Euler–Poincaré Principle, for characters, and Theorem 2.25, we have

$$\begin{aligned} \dim(W^{\mathscr{G}^+}[\lambda]/\mathrm{rad}) &= \sum_{n \in \mathbf{Z}} (-1)^n \dim C^n_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes L) \\ &= \sum_{\substack{n \in \mathbf{Z} \\ \alpha + \beta + \gamma = 0}} (-1)^n \dim W[\alpha] \cdot \dim L[\beta] \cdot \dim \Omega_\infty[\gamma] \\ &= \left[ch_q W \cdot ch_q L(-\bar{\beta} - \lambda) \cdot \sum_{n \in \mathbf{Z}} (-1)^n ch_q \Omega^n_\infty(\mathscr{G}/\mathscr{G}_0) \right]_{q^0}. \end{aligned} \tag{1}$$

A simple computation shows that

$$\sum_{n \in \mathbf{Z}} (-1)^n ch_q \Omega^n_\infty(\mathscr{G}/\mathscr{G}_0) = q^{-\bar{\beta}} \prod_{\alpha \in \Delta_+} (1 - q^\alpha)^{2 \dim \mathscr{G}^{(0)}_\alpha} \cdot (1 + q^\alpha)^{-2 \dim \mathscr{G}^{(1)}_\alpha}. \tag{2}$$

(ii) Now apply the Euler–Poincaré Principle, for signatures, and Theorem 2.25:

$$\begin{aligned} \mathrm{sign}(W^{\mathscr{G}^+}[\lambda]/\mathrm{rad}) &= \sum_{n \in \mathbf{Z}} \mathrm{sign}\, C^n_\infty(\mathscr{G}, \mathscr{G}_0; W \otimes L) \\ &= \left[\mathrm{sign}_q W \cdot \mathrm{sign}_q L(-\bar{\beta} - \lambda) \cdot \sum_n \mathrm{sign}_q \Omega^n_\infty(\mathscr{G}/\mathscr{G}_0) \right]_{q^0}. \end{aligned} \tag{3}$$

Computing the third term in the brackets gives

$$\sum_n \mathrm{sign}\, \Omega^n_\infty(\mathscr{G}/\mathscr{G}_0) = q^{-\bar{\beta}} \prod_{\alpha \in \Delta_+} [(1 - q^\alpha)(1 + q^\alpha)]^{\dim \mathscr{G}^{(0)}_\alpha - \dim \mathscr{G}^{(1)}_\alpha}. \quad ■ \tag{4}$$

Corollary 2.28. ("*No-ghost Theorem*") $H^0_\infty(\mathscr{G},\mathscr{G}_0;W)$, $W^{\mathscr{G}_+}[\bar{\beta}]/\mathrm{rad}$ *are unitary* (*positive definite*) iff

$$\left[ch_q W\cdot q^{-\bar{\beta}} \prod_{\alpha\in\Delta_+}(1-q^\alpha)^{2\dim\mathscr{G}^{(0)}_\alpha}(1+q^\alpha)^{-2\dim\mathscr{G}^{(1)}_\alpha}\right]_{q^0}$$

$$=[\mathrm{sign}_q W\cdot q^{-\bar{\beta}}\prod_{\alpha\in\Delta_+}[(1-q^\alpha)(1+q^\alpha)]^{\dim\mathscr{G}^{(0)}_\alpha-\dim\mathscr{G}^{(1)}_\alpha}]_{q^0}.$$

Proof. Use $ch_q L(0)=q^0$ and apply Theorem 2.25 and Corollary 2.27. ■

We now consider another special case of Theorem 2.13.

Proposition 2.29. *Let* $W\in\mathcal{O}$, $\lambda\in\mathrm{Spec}_W\mathscr{G}_0$. *If* $M(-\bar{\beta}-\lambda)$ *is irreducible, then*

$$\begin{aligned} H^n_\infty(\mathscr{G},\mathscr{G}_0;W\otimes M(-\bar{\beta}-\lambda)) &= 0 \qquad n\neq 0\\ &\cong W^{\mathscr{G}_+}[\lambda] \quad n=0. \end{aligned}$$

Proof. We know that $M(-\bar{\beta}-\lambda)$ is $\mathscr{G}_-$-free. By hypothesis, it is also hermitian. Set $V_+=M(-\bar{\beta}-\lambda)$ and $V_-=W$. Then Lemma 2.21 gives the vanishing part. The isomorphism is just given by Corollary 2.17. ■

Corollary 2.30. *For the same hypotheses as above,*

$$W^{\mathscr{G}_+}[\lambda]\cong(W/I\cdot W)[\lambda],$$

where I is the maximal proper ideal of $U(\mathscr{G}_-)$. *Thus* $W[\lambda]=W^{\mathscr{G}_+}[\lambda]+(I\cdot W)[\lambda]$ (*direct sum*).

Proof. By interchanging the role of W and $M(-\bar{\beta}-\lambda)$, we can define a second filtration $\{B'^q\}_{q\in\mathbb{Z}}$ as we did in the proof of Lemma 2.21. A similar argument as in Theorem 2.13 shows that

$$H^*_\infty(\mathscr{G},\mathscr{G}_0;W\otimes M(-\bar{\beta}-\lambda))\cong H^*(\mathscr{G}_-,W)[\bar{\beta}+\lambda]. \tag{1}$$

Now Proposition 2.29 and Eq. (1) imply that

$$W^{\mathscr{G}_+}[\lambda]\cong H^0(\mathscr{G}_-,W][\bar{\beta}+\lambda]. \tag{2}$$

But this right-hand side is precisely $(W\otimes 1\otimes 1/(I\cdot W)\otimes 1\otimes 1)[\lambda]$ with the convention that $1\otimes 1\in\wedge\mathscr{G}^{(0)}_-\otimes\vee\mathscr{G}^{(1)}_-$ carries the weight $\bar{\beta}$. Thus Eq. (2) gives a canonical isomorphism

$$W^{\mathscr{G}_+}[\lambda]\cong(W/I\cdot W)[\lambda], \tag{3}$$

where $\omega\mapsto\omega+I\cdot W$. This means that we have a direct sum decomposition as given above. ■

3. Examples

In this section, we present a few applications which play an important role in string theory and recently, conformal field theory.

First, we will recover a result which has been known to physicists for quite some time – the No-ghost theorem of the so-called bosonic Fock space in D dimensions, as well as its superstring analogue [GT, T1, LZ, FGZ]. We will show

how the no-ghost condition is closely related to the unitarity of some Verma modules over Vir (Theorem 3.5).

Unitary conformal field theories are a very important class of conformal field theories. Recently Dixon, Peskin and Lykken have demonstrated that the unitary representations of the $N=2$ superconformal algebra for $c>3$, can be obtained from a "coset construction" using the $s\hat{o}(2,1)$ current algebra and its $\hat{u}(1)$ subalgebra. They have identified the "unitary domain" in the parameter space of the $s\hat{o}(2,1)$ representations. Later in this section, we will show that the $\hat{u}(1)_+$-invariant subspace (or "$SO(2,1)/U(1)$ coset module" as the authors called it) is precisely a zeroth relative cohomology group; and that the description of the unitary domain is a result of our No-ghost theorem corollary 2.28. We will make use of the Kac–Kazhdan determinant formula (in a different way than Dixon et al.).

We now briefly review some notations for the free bosonic string theory [GSW]. Consider a Heisenberg algebra over $\mathbf{C}$, spanned by $\{\alpha^\mu_n,\ \mu=1,\dots,D, n\in\mathbf{Z}\}\cup\{1\}$, with the Lie bracket

$$[\alpha^\mu_m,\alpha^\nu_n]=mg^{\mu\nu}\delta_{m,-n}\cdot 1, \tag{3.1}$$

where $g^{\mu\nu}$ is the Lorentz metric. Define the fock module $V(D,p)$ of this algebra:

$$V(D,p)=\mathrm{sym}\bigoplus_{\mu=1}^{D}\bigoplus_{n=1}^{\infty}\mathbf{C}\alpha^\mu_{-n}. \tag{3.2}$$

The vacuum of $V(D,p)$ is denoted by v_p, $p\in\mathrm{cent}'$, where $x\in\mathrm{cent}$ acts by

$$x\cdot v_p=\langle p,x\rangle v_p. \tag{3.3}$$

We will assume that $\langle p,1\rangle=1$, that $\langle p,x\rangle\neq 0$ for some $x\notin\mathbf{C}1$, that $\langle p,\alpha^\mu_0\rangle\in\mathbf{R}$ for all μ, and that $D\geqq 2$. The Virasoro algebra Vir is given by

$$\mathrm{Vir}=\sum_{n\in\mathbf{Z}}\mathbf{C}L_n+\mathbf{C}z,$$

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{z}{12}(m^3-m)\delta_{m,-n},$$

$$[z,\mathrm{Vir}]=0. \tag{3.4}$$

For $m,n\in\mathbf{Z}$, set $\mathscr{G}=\mathrm{Vir}, \mathscr{G}_0=\mathbf{C}L_0+\mathbf{C}z, \mathscr{H}_0=\mathbf{C}L_0, \Gamma=\mathbf{Z}L'_0$. Then clearly $\mathscr{G}$ is Γ-graded. $V(D,p)$ is a $\mathscr{G}$-module whose properties are summarized as follows.

Proposition 3.1. [GSW, FGZ].

(i) *$V(D,p)$ is a $\mathscr{G}$-module in $\mathscr{O}_0$, where the action is given by* $\pi(L_m)=\frac{1}{2}\sum_{m\in\mathbf{Z}}:\alpha_m\cdot\alpha_{n-m}:$, $\pi(z)=D\cdot Id_V$. *Here* $\alpha_m\cdot\alpha_n=\alpha^\mu_m\alpha^\nu_n g_{\mu\nu}$ *and:* $\alpha_m\cdot\alpha_n:=\alpha_m\cdot\alpha_n$ *if* $n\geqq m$ and $=\alpha_n\cdot\alpha_m$ *otherwise.*

(ii) *$V(D,p)$ is hermitian with*

$$\langle v_p,v_p\rangle=1,\quad \alpha^{\mu\dagger}_n=\alpha^\mu_{-n},\quad \pi(L_n)^\dagger=\pi(L_{-n}).$$

(iii) $\mathrm{Spec}_{V(D,p)}\mathscr{G}_0=\left\{Dz'+\left(n+\frac{p\cdot p}{2}\right)L'_0\right\}_{n\in\mathbf{Z}_+}$, *where* $p\cdot p=g_{\mu\nu}\langle p,\alpha^\mu_0\rangle\langle p,\alpha^\nu_0\rangle$.

(iv) $\dim V(D,p)[\lambda]<+\infty$ *for all* $\lambda\in\mathrm{Spec}_{V(D,p)}\mathscr{G}_0$.

(v) *$V(D,p)$ is $\mathscr{G}_-$-free.*

Remark 3.2.
(i) Because the central element z acts by a constant, it is customary to use the eigenvalues of $-L_0$ alone as the set of weights, i.e. one identifies $\mathrm{Spec}_{V(D,p)}\mathscr{G}_0$ with $\left\{-n-\frac{p\cdot p}{2}\right\}_{n\in \mathbf{Z}_+}$.

(ii) The Verma module of highest weight $cz'+hL'_0$ is denoted by $M(c,h)$. Its irreducible quotient is denoted by $L(c,h)$. As in (i), one uses the eigenvalues of $-L_0$ as the $\mathscr{G}'_0$-weights of the module.

We now summarize some well-known results about $\mathscr{G}=\mathrm{Vir}$, $\wedge_\infty\mathscr{G}$ and the $\mathscr{G}$-action η on $\wedge_\infty\mathscr{G}$.

Proposition 3.3.
(i) (cf. *Proposition 1.8*) $H^2(\mathscr{G})=0$.
(ii) $\sigma(L_n)=-L_{-n}$ $\sigma(z)=-z$ *defines an antilinear automorphism.*
(iii) $(\wedge_\infty\mathscr{G},\eta)$ *is a* $\mathscr{G}$*-module if* $\bar\beta=-26z'-L'_0$. *Note that* $\sigma(\bar\beta)=-\bar\beta$. *Thus* $\wedge_\infty\mathscr{G}$ *is hermitian.*
(iv) $\mathrm{Spec}_{\wedge_\infty}\mathscr{G}_0=\{-26z'-L'_0+nL'_0\}_{n\in\mathbf{Z}_+}$; *as before, we will identify this set with* $\{1-n\}_{n\in\mathbf{Z}_+}$, i.e. *the eigenvalues of* $-L_0$.

Proposition 3.4. *With the above identifications*:
(i) $\dim V(D,p)\left[-n-\frac{p\cdot p}{2}\right]<+\infty$,
(ii) $ch_q V(D,p)=q^{p\cdot p/2}\phi(q)^{-D}$, *where* $\phi(q)=\prod_{n>0}(1-q^n)$,
(iii) $\mathrm{sign}_q V(D,p)=q^{p\cdot p/2}\phi(q)^{-D+1}\prod_{n>\infty}(1+q^n)^{-1}$.

The reader may consult [GSW] for detailed derivations. Recall that for a (non-super) Lie algebra $\mathscr{G},\Omega_\infty\mathscr{G}=\wedge_\infty\mathscr{G}$ and $\Omega_\infty(\mathscr{G}/\mathscr{G}_0)=\wedge_\infty(\mathscr{G}/\mathscr{G}_0)$. When $\mathscr{G}=\mathrm{Vir}$, the normal ordering defined in Definition 1.6 coincides with that in [GSW], where they use the notation $\varepsilon(L'_n)=c_{-n}$, $\iota(L'_n)=b_n$, $n\in Z$.

Theorem 3.5. *Let* $D\geqq 2$, $h\in\frac{p\cdot p}{2}+\mathbf{Z}_+$:
(i) *There is a canonical isometry*

$$V(D,p)^{\mathscr{G}_+}[-h]/\mathrm{rad}\cong H^0_\infty(\mathscr{G},\mathscr{G}_0;V(D,p)\otimes L(26-D,1-h)).$$

(ii) *The above space is unitary for each* p iff (D,h) *takes the following values*

$$D=26,\quad h=1\quad or\quad 26>D\geqq 2,\quad h\leqq 1.$$

(The number h *is called the "Regge slope" in the physics literature.)*

Proof.
(i) We remind the reader of the convention that $V(D,p)[-h]$ is the eigenspace of $-L_0$ with eigenvalue $-h$. Then (i) is simply a direct application of Theorem 2.25.
(ii) Now by Corollary 2.27 and Proposition 3.4,

$$\dim V(D,p)^{\mathscr{G}_+}[-h]/\mathrm{rad}=\left[q^{p\cdot p/2}\cdot\prod_{n\in N}(1-q^n)^{-D+2}ch_qL(26-D,1-h)\right]_{q^0},$$

$$\mathrm{sign}\, V(D,p)^{\mathscr{G}_+}[-h]/\mathrm{rad}=\left[q^{(p\cdot p/2)-1}\cdot\prod_{n\in N}(1-q^n)^{-D+2}\,\mathrm{sign}_qL(26-D,1-h)\right]_{q^0}.$$

Thus $V(D,p)^{\mathscr{G}+}[-h]/\mathrm{rad}$ is unitary for all p iff $ch_q L(26-D, 1-h) = \mathrm{sign}_q L(26-D, 1-h)$, i.e. iff the Verma module $M(26-D, 1-h)$ is positive semi-definite. The latter is equivalent to precisely the conditions on (D,h), as given above. ■

Remark 3.6.
(i) Since Theorem 2.25, Corollary 2.27 and Proposition 3.4 cover the case of Lie superalgebra as well, the super analogue of Theorem 3.5 also holds. In particular, it turns out that $\bar{\beta} = -10z' - \frac{1}{2}L'_0$ when $\mathscr{G}$ is the super Virasoro algebra $\mathrm{Vir}_{1/2}$. Thus Theorem 3.5 can be restated for $\mathscr{G} = \mathrm{Vir}_{1/2}$ by replacing $h \in \frac{p \cdot p}{2} + \mathbf{Z}_+$ with $h \in \frac{p \cdot p}{2} + \frac{1}{2}\mathbf{Z}_+$, 26 with 10, $1-h$ with $\frac{1}{2}-h$ etc. For details of the notation, the reader may consult [LZ].
(ii) Theorem 2.25 can be applied to a general class of string theories, known as the "compactified" string models in a Minkowski background. These models are described by modules of the form $V(D,p)\otimes W$, where $W \in \mathcal{O}$ and z acts by $(26-D)Id_W$. Many such theories have been constructed (see [GW], [G1], for example). Given such a theory, one postulates that the "particle spectrum" be given by the cohomology classes of $H^0_\infty(\mathscr{G}, \mathscr{G}_0; V(D,p)\otimes W)$. Thus the tools discussed in Sect. 2 can be powerful methods for analyzing some of the properties of the compactified strings. For example, by Corollary 2.26, we have

$$(V(D,p)\otimes W)^{\mathscr{G}+}[-1]/\mathrm{rad} \cong H^0_\infty(\mathscr{G}, \mathscr{G}_0; V(D,p)\otimes W).$$

Corollary 2.27 then gives us the dimensionality of physical space in such a theory. Now using Corollary 2.28, we can determine whether this model is ghost-free. It turns out that the physical space is ghost-free if $ch_q W = \mathrm{sign}_q W$, i.e. if W itself is a unitary $\mathscr{G}$-module. As expected, the unitarity condition for the compactified superstring is precisely the same. For detailed discussion of the latter, the reader is referred to [LZ].

Next we discuss the $SO(2,1)/U(1)$ "Coset construction" of Dixon, Peskin and Lykken using the tools we have developed in the previous section. We begin by a brief review of the notations. Since the following discussion is motivated by [DPL], we will try to follow the notations used by those authors.

Let $G = so(2,1)_{\mathbf{C}} = sl_2(\mathbf{C})$ denote the Kac-Moody Lie algebra given by $G = \sum_{n\in\mathbf{Z}} G_n + \mathbf{C}d + \mathbf{C}z$, $G_n = \mathbf{C}J_n^+ + \mathbf{C}J_n^- + \mathbf{C}J_n^3$, with the Lie bracket:

$$\begin{aligned}
[J_n^3, J_m^3] &= \tfrac{1}{2} z n \delta_{n,-m} \\
[J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm, \\
[J_n^+, J_m^-] &= -zn\delta_{n,-m} - 2J_{n+m}^3, \\
[d, J_n^a] &= nJ_n^a, a = \pm, 3, \quad n, m \in \mathbf{Z}, \\
[z, G] &= 0.
\end{aligned} \tag{3.1}$$

Two important subalgebra are

$$G_0 = \mathbf{C}J_0^+ + \mathbf{C}J_0^- + \mathbf{C}J_0^3 = so(2,1)_{\mathbf{C}} \tag{3.2}$$

and

$$\mathscr{G} = \sum_{n\in\mathbf{Z}} \mathbf{C}J_n^3 + \mathbf{C}d + \mathbf{C}z = \hat{u}(1). \tag{3.3}$$

We will first focus on $\mathscr{G}$. Let

$$\begin{aligned}\mathscr{G}_0 &= \mathbf{C}J_0^3 + \mathbf{C}d + \mathbf{C}z,\\ \tilde{\Gamma} &= \mathbf{Z}J_0^{3'} + \mathbf{Z}d' + \mathbf{Z}z' \subseteqq \mathscr{G}'_0,\\ \mathscr{H}_0 &= \mathbf{C}d,\\ \Gamma &= \mathbf{Z}d' \subseteqq \mathscr{H}'_0,\\ \tilde{\Gamma}_\pm &= \mathbf{Z}_\pm J_0^{3'} + \mathbf{Z}_\pm d' + \mathbf{Z}_\pm z',\\ \Gamma_\pm &= \mathbf{Z}_\pm d'.\end{aligned} \tag{3.4}$$

Thus $\mathscr{G}$ is Γ-graded with $\mathscr{G}_\pm = \sum_{\pm n>0} \mathbf{C}J_n^3$ and $\Delta_\pm = \{nd' \mid \pm n \in \mathbf{N}\} \subseteqq \Gamma_\pm$.

Proposition 3.7.
(i) $\sigma(J_n^3) = -J_{-n}^3, \sigma(z) = -z$ *and* $\sigma(d) = -d$ *defines an antilinear automorphism of* $\mathscr{G}$.
(ii) *If* $\bar{\beta} = \mu J_0^{3'} + \nu d'$, *for* $\mu, \nu \in \mathbf{C}$, *then* $\gamma = 0$ (cf. *Proposition 1.8*). *Thus* $(\wedge_\infty \mathscr{G}, \eta)$ *is a hermitian* $\mathscr{G}$*-module if* $\mu, \nu \in \mathbf{R}$.

Proof.
(i) is easily checked.
(ii) Definition 1.6 gives

$$\eta(J_n^3) = -\frac{n}{2}\varepsilon(J_n^{3'})\iota(z) + n\cdot\varepsilon(d')\iota(J_n^3) + \langle \bar{\beta}, J_n^3 \rangle.$$

Thus $[\eta(J_n^3), \eta(J_m^3)] = 0$. To get $\gamma = 0$, it is enough to choose $\langle \beta, z \rangle = 0$. By Remark 1.9 (i), $(\wedge_\infty \mathscr{G}, \eta)$ is hermitian if $\sigma(\bar{\beta}) = -\bar{\beta}$. Thus we need $\mu, \nu \in \mathbf{R}$. ■

Remark 3.8.
(i) For convenience, we will always assume that $\bar{\beta} = 0$.
(ii) From now on we will consider $\mathscr{G}$-modules $W \in \mathscr{O}$ in which the central element z acts by $k \cdot Id_W$ for $k \in \mathbf{C}\backslash 0$, and $\dim W[\alpha] < +\infty$ for all $\alpha \in \mathscr{G}'_0$.
(iii) We observe that any Verma module $M(\lambda)$ of $\mathscr{G} = \hat{u}(1)$ such that $\langle \lambda, z \rangle \neq 0$ is irreducible. Together with Corollary 2.30, this will give us a further structure on W(the $\mathscr{G}$ module in (ii) above). This structure was implicitly assumed in [DPL, cf. Eq. 3.19], in one of the examples that the authors gave.

Proposition 3.9. *If* $W \in \mathscr{O}$ *satisfies Remark 3.8*(ii) *then* $W = W^{\mathscr{G}_+} + I \cdot W^{\mathscr{G}_+}$ *(direct), where I is the maximal proper ideal of* $\mathscr{U}(\mathscr{G}_-)$. *If, furthermore, W is hermitian, then we have an orthogonal decomposition. In particular,* $\langle , \rangle|_{W^{\mathscr{G}_+} \times W^{\mathscr{G}_+}}$ *is non-degenerate in this case.*

Proof. Suppose W satisfies Remark 3.8(ii). Then for any $\lambda \in \mathrm{Spec}_W \mathscr{G}_0$, $\langle \lambda, z \rangle \neq 0$ and $M(-\lambda)$ is irreducible. Since $\bar{\beta} = 0$, Corollary 2.30 implies that

$$W[\lambda] = W^{\mathscr{G}_+}[\lambda] + (I \cdot W)[\lambda] \tag{1}$$

(direct sum). Thus it is enough to show that $ch_q I \cdot W^{\mathscr{G}_+} = ch_q I \cdot W$. It is clear that I has a canonical basis of the form

$$P_L = (J_{-1}^3)^{l_1}(J_{-2}^3)^{l_2}\cdots \quad \text{with} \quad 0 < \sum_{i=1}^{\infty} l_i < +\infty, \tag{3}$$

where $L=\{l_1,l_2\cdots\}$. Let $\{w_i\}_{i\in\mathbf{Z}}$ be any basis of $W^{\mathscr{G}_+}$. Then the vectors $P_L\cdot w_i$ are linearly independent. For if $\sum_{L,i} a_{L_i}P_L\cdot w_i=0$, then we get, for each J, $\sum_{L,i} a_{L_i}\sigma(P_J)P_L\cdot w_i=C_J\sum_i a_{J_i}w_i=0$, for some $C_J\in\mathbf{C}\backslash 0$. This implies that $a_{J_i}=0$. Therefore, we have

$$ch_q I\cdot W^{\mathscr{G}_+}=ch_q I\cdot ch_q W^{\mathscr{G}_+}=ch_q W^{\mathscr{G}_+}\left(-1+\prod_{\alpha\in\Delta_+}(1-q^\alpha)^{-1}\right). \tag{4}$$

Now using Proposition 2.29, the Euler-Poincaré Principle, and the fact that

$$ch_q M(\lambda)=q^{-\lambda}\prod_{\alpha\in\Delta_+}(1-q^\alpha)^{-1}$$

we get

$$ch_q W^{\mathscr{G}_+}=\prod_{\alpha\in\Delta_+}(1-q^\alpha)ch_q W. \tag{5}$$

Thus (4) becomes

$$ch_q I\cdot W^{\mathscr{G}_+}=-ch_q W^{\mathscr{G}_+}+ch_q W. \tag{6}$$

But by (1), the right-hand side of (6) is just $ch_q I\cdot W$.

The fact that $\langle W^{\mathscr{G}_+}, I\cdot W^{\mathscr{G}_+}\rangle_W=0$ follows from $\mathscr{G}_+\cdot W^{\mathscr{G}_+}=0$. ■

Proposition 3.10. *For W as in Remark 3.8*(ii), *there is an isometry*

$$W^{\mathscr{G}_+}\cong\prod_{\lambda\in\mathrm{Spec}_W\mathscr{G}_0}H^0_\infty(\mathscr{G},\mathscr{G}_0;W\otimes M(-\lambda)).$$

Proof. This is an immediate consequence of Proposition 2.29. ■

We now return to the algebra $G=so(2,1)_{\mathbf{C}}$ and discuss an important class of G-modules – the induced modules. Let $H=\mathbf{C}d+\mathbf{C}z$. Suppose N is a left G_0-module ($G_0=so(2,1)_{\mathbf{C}}$). Turn it into a left B-module $N(\lambda)$ as follows, where B is the subalgebra $\sum_{n\geq 0}G_n+H$. Fix $\lambda\in H'$ and let H act by the weight λ; let each $G_n, n>0$, act trivially. Now take the right B-module $U(G)$ and form the tensor product: $\mathrm{Ind}_{U(B)}^{U(G)}N(\lambda)\equiv U(G)\otimes_{U(B)}N(\lambda)$. This is a left G-module called the module induced by $N(\lambda)$.

Remark 3.11.

(i) We will be primarily interested in G-modules induced by (semi)-unitary G_0-modules.

(ii) In [DPL], the authors have reviewed the construction of all irreducible unitary G_0-modules, which basically come in four families (for details, consult [DPL]):

G1. The trivial module $\mathbf{C}$.

G2. $l<0$: $\mathscr{D}_-(l)=\sum_{n\geq 0}\mathbf{C}(J_0^-)^n v_0$, with $J_0^+v_0=0$, $J_0^3v_0=lv_0$.

G3. $l>0$: $\mathscr{D}_+(l)=\sum_{n\geq 0}\mathbf{C}(J_0^+)^n v_0$, with $J_0^-v_0=0$, $J_0^3v_0=lv_0$.

G4. $\mu\in[0,1)$, $v>\mu(1-\mu)$: $\mathscr{D}_0(\mu,v)=\sum_{n\geq 0}\mathbf{C}(J_0^-)^n v_0+\sum_{n>0}\mathbf{C}(J_0^+)^n v_0$, with $\Omega v_0=vv_0$, $J_0^3v_0=\mu v_0$, where $\Omega\equiv\frac{1}{2}(J_0^+J_0^-+J_0^-J_0^+)-(J_0^3)^2$ is the quadratic Casimir of G_0.

For future convenience, we will include in each of the families G2, G3, G4 those possibly reducible G_0-modules. Namely, $\mathscr{D}_-(l)$, $l \geqq 0$; $\mathscr{D}_-(l)$, $l \leqq 0$; and $\mathscr{D}_0(\mu, \nu)$, $\nu \leqq \mu(1-\mu)$ respectively. We will refer to these (enlarged) classes as G1, G2 etc. in the future. Note that in G2, G3, $\Omega = -l(l \pm 1)Id$ respectively is determined by l, while ν is independent of μ in G4.

(iii) To define the induced modules, we will always assume that $H = \mathbf{C}d + \mathbf{C}z$ acts by the weight $-kz'$, $k \in \mathbf{R}$. We will denote the induced modules in G1, ..., G4 respectively as $V = V(k)$, $V_- = V_-(l,k)$, $V_+ = V_+(l,k)$, $V_0 = V_0(\mu, \nu, k)$. Note that as vector spaces, they are isomorphic respectively to $U(G_-)$, $U(G_-) \otimes \mathscr{D}_-(l)$, $U(G_-) \otimes \mathscr{D}_+(l)$, $U(G_-) \otimes \mathscr{D}_0(\mu, \nu)$, where $G_- = \sum_{n<0} G_n$.

(iv) Observe that each of the $W = V, V_\pm, V_0$ has a unique sesquilinear form $\langle , \rangle_W$ such that

$$\langle v_0, v_0 \rangle = 1,$$
$$\langle J_n^{\pm} \cdot, \cdot \rangle = \langle \cdot, J_{-n}^{\mp} \cdot \rangle,$$
$$\langle J_n^3 \cdot, \cdot \rangle = \langle \cdot, J_{-n}^3 \cdot \rangle,$$

for $n \in \mathbf{Z}$. Also each of these spaces is irreducible as a G-module iff this form is nondegenerate. Equivalently, W is irreducible iff given a basis for each weight space $W[\lambda]$, the (Shapovalov) determinant of the form is non-zero.

(v) We will argue that the $V_\mp(l,k)$ are essentially Verma modules over G and that $V(k)$ is a quotient of a Verma module.

G2: Set $\alpha_0^\vee = -2J_0^3 + z$, $\alpha_1^\vee = 2J_0^3$, $\alpha_0 = -J_0^{3'} + d'$, $\alpha_1 = J_0^{3'}$. Then one can check that α_0, α_1 are the simple roots of G, and that $(\langle \alpha_i, \alpha_j^\vee \rangle) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is the Cartan matrix. In this presentation of G, one can show that $V_-(l,k)$ is the Verma module $M(\lambda)$ over G with highest weight $\lambda = lJ_0^{3'} - kz'$.

G1: Now using the above presentation of G, we find that $V(k)$ is a module with highest weight $\lambda = -kz'$. Thus it is a quotient of $V_-(0,k)$.

G3: Set $\alpha_0^\vee = 2J_0^3 + z$, $\alpha_1^\vee = -2J_0^3$, $\alpha_0 = J_0^{3'} + d'$, $\alpha_1 = -J_0^{3'}$. Then in this new presentation of G, $V_+(l,k)$ is the Verma module $M(\lambda)$ with $\lambda = lJ_0^{3'} - kz'$.

(vi) We will now consider V, $V_\pm$, V_0 as modules over $\mathscr{G} = \hat{u}(1)$. Since $\mathscr{G}_0 = \mathbf{C}J_3^0 + \mathbf{C}d + \mathbf{C}z$, they are clearly $\mathscr{G}_0'$-graded with finite dimensional weight spaces. Thus they satisfy the condition in Remark 3.8(ii). Recall that $\mathscr{H}_0 = \mathbf{C}d$. Thus (cf. (iii) above) $\mathrm{Spec}_W \mathscr{H}_0$, for $W = V$, $V_\pm$, V_0, is a subset of $\Gamma_- = \mathbf{Z}_- d'$. Hence V, $V_\pm$, $V_0 \in \mathscr{O}$. In fact, V, $V_\pm$ are $\mathscr{G}_0$-bounded above and thus are in $\mathscr{O}_0$.

Proposition 3.12.

(i) $ch_q V(k) = \psi(q)^{-1}$,

(ii) $ch_q V_1(l,k) = q^{-l\alpha}(1-q^{\alpha})^{-1}\psi(q)^{-1}$,

(iii) $ch_q V_+(l,k) = q^{-l\alpha}(1-q^{-\alpha})^{-1}\psi(q)^{-1}$,

(iv) $ch_q V_0(\mu,\nu,k) = q^{-\mu\alpha}\chi(q)\psi(q)^{-1}$,

where

$$\psi(q) = q^{-kz'} \prod_{n=1}^{\infty} (1 - q^{n\delta + \alpha})(1 - q^{n\delta - \alpha})(1 - q^{n\delta}),$$

$$\chi(q) = \sum_{n \in \mathbf{Z}} q^{n\alpha}, \delta = d', \alpha = J_0^{3'}.$$

Proof. Using Eq. (3.1), we have for $h \in \mathscr{G}_0$, $n \in \mathbf{Z}$,

$$[h, J^{\pm}_{-n}] = \langle -(n\delta \mp \alpha), h \rangle J^{\pm}_{-n},$$
$$[h, J^3_{-n}] = \langle -n\delta, h \rangle J^3_{-n}.$$

This gives $ch_q U(G_-) = \psi(q)^{-1}$, where $G_- = \sum_{n>0} (\mathbf{C}J^+_{-n} + \mathbf{C}J^-_{-n} + \mathbf{C}J^3_{-n})$, $ch_q D_{\pm}(l) = q^{-l\alpha}(1-q^{\mp\alpha})^{-1} ch_q D_0(\mu, v) = q^{-\mu\alpha} \sum_{n \in \mathscr{Z}} q^{n\alpha}$. Now the observation in Remark (iii) gives the desired result. ■

Theorem 3.13. *Let $W^{\mathscr{G}_+}$ be the subspace of $\mathscr{G}_+$-invariants in W, where W is one of $V(k)$, $V_{\pm}(l,k)$ or $V_0(\mu, v, k)$. Then precisely for the following values of the parameters, $W^{\mathscr{G}_+}$ is positive definite:*
case G1: $k > 2$,
case G2: $k > 2, k > -2l, l < 0$,
case G3: $k > 2, k > 2l, l > 0$,
case G4: $k > 2, v > \mu(1-\mu)$.

We will return to the proof later. The result is also obtained in [DPL], using a different approach.

Proposition 3.14. *For the parameter values given in Theorem 3.13, we have*
(i) $\operatorname{sign}_q V(k) = \phi(q)^{-1}$,
(ii) $\operatorname{sign}_q V_-(l,k) = q^{-l\alpha}(1-q^{\alpha})^{-1}\phi(q)^{-1}$,
(iii) $\operatorname{sign}_q V_+(l,k) = q^{-l\alpha}(1-q^{-\alpha})^{-1}\phi(q)^{-1}$,
(iv) $\operatorname{sign}_q V_0(\mu, v, k) = q^{-\mu\alpha}\chi(q)\phi(q)^{-1}$,
where

$$\phi(q) = q^{-kz'} \prod_{n \in N} (1 - q^{n\delta+\alpha})(1 - q^{n\delta-\alpha})(1 + q^{n\delta}).$$

Remark 3.15. These signature formulae are intuitively clear from the commutation relations (3.1), with $z = -k$ and $k \to +\infty$ ("classical limit"), i.e. we replace $\frac{1}{k}[,]$ by $[,]$ in Eq. (3.1), and we get for "$k \to +\infty$",

$$[J^3_n, J^3_m) = -\tfrac{1}{2} n\delta_{n,-m},$$
$$[J^3_n, J^{\pm}_m] = 0,$$
$$[J^+_n, J^-_m] = n\delta_{n,-m}.$$

But for a rigorous proof, we will invoke the Kac–Kazhdan [KK] formula and a method of Wallach's [W].

Lemma 3.16. *For $k > 2$, $V(k)$ is irreducible.*

Proof. Let Ω be the generalized Casimir of $G = \widehat{sl}_2$.

If $v \in V(k)$ is a G_+-invariant with weight $\lambda - n\alpha_0 - m\alpha_1, \lambda = -kz'$ (see notation in Remark 3.11(v), G2), $n, m \in \mathbf{Z}_+$, one has

$$\Omega v = (\lambda - \beta + 2\rho \,|\, \lambda - \beta)v = (\lambda + 2\rho \,|\, \lambda)v.$$

Simplifying this, we get

$$-n(k-2) = (n-m)(n-m+1) \geqq 0$$

for $n, m \in \mathbf{Z}_+$. Thus for $k > 2$, $(n, m) = (0,0), (0,1)$ are the only solutions. That is, the only weight spaces of $V(k)$ with G_+-invariant are $V(k)[\lambda]$ and $V(k)[\lambda - \alpha_1]$. But the former one is the highest weight space of $V(k)$; the latter is zero. ■

Lemma 3.17. *For the parameter values given in Theorem 3.13, the Shapovalov determinant of W in each case is non-vanishing.*

Proof.
Case G1: This follows from Lemma 3.16.
Case G2, G3: Recall that (Remark 3.11(v)) both $V_\pm(l; k)$ are Verma modules. Thus one can use the Kac–Kazhdan formula to check that, indeed, for the given parameter values, the determinant is non-zero.
G4: Similarly, this follows from the determinant formula for $V_0(\mu, \nu, k)$ given in [DPL]. ■

Remark 3.18.
(i) Let $v(n_0, m_0) = (J_0^+)^{n_0}(J_0^-)^{m_0}v_0$ be a basis of $\mathbf{C}$, $\mathscr{D}_-(l)$, $\mathscr{D}_+(l)$ or $\mathscr{D}_0(\mu, \nu)$, where (n_0, m_0) ranges over $(0,0), (0, \mathbf{Z}_+)$, $(\mathbf{Z}_+, 0)$ or $(\mathbf{Z}_+, 0) \cup (0, \mathbf{Z}_+)$ respectively (see Remark 3.11). Fix l, μ, ν. Recall that $V(k)$, $V_-(l, k)$, $V_+(l, k)$ or $V_0(\mu, \nu, k)$ has basis vectors of the form:

$$u_I(k) = (J_{-1}^3)^{l_1}(J_{-2}^3)^{l_2}\cdots(J_{-1}^-)^{m_1}(J_{-2}^-)^{m_2}\cdots(J_{-1}^+)^{n_1}(J_{-2}^+)^{n_2}\cdots v(n_0, m_0),$$

where $l_i, n_i, m_i \in \mathbf{Z}_+$ with $|I| = \sum_{i=1}^{\infty}(l_i + n_i + m_i) < +\infty$ and I denotes the index sequence $\{l_{i+1}, n_i, m_i\}_{i \geq 0}$. Given $m, N \in \mathbf{Z}$, we call such an I admissible if

$$N = -\sum_{i=1}^{\infty}(l_i + m_i + n_i)\cdot i,$$

$$m = \sum_{i=0}^{\infty}(n_i - m_i).$$

Note that in this case,

$$d \cdot u_I(k) = N \cdot u_I(k),$$

$$J_0^3 u_I(k) = \left(m + \frac{\langle v_0, J_0^3 v_0\rangle}{\langle v_0, v_0\rangle}\right)u_I(k).$$

(ii) The matrix

$$F_{m,N}(k) = (\langle u_I(k), u_{I'}(k)\rangle),$$

where I, I' ranges over the admissible index sequences (for given $m, N \in \mathbf{Z}$), is the Shapovalov matrix. Note that this matrix is finite in all four cases.

Lemma 3.19. *In each of the four cases* G1, ..., G4, *for I, I' admissible, we have*
(i) $\lim_{k \to +\infty} \langle u_I(k), u_I(k)\rangle k^{-|I|} = c_I$, *for some non-zero constant* c_I.
(ii) $\lim_{k \to +\infty} \langle u_I(k), u_{I'}(k)\rangle k^{-(|I| + |I'|)/2} = 0$, where $I \neq I'$.

Proof. It is obvious from the commutation relations that the matrix elements are polynomials in k. Let $\mathcal{O}(k^p)$ denote an arbitrary polynomial in k of degree at most

p. We will only sketch the argument, leaving out the straightforward inductive argument. By induction on $|I|$, one can show that

$$\langle u_I(k), u_I(k)\rangle = \langle v(n_0, m_0), v(n_0, m_0)\rangle \prod_{i=1}^{\infty} l_i! m_i! n_i! \left(\frac{-ik}{2}\right)^{l_i} (ik)^{m_i}(ik)^{n_i} + \mathcal{O}(k^{|I|-1}).$$

Thus (i) follows. To prove (ii), first let $n = |I| \neq |I'| = p$. By induction again, one has

$$\langle v(n_0, m_0), A_1 \cdots A_n C_p \cdots C_1 v(n'_0, m'_0)\rangle = \mathcal{O}(k^{\min(n,p)}),$$

where $A_1, \ldots, A_n \in \{J^a_n | a = \pm, 3, n > 0\}$ (annihilators) and $C_1, \ldots, C_P \in \{J^a_n | a = \pm, 3, n < 0\}$ (creators). This implies that

$$\langle u_I(k), u_{I'}(k)\rangle = \mathcal{O}(k^{\min(|I|,|I'|)}).$$

Thus (ii) follows in this case. To consider the case $|I| = |I'|$, $I \neq I'$, one writes $\langle u_I(k), u_{I'}(k)\rangle$ in the form

$$\langle v(n_0, m_0), A_1 \cdots A_m C_m \cdots C_1 v(n'_0, m'_0)\rangle, m = |I|.$$

Here the A's and the C's are annihilators and creators as before. Then using the fact that $A_i \neq C_i^\dagger$ for some i (because $I \neq I'$) one can show that the above inner product is $\mathcal{O}(k^{m-1})$. Thus (ii) also holds in this case. ■

Lemma 3.20. *For the same assumptions as in Lemma 3.17,*

$$\operatorname{sign} F_{m,N}(k) = \Sigma \operatorname{sign} c_I,$$

where the sum is over all admissible I, and c_I is given in Lemma 3.19.

Proof. Let $M(k)$ be the matrix whose elements are $\langle u_I(k)k^{|I|/2}, u_{I'}(k)k^{-|I'|/2}\rangle$. Thus $\operatorname{sign} F_{m,N}(k) = \operatorname{sign} M(k)$ for $k > 0$. By definition $\operatorname{sign} M(k)$ is the sum of the signs of the eigenvalues of $M(k)$. These eigenvalues are continuous functions of k at least in the region $k > 2$. Lemma 3.17 implies that they never vanish under the conditions of Theorem 3.13. By Lemma 3.19, the c_I's are the limits of these eigenvalues, as $k \to +\infty$. Since all $c_I \neq 0$, we have $\operatorname{sign} M(k) = \Sigma \operatorname{sign} c_I$. ■

Proof of Proposition 3.14. We will do only $\operatorname{sign}_q V(k)$. The other three cases are very similar. By definition,

$$\operatorname{sign}_q V(k) = \sum_{N \in \mathbf{Z}_-} \sum_{m \in \mathbf{Z}} \operatorname{sign} F_{m,N}(k) q^{-N\delta - m\alpha - \lambda}, \tag{1}$$

where $\delta = d'$, $\alpha = J_0^{3'}$, $\lambda = -kz'$. Recall (Proof of Lemma 3.19) that

$$\operatorname{sign} c_I = (-1)^{\sum_{i=1}^{\infty} l_i}. \tag{2}$$

Using (2), Lemma 3.20 and the notation in Remark 3.18, we can write

$$\operatorname{sign}_q V(k) = q^{-\lambda} \cdot \sum_{N \in \mathbf{Z}_-} \sum_{m \in \mathbf{Z}} \sum{}' \{(-1)^{\sum_{i=1}^{\infty} l_i} \cdot q^{\sum_{i=1}^{\infty} (l_i + n_i + m_i) \cdot i\delta + \sum_{i=0}^{\infty} (m_i - n_i)\alpha}\}, \tag{3}$$

where $\sum'$ sums over all admissible $I = \{l_{i+1}, m_i, n_i\}_{i \geqq 0}$ with $N = -\sum_{i=0}^{\infty} (l_i + m_i + n_i) \cdot i$, $m = \sum_{i=0}^{\infty} (n_i - m_i)$ and $l_i, m_i, n_i \in \mathbf{Z}_+$ for $i > 0$. In the case $V(k)$, $(m_0, n_0) = (0, 0)$. Thus

(3) becomes

$$\mathrm{sign}_q V(k) = q^{-\lambda} \prod_{i=1}^{\infty} (1 - q^{i\delta+\alpha})^{-1}(1 - q^{i\delta-\alpha})^{-1}(1 + q^{i\delta}) = \phi(q)^{-1},$$

as given in Proposition 3.14. In each of the other cases, G2, G3 or G4, one has an extra factor $q^{-l\alpha}(1-q^{\alpha})^{-1}, q^{-l\alpha}(1-q^{-\alpha})^{-1}$ or $q^{-\mu\alpha}\sum_{n\in\mathbf{Z}} q^{n\alpha}$. It corresponds to the range $(\mathbf{0}, \mathbf{Z}_+)$, $(\mathbf{Z}_+, 0)$ or $(\mathbf{Z}_+, 0) \cup (0, \mathbf{Z}_+)$ of (n_0, m_0). ■

We are now ready to prove Theorem 3.13.

Proof of Theorem 3.13. Using the Euler–Poincaré Principle and Proposition 3.10, we have (see also Corollary 2.27)

$$\begin{aligned} ch_q W^{\mathscr{G}_+} &= \sum_{\lambda} q^{-\lambda}\left[ch_q W \cdot ch_q M(-\lambda) \prod_{\gamma\in\Delta_+} (1-q^{\gamma})^2 \right]_{q_0} \\ &= \sum_{\lambda} q^{-\lambda}\left[ch_q W \prod_{\gamma\in\Delta_+} (1-q^{\gamma}) \right]_{q^{-\lambda}} \\ &= ch_q W \cdot \prod_{\gamma\in\Delta_+} (1-q^{\gamma}), \end{aligned} \tag{1}$$

where W is in one of the four classes, and $\Delta_+ = \{n\delta, n\in\mathbf{N}\}$ is the positive roots of $\mathscr{G} = \hat{u}(1)$. Similarly,

$$\mathrm{sign}_q W^{\mathscr{G}_+} = \mathrm{sign}_q W \prod_{\gamma\in\Delta_+} (1+q^{\gamma}). \tag{2}$$

Proposition 3.12(i) and Eq. (1) give

$$ch_q V(k)^{\mathscr{G}_+} = q^{kz'} \prod_{n\in\mathbf{Z}} (1-q^{n\delta+\alpha})(1-q^{n\delta-\alpha}), \tag{3}$$

which is identical to $\mathrm{sign}_q V(k)^{\mathscr{G}_+}$, following Proposition 3.14(i) and Eq. (2). Cases G2, G3 and G4 are similar, i.e.

$$ch_q W^{\mathscr{G}_+} = \mathrm{sign}_q W^{\mathscr{G}_+}$$

in all four cases. This establishes that the parameter values given in Theorem 3.13 are sufficient to guarantee unitarity.

To see that these values are necessary, it is enough to give $\mathscr{G}_+$-invariants in W that require these parameter values. One finds that in $V(k)$,

$$\begin{aligned} v_1 &= kJ^3_{-2}v_0 + kJ^+_{-1}J^-_{-1}v_0 - 2J^3_{-1}J^3_{-1}v_0 \\ v_2 &= J^+_{-1}v_0 \end{aligned}$$

are $\mathscr{G}_+$-invariants with

$$\begin{aligned} \langle v_1, v_1\rangle &= k^2(k-2)(k+1)\langle v_0, v_0\rangle, \\ \langle v_2, v_2\rangle &= k\langle v_0, v_0\rangle. \end{aligned}$$

Thus unitarity requires $k > 2$ in case G1. Similarly, in case G2,

$$\begin{aligned} v_1 &= J_0^- v_0, \\ v_2 &= J^+_{-1}v_0, \\ v_3 &= 4lJ^3_{-1}v_0 - kJ^+_{-1}J_0^- v_0 \end{aligned}$$

are $\mathscr{G}_+$-invariants in $V_-(l,k)$ and

$$\langle v_1, v_1 \rangle = -2l\langle v_0, v_0 \rangle,$$
$$\langle v_2, v_2 \rangle = (k+2l)\langle v_0, v_0 \rangle,$$
$$\langle v_3, v_3 \rangle = -2lk(k+2l)(k-2)\langle v_0, v_0 \rangle.$$

Thus unitarity of $V_-(l,k)^{\mathscr{G}_+}$ requires $l<0, k>-2l, k>2$. There are similar vectors in case G3 which require $l>0, k>2l, k>2$, for unitarity of $V_+(l,k)^{\mathscr{G}_+}$. Finally, in case G4, the following $\mathscr{G}_+$-invariants in $V_0(\mu,\nu,k)$ have positive norm only if $\nu>\mu(1-\mu)$ and $k>2$:

$$v_1 = J_0^- v_0,$$
$$v_2 = kJ_{-1}^+ J_0^- (J_0^+)^m v_0 + kJ_{-1}^- (J_0^+)^{m+1} v_0 - 4mJ_{-1}^3 (J_0^+)^m v_0, m>0.$$

This completes the proof of Theorem 3.13. ■

4. Discussions

In Chaps. 1 and 2, we have dealt with the functor

$$W \rightsquigarrow H_\infty^*(\mathscr{G}, \mathscr{G}_0; W \otimes V),$$

where $\mathscr{G}_0$ is ad-diagonalizable. One can generalize to the case in which (A) $\mathscr{G}_0$ is not ad-diagonalizable or (B) $\mathscr{G}_0$ is non-abelian. In these cases, one has to deal with certain technical problems in computing H_∞^*. Another interesting problem that arises in Chap. 1 is (C) What is the role of the representation of the super Heisenberg algebra? To explain this, we recall some of Chap. 1.

The construction of the BRST complex can be viewed abstractly as follows. Given a graded super Lie algebra $\mathscr{G}$, there is a natural graded super-Heisenberg algebra $\mathscr{H}(\mathscr{G} \oplus \mathscr{G}')$ associated with $\mathscr{G}$. To define a semi-infinite complex, we have chosen a particular irreducible representation of the associated super Weyl algebra $\mathscr{W}(\mathscr{G} \oplus \mathscr{G}')$. However, it is well-known that there are many inequivalent irreps of the Weyl algebra (even when $\dim \mathscr{G} < +\infty$!). The problem is to decide when the functors $W \rightsquigarrow H_\infty^*(\mathscr{G}, W)$ resulting from different irreps of $\mathscr{W}$ are naturally isomorphic. In [LZ1], we have had a glimpse at this problem in the case of the super Virasoro algebra. We have shown that in some cases, two distinct irreps of $\mathscr{W}$ do result in the same cohomology. Note that the above problem does not arise when $\mathscr{G}$ is non-super. In this case $\mathscr{W}$ reduces to a graded Clifford algebra $C(\mathscr{G} \oplus \mathscr{G}')$ which has a unique irrep.

We note that problem (C) is in fact physically relevant. It is equivalent to the question of whether the physical states in a model are dependent on the choice of (super) ghost states. In the context of superstring theories, Friedan, Martinec and Shenker [FMS] have shown that the answer is negative, at least for a certain class of superghost representations.

Problems (A), (B), (C) will be the subject of our future investigations. We note here that problems (B) and (C) are also of direct relevance to the problems listed in the introduction.

We emphasize here that the material in Chapter 3 serves merely as an exercise to illustrate our fundamental results developed in Chap. 2. Our on-going work now is to apply our machinery to higher-rank algebras. The new results will be reported soon.

Acknowledgements. We would like to thank I. Frenkel, H. Garland, G. Moore, and G. Seligman for helpful discussions during the course of our work. The second author would like to thank L. Dixon and M. Peskin for bringing their work [DPL] to his attention, and for a helpful introduction to that work.

References

[FGZ] Frenkel, I. B., Garland H., Zuckerman, G. J.: Semi-infinite cohomology and string theory. Proc. Natl. Acad. Sci. USA **83**, 8442 (1986)
[K1] Kac V. G.: Contravariant form for infinite-dimensional Lie algebras and superalgebras. Lecture Notes in Physics, Vol. **94**, p. 441. Berlin, Heidelberg, New York: Springer 1979
[K2] Kac, V. G.: Infinite dimensional Lie algebras. Cambridge: Cambridge University Press, 2nd Ed., 1985.
[Fe] Feigin, B. L.: The semi-infinite homology of Kac–Moody and Virasoro Lie algebras. Russ. Math. Surv. **39**, 155 (1984)
[LZ] Lian, B., Zuckerman, G. J.: BRST-cohomology of the Super Virasoro algebras, Commun. Math. Phys. **125**, 301–335 (1989)
[Zu] Zuckerman, G. J.: Modular forms, strings, and Ghosts. Proceedings of Symposia in Pure Math., Vol. **49**, Gunning R. Ehrenpreis L. (eds.) (1989), Part I, pp. 273–284
[Le] Leites, D. A.: Cohomologies of Lie superalgebras. Funct. Anal. Appl. **9**, 340 (1975)
[HS] Hilton P. J., Stammbach, U.: A course in Homological algebra, Berlin, Heidelberg, New York: Springer 1970, Chap. 7
[GSW] Green, M. B., Schwarz, J. H., Witten, E.: Superstring theory, Vol. 1, Cambridge: Cambridge University Press 1987
[GT] Goddard, P., Thorn, C. B.: Compatibility of the Dual Pomeron with Unitarity and Absence of Ghosts in the Dual Resonance Model, Phys. Lett. **40B**, 235–238 (1972)
[T1] Thorn, C. B.: Computing the Kac determinant using dual model techniques and more about the no-ghost theorem. Nucl. Phys. **B248**, 551 (1984)
[GW] Gepner, D., Witten, E.: String theory on Group manifolds, Nucl. Phys. **B278**, 493 (1986)
[G1] Gepner, D.: Spacetime supersymmetry in compactified string theory and superconformal models.' PUPT-1056 (1987)
[DPL] Dixon, L. J., Peskin, M. E., Lykken, J.: $N = 2$ superconformal symmetry and $SO(2,1)$ current Algebra. SLAC-PUB-4884 (1989)
[KK] Kac, V. G., Kazhdan, D. A.: Structure of representations with highest weights of infinite dimensional Lie algebras, Adv. Math. **34**, 97 (1979)
[W] Wallach, N. R., On the unitarizability of derived functor modules. Invent. Math. **78**, 131–141 (1984)
[F] Farnsteiner, R.: Derivations and central extensions of Finitely Generated Lie Algebra, J. Alg. **118**, 33–45 (1988)
[BO] Bershadsky, M., Ooguri, H. Hidden $SL(n)$ Symmetry in CFT, Commun. Math. Phys. **126**, 49–83 (1989)
[KS] Kazama, Y., Suzuki, H.: Tokyo preprints UT-KOMABA-88-8, 88-13
[FF] Frenkel, E., Feigin, B.: Moscow preprint April-89, May-89
[Ba] Itzhak Bars.: Heterotic superstring vacua in 4-D based on non-compact affine Current algebras. USC-90/HEP02
[KPZ] Knizhnik, V. G., Polyakov, A. M., Zamolodchikov, A. B.: Fractal structure of 2*d*-quantum gravity. Mod. Phys. Lett. A, Vol. **3**, **8**, 819–826 (1988)
[HPV] Horvath, Z., Palla, L., Vecsernyes, P.: BRS Cohomology and 2*d* gravity. ITP Budapest Report No. 468, 1989

Communicated by A. Jaffe