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Concentration Properties of Blow-Up Solutions and Instability Results for Zakharov Equation in Dimension Two. Part II

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Abstract. We consider the Zakharov equation in space dimension two

$$\begin{cases} iu_t = -\Delta u + nu\,, \\ \dfrac{1}{c_0^2}\,n_{tt} = \Delta n + \Delta|u|^2\,. \end{cases}$$

In the first part of the paper, we consider blow-up solutions of this equation. We prove various concentration properties of these solutions: existence, characterization of concentration mass, non existence of minimal concentration mass.

In the second part, we prove instability of periodic solutions.

I. Introduction

We consider as in Part I [7] the Zakharov system in space dimension two,

$$(\mathrm{I}_{c_0}) \qquad \begin{cases} iu_t = -\Delta u + nu\,, & (1.1) \\ \dfrac{1}{c_0^2}\,n_{tt} = \Delta n + \Delta|u|^2\,, & (1.2) \\ u(0) = \phi_0\,, \quad n(0) = n_0\,, \quad n_t(0) = n_1\,, \end{cases}$$

where Δ is the Laplace operator on $\mathbb{R}^2$, $u:[0,T)\times\mathbb{R}^2 \to \mathbb{C}$, $n:[0,T)\times\mathbb{R}^2 \to \mathbb{R}$ and ϕ_0, n_0, n_1 are the initial data. $c_0 > 0$ is a fixed number.

Let us recall the main results of part I.

It is known that the local-in-time Cauchy problem of (I_{c_0}) is solvable in various function spaces. Existence of strong solutions of (I_{c_0}) for regular initial data has been investigated by several authors (see Part I [7]). You can show that, for initial data

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(ϕ_0, n_0, n_1) in $H_2 = H^2 \times H^1 \times L^2$, there is a unique solution (u, n, n_t) in H_2 on $[0, T_2)$ and

– $T_2 = +\infty$ or

– $|u(t)|_{H^2} + |n(t)|_{H^1} + |n_t(t)|_{L^2} \to +\infty$ as $t \to T_2$.

The question to know if this space is optimal for local existence is open. For example the case of the energy space $\hat{H}_1 = H^1 \times L^2 \times \hat{H}^{-1}$, (or even $H_1 = H^1 \times L^2 \times H^{-1}$) for the Cauchy problem is unknown, where $\hat{H}^{-1}$ is the space of functions u such that $\exists v : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$n = -\nabla \cdot v \quad \text{and} \quad v \in L^2$$

and

$$|n|_{\hat{H}^{-1}} = |v|_{L^2}\,.$$

The main result of Part I was to show existence of blow-up solutions of equation (I_{c_0}) in H_2 and $\hat{H}_1$ of a special form which we call self-similar

$$u(t,x) = \frac{\omega}{T-t}\, e^{i\left(\theta + \frac{|x|^2}{4(-T+4)} - \frac{\omega^2}{(-T+t)}\right)} P\left(\frac{x\omega}{T-t}\right), \tag{1.3}$$

$$n(t,x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{x\omega}{T-t}\right), \tag{1.4}$$

where P, N are radial functions and $\theta \in S^1$, $\omega > 0$, $T > 0$ are fixed parameters.

(u, n) is a solution of (I_{c_0}) is equivalent to saying that (P, N) satisfies the following equation:

$$(\mathrm{II}_\lambda) \qquad \begin{cases} \Delta P - P = NP\,, & (1.5)\\ \lambda^2(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta|P|^2\,, & (1.6)\end{cases}$$

where

$$\lambda = \frac{1}{c_0\omega}$$

and $r = |x|$, $W_r = \dfrac{\partial W}{\partial r}$, $\Delta W = W_{rr} + \dfrac{1}{r} W_r$.

We show in part I that there exist solutions (P_λ, N_λ) of the system (II_λ) for $0 < \lambda < \lambda^*$ such that

– $(P_\lambda, N_\lambda) \to (Q, -Q^2)$ as $\lambda \to 0$.

– $\mathscr{C}_1$ is unbounded in $\mathbb{R}^+ \times H^1_r \times L^2_r$, where $\mathscr{C}_1$ is the connected component of the set

$$\{\lambda, P_\lambda, N_\lambda),\ (P_\lambda, N_\lambda) \text{ solution of } (\mathrm{II}_\lambda)\}$$

containing $(0, Q, -Q^2) \in \mathbb{R}^+ \times H^1_r \times L^2_r$ $(L^2_r = L^2 \cap \{u(x) = u(|x|)\})$ and Q is the unique radial solution of

$$(\mathrm{V}^+) \qquad Q = \Delta Q + |Q|^2 Q\,, \qquad Q > 0\,, \qquad Q(x) = Q(|x|)\,,$$

– $P_\lambda > 0$.

Solutions of the corresponding Zakharov equation (u, n) defined by (1.3)–(1.4) are such that

– for $t < T$, $(u, n, n_t) \in \hat{H}_1 \cap H_2$

– $- \lim\limits_{t \to T} |(u, n, n_t)|_{\hat{H}_1} = \lim\limits_{t \to T} |(u, n, n_t)|_{H_2} = +\infty$.

In particular we have the existence of blow-up solutions of (I_{c_0}).

In this paper, we are interested in two types of results.

A) Qualitative properties of blow-up solutions of (I_{c_0}). That is, if we consider a solution of the equation (I_{c_0}) $(u(t), n(t))$ which blows up at $t = T$, what can be said about the structure of the formation of the singularities?

B) Instability by blow-up of periodic solutions of (I_{c_0}) of the form $(u(t), n(t)) = e^{i(\omega t+\theta)}V(x - x_0), -V^2(x - x_0))$, where $x_0 \in \mathbb{R}^2$, $\omega > 0$, $\theta \in S^1$ are parameters and V is a solution of the equation

$$(\mathrm{V}_\omega) \qquad \omega V = \Delta V + |V|^2 V \quad \text{in } \mathbb{R}^2 .$$

I.A Qualitative Properties of Blow-Up Solutions of Zakharow Equation. Let us consider $(\phi_0, n_0, n_1) \in H_k$ $(k \geq 1)$ such that the solution $(u(t), n(t))$ of (I_{c_0}) blows up in finite time $T > 0$ in H, where $H_k = H^k \times H^{k-1} \times H^{k-2}$, that is

$$|(u(t), n(t), n_t(t))|_{H_k} \to +\infty \quad \text{as } t \to T .$$

We are interested in this section in the behavior of $(u(t), n(t))$ at the blow-up time in various spaces and in particular in L^2 for physical reasons.

In the case of the special blow-up solution (u_λ, n_λ) of (1.3)–(1.4) associated with (P_λ, N_λ), we remark that

– $T = T_1 = T_2 = \ldots = T_k = \ldots$, where T_k is the blow-up time in H_k.

– $|u_\lambda(t,x)|^2 \to |P_\lambda|^2_{L^2}\delta_{x=0}$ and $|n_\lambda(t,x)| \to |N_\lambda|_{L^1}\delta_{x=0}$ as $T \to T$.

The question is to know if this concentration phenomenon of the self similar solutions is a general behavior for blow-up solutions or not. That is, given any initial data (ϕ_0, n_0, n_1), are the parameters $m_u > 0$, $m_n > 0$ and a function $t \to x(t) \in \mathbb{R}^2$ such that

$$\forall R, \quad \liminf_{t\to T} |u(t,x)|_{L^2(B(x(t),R))} \geq m_u ,$$
$$\liminf_{t\to T} |u(t,x)|_{L^1(B(x(t),R))} \geq m_n ,$$

where $|u|_{L^2(B(x,R))}$ [resp. $L^1(B(x,R))$] represents the L^2 norm [resp. L^1 norm] of the restriction of u to the ball of center x and radius R?

This phenomenon is known for the nonlinear Schrödinger equation (formal limit of (I_{c_0}) as $c_0 \to +\infty$):

$$(\mathrm{I}_\infty) \qquad \begin{cases} iu_t = -\Delta u - |u|^2 u , & (1.7) \\ u(0) = \phi_0 . & (1.8) \end{cases}$$

Let us consider a solution of (I_∞) which blows up at time $T > 0$ in H^1. Various properties are known:

1) Mass concentration at the blow-up time (Merle, Tsusumi [18] and Weinstein [29]).

– In the case where $\phi_0(x) = \phi_0(|x|)$ we have

$$\forall R, \liminf_{t\to T} |u(t,x)^2_{L^2(B(0,R))} \geq |Q|^2_{L^2} .$$

– In the general case there is a function $x(t)$ such that

$$\forall R, \quad \liminf_{t\to T} |u(t,x)|^2_{L^2(B(x(t),R))} \geq |Q|^2_{L^2} ,$$

where Q is the unique radial solution of (V^+).

2) Lower bound for blow-up solutions. As a corollary of property 1) (see also Weinstein [28]), we have if $|\phi_0|_{L^2} < |Q|_{L^2}$, the solution $u(t)$ is globally defined in time.
3) Characterization of minimal blow-up solutions and optimality of lower bound (Merle [14, 15]).

If $u(t)$ blows up in finite finite time $T > 0$ and $|\phi_0|_{L^2} = |Q|_{L^2}$, then there are $\theta \in S^1$, $x_0, x_1 \in \mathbb{R}^2$, $\omega > 0$ such that

$$u(t,x) = S_{\theta,\omega,x_0,x_1}(t,x) = \left(\frac{\omega}{T-t}\right) e^{i\left(\theta + \frac{|x-x_1|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)} Q\left(\frac{(x-x_1)\omega}{T-t} - x_0\right).$$

Our goal in this section is to prove similar results for Zakharov equation (I_{c_0}) with $0 < c_0 < +\infty$.

The first result is about the relation between the different blow-up times of a solution in various spaces where a Cauchy theory can be done. We have the following proposition.

Proposition 1 (H_1 Control on Higher Derivatives). *If $(\phi_0, n_0, n_1) \in H_k$ for $k \geq 2$, then there is a unique solution $(u(t), n(t))$ of (I_{c_0}) in H_k on $[0, T_k)$ and if $T_k < +\infty$, $|(u(t), n(t), n_t(t))|_{H_k} \to +\infty$ as $t \to T_k$.*

Moreover, $(u(t), n(t), v(t))$ is bounded in H_1 on compact sets of $[0, T_k)$ and if $T_k < +\infty$, $|(u(t), n(t), n_t(t))|_{H_1} \to +\infty$ as $t \to T_k$.

Remark. The uniqueness of the weak solutions is still an open problem.

Assuming that $(\phi_0, n_0, n_1) \in H_k$, and that we can apply different Cauchy theories in $H_1, \ldots, H_k$, let T_i be the blow-up time of $(u(t), n(t), n_t(t))$ in H_i. From Proposition 1, it then follows that $T_1 = T_2 = \ldots = T_k$. Thus, we can restrict ourselves without loss of generality, to the study of blow-up solutions of (I_{c_0}) in H_1. That is, we consider (u, n, n_t), solution of (I_{c_0}) such that for $T > +\infty$

$$|(u(t), n(t), n_t(t))|_{H_1} \to +\infty \quad \text{as } t \to T\,.$$

In fact, for the Zakharov equation as $t \to T$, we have a phenomenon of mass concentration of u and n.

Theorem 1 (L^2-concentration of Blow-Up Solutions). *Let (u, n) be a blow-up solution of equation (I_{c_0}) in H_1, That is*

$$|u|_{H^1} + |n|_{L^2} + |n_t|_{H^{-1}} \to +\infty \quad \text{as } t \to T\,.$$

Then, there is a constant $m_n > 0$ depending on the initial data such that the following properties are true:

1) *If $n_1 \in \hat{H}^{-1}$.*

i) *Radial case.*

$$\forall R > 0\,, \quad \liminf_{t \to T} |u(t,x)|_{L^2(B(0,R))} \geq |Q|_{L^2}$$

and

$$\liminf_{t \to T} |n(t,x)|_{L^1(B(0,R))} \geq m_n\,.$$

ii) *Nonradial case.*

There is a function $t \to x(t)$ such that $\forall R > 0$

$$\liminf_{t\to T} |u(t,x)|_{L^2(B(x(t),R))} \geq |Q|_{L^2} \quad \textit{and} \quad \liminf_{t\to T} |n(t,x)|_{L^1(B(x(t),R))} \geq m_n\,.$$

2) *If* $n_1 \in H^{-1}$ *and* $n_1 \notin \hat{H}^{-1}$. *There is a sequence* $t_k \to T$ *as* $k \to +\infty$ *such that:*
i) *Radial case.*

$$\forall R > 0\,, \quad \liminf_{k\to+\infty} |u(t_k,x)|_{L^2(B(0,R))} \geq |Q|_{L^2}$$

and

$$\liminf_{k\to+\infty} |n(t_k,x)|_{L^1(B(0,R))} \geq m_n\,.$$

ii) *Nonradial case.*
There is x_k *such that* $\forall R > 0$,

$$\liminf_{k\to+\infty} |u(t_k,x)|_{L^2(B(x_k,R))} \geq |Q|_{L^2} \quad \textit{and} \quad \liminf_{k\to+\infty} |n(t_k,x)|_{L^1(B(x_k,R))} \geq m_n\,.$$

Remark. We are not able to find a non-zero lower bound of m_n independent of the initial data. We remark that in the case of the self-similar solution (u_λ, n_λ) defined by (1.3)–(1.4):
- m_n is not necessarily equal to m_u.
- $m_n \to |Q|^2_{L^2}$ as $\lambda \to 0$ and $m_n \to +\infty$ as $\lambda \to +\infty$.

However, it can be shown using variational arguments that if $|\phi_0|^2_{L^2} \leq M_0$ and $(u(t), n(t))$ blows up in finite time, then $m_n \geq K(M_0) > 0$, where $K(M_0) \to 0$ (resp. $K(M_0) \to |Q|^2_{L^2}$) as $M_0 \to +\infty$ (resp. $M_0 \to |Q|^2_{L^2}$). We do not know if these results are optimal or not.

Remarks. i) The proof of part i) of the Theorem 1 follows directly from techniques of [17, 18].
ii) We point out that in the case where $n_1 \notin \hat{H}^{-1}$, we do not have the conservation in time of the energy $\mathscr{H}$, which does not allow us to prove the result for the full sequence. However, we suspect strongly that the result is true for $t \to T$.

This property of the L^2-concentration of $u(t,x)$ at the blow-up time raises an important question namely, which amount of mass can be concentrated at a blow-up point.

More precisely, the following question can be asked:

Characterize the set $\{m\}$ with the property $(\mathscr{P})$, where

$(\mathscr{P})$ There is a initial data (ϕ_0, n_0, n_1) such that
- the solution $(u(t), n(t))$ of (I_{c_0}) blows up in finite time $T > 0$,
- $|u(t,x)|^2 \rightharpoonup m\delta_{x=0}$ where $t \to T$ with $m = |u(t,x)|^2_{L^2} = |\phi_0|^2_{L^2}$.

In fact, using the explicit blow-up solutions constructed in Part I and the concentration results, we are able to give a complete answer to this problem.

Theorem 2. *m has the property $(\mathscr{P})$ if and only if $m > |Q|^2_{L^2}$.*

Remark. In particular, there is no quantification of the concentration of mass. Indeed, the set we obtain in Theorem 2 is $(|Q|^2_{L^2}, +\infty)$ which has no isolated points.

We want to point out the same problem for the limit equation as $c_0 \to +\infty$: (I_∞) (nonlinear Schrödinger equation with critical exponent). This problem is open and the structure of the set of mass concentration is unknown. In particular, we don't know whether there is or not a quantification of the concentration of mass.

Remark. The result is independent of the Cauchy space in the sense that for a given $m > 0$, we exhibit a solution in H_k for all $k \geq 1$. In addition, the result we obtain in Theorem 2 is independent of the parameter c_0 because of the scaling property of the equation. Indeed, if $(u(t,x), n(t,x))$ is a solution of (I_{c_0}) on $[0, T)$, then $\forall \mu > 0$,

$$(u_\mu(t,x), n_\mu(t,x)) = (\mu u(\mu^2 t, \mu x), \mu^2 n(\mu^2 t, \mu x))$$

is a solution of $(\mathrm{I}_{c_0 \mu^2})$ defined on $[0, T/\mu^2)$. It is then easy to check that if

$$|u(t,x)|^2 \rightharpoonup m_u \delta_{x=0} \quad \text{and} \quad |n(t,x)| \rightharpoonup m_n \delta_{x=0} \quad \text{as} \quad t \to T\,,$$

then

$$|u_\mu(t,x)|^2 \rightharpoonup m_u \delta_{x=0} \quad \text{and} \quad |n_\mu(t,x)| \rightharpoonup m_n \delta_{x=0} \quad \text{as} \quad t \to T/\mu^2\,,$$

Theorem 2 will be a consequence of the following propositions.

Proposition 2. (Global Existence for $|\phi_0|_{L^2} < |Q|_{L^2}$). *Assume $|\phi_0|_{L^2} < |Q|_{L^2}$. Then the solution $(u(t), n(t))$ is globally defined in time.*

Remark. In the case where $n_1 \in \hat{H}^{-1}$ (which is not assumed in this proposition), the result has been proved by C. Sulem, P. L. Sulem [25] and H. Added, S. Added [1].

Proposition 3 (Non-Existence of Minimal Blowing-Up Solutions and Global Existence for $|\phi_0|_{L^2} = |Q|_{L^2}$) *Assume $|\phi_0|_{L^2} = |Q|_{L^2}$. Then the solution $(u(t), n(t))$ is globally defined in time.*

Remark. As before, we do not assume that $n_1 \in \hat{H}^{-1}$.

Remark. The result is completely different from the one in the case of Schrödinger equation. Indeed, for the nonlinear Schrödinger equation, there are minimal blow-up solutions in L^2, that is blowing-up solutions which have minimal mass in L^2 norm among the set of blow-up solutions ([14]).

Remark. Let us point out an important corollary of this proposition. Let ϕ_0 be such that the solution of (I_∞) (nonlinear Schrödinger equation), $u(t)$ blows up in finite time and $|\phi_0|_{L^2} = |Q|_{L^2}$. For all $c_0 > 0$ and n_0, n_1, the solution $(u(t), n(t))$ is globally defined in time and does not blow up in finite time.

Let us consider now the explicit solution constructed in Part I,

$$u_\lambda(t,x) = \frac{1}{c_0 \lambda (T-1)} e^{i\left(\frac{|x|^2}{4(-T+t)} - \frac{1}{c_0^2 \lambda^2 (-T+t)}\right)} P_\lambda\left(\frac{x}{c_0 \lambda (T-t)}\right), \tag{1.9}$$

$$n_\lambda(t,x) = \left(\frac{1}{c_0 \lambda (T-t)}\right)^2 N_\lambda\left(\frac{x}{c_0 \lambda (T-t)}\right), \tag{1.10}$$

where $T \geq 0$ and (P_λ, N_λ) satisfies (II_λ) and $\mathscr{C}_1$ be the connected component of $(\lambda, P_\lambda, N_\lambda)$ in $\mathbb{R}^+ \times H_r^1 \times L_r^2$ of solutions of (II_λ) containing $(0, Q, -Q^2)$.

We claim that $\forall m > |Q|_{L^2}^2$, there is a $\lambda = \lambda_m$ such that $(u_{\lambda_m}, n_{\lambda_m})$ is a blow-up solution which has the following property:

$$|u_{\lambda_m}(x,t)|^2 \rightharpoonup m\delta_{x=0} \quad \text{as} \quad t \to T \quad \text{with} \quad m = |u_{\lambda_m}(t,x)|_{L^2}^2 = |\phi_0|_{L^2}^2\,.$$

Proposition 4. 1) *There is a sequence (λ_n, P_n, N_n) of $\mathscr{C}_1$ such that $|P_n|_{L^2} \to |Q|_{L2}$ as $n \to +\infty$.*

– *There is a sequence (λ_n, P_n, N_n) of $\mathscr{C}_1$ such that $|P_n|_{L^2} \to +\infty$ as $n \to +\infty$.*

2) $\forall m > |Q|_{L^2}^2$, *there is* λ_m *such that* $(u_{\lambda_m}, n_{\lambda_m})$ *defined by* (1.9)–(1.10) *is a blow-up solution of* (I_{c_0}) *which has the following properties:*
– *for all* $t \in [0,T)$, $\left(u_{\lambda_m}(t), n_{\lambda_m}(t), \frac{\partial}{\partial t} n_{\lambda_m}(t)\right) \in H_k$, $\forall k \geq 1$,
– $|u_{\lambda_m}(t,x)|^2 \rightharpoonup m\delta_{x=0}$ *as* $t \to T$ *with* $|u_{\lambda_m}(0)|_{L^2}^2 = m$.

It is then easy to see that Theorem 2 follows from Propositions 2, 3, 4. Indeed, from Proposition 4, if $m > |Q|_{L^2}^2$, then m satisfies property $(\mathscr{B})$ and if $m \leq |Q|_{L^2}^2$ then m does not satisfy property $(\mathscr{B})$.

I.B Strong Instability of Periodic Solutions of (I_{c_0}). We recall from Part I, that equation (I_{c_0}) has periodic solutions of the form

$$(u(t), n(t)) = (e^{i\omega t}V(x), -|V(x)|^2),$$

where V satisfies the elliptic equation (V_ω) in $\mathbb{R}^2$. The set of solutions of (V_ω) for $\omega > 0$ has a minimal element in L^2, Q the unique solution of (V^+).

More precisely,
– If $V \not\equiv 0$ satisfy (V_ω) for some $\omega > 0$ then $|V|_{L^2} \geq |Q|_{L^2}$.
– If $V \not\equiv 0$ is a solution of (V_ω) for some $\omega > 0$ such that $|V|_{L^2} = |Q|_{L^2}$, then there are $\theta \in S^1$, $x_0 \in \mathbb{R}^2$ such that

$$V(x) = e^{i\theta}\omega^{1/2}Q(\omega(x - x_0)).$$

The question we are interested in this section is to know whether these periodic solutions are orbitally stable or not in spaces where the Cauchy Problem of (I_{c_0}) can be solved locally in time:
– $\hat{H}_1 = H^1 \times L^2 \times \hat{H}^{-1}$ for weak solutions,
– $H_2 = H^2 \times H^1 \times L^2$ for strong solutions,
– $H_k = H^k \times H^{k-1} \times H^{k-2}$ for $k > 2$ for solutions with additional regularity.

That is $\forall i \geq 1$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|(\phi_0, n_0, n_1) - (V(x), -|V(x)|^2, 0)|_{H_i} \leq \delta.$$

Then $\forall t \in \mathbb{R}^2$,

$$\min_{\substack{\theta \in S^1 \\ x_0 \in \mathbb{R}^2}} |(u(t), n(t), n_t(t)) - (e^{i\theta}V(x - x_0), -|V(x - x_0)|^2, 0)|_{H_i} \leq \varepsilon.$$

We first show that any minimal periodic solution V is orbitally unstable in $\hat{H}_1$ and H_i ($\forall i \geq 1$). That is, if V is such that there are $\theta_o \in S^1$, $\omega_0 > 0$, $x_0 \in \mathbb{R}^2$,

$$V(x) = e^{i\theta_0}\omega_0^{1/2}Q(\omega_0^{1/2}(x - x_0)),$$

then V is unstable.

We then give a similar instability result for a general periodic solution of equation (I_{c_0}) $(e^{i\omega t}V(x), -|V(x)|^2, 0)$ under some nondegeneracy conditions on V.

More precisely, we want to prove that for a given periodic solution of the form

$$(e^{i\omega t}V(x), -|V(x)|^2, 0)$$

and $i \geq 1$, there is δ_i such that $\forall \varepsilon > 0$, $\exists(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$ and t_ε such that

$$|(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) - (V(x), -|V(x)|^2, 0)|_{H_i} \leq \varepsilon$$

and

$$\min_{\substack{\theta\in S^1\\ x_0\in\mathbb{R}^2}}\left|\left(u_\varepsilon(t_\varepsilon), n_\varepsilon(t_\varepsilon), \frac{\partial n_\varepsilon}{\partial t}(t_\varepsilon)\right) - (e^{i\theta}V(x-x_0), -|V(x-x_0)|^2, 0)\right|_{H_i} \geq \delta_i\,,$$

where $(u_\varepsilon, n_\varepsilon)$ is the solution of (I_{c_0}) with initial data $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$.

In fact, we show a stronger result (strong instability or instability by blow-up): There is a sequence of initial data such that

$$(\phi_{0k}, n_{0k}, n_{1k}) \to (V(x), -|V(x)|^2, 0) \quad \text{in } H_i \text{ for } i\geq 1\,,$$

such that (u_k, n_k) blows up in finite time $T_k < +\infty$ in H_i for $i \geq 1$ (in other words, δ_i can be taken arbitrary in the definition of instability).

Such results are well known in the case of the nonlinear Schrödinger equation (I_∞). Indeed if V is a solution of (V_ω) then Pohozaev identity yields that $\mathscr{E}(V) = 0$ where

$$\mathscr{E}(V) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla V(x)|^2dx - \frac{1}{4}\int_{\mathbb{R}^2}|V(x)|^4dx\,.$$

Now considering $\phi_{0\varepsilon} = (1+\varepsilon)V$, we have $\phi_{0\varepsilon} \to V$ in H^1 and

$$\mathscr{E}(\phi_{0\varepsilon}) < 0\,, \qquad \int_{\mathbb{R}^2}|x|^2|\phi_{0\varepsilon}(x)|^2dx < +\infty\,.$$

Therefore, the solution of (I_∞) with initial data $\phi_{0\varepsilon}u_\varepsilon(t)$ blows up in finite time (see [8] and [23]). We can also mention a similar result obtained by Berestycki and Cazenave [3] for nonlinear Schrödinger equation for the ground state solution

$$iu_t = -\Delta u - |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N$$

with $1 + \dfrac{4}{N} < p < \dfrac{N+2}{N-2}$.

The argument will be quite different (argument such as in [3] does not apply) and uses strongly self-similar solutions of (I_{c_0}) constructed in Part I and their asymptotics.

Theorem 3 (Strong Instability of Minimal Periodic Solutions). *Let $(u(t), n(t))$ a non-zero minimal periodic solution of (I_{c_0}), that is there are $\theta_0 \in S^1$, $\omega_0 > 0$, $x_0 \in \mathbb{R}^2$ such that*

$$V(x) = e^{i\theta_0}\omega_0^{1/2}Q(\omega_0^{1/2}(x-x_0))\,.$$

i) *There is a sequence $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (V, -|V|^2, 0)$ in H_k $\forall k \geq 1$ as $\varepsilon \to 0$ such that $(u_\varepsilon(t), n_\varepsilon(t))$ blows up in finite time T_ε in H_1, where $(u_\varepsilon, n_\varepsilon)$ is the solution of (I_{c_0}) with initial data $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$.*

ii) *$(V(x), -|V(x)|^2, 0)$ is orbitally unstable in H_i, $\forall i \geq 1$.*

Remark. Part ii) is a direct consequence of part i). In addition, if $(u_\varepsilon, n_\varepsilon)$ blows up in H_1 in finite time, then it blows up in H_k, $\forall k \geq 1$.

For a general periodic solution, we have the following result.

Theorem 4 (Strong Instability of Periodic Solutions). *Let V a radial solution of equation (V_ω) and $(e^{i\omega t}V(x), -|V(x)|^2, 0)$ the associated periodic solution of (I_{c_0}).*

Assume in addition that V is a nondegenerated critical point of the functional

$$\mathscr{E}_\omega(V) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla V(x)|^2 dx - \frac{1}{4}\int_{\mathbb{R}^2} |V(x)|^4 dx + \frac{\omega^2}{2}\int_{\mathbb{R}^2} |V(x)|^2 dx$$

in $H_r^1 = \{v \in H^1$ such that $v(x) = v(|x|)\}$ in the following sense: the operator

$$W \to -\Delta W + \omega W - 3V^2 W$$

is a continuous one to one application from $H^2 \cap H_r^1$ to L_r^2, where

$$L_r^2 = \{v \in L^2 \text{ such that } v(x) = v(|x|)\}\,.$$

Then the conclusions of Theorem 3 hold.

Remark. We strongly suspect that the result is still true without the nondegeneracy condition (we in fact conjecture that the set of degenerate solutions of (V_ω) is empty).

Remark. In Part I, we have shown that Q is a nondegenerate critical point of $\mathscr{E}_\omega$ in $H^2 \cap H_r^1$. Therefore, Theorem 3 can be seen as a consequence of Theorem 4.

A. Qualitative Properties of Blowing-Up Solutions of Zakharov Equation (I_{c_0})

Let us consider in this section a solution of equation (I_{c_0}) (u, n, n_t) which blows up in finite time $T < +\infty$ in H_k for $k \geq 1$. Existence of such a solution has been proved in Part I [7]. We show in this section various properties of (u, n, n_t) at the blow-up time T. We first give some general properties of solutions of equation (I_{c_0}). For blow-up solutions in H_1, we then show some concentration properties at the blow-up time in Sect. A.1. In Sect. A.2, A.3 we show some properties of the concentration mass. We conclude Sect. A showing that a solution which blows up in H_k for $k \geq 2$ blows up in H_1.

Let (u, n, n_t) solution of equation (I_{c_0}) in H_k for $k \geq 1$. Let first give a different formulation of equation (I_{c_0}).

If $n_t \in H^{-1}$, there is a $(v_0, w_0) \in L^2 \times L^2$ such that

$$n_1 = -\nabla \cdot v_0 + w_0\,. \tag{A.1}$$

We remark in addition that

- if $n_1 \in H^{k-1}$ for $k \geq 2$, we can choose $(v_0, w_0) \in H^{k-2} \times H^{k-2}$,
- if $n_1 \in \hat{H}^{-1}$, we can choose $w_0 = 0$.

We can check that (I_{c_0}) can be written in the form

$$(\mathrm{I}'_{c_0}) \qquad \begin{cases} iu_t = \Delta u + nu\,, \\ n_t = -\nabla \cdot v + w_0\,, \\ \dfrac{1}{c_0^2} v_t + \nabla n = -\nabla |u|^2\,, \end{cases}$$

with the initial data $u(0) = \phi_0$, $n(0) = n_0$, $v(0) = v_0$.

We can remark (u, n, n_t) is a solution of (I_{c_0}) in H_k if and only if (u, n, v) is a solution of (I'_{c_0}) in $H'_k = H^k \times H^{k-1} \times H^{k-1}$.

We have the following properties for a regular solution

Lemma A.1. i) $\forall t \in [0,T)$, $|u(t)|_{L^2} = |\phi_0|_{L^2}$.
ii) $\forall t \in [0,T)$, $\dfrac{d\mathscr{H}(t)}{dt} = \int\limits_{\mathbb{R}^2} w_0(n+|u|^2)$,
where $\mathscr{H}(t) = \mathscr{H}(u(t), n(t), v(t))$ *and*

$$\mathscr{H}(u,n,v) = \int\limits_{\mathbb{R}^2} \left(|\nabla u|^2 + n|u|^2 + \frac{1}{2c_0^2}|v|^2 + \frac{1}{2}n^2 \right). \tag{A.2}$$

Remark. For weak solutions of (I_{c_0}), we can show inequalities, and we can check that all proofs in this section can be carried out. For simplicity, we will assume that the solution is regular enough to prove properties i) and ii). We can check directly from the local uniqueness in time of the solution in H_2 that if $(u(0), n(0), n_t(0)) \in H_2$, the solution satisfies these identities.

Remark. In the case $w_0 = 0$ $(n_t(0) = n_1 \in \hat{H}^{-1})$, we remark that $\mathscr{H}(t)$ is a conserved quantity in time, otherwise it is not.

Proof. Proofs of i) and ii) follow from direct calculations.

A.1 L^2-Concentration of Blow-Up Solutions. We consider in this section a solution $(u(t), n(t))$ of (I_{c_0}) such that

$$|u(t)|_{H^1} + |n(t)|_{L^2} + |n_t(t)|_{H^{-1}} \to +\infty \quad \text{as } t \to T\,,$$

or equivalently

$$|u(t)|_{H^1} + |n(t)|_{L^2} + |v(t)|_{L^2} \to +\infty \quad \text{as } t \to T\,.$$

We want to show concentrations properties of (u, n) in suitable spaces. This result is obtained using methods similar to those in Merle Tsutsumi [18], Merle [17] and Weinstein [28].

Proof of Theorem 1. It follows from energy arguments. Let us recall the energy identity:

Lemma A.2. ([28])

$$\forall u \in H^1\,, \quad \frac{1}{2}|u|_{L^4}^4 \le \left(\frac{|u|_{L^2}^2}{|Q|_{L^2}^2} \right) |\nabla u|_{L^2}^2\,, \tag{A.3}$$

where Q is the unique solution of (V^+).

Define for $(u,n) \in H^1 \times L^2$,

$$\mathscr{E}(u) = \int\limits_{\mathbb{R}^2} |\nabla u|^2 - \frac{1}{2}\int\limits_{\mathbb{R}^2} |u|^4$$

and

$$\mathscr{H}_1(u,n) = \int\limits_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{2}\int\limits_{\mathbb{R}^2} |n|^2 + \int\limits_{\mathbb{R}^2} n|u|^2 = \mathscr{E}(u) + \frac{1}{2}\int\limits_{\mathbb{R}^2} (n+|u|^2)^2\,. \tag{A.4}$$

We consider several cases.

We first consider the case where (u,n) radial functions and $n_1 \in \hat{H}^{-1}$. We can remark (u,n,v) are radial functions for all $t \in [0,T)$ if (ϕ_0, n_0, n_1) belongs to H_2 and is a radial function (uniqueness of the Cauchy in suitable space H_k, $k \ge 2$).

We then consider the general case.

Case 1. $n_1 \in \hat{H}^{-1}$ and (u,n) radial.

In this case, the functions are radial and $\forall t \in [0,T)$, $\mathscr{H}(t) = \mathscr{H}(0)$. We argue by contradiction following [17]. Assume there are $\delta_0 > 0$ and $R_0 > 0$ and a sequence $t_k \to T$ as $k \to +\infty$ such that

$$\int_{|x|<R_0} |u(t_k,x)|^2 dx \le |Q|_{L^2}^2 - \delta_0 \tag{A.5}$$

or

$$\left[\liminf_{k\to+\infty}\left(\int_{|x|<R} |n(t_k,x)|\, dx\right)\right] \to 0 \quad \text{as } R \to 0\,. \tag{A.6}$$

Step 1. Scaling arguments. We consider

$$u_k(x) = \lambda_k^{-1} u(t_k, x\lambda_k^{-1})\,,$$
$$n_k(x) = \lambda_k^{-2} n(t_k, x\lambda_k^{-1})\,,$$

where $\lambda_k = |\nabla u(t_k,x)|_{L^2}$. By direct calculations, we have

$$\begin{gathered}\int_{\mathbb{R}^2} |\nabla u_k|^2 = 1\,, \qquad \int_{\mathbb{R}^2} |u_k|^2 = |\phi_0|_{L^2}^2\,,\\ \mathscr{E}(u_k) = \frac{1}{\lambda_k^2}\left(\int_{\mathbb{R}^2} |\nabla u(t_k,x)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} |u(t_k,x)|^4 dx\right) = \frac{1}{\lambda_k^2}\,\mathscr{E}(u(t_k))\end{gathered} \tag{A.7}$$

and

$$\mathscr{H}_1(u_k,n_k) = \frac{1}{\lambda_k^2}\,\mathscr{H}_1(u(t_k),n(t_k))\,.$$

From the fact that $\mathscr{H}(t) = \mathscr{H}(0)$ where

$$\mathscr{H}(t) = \mathscr{E}(u(t)) + \frac{1}{2}\int_{\mathbb{R}^2} (n(t) + |u(t)|^2)^2 + \frac{c_0^2}{2}\int_{\mathbb{R}^2} |v(t)|^2\,,$$

we have

$$\mathscr{E}(u(t_k)) \le \mathscr{H}_1(u(t_k),n(t_k)) \le \mathscr{H}(t_k) \le \mathscr{H}(0)$$

and

$$\mathscr{E}(u_k) \le \mathscr{H}_1(u_k,n_k) \le \frac{1}{\lambda_k^2}\,\mathscr{H}(0) \to 0 \quad \text{as} \quad k \to +\infty\,. \tag{A.8}$$

In particular, (A.8) yields

$$\limsup_{k\to+\infty} \mathscr{E}(u_k) \le 0 \quad \text{and} \quad \limsup_{k\to+\infty} \mathscr{H}_1(u_k,n_k) \le 0\,.$$

Therefore,

$$\liminf_{k\to+\infty}\int_{\mathbb{R}^2} |u_k|^4 \ge 2\liminf_{k\to+\infty}\left[\int_{\mathbb{R}^2} |\nabla u_k|^2 - \mathscr{E}(u_k)\right] \ge 2 \tag{A.9}$$

and

$$\limsup_{k\to+\infty}\left[\int_{\mathbb{R}^2}(n_k+|u_k|^2)^2-\int_{\mathbb{R}^2}|u_k|^4\right]\le 0$$

or

$$\limsup_{k\to+\infty}\int_{\mathbb{R}^2}|n_k|^2\le 2\limsup_{k\to+\infty}\int_{\mathbb{R}^2}|u_k|^4\le c\,. \tag{A.10}$$

Moreover, property (A.5)–(A.6) and the fact that $\lambda_k\to+\infty$ as $k\to+\infty$ imply $\forall R>0$,

$$\limsup_{k\to+\infty}\int_{|x|<R}|u_k|^2\le|Q|_{L^2}^2-\delta_0\quad\text{or}\quad\limsup_{k\to+\infty}\int_{|x|<R}|n_k|=0\,. \tag{A.11}$$

Step 2. Compactness procedure. Let us obtain a contradiction by compactness procedures. Using classical compactness procedures from (A.7), (A.10), we can assume that there is a $(U,N)\in H^1\times L^2$ such that $u_k\rightharpoonup U$ in H^1, $n_k\rightharpoonup N$ in L^2. Since u_k is a radial function, a compactness lemma (see Strauss [24] yields $u_k\to U$ in L^4.

We then have from (A.9)

$$\int_{\mathbb{R}^2}|U|^4\ge 2\quad\text{and}\quad U\not\equiv 0\,. \tag{A.12}$$

Let k going to $+\infty$ in (A.11), we have

$$\int_{\mathbb{R}^2}|U|^2<|Q|_{L^2}^2\quad\text{or}\quad N=0\,. \tag{A.13}$$

Indeed, $\forall R>0$,

$$\int_{|x|<R}|U|^2\le\liminf_{k\to+\infty}\int_{|x|<R}|u_k|^2\le|Q|_{L^2}^2-\delta_0$$

or

$$\int_{|x|<R}|N|\le\liminf_{k\to+\infty}\int_{|x|<R}|n_k|=0\,.$$

Letting $R\to+\infty$, we obtain (A.13).

Furthermore, since $u_k^2\to U^2$ ($u_k\to U$ and L^4) and $n_k\rightharpoonup N$ in L^2, we have

$$\int_{\mathbb{R}^2}n_k|u_k|^2\to\int_{\mathbb{R}^2}N|U|^2\quad\text{as } k\to+\infty\,. \tag{A.14}$$

We deduce from (A.8), (A.14) that

$$\mathscr{H}_1(U,N) \le \liminf_{k\to+\infty} \mathscr{H}_1(u_k,n_k) = \liminf_{k\to+\infty} \int_{\mathbb{R}^2} |\nabla u_k|^2 + \frac{1}{2}\int_{\mathbb{R}^2} n_k^2 + \int_{\mathbb{R}^2} n_k|u_k|^2 \le 0$$

or equivalently

$$\mathscr{E}(U) + \frac{1}{2}\int_{\mathbb{R}^2} (N+|U|^2)^2 \le 0\,. \tag{A.15}$$

– If $\int_{\mathbb{R}^2} |U|^2 \le |Q|_{L^2}^2 - \delta_0$, we have

$$\mathscr{E}(U) = \int_{\mathbb{R}^2} |\nabla U|^2 - \frac{1}{2}\int_{\mathbb{R}^2} |U|^4 \le 0 \quad \text{and} \quad U \not\equiv 0$$

which is a contradiction (Lemma A.2).
– If $N=0$ then $\mathscr{H}_1(U,N) = \int_{\mathbb{R}^2} |\nabla U|^2 \le 0$ and $U \not\equiv 0$, which is a contradiction.

Therefore, there exists a constant $m_n > 0$ such that $\forall R>0$,

$$\liminf_{t\to T}\left(\int_{|x|<R} |u(t,x)|^2\,dx\right) \ge |Q|_{L^2}^2 \quad \text{and} \quad \liminf_{t\to T}\left(\int_{|x|<R} |n(t,x)|\,dx\right) \ge m_n\,.$$

General case. We now do not assume that $\mathscr{H}(t)$ is a conserved quantity nor the functions $(u(t),n(t),v(t))$ are radial.

Let us give a crucial estimate.

Proposition A.3. *There is $m_n = m_n(|\phi_0|_{L^2}) > 0$ such that the following property is true: Let $u_k \in H^1$, $v_k \in L^2$, $n_k \in L^2$ a sequence such that*

$$|u_k|_{L^2}^2 = |\phi_0|_{L^2}^2\,.$$

Let assume in addition that there are $R_0 > 0$ and $\delta_0 > 0$ such that

$$\sup_y \int_{|y-x|<R_0} |u_k|^2 \le |Q|_{L^2}^2 - \delta_0$$

or

$$\left(\sup_y \int_{|x-y|<R_0} |n_k(x)|\,dx\right) \le m_n(|\phi_0|_{L^2}) - \delta_0\,.$$

There are then constants $c_1>0$, $c_2>0$ such that

$$\forall k\,, \quad -c_1 + c_2\left(\int_{\mathbb{R}^2} |\nabla u_k|^2 + |n_k|^2 + |v_k|^2\right) \le \mathscr{H}(u_k,n_k,v_k)\,.$$

Remark. We can replace the condition $|u_k|_{L^2}^2 = |\phi_0|_{L^2}^2$ by $|u_k|_{L^2}^2 \le |\phi_0|_{L^2}^2$.

Before proving this crucial estimates, let us conclude the proof of Theorem 1.

Case $n_1 \in \hat{H}^{-1}$.

In this case we have $\forall t$, $\mathscr{H}(t) = \mathscr{H}(u(t),n(t),v(t)) = \mathscr{H}(0)$.

Let $m_n(|\phi_0|_{L^2})$ defined by Proposition A.3. Assume there is a sequence $t_k \to T$ as $k \to +\infty$, $R_0 > 0$, $\delta_0 > 0$ such that

$$\liminf_{k\to+\infty}\left(\sup_y \int\limits_{|x-y|<R_0} |u(t_k,x)|^2\,dx\right) \le |Q|_{L^2}^2 - \delta_0 \tag{A.16}$$

or

$$\left[\liminf_{k\to+\infty}\left(\sup_y \int\limits_{|x-y|<R} |n(t_k,x)|\,dx\right)\right] \le m_n(|\phi_0|_{L^2}) - \delta_0\,. \tag{A.17}$$

We then apply Proposition A.3 with $(u(t_k), n(t_k), v(t_k))$ and we obtain

$$\int\limits_{\mathbb{R}^2} |\nabla u(t_k)|^2 + |n(t_k)|^2 + |v(t_k)|^2 \le c \quad \text{and} \quad t_k \to T\,,$$

which is a contradiction. Thus, there is $x_k, y_k, R_k \to 0$ such that

$$\liminf_{k\to+\infty}\left(\int\limits_{|x-x_k|<R_k} |u(t_k,x)|^2\,dx\right) \ge |Q|_{L^2}^2$$

and

$$\left[\liminf_{k\to+\infty}\left(\int\limits_{|x-y_k|<R} |n(t_k,x)|\,dx\right)\right] \ge m_n(|\phi_0|_{L^2}) > 0\,,$$

which concludes the proof.

Case $n_1 \notin \hat{H}^{-1}$.

Assume that there is no sequence $t_k \to T$ such that $\forall R > 0$,

$$\liminf_{k\to+\infty}\left(\sup_y \int\limits_{|x-y|<R} |u(t_k,x)|^2\,dx\right) \ge |Q|_{L^2}^2 \tag{A.18}$$

or

$$\left[\liminf_{k\to+\infty}\left(\sup_y \int\limits_{|x-y|<R} |n(t_k,x)|\,dx\right)\right] \ge m_n(|\phi_0|_{L^2})\,. \tag{A.19}$$

Then there are $R_0 > 0$, $\delta_0 > 0$ such that $\forall t \in [0,T)$,

$$\left(\sup_y \int\limits_{|x-y|<R_0} |u(t,x)|^2\,dx\right) \le |Q|_{L^2}^2 - \delta_0 \tag{A.20}$$

or

$$\left(\sup_y \int\limits_{|x-y|<R_0} |n(t,x)|\,dx\right) \le m_n(|\phi_0|_{L^2}) - \delta_0\,. \tag{A.21}$$

We apply Proposition A.3 and we obtain $\forall t \in [0,T)$,

$$\int\limits_{\mathbb{R}^2} |\nabla u(t)|^2 + |n(t)|^2 + |v(t)|^2 \le c_0\mathscr{H}(t) + c_1\,. \tag{A.22}$$

In addition, from Lemmas A.1, A.2, $\forall t \in [0,T)$,

$$\begin{aligned}
\mathscr{H}(t) &\leq \mathscr{H}(0) + \int_0^t \frac{d\mathscr{H}}{dt}(s)ds \leq c\left(1 + \int_0^t \left(\int_{\mathbb{R}^2} |w_0|(n(s) + |u(s)|^2)\right) ds\right) \\
&\leq c\left(1 + \int_0^t |w_0|_{L^2}^2 + |n(s) + |u(s)|^2|_{L^2}^2 \, ds\right) \\
&\leq c\left(1 + \int_0^t (|n(s)|_{L^2}^2 + |\nabla u(s)|_{L^2}^2) \, ds\right) \\
&\leq c\left(1 + \int_0^t M(s) \, ds\right),
\end{aligned} \tag{A.23}$$

where

$$M(t) = |\nabla u(t)|_{L^2}^2 + |n(t)|_{L^2}^2 + |v(t)|_{L^2}^2 .$$

From (A.22)–(A.23) we have

$$\forall t \in [0,T)\,, \quad M(t) \leq c\left(1 + \int_0^t M(s) \, ds\right),$$

this implies from Gronwall lemma that $\forall t \in [0,T)$, $M(t) \leq c$ or equivalently

$$\forall t \in [0,T)\,, \quad |(u(t), n(t), n_t(t))|_{H_1} \leq c\,,$$

which is a contradiction.

We remark that in the radial case, obvious symmetry reasons and the conservation of the L^2 norm implies that we choose $x_k = 0$ in Theorem 1.

This concludes the proof of Theorem 1.

Proof of Proposition A.3. It is based on similar ideas of Lieb [12] and Weinstein [28] for the nonlinear Schrödinger equation. The proof we present here is based on a lemma which was presented by Merle in a seminar as an alternative proof of the result of Weinstein in [28].

We first remark that it is sufficient to prove that there are constants c_1, c_2 such that

$$\forall k\,, \quad -c_1 + c_2\left(\int_{\mathbb{R}^2} |\nabla u_k|^2 + |n_k|^2\right) \leq \mathscr{H}_1(u_k, n_k)\,,$$

where $\mathscr{H}_1(u,n)$ is defined by (A.4).

Step 1. Scaling arguments.

We argue by contradiction. Assume that the conclusion does not hold for a subsequence (u_k, n_k). Then

$$\lambda_k^2 = \int_{\mathbb{R}^2} |\nabla u_k|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |n_k|^2 \to +\infty$$

and

$$\limsup_{k\to+\infty} \frac{\mathscr{H}_1(u_k, n_k)}{\lambda_k^2} \le 0 \quad \text{as} \quad k \to +\infty .$$

Indeed,
– if $\lambda_k \le c$, then $|\mathscr{H}_1| \le c$ and the conclusion is obvious,
– if $\dfrac{\mathscr{H}_1(n_k, n_k)}{\lambda_k^2} \to c > 0$ as $k \to +\infty$, then for k large,

$$\mathscr{H}_1(u_k, n_k) \ge \frac{c}{2}\left(\int_{\mathbb{R}^2} |\nabla u_k|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |n_k|^2\right),$$

which is a contradiction $\left(\text{since Proposition A.3 will be satisfied with } c_1 = 0 \text{ and } c_2 = \frac{c}{2}\right)$. Consider

$$U_k(x) = \lambda_k^{-1} u_k(x\lambda_k^{-1}) \quad \text{and} \quad N_k(x) = \lambda_k^{-2} n_k(x\lambda_k^{-1}) .$$

We have by direct calculations

$$\int_{\mathbb{R}^2} |U_k|^2 = \int_{\mathbb{R}^2} |\phi_0|^2 \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla U_k|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |N_k|^2 = 1 . \tag{A.24}$$

We remark that

$$\begin{aligned} \limsup_{k\to+\infty}\left(1 + \int_{\mathbb{R}^2} N_k |U_k|^2\right) &= \limsup_{k\to+\infty} \mathscr{H}_1(U_k, N_k) \\ &= \limsup_{k\to+\infty} \frac{\mathscr{H}_1(u_k, n_k)}{\lambda_k^2} \le 0 . \end{aligned} \tag{A.25}$$

Since $\left|\int_{\mathbb{R}^2} N_k |U_k|^2\right| \le c$ by Sobolev estimates, we can assume that

$$\int_{\mathbb{R}^2} N_k |U_k|^2 \to c \le -1 \quad \text{as} \quad k \to +\infty . \tag{A.26}$$

In addition, we have from the assumptions of the proposition,

$$\forall R > 0, \quad \liminf_{k\to+\infty}\left(\sup_y \int_{|x-y|<R} |U_k|^2\right) \le |Q|_{L^2}^2 - \delta_0 \tag{A.27}$$

or

$$\left[\liminf_{k\to+\infty}\left(\sup_y \int_{|x-y|<R} |N_k|\right)\right] \to 0 \quad \text{as} \quad R \to 0 . \tag{A.28}$$

Step 2. A non-vanishing property of (U_k, N_k).

Let us give a crucial lemma which rules out the case of a vanishing sequence (U_n, N_n) in $L^2 \times L^1$.

Lemma A.4. *(Merle) Assume there is a sequence $(U_n, N_n) \in H^1 \times L^2$ such that*

$$\int_{\mathbb{R}^2} |U_k|^2 \to c_1 > 0\,, \quad \int_{\mathbb{R}^2} |\nabla U_k|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |N_k|^2 \to c_2 > 0\,, \quad \int_{\mathbb{R}^2} N_k |U_k|^2 \to -c_3 < 0$$

as $k \to +\infty$.

Then there are a constant $c_4 = c_4(c_1, c_2, c_3) > 0$ and a sequence x_k such that

$$\int_{|x-x_k|<1} |N_k| > c_4\,.$$

Remark. For Schrödinger equation, we apply this lemma with $N_k = -|U_k|^2$.

Proof. We use here some ideas of Lieb in [12]. Clearly, there exists some x_k such that for large k,

$$\int_{C_k} -N_k |U_k|^2 \geq a\left(\int_{C_k}\left(|\nabla U_k|^2 + |U_k|^2 + \frac{1}{2}\,|N_k|^2\right)\right), \tag{A.29}$$

where C_k is the square of center x_k and $a = \dfrac{c_3}{2(c_1 + c_2)}$. Indeed, by contradiction we obtain from (A.29),

$$\int_{\mathbb{R}_2} -N_k |U_k|^2 \leq a\left(\int_{\mathbb{R}^2}\left(|\nabla U_k|^2 + |U_k|^2 + \frac{1}{2}\,|N_k|^2\right)\right).$$

As $k \to +\infty$, we deduce $c_3 \leq a(c_1 + c_2) \leq \dfrac{c_3}{2}$ which is a contradiction.

We claim now that there exists $c > 0$ such that

$$\int_{C_k} -N_k |U_k|^2 \geq c \quad \text{and} \quad \int_{C_k} |U_k|^4 \geq c\,. \tag{A.30}$$

Indeed, by Sobolev identity on C_k there is $s_0 > 0$ independent of k such that

$$\int_{C_k} |\nabla U_k|^2 + |U_k|^2 \geq s_0 \left(\int_{C_k} |U_k|^4\right)^{1/2}.$$

Equation (A.29) gives then

$$a s_0 |U_k|^2_{L^4(C_k)} + \frac{a}{2}\,|N_k|^2_{L^2(C_k)} \leq \int_{C_k} -N_k |U_k|^2 \leq |N_k|_{L^2(C_k)} |U_k|^2_{L^4(C_k)}\,.$$

Thus $|U_k|_{L^4(C_k)} \geq \sqrt{8 s_0 a}$ and $\int_{C_k} -N_k |U_k|^2 \geq c > 0$.

Assume by contradiction for a subsequence N_k,

$$\int_{C_k} |N_k| \to 0 \quad \text{as} \quad k \to +\infty\,. \tag{A.31}$$

We can assume that

$$N_k(x_k+\cdot) \rightharpoonup N \quad \text{in } L^2 \quad \text{and} \quad U_k(x_k+\cdot) \rightharpoonup U \quad \text{in } H^1 .$$

Then $U_k(x_k+\cdot) \to U$ in L^4_{loc} and $|U_k|^2 \to |U|^2$ in L^2_{loc}. From (A.31), $N_k(x_k+\cdot) \rightharpoonup 0$ in $L^2(C_0)$ and

$$\int_{C_k} N_k|U_k|^2 = \int_{C_0} N_k(x_k+x)\,|U_k(x_k+x)|^2 \to 0 \quad \text{as} \quad k \to +\infty .$$

A contradiction follows from (A.30) and the lemma is proved.

Step 3. Conclusion of the proof.

Let us now conclude the proof of the proposition.

Case A.

$$\forall R>0, \quad \left[\liminf_{k\to+\infty}\left(\sup_y \int_{|x-y|<R} |N_k|\right)\right] \to 0 . \tag{A.32}$$

We apply Lemma A.4 and we obtain a contraiction with (A.32) with $R=1$.

Case B.

$$\forall R>0, \quad \liminf_{k\to+\infty}\left(\sup_y \int_{|x-y|<R} |U_k|^2\right) \le |Q|^2_{L^2} - \delta_0 . \tag{A.33}$$

We apply the same procedure then in [28] to obtain a contradiction. In this case, we have from (A.25) and from the fact that

$$\mathscr{H}_1(U_k, N_k) = \mathscr{E}(U_k) + \frac{1}{2}\int_{\mathbb{R}^2} (|N_k| + |U_k|^2)^2 ,$$

$$\limsup_{k\to+\infty} \mathscr{E}(U_k) \le \limsup_{k\to+\infty} \mathscr{H}_1(U_k, N_k) \le 0 .$$

We now can conclude the proof. Indeed, we apply Lemma A.4 (and proof of Lemma A.4), and we obtain dichotomy

$$U_k = U_k^1 + U_{kR}^1 ,$$

where for a sequence x_k^1,

$$U_k^1(x_k^1+x) \rightharpoonup \psi_1 \quad \text{in } H^1 \quad \text{and} \quad |U_k^1|_{L^4(|x-x_k|<1)} \ge c > 0 .$$

Therefore, by Sobolev estimates, there is a $\delta_1 > 0$ (depending only of $|\phi_0|_{L^2}$) such that

$$|U_k^1|_{L^2(|x-x_k^1|<1)} \ge \delta_1 .$$

On one hand, from (A.33),

$$\forall R>0, \quad \liminf_{k\to+\infty} |U_k^1(x_k+\cdot)|_{L^2(B_R)} \le |Q|^2_{L^2} - \delta_0 .$$

By usual techniques of concentration compactness method (see Lions [13]), we have by a suitable choice of U_k^1,

$$|U_k^1|^2_{L^2} + |U_{kR}^1|^2_{L^2} \to |\phi_0|^2_{L^2} \quad \text{and} \quad \delta_1 \le \lim_{k\to+\infty} |U_k^1|^2_{L^2} \le |Q|^2_{L^2} - \delta_0 .$$

On the other hand,

$$\mathscr{E}(\psi_1) + \limsup_{k\to+\infty} \mathscr{E}(U^1_{kR}) \le \limsup_{k\to+\infty} \mathscr{E}(U^1_k) + \limsup_{k\to+\infty} \mathscr{E}(U^1_{kR}) \le \limsup_{k\to+\infty} \mathscr{E}(U_k) \le 0\,.$$

Therefore, from Lemma A.2, since $\delta_1 \le |\psi_1|^2_{L^2} \le |Q|^2_{L^2} - \delta_0$,

$$\limsup_{k\to+\infty} \mathscr{E}(U^1_{kR}) \le -\mathscr{E}(\psi_1) < 0\,.$$

Thus, extracting a subsequence, we have

$$|U^1_{kR}|^2_{L^2} \to c_1 < |Q|^2_{L2} - \delta_1 \quad \text{and} \quad \limsup_{k\to+\infty} \mathscr{E}(U^1_{kR}) \le -\mathscr{E}(\psi_1) < 0\,.$$

We iterate the same procedure and define

$$U^1_{kR} = U^2_k + U^2_{kR}$$

with $|U_k(x^2_k + \cdot)|_{L^2(|x-x^2_k|<1)} \ge \delta_1$.

Let us define p such that $-p\delta_1 + |\phi_0|^2_{L^2} < |Q|^2_{L^2}$. Applying the same procedure at most p times, we find for a $i \le p$ and k large a function U^i_{kR} such that

$$\mathscr{E}(U^i_{kR}) \le \frac{-1}{2}\,\mathscr{E}(\psi_1) < 0 \quad \text{and} \quad |U^i_{kR}|^2_{L^2} < |Q|^2_{L^2}\,,$$

which is a contradiction with Lemma A.2. This concludes the proof of Proposition A.3.

A.2 Non-Existence of Minimal Blowing-Up Solutions in L^2. Let $(\phi_0, n_0, n_1) \in H_1$ and (u, n, n_t) the associated solution of (I_{c_0}). From Theorem 1 and the conservation in time of the $|u(t)|_{L^2}$, we derive easily that if

$$|\phi_0|_{L^2} < |Q|_{L^2}\,,$$

there is no blow-up in time in H_1 of (u, n, n_t) and the solution is globally defined in time (see also Sect. A.4).

The Question is to know if there are solutions which blow-up in H_1 such that

$$|\phi_0|_{L^2} = |Q|_{L^2}\,.$$

We see in the next section that for all $m > |Q|_{L^2}$ there is $(\phi_{0m}, n_{0m}, n_{1m})$ such that
– $|\phi_{0m}|_{L^2} = m$,
– (u_m, n_m) blows up in time [where (u_m, n_m) is the solution of (I_{c_0}) with initial data $(\phi_{0m}, n_{0m}, n_{1m}]$. Then the question is to know if there are minimal blow-up solutions of the Zakharov equation and to characterize them (if they exist). In fact, we claim that there are *no blow-up minimal solutions in L^2*.

Proof of Proposition 3. Let us prove that if $|\phi_0|_{L^2} = |Q|_{L^2}$, then the solution does now blow-up in H_1 (Sect. A.4 will imply the result). Let us argue by contradiction: assume there is $T > 0$ such that

$$|\nabla u(t)|_{L^2} + |n(t)|_{L^2} + |n_t(t)|_{H^{-1}} \to +\infty \quad \text{as} \quad t \to T\,,$$

or equivalently

$$|\nabla u(t)|_{L^2} + |n(t)|_{L^2} + |v(t)|_{L^2} \to +\infty \quad \text{as} \quad t \to T\,. \tag{A.34}$$

Step 1. We claim there is a $c > 0$ such that

$$\forall t \in [0,T), \quad \mathscr{E}(u(t)) \le c, \quad \int_{\mathbb{R}^2} |v(t,x)|^2 \, dx \le c,$$
$$\int_{\mathbb{R}^2} (n(x,t) + |u(t,x)|^2)^2 \, dx \le c. \tag{A.35}$$

– If $n_1 \in \hat{H}^{-1}$, then $\mathscr{H}(0) = \mathscr{H}(t)$, where

$$\mathscr{H}(t) = \mathscr{E}(u(t)) + \frac{1}{2}\int_{\mathbb{R}^2} (n(t) + |u(t)|^2)^2 + \frac{c_0^2}{2}\int_{\mathbb{R}^2} |v(t)|^2 . \tag{A.36}$$

Since from Lemma A.2,

$$\mathscr{E}(u(t)) \ge \left(1 - \frac{|u(t)|_{L^2}^2}{|Q|_{L^2}^2}\right)|\nabla u(t)|_{L^2}^2 \ge 0, \tag{A.37}$$

we have from (A.36), $\forall t \in [0,T)$,

$$\frac{c_0^2}{2}\int_{\mathbb{R}^2} |v(t)|^2 \le \mathscr{H}(0), \quad \frac{1}{2}\int_{\mathbb{R}^2} (n(t) + |u(t)|^2)^2 \le \mathscr{H}(0), \quad \mathscr{E}(u(t)) \le \mathscr{H}(0).$$

– If $n_1 \notin \hat{H}^{-1}$, let us show that $\forall t \in [0,T)$, $\mathscr{H}(t) \le c$ and then conclude as before. We have from Lemma A.1, $\forall t \in [0,T)$,

$$\frac{d\mathscr{H}(t)}{dt} = \int_{\mathbb{R}^2} w_0(n(t) + |u(t)|^2).$$

Thus by Cauchy-Schwarz,

$$\left|\frac{d\mathscr{H}(t)}{dt}\right| \le c + \int_{\mathbb{R}^2} (n(t) + |u(t)|^2)^2$$

and

$$\mathscr{H}(t) \le |\mathscr{H}(t)| \le c + \int_0^t \int_{\mathbb{R}^2} (n(s) + |u(s)|^2)^2 \, ds. \tag{A.38}$$

In particular, from (A.36)–(A.37),

$$\forall t \in [0,T), \quad \int_{\mathbb{R}^2} (n(t) + |u(t)|^2)^2 \le c + \int_0^t \int_{\mathbb{R}^2} (n(s) + |u(s)|^2)^2 ds$$

and the Gronwall lemma yields

$$\forall t \in [0,T), \quad \int_{\mathbb{R}^2} (n(t) + |u(t)|^2)^2 \le c.$$

Using again (A.38), $\forall t \in [0,T)$, $\mathscr{H}(t) \le c$.

Thus as before there is a $c>0$ such that $\forall t\in[0,T)$,

$$\mathscr{E}(u(t))\le c\,,\quad \int\limits_{\mathbb{R}^2}|v(t,x)|^2\,dx\le c\,,\quad \int\limits_{\mathbb{R}^2}(n(t,x)+|u(t,x)|^2)^2\,dx\le c\,.$$

Step 2. Let us show that there is a $c>0$ such that $\forall t\in[0,T)$, $||u(t)|^2|_{H^{-1}}\le c$.We have $\forall t\in[0,T)$, $n_t=\nabla\cdot v+w_0$, where $w_0\in L^2$,

$$\begin{aligned}\forall t\in[0,T)\,,\quad |n(t)|_{H^{-1}}&\le c+\int_0^t|n_t(s)|_{H^{-1}}\,ds\\&\le c+\int_0^t(|\nabla\cdot v(s)|_{H^{-1}}+|w_0|_{H_{-1}})\,ds\\&\le c+\int_0^t(|v(s)|_{L^2}+|w_0|_{L^2})\,ds\le c\end{aligned}\tag{A.39}$$

from Step 1.

Since $|u(t)|^2=(n(t)+|u(t)|^2)-n(t)$, we have from (A.35) and (A.39),

$$\begin{aligned}\forall t\in[0,T)\,,\quad ||u(t)|^2|_{H^{-1}}&\le|n(t)|_{H^{-1}}+|n(t)+|u(t)|^2|_{H^{-1}}\\&\le c+|n(t)+|u(t)|^2|_{L^2}\le c\,.\end{aligned}$$

Step 3. Let us obtain a contradiction with the concentration property of u proved in Theorem 2.

We have $\forall t\in[0,T)$, $\mathscr{E}(u(t))\le c$, $|u(t)|_{L^2}=|Q|_{L^2}$. We claim that

$$|\nabla u(t)|_{L^2}\to+\infty\quad\text{as}\quad t\to T\,.$$

Indeed, assume there is a $c>0$ such that $\int\limits_{\mathbb{R}^2}|\nabla u(t)|^2\le c$, then $\int\limits_{\mathbb{R}^2}|u(t)|^4\le c$, and from (A.35), $\int\limits_{\mathbb{R}^2}(|\nabla u(t)|^2+n^2(t)+v^2(t))\le c$, which is a contradiction with (A.34). Thus from Proposition A.3, there is a $x(t)$ such that

$$|u(t,x(t)+x)|^2\rightharpoonup|Q|_{L^2}^2\delta_{x=0}\quad\text{as}\quad t\to T\,,$$

in the distribution sense.

Let $h(t,x)=|u(t,x(t)+x)|^2$. We then have

$$|h(t,x)|_{H^{-1}}\le c\quad\text{and}\quad h(t,x)\rightharpoonup|Q|_{L^2}^2\delta_{x=0}\quad\text{as}\quad t\to T$$

in the distribution sense. Therefore considering weak limit of $h(t)$,

$$|Q|_{L^2}^2\delta_{x=0}\in H^{-1}$$

which is a contradiction since there is a bounded sequence of continuous functions z_k in H^1 such that $z_k(0)\to+\infty$. Therefore there is no blow-up solutions of minimal mass and Proposition 3 is proved.

A.3 Proof of Proposition 4 and Theorem 2. Let us prove now Proposition 4.

Proof of Proposition 4. 1) Let us consider $\mathscr{C}_1$ be the connected component of $(\lambda, P_\lambda, N_\lambda)$ in $\mathbb{R}^+ \times H^1_r \times L^2_r$ of solutions of equation (II_λ) containing $(0, Q, -Q^2)$. From Sect. 3 of Part I, we know

$$(P_\lambda, N_\lambda) \to (Q, -Q^2) \quad \text{in} \quad H^1 \times L^2 \quad \text{as} \quad \lambda \to 0\,,$$

and in particular $|P_\lambda|_{L^2} \to |Q|_{L^2}$ as $\lambda \to 0$. Moreover from Sect. 5 of Part I, we know that $\mathscr{C}_1$ is unbounded in $\mathbb{R}^+ \times H^1 \times L^2$. This yields to the following alternative: There is a sequence $(\lambda_n, P_n, N_n) \in \mathscr{C}_1$ such that
– Case 1: There is a $0 \le \lambda^{**} < +\infty$, $\lambda_n \to \lambda^{**}$, and $|P_n|_{H^1} + |N_n|_{L^2} \to +\infty$.
– Case 2: $\lambda_n \to +\infty$.

We recall from Part I (Sects. 2 and 5), that we have the following identities.

Lemma A.5. *Let* $(P_\lambda, N_\lambda) \neq 0$ *a solution of the equation* (II_λ) *with* $\lambda > 0$. *We then have*

i) $\int_{\mathbb{R}^2} |\nabla P_\lambda(x)|^2\,dx + \int_{\mathbb{R}^2} P^2_\lambda(x)\,dx = \int_{\mathbb{R}^2} -N_\lambda(x) P^2_\lambda(x)\,dx,$

ii) $\int_{\mathbb{R}^2} P^2_\lambda(x)\,dx = \frac{1}{2}\int_{\mathbb{R}^2} (\lambda^2|x|^2 + 1) N^2_\lambda(x)\,dx,$

iii) $\int_{\mathbb{R}^2} P^2_\lambda(x)\,dx > \int_{\mathbb{R}^2} Q^2(x)\,dx.$

We claim that

Lemma A.6. *We have*

$$|P_n|_{L^2} \to +\infty \quad \text{as} \quad n \to +\infty\,.$$

Proof. Assume by contradiction that for a subsequence also denoted (λ_n, P_n, N_n), we have

$$\int_{\mathbb{R}^2} P_n^2 \le c\,. \tag{A.40}$$

We claim from Pohozaev and energy identities (Lemma A.5) that

$$\int_{\mathbb{R}^2} |\nabla P_n|^2 + \int_{\mathbb{R}^2} P_n^2 + \int_{\mathbb{R}^2} N_n^2 \le c\,. \tag{A.41}$$

Indeed, we have from Lemma A.5 ii)

$$\int_{\mathbb{R}^2} N_n^2 \le \int_{\mathbb{R}^2} P_n^2\,. \tag{A.42}$$

In addition, from Lemma A.5 i) and Gagliardo-Nirenberg inequality,

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla P_n|^2 + \int_{\mathbb{R}^2} P_n^2 &\le \left(\int_{\mathbb{R}^2} N_n^2\right)^{1/2} \left(\int_{\mathbb{R}^2} P_n^4\right)^{1/2} \le c\left(\int_{\mathbb{R}^2} P_n^4\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^2} P_n^2\right)^{1/2} \left(\int_{\mathbb{R}^2} |\nabla P_n|^2\right)^{1/2} \le c\left(\int_{\mathbb{R}^2} |\nabla P_n|^2\right)^{1/2}\end{aligned}$$

and

$$\int_{\mathbb{R}^2} |\nabla P_n|^2 \le c. \tag{A.43}$$

From (A.40), (A.42), and (A.43) we have

$$\int_{\mathbb{R}^2} |\nabla P_n^2| + \int_{\mathbb{R}^2} P_n^2 + \int_{\mathbb{R}^2} N_n^2 \le c. \tag{A.44}$$

Let us consider the two cases.
– Case 1: ($0 \le \lambda^{**} < +\infty$). We have directly a contradiction since

$$\int_{\mathbb{R}^2} |\nabla P_n^2| + \int_{\mathbb{R}^2} P_n^2 + \int_{\mathbb{R}^2} N_n^2 \to +\infty \quad \text{as} \quad n \to +\infty.$$

– Case 2: Using Lemma 2.2 of Part I, from (A.44) we have that $|P_n|_{L^\infty} \le c$. Moreover from Lemma A.5, $\forall \varepsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla P_n^2| + \int_{\mathbb{R}^2} P_n^2 &\le \int_{|x|<\varepsilon} -N_n P_n^2 + \int_{|x|>\varepsilon} -N_n P_n^2 \\ &\le c \int_{|x|<\varepsilon} |N_n| + \Bigg(\int_{|x|>\varepsilon} N_n^2\Bigg)^{1/2} \Bigg(\int_{|x|>\varepsilon} P_n^4\Bigg)^{1/2} \\ &\le c\varepsilon \Bigg(\int_{|x|<\varepsilon} N_n^2\Bigg)^{1/2} + \frac{c}{\varepsilon}\Bigg(\int_{|x|>\varepsilon} |x|^2 N_n^2\Bigg)^{1/2} \\ &\le c\varepsilon + \frac{c}{\varepsilon\lambda_n}\Bigg(\int_{|x|>\varepsilon} \lambda_n^2 |x|^2 N_n^2\Bigg)^{1/2} \le c\varepsilon + \frac{c}{\varepsilon\lambda_n}.\end{aligned}$$

Since $\lambda_n \to +\infty$, we have $\forall \varepsilon > 0$, $\limsup_{n\to+\infty} \int_{\mathbb{R}^2} P_n^2 \le c\varepsilon$ and

$$\int_{\mathbb{R}^2} P_n^2 \to 0 \quad \text{as} \quad n \to +\infty.$$

This is a contradiction with Lemma A.5 iii). This concludes the proof of the lemma and Part 1) of Proposition 4.
2) We consider (u_λ, n_λ) defined in (1.9)–(1.10) for $(\lambda, P_\lambda, N_\lambda) \in \mathscr{C}_1^*$, where $\mathscr{C}_1^* = \mathscr{C}_1 \backslash \{(0, Q, -Q^2)\}$. We have from Part I, (or we can check directly)

$$\forall t < T, \quad \forall k, \quad \left(u_\lambda, n_\lambda, \frac{\partial n_\lambda}{\partial t}\right) \in H_k, \tag{A.45}$$

$$|u_\lambda(t,x)|^2 \rightharpoonup |P_\lambda|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \to T \quad \text{and} \quad |u_\lambda(t)|_{L^2} = |P_\lambda|_{L^2}. \tag{A.46}$$

Let us consider $I = \{|P_\lambda|_{L^2}$, where $(\lambda, P_\lambda, N_\lambda) \in \mathscr{C}_1^*\}$. We want to show that $I = (|Q|_{L^2}, +\infty)$.

Since $\mathscr{E}_1^*$ is connected set in $\mathbb{R}^+ \times H^1 \times L^2$ (see Sect. 4 Part I) and the application $(\lambda, P, N) \in \mathbb{R}^+ \times H^1 \times L^2 \to |P|_{L^2}$ is continuous, we have that I is a connected set of $\mathbb{R}$, thus an interval.

From the facts that

– $\forall(\lambda, P_\lambda, N_\lambda) \in \mathscr{E}_1^*$, $\int\limits_{\mathbb{R}^2} P_\lambda^2 > \int\limits_{\mathbb{R}^2} Q^2$ (Lemma A.5),

– there is a sequence $(\lambda_n, P_n, N_n) \in \mathscr{E}_1^*$ such that $|P_n|_{L^2} \to |Q|_{L^2}$,

– there is a sequence $(\lambda_n, P_n, N_n) \in \mathscr{E}_1^*$ such that $|P_n|_{L^2} \to +\infty$, we have that

$$I = (|Q|_{L^2}, +\infty),$$

and Part 2) follows from the properties of (u_λ, n_λ) (A.45)–(A.46). This concludes the proof of Proposition 4 and Theorem 2 follows from Proposition 2, 3, 4.

A.4 H_1 Control on Higher Derivatives. We assume in this section that different Cauchy theory can be done $(H_k, k \geq 2)$ and we show that the blow-up times in H_k for all k are the same. More precisely, if a solution blows up in H_k (for $k \geq 2$), it blows up in H_1:

$$\lim_{t \to T} |(u(t), n(t), n_t(t))|_{H_1} = \lim_{t \to T} |(u(t), n(t), v_t(t))|_{H_1'} = +\infty.$$

Proof of Proposition 1. The existence and uniqueness and the alternative in H_k for $k \geq 2$ has been proved by Ozawa and Tsutsumi [20]. The H_1 control of H_k norms follows from the two next lemmas.

Lemma A.7. *Let $(u(t), n(t), n_t(t))$ a solution of (I_{c_0}) on $[0, t_0]$ such that*

$$\forall t \in [0, t_0], \quad |u(t), n(t), n_t(t))|_{H_1} \leq c$$

and

$$((u(0), n(0), n_t(0)) \in H_k \quad \text{for} \quad k \geq 2.$$

There is a constant $c > 0$ such that $\forall t \in [0, t_0]$, $|(u(t), n(t), n_t(t))|_{H_k} \leq c$.

Lemma A.8. *Let (u, n, n_t) a solution of (I_{c_0}) in H_2 such that*

$$|(u(0), n(0), n_t(0)|_{H_1} \leq c_1.$$

There is a $\delta_1 > 0$ and c_2 depending on c_1 such that

$$\forall t \in [0, \delta_1], \quad |(u(t), n(t), n_t(t))|_{H_1} \leq c_2.$$

Proof of Lemma A.7. The lemma follows from the following property: Assume there is $t_0 > 0$ and $c > 0$ such that

$$\forall t \in [0, t_0], \quad |(u(t), n(t), n_t(t))|_{H_{k-1}} \leq c \quad \text{and} \quad (u(0), n(0), n_t(0)) \in H_k. \tag{A.47}$$

Then there is a constant c such that $\forall t \in [0, t_0]$, $|(u(t), n(t), n_t(t))|_{H_k} \leq c$.

The cases $k = 2, 3$ follow directly from [1]. The cases $k \geq 4$ use similar techniques than in [1].

a) *Uniform bounds in H_1 imply uniform bounds in H_2 ($k = 2$).* From the same argument as H. Added and S. Added in [1], we show that if

$$\forall t \in [0, t_0], \quad |(u(t), n(t), n_t(t))|_{H_1} \leq c \quad \text{and} \quad (u(0), n(0), n_t(0)|_{H_2} \leq c$$

then

$$\forall t \in [0, t_0], \quad |(u(t), n(t), n_t(t))|_{H_2} \leq c.$$

We recall that H. Added and S. Added use some energy estimates of C. Sulem and P. Sulem [25] and the following lemma:

Lemma A.9. (Brezis and Gallouet [5]). *For* $u \in H^2$ *we have*
i) $|u|_{L^\infty} \le c(1 + |u|_{H^1}\sqrt{\log(1 + |\Delta u|_{L^2})})$,
ii) *if* $|u|_{H^1} \le c$, *then* $|u|_{L^\infty} \le c(1 + \sqrt{\log(1 + |\Delta u|_{L^2})})$.
b) *Uniform bounds in* H_2 *imply uniform bounds in* H_3 ($k = 3$) (see [11]). Let us consider now the case $k = 2p$ and $k = 2p + 1$ where $p \ge 2$.
c) *Uniform bounds in* H_{2p-1} *imply uniform bounds in* H_{2p} ($k = 2p$). Let us first assume that $k = 2p$. We have by a recurrence and direct calculations that $\left(\text{with the notation } u^{(k)} = \dfrac{\partial^k}{\partial t^k}\right)$ $\forall t \in [0, t_0]$,

$$|\Delta^p u(t)|_{L^2} \le a|u^{(p)}(t)|_{L^2} + b \quad \text{and} \quad |u^{(p)}(t)|_{L^2} \le a|\Delta^p u(t)|_{L^2} + b\,. \tag{A.48}$$

We then remark that

$$iu^{(p+1)} = \Delta u^{(p)} + nu^{(p)} + pn_t u^{(p-1)} + \sum_{k=0}^{p-2} c_k n^{(p-k)} u^{(k)}\,. \tag{A.49}$$

Thus

$$\frac{d}{dt}|u^{(p)}(t)|^2_{L^2} \le c\left(\left|\int_{\mathbb{R}^2} n_t u^{(p-1)} u^{(p)}\right| + \sum_{k=0}^{p-2}\left|\int_{\mathbb{R}^2} n^{(p-k)} u^{(k)} u^{(p)}\right|\right),$$

$$\sum_0^{p-2} |u^{(k)}(t)|_{L^\infty} + |n_t(t)|_{L^4} + |u^{(p-1)}(t)|_{L^4} \le c$$

yields with (A.50)

$$\forall t \in [0, t_0]\,, \quad \frac{d}{dt}|u^{(p)}(t)|^2_{L^2} \le c|u^{(p)}(t)|^2_{L^2}\,,$$

and we conclude from Gronwall's lemma that $\forall t \in [0, t_0]$, $|u^{(p)}(t)|^2_{L^2} \le c$ and in particular from (A.48),

$$\forall t \in [0, t_0]\,, \quad |u(t)|_{H^{2p}} \le c\,. \tag{A.51}$$

To conclude, we have from (1.2) that

$$\forall t \in [0, t_0]\,, \quad \frac{1}{c_0^2}(\Delta^{2p-2} n)_{tt} - \Delta(\Delta^{2p-2} n) = \Delta^{2p}|u|^2\,.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\left(\frac{1}{c_0^2}|\Delta^{2p-2} n_t|^2 + |\nabla(\Delta^{2p-2} n)|^2\right) &\le 2\int_{\mathbb{R}^2} |\Delta^{2p}|u|^2 \Delta^{2p-2} n_t| \\ &\le 2|\Delta^{2p}|u|^2|_{L^2}|\Delta^{2p-2} n_t|_{L^2}\,.\end{aligned}$$

From Galiardo-Nirenberg and the Gronwall lemma,

$$\frac{d}{dt}\int_{\mathbb{R}^2}(|\Delta^{2p-2}n_t|^2+|\nabla(\Delta^{2p-2}n)|^2)\le c|\Delta^{2p-2}n_t|_{L^2},$$

$$\forall t\in[0,t_0],\quad \int_{\mathbb{R}^2}(|\Delta^{2p-2}n_t|^2+|\nabla(\Delta^{2p-2}n)|^2)\le c \quad\text{and}\quad |(u(t),v(t),n_t(t)|_{H_{2p}}\le c.$$

d) *Uniform bounds in H_{2p} imply uniform bounds in H_{2p+1}* ($k=2p+1$). From (A.49), we have $\left(\text{with the notation } v^{(k)}=\dfrac{\partial^k}{\partial t^k}\right)$

$$-\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\nabla u^{(p)}|^2+\int_{\mathbb{R}^2}n|u^{(p)}|^2\right)$$

$$=\mathrm{Re}\left(p\int_{\mathbb{R}^2}n^{(1)}u^{(p-1)}\bar{u}^{(p+1)}+\sum_0^{k-2}c_k\int_{\mathbb{R}^2}n^{(p-k)}u^{(k)}\bar{u}^{(p+1)}\right)$$

or

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\nabla u^{(p)}|^2+\int_{\mathbb{R}^2}n|u^{(p)}|^2\right.$$

$$\left.+p\int_{\mathbb{R}^2}n^{(1)}u^{(p-1)}\bar{u}^{(p)}+\sum_0^{k-2}c_k\int_{\mathbb{R}^2}n^{(p-k)}u^{(k)}\bar{u}^{(p)}\right)$$

$$=\mathrm{Re}\left(\int_{\mathbb{R}^2}p(n^{(2)}u^{(p-1)}+n^{(1)}u^{(p)})\bar{u}^{(p)}\right.$$

$$\left.+\sum_0^{k-2}c_k\int_{\mathbb{R}^2}(n^{(p-k)}u^{(k+1)}+n^{(p-k+1)}u^{(k)})\bar{u}^{(p)}\right).$$

Using direct estimates as in [1], we can easily conclude that

$$\forall t\in[0,t_0],\quad \int_{\mathbb{R}^2}|\nabla u^{(p)}(t)|^2\le c.$$

Thus using the equation,

$$\forall t\in[0,t_0],\quad |u(t)|_{H^{2p+1}}\le c.$$

From the fact that

$$\frac{1}{c_0^2}\nabla(\Delta^{2p-1}n_{tt})-\nabla(\Delta^{2p}n)=\nabla(\Delta^{2p}|u|^2),$$

$$\frac{d}{dt}\int_{\mathbb{R}^2}\frac{1}{c_0^2}|\nabla(\Delta^{2p-1}n_t)|^2+\int_{\mathbb{R}^2}|\Delta^{(2p)}n|^2\le\int_{\mathbb{R}^2}\nabla(\Delta^{2(p-1)}n_t)\Delta^{2p+1}|u|^2$$

$$\le c\left(\int_{\mathbb{R}^2}|\nabla(\Delta^{2(p-1)}n_t)|\right)^{1/2},$$

we obtain $\forall t \in [0, t_0]$, $|n(t)|_{H^{2p}} + |n_t(t)|_{H^{2p-1}} \leq c$, which concludes the proof for $k = 2p + 1$ and Lemma A.7 follows.

Proof of Lemma A.8. It follows directly from the techniques used by Merle in [16]. Assume $c_1 \geq \max(1, |\phi_0|^2_{L^2})$ and $|(u(0), n(0), v(0))|_{H_1'} \leq c_1$. Let t_0 such that

$$\forall t \in [0, t_0], \quad |(u(t), n(t), v(t))|^2_{H_1'} \leq c' \quad \text{and} \quad |(u(t_0), n(t_0), v(t_0))|^2_{H_1'} \leq c',$$

where $c' = \alpha c_1^2$ and $\alpha = \max\left(100, \frac{100}{|Q|^2_{L^2}}\right)$. From Lemma A.7, (u, n) is defined on $[0, t_0]$ and let us show that $t_0 > \delta_1 > 0$ where δ_1 depends only on c_1, which will concludes the proof of Lemma A.8.

Step 1. Estimates on $|u(t)|_{L^4}$.

Let $S(t)$ the Schrödinger group. We have

Lemma A.10. *For $\phi \in H^1$, we have*

i) $|S(t)\phi|_{L^2} = |\phi|_{L^2}$,

ii) $|\nabla S(t)\phi|_{L^2} = |\nabla \phi|_{L^2}$,

iii) $|S(t)\phi|_{L^4} \leq \frac{1}{|t|^{1/4}} |\phi|_{L^{4/3}}$.

From (1.1) and Lemma A.10, we have $\forall t \in [0, t_0]$,

$$u(t) = S(t) - i \int_0^t S(t-s)n(s)u(s)\, ds$$

and $\forall t \in [0, t_0]$,

$$\begin{aligned}
|u(t)|_{L^4} &\leq |S(t)\phi_0|_{L^4} + \int_0^t |S(t-s)n(s)u(s)|_{L^4}\, ds \\
&\leq \frac{c_1^{1/2}}{|Q|^{1/2}_{L^2}} + \int_0^t \frac{1}{(t-s)^{1/4}} |n(s)u(s)|_{L^{4/3}}\, ds \\
&\leq \frac{c_1^{1/2}}{|Q|^{1/2}_{L^2}} + \int_0^t \frac{1}{(t-s)^{1/4}} |n(s)|_{L^2} |u(s)|_{L^4}\, ds \\
&\leq \frac{1}{|Q|^{1/2}_{L^2}} \left(c_1^{1/2} + \int_0^t \frac{1}{(t-s)^{1/4}} |n(s)|_{L^2} |u(s)|^{1/2}_{L^2} |\nabla u(s)|^{1/2}_{L^2}\, ds \right) \\
&\leq \frac{1}{|Q|^{1/2}_{L^2}} \left(c_1^{1/2} + \alpha^3 c_1^2 + \int_0^t \frac{1}{(t-s)^{1/4}}\, ds \right) \leq \frac{1}{|Q|^{1/2}_{L^2}} (c_1^{1/2} + \alpha^3 c_1^2 t_0^{3/4}) \\
&\leq \frac{1}{|Q|^{1/2}_{L^2}} + c c_1^2 t_0^{3/4}. \qquad \text{(A.52)}
\end{aligned}$$

Step 2. Conclusion.

For $t \in [0, t_0]$, we have

$$\begin{aligned}
\mathscr{H}(t) &= \mathscr{H}(0) + \int_0^t \int_{\mathbb{R}^2} w_0(n(s) + |u(s)|^2)\, ds \\
&\le \mathscr{H}(0) + \int_0^t c(|n(s)|_{L^2} + |u(s)|^2_{L^4})\, ds \\
&\le \mathscr{H}(0) + \int_0^t c(c_1 \alpha^{1/2} + c_1^2)\alpha\, ds \\
&\le \mathscr{H}(0) + ct_0(c_1\alpha^{1/2} + c_1^2\alpha) \le c_1^2(1 + ct_0)\,. \qquad \text{(A.53)}
\end{aligned}$$

From (A.52)–(A.53), for $t \in [0, t_0]$,

$$\begin{aligned}
|(u(t), n(t), v(t))|^2_{H_1'} &\le |\phi_0|^2_{L^2} + \mathscr{H}(t) + \frac{1}{2}\,|u(t)|^4_{L^4} \\
&\le c_1^2(1 + ct_0) + \frac{1}{|Q|^2_{L^2}}\,(c_1^{1/2} + ct_0^{3/4} c_1^2)^4 \\
&\le c_1^2(1 + ct_0) + \frac{16}{|Q|^2_{L^2}}\,(c_1^{1/2} + ct_0^{3/4} c_1^8) \\
&\le \frac{1}{2}\, c_2 + c(t_0 c_1^2 + t_0^{3/4} c_1^8)\,.
\end{aligned}$$

Therefore, $t_0 > \delta_1$, where $\delta_1 > 0$ depends only on c_1 $\left[\delta_1 \text{ such that } \frac{1}{2}\, c_2 = c(\delta_1 c_1^2 + \delta_1^{3/4} c_1^8)\right]$. Thus, $\forall t \in [0, \delta_1]$,

$$|(u(t), n(t), v(t))|^2_{H_1'} \le c_2\,,$$

which concludes the proof of the lemma and Proposition 4.

B. Strong Instabilities of Periodic Solutions of (I_{c_0})

We consider in this section the periodic solutions of (I_{c_0}) of the form

$$(u(t), n(t)) = (e^{i\omega_0 t} V(x), -|V(x)|^2)\,,$$

where V is a radial solution of the elliptic equation (V_{ω_0}). We want to prove the strong instability (instability by blow-up) of this periodic solution in H_k for $k \ge 1$.

We consider two cases:

– The case of minimal periodic solutions (in L^2 sense) that is there are $\theta_0 \in S^1$, $\omega_0 > 0$, $x_0 \in \mathbb{R}^2$ such that

$$V(x) = e^{i\theta_0} \omega_0^{1/2} Q(\omega_0^{1/2}(x - x_0))\,.$$

– The case of multiple solutions where an extra nondegeneracy condition is needed.

B.1. Case of Minimal Periodic Solutions. This section is devoted to the proof of Theorem 3.

Let us assume that V is a minimal solution of (V_{ω_0}), that is there are $\theta_0 \in S^1$, $\omega_0 > 0$, $x_0 \in \mathbb{R}^2$ such that

$$V(x) = e^{i\theta_0}\omega_0^{1/2} Q(\omega_0^{1/2}(x - x_0)).$$

Part ii) of Theorem 3 follows directly from Part i). Therefore, we restrict ourselves to the proof of Part i). We want to show that there is a sequence

$$(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (V, -|V|^2, 0) \quad \text{in} \quad H_k,\ \forall k \geq 1 \quad \text{as} \quad \varepsilon \to 0$$

such that $(u_\varepsilon(t), n_\varepsilon(t))$ blows up in finite time T_ε in H_1, where $(u_\varepsilon, n_\varepsilon)$ is the solution of (I_{c_0}) with initial data $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$. To prove this instability result, we use in fact the explicit blow-up solutions constructed in Part I; for a fixed $\varepsilon > 0$,

$$u_\varepsilon(t,x) = \frac{\omega_\varepsilon}{(T_\varepsilon - t)} e^{i\left(\theta_\varepsilon + \frac{|x|^2}{4(-T_\varepsilon + t)} - \frac{\omega_\varepsilon^2}{(-T_\varepsilon + t)}\right)} P_\varepsilon\left(\frac{\omega_\varepsilon x}{(T_\varepsilon - t)}\right), \tag{B.1}$$

$$n_\varepsilon(t,x) = \left(\frac{\omega_\varepsilon}{(T_\varepsilon - t)}\right)^2 N_\varepsilon\left(\frac{\omega_\varepsilon x}{(T_\varepsilon - t)}\right), \tag{B.2}$$

where $\omega_\varepsilon = \dfrac{1}{c_0\varepsilon}$ and the parameters $\theta_\varepsilon, T_\varepsilon$ will be carefully chosen and $(P_\varepsilon, N_\varepsilon)$ satisfies the following equation

$$(\mathrm{II}_\varepsilon) \qquad \begin{cases} \Delta P - P = NP\,, & \text{(B.3)} \\ \varepsilon^2(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta |P|^2\,. & \text{(B.4)} \end{cases}$$

Indeed, if we can show that as $\varepsilon \to 0$

$$(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (V, -|V|^2, 0)$$

in H_k, $\forall k \geq 1$, where

$$\phi_{0\varepsilon}(x) = \frac{\omega_\varepsilon}{T_\varepsilon} e^{i\left(\theta_\varepsilon + \frac{|x|^2}{4(-T_\varepsilon)} - \frac{\omega_\varepsilon^2}{(-T_\varepsilon)}\right)} P_\varepsilon\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right), \tag{B.5}$$

$$n_{0\varepsilon}(x) = \left(\frac{\omega_\varepsilon}{T_\varepsilon}\right)^2 N_\varepsilon\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right), \tag{B.6}$$

$$n_{1\varepsilon}(x) = \frac{1}{T_\varepsilon}\left(\frac{\omega_\varepsilon}{T_\varepsilon}\right)^3 \left[|x| N_\varepsilon'\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right) + \frac{2\omega_\omega^2}{T_\varepsilon^3} N_\varepsilon\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right)\right]. \tag{B.7}$$

Then the uniqueness in time of solutions of equation (I_{c_0}) implies that $(u_\varepsilon, n_\varepsilon)$ defined by (B.1)–(B.2) is the solution with initial data $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$. The result follows from the fact $(u_\varepsilon, n_\varepsilon)$ blows up in finite time T_ε.

The proof of this result is done in several steps.

– Step 1: Reduction to the case $V(x) = Q(x)$.
– Step 2: Choice of the parameters $\theta_\varepsilon, \omega_\varepsilon, T_\varepsilon$.

Several steps and then needed to the proof of the convergence of $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$ to $(Q(x), -Q^2(x), 0)$ in H_k, $\forall k \geq 1$ as $\varepsilon \to 0$.

– Step 3: Uniform convergence on bounded sets of $\mathbb{R}^2$ as $\varepsilon \to 0$.
– Step 4: Uniform estimates at infinity in $\mathbb{R}^2$ as $\varepsilon \to 0$.
– Step 5: Conclusion of the proof of Theorem 3.

Step 1. Reduction to the case $V(x) = Q(x)$. We claim that showing the result for $V(x) = Q(x)$ is enough from the scaling properties of the equation. Indeed, assume for all $c_1 > 0$ there is a sequence $(\phi_{0\varepsilon}^{c_1}, n_{0\varepsilon}^{c_1}, n_{1\varepsilon}^{c_1}) \in H_k$, $\forall k \geq 1$, depending on c_1 such that
– $(\phi_{0\varepsilon}^{c_1}, n_{0\varepsilon}^{c_1}, n_{1\varepsilon}^{c_1}) \to (Q, -Q^2, 0)$ in H_k, for $k \geq 1$.
– $(u_{\varepsilon c_1}(t), n_{\varepsilon c_1}(t))$ blows up in finite time T_ε in H_1 where $(u_{\varepsilon c_1}(t), n_{\varepsilon c_1}(t))$ is the solution of (I_{c_0}) with initial data $(\phi_{0\varepsilon}^{c_1}, n_{0\varepsilon}^{c_1}, n_{1\varepsilon}^{c_1})$.

Let consider a given $V(x) = e^{i\theta_0}\omega_0^{1/2}Q(\omega_0^{1/2}(x - x_0))$. We fix $c_1 = c_0\omega_0$ and let

$$\begin{aligned}
\phi_{0\varepsilon}(x) &= e^{i\theta_0}\omega_0^{1/2}\phi_{0\varepsilon}^{c_1}(\omega_0^{1/2}(x - x_0)),\\
n_{0\varepsilon}(x) &= \omega_0 n_{0\varepsilon}^{c_1}(\omega_0^{1/2}(x - x_0)),\\
n_{1\varepsilon}(x) &= \omega_0^{3/2} n_{1\varepsilon}^{c_1}(\omega_0^{1/2}(x - x_0)).
\end{aligned}$$

We have by direct calculations that

$$(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (V, -|V|^2, 0)$$

in H_k, $\forall k \geq 1$ and

$$\begin{aligned}
u_\varepsilon(t,x) &= e^{i\theta_0}\omega_0^{1/2}u_{\varepsilon c_1}(\omega_0 t, \omega_0^{1/2}(x - x_0)),\\
n_\varepsilon(t,x) &= \omega_0 n_{\varepsilon c_1}(\omega_0 t, \omega_0^{1/2}(x - x_0)),
\end{aligned}$$

is solution of equation (I_{c_0}) with initial data $\left(u_\varepsilon(0), n_\varepsilon(0), \frac{\partial n_\varepsilon}{\partial t}(0)\right) = (\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$ which blows up in H_1 at $\dfrac{T_\varepsilon}{\omega_0} < +\infty$.

We now consider the case $V(x) = Q(x)$.

Step 2. Choice of the parameters $\theta_\varepsilon, \omega_\varepsilon, T_\varepsilon$ in formula (B.5)–(B.7).

We consider solutions $(u_\varepsilon, n_\varepsilon)$ of (I_{c_0}) of the form (B.1)–(B.2) with $\omega_\varepsilon = \dfrac{1}{c_0\varepsilon}$ and $(P_\varepsilon, N_\varepsilon)$ solution of $(\mathrm{II}_\varepsilon)$ such that $(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2)$ in $H^1 \times L^2$ as $\varepsilon \to 0$. We have to choose $T_\varepsilon, \theta_\varepsilon$ such that the initial data of $(u_\varepsilon, n_\varepsilon)$:

$$\begin{aligned}
\phi_{0\varepsilon}(x) &= \frac{\omega_\varepsilon}{T_\varepsilon} e^{i\left(\theta_\varepsilon + \frac{|x|^2}{-4T_\varepsilon} + \frac{\omega_\varepsilon^2}{T_\varepsilon}\right)} P_\varepsilon\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right),\\
n_{0\varepsilon}(x) &= \left(\frac{\omega_\varepsilon}{T_\varepsilon}\right)^2 N_\varepsilon\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right),\\
n_{1\varepsilon}(x) &= \frac{1}{T_\varepsilon}\left(\frac{\omega_\varepsilon}{T_\varepsilon}\right)^2\left[\left(\frac{\omega_\varepsilon}{T_\varepsilon}\right)|x|N_\varepsilon'\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right) + 2N_\varepsilon\left(\frac{\omega_\varepsilon x}{T_\varepsilon}\right)\right],
\end{aligned}$$

converges in H_k as $\varepsilon \to 0$ to the initial data $(Q, -Q^2, 0)$ of the periodic solution. Let

$$T_\varepsilon = \omega_\varepsilon = \frac{1}{c_0\varepsilon} \quad\text{and}\quad \theta_\varepsilon = \frac{-\omega_\varepsilon^2}{T_\varepsilon} = \frac{-1}{c_0\varepsilon}.$$

We have

$$\phi_{0\varepsilon}(x) = \frac{\omega_\varepsilon}{(T_\varepsilon)} e^{i\frac{|x|^2}{4}c_0\varepsilon} P_\varepsilon(x)\,, \tag{B.8}$$

$$n_{0\varepsilon}(x) = N_\varepsilon(x)\,, \tag{B.9}$$

$$n_{1\varepsilon}(x) = c_0\varepsilon(|x|N'_\varepsilon(x) + 2N_\varepsilon(x))\,, \tag{B.10}$$

and the associated solution of (I_{c_0})

$$u_\varepsilon(t,x) = \left(\frac{1}{1-c_0\varepsilon t}\right) e^{i\left(\frac{|x|^2 c_0\varepsilon}{4(c_0\varepsilon t-1)} + \frac{t}{(1-c_0\varepsilon t)}\right)} P_\varepsilon\left(\frac{x}{1-c_0\varepsilon t}\right), \tag{B.11}$$

$$n_\varepsilon(t,x) = \left(\frac{1}{1-c_0\varepsilon t}\right)^2 N_\varepsilon\left(\frac{x}{1-c_0\varepsilon t}\right). \tag{B.12}$$

We know from Part I that $(P_\varepsilon, N_\varepsilon)$ converges in $H^1 \times L^2$ to $(Q, -Q^2)$ as $\varepsilon \to 0$. Therefore, $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$ converges to $(Q, -Q^2, 0)$ in distribution sense. From the fact that $(P_\varepsilon, N_\varepsilon)$ satisfies equation $(\mathrm{II}_\varepsilon)$ we are able to prove a more accurate convergence of $(\phi_{0,\varepsilon}, n_{0,\varepsilon}, n_{1,\varepsilon})$. Indeed we show in the following steps that

$$(\phi_{0,\varepsilon}, n_{0,\varepsilon}, n_{1,\varepsilon}) \to (Q, -Q^2, 0) \quad \text{in} \quad H^k,\ \forall k \geq 1\,.$$

This allows us to conclude the proof of Theorem 3.

Step 3. Uniform convergence $(P_\varepsilon, N_\varepsilon)$ to $(Q, -Q^2)$ on compact set of $\mathbb{R}^2$.

Let us prove some uniform estimates in ε on $(P_\varepsilon, N_\varepsilon)$ in $H^k(B_A)$, where $B_A = \{x \in \mathbb{R}^2,\ |x| < A\}$. We then conclude by compactness arguments that $(P_\varepsilon, N_\varepsilon)$ converges to $(Q, -Q^2)$ in $H^k(B_A)$ for all $A > 0$, $k \geq 1$.

Proposition B.1. i) *For $A > 0$ and $k \geq 1$, there is a $c_{k,A}$ and ε_A such that for $0 < \varepsilon \leq \varepsilon_A$,*

$$|P_\varepsilon|_{H^k(B_A)} + |N_\varepsilon|_{H^k(B_A)} \leq c_{k,A}\,. \tag{B.13}$$

ii) *$\forall A > 0$ and $k \geq 1$, we have that $(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2)$ in $H^k(B_A)$.*

Proof. i) Let us fix $A > 0$ and prove the result by recurrence on k. We know from Part I (see Theorem 4.2 and Corollary 4.3):

– There exist $\varepsilon_0 > 0$ and a constant c such that for $0 < \varepsilon < \varepsilon_0$ the solution of $(\mathrm{II}_\varepsilon)$ constructed in Part I is such that

$$|P_\varepsilon|_{H^2} + |N_\varepsilon| L^2 \leq c\,, \tag{B.14}$$

– $(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2)$ in $H^1 \times L^2$ as $\varepsilon \to 0$.

Define $\varepsilon_A = \min\left(\varepsilon_0, \frac{1}{2A}\right)$. From (B.14) and Lemma 4.8 in Part I, we have that (B.13) is true for $k = 1$,

$$|P_\varepsilon|_{H^1(B_A)} + |N_\varepsilon|_{H^1(B_A)} \leq c \quad \text{and} \quad |P_\varepsilon|_{L^\infty} + |N_\varepsilon|_{L^\infty} \leq c\,. \tag{B.15}$$

By recurrence, assuming (B.13) for $k \geq 1$, let us prove the property for $k+1$. We estimate $|P_\varepsilon|_{H^{k+1}(B_A)}$ using the elliptic regularity theory. We have

$$\Delta P_\varepsilon = P_\varepsilon(N_\varepsilon + 1) \quad \text{in} \quad B_A\,,$$
$$P_\varepsilon = P_\varepsilon(A) \quad \text{on} \quad \partial B_A\,.$$

We then deduce

$$|P_\varepsilon|_{H^{k+1}(B_A)} \le c(|P_\varepsilon N_\varepsilon|_{H^{k-1}(B_A)} + |P_\varepsilon|_{H^{k-1}(B_A)})$$
$$\le c(|P_\varepsilon|_{H^{k-1}(B_A)}|N_\varepsilon|_{H^{k-1}(B_A)} + |P_\varepsilon|_{H^{k-1}(B_A)}) \le c\,,$$

since (B.15).

We now estimate $|N_\varepsilon|_{H^{k-1}(B_A)}$ from the integral formula (2.4) given in Part I:

$$N_\varepsilon(r) = \frac{1}{(\varepsilon^2 r^2 - 1)^{3/2}} \int_{1/\varepsilon}^{r} 2P_\varepsilon(s)P_\varepsilon'(s)\,(\varepsilon^2 s^2 - 1)^{1/2}\,ds\,. \tag{B.16}$$

Thus the Leibnitz formula gives $\forall r \in [0, A]$,

$$|N_\varepsilon^{(k)}(r)| \le \left|\left(\frac{1}{(1-\varepsilon^2 r^2)^{3/2}}\right)^{(k)}\right| \left|\int_r^{1/\varepsilon} P_\varepsilon(s)P_\varepsilon'(s)\,(1-\varepsilon^2 s^2)^{1/2} s\,ds\right|$$
$$+ c\sum_{i=0}^{k-1} \left|\left(\frac{1}{(1-\varepsilon^2 r^2)^{3/2}}\right)^{(k-1-i)}\right| |(P_\varepsilon(r)P_\varepsilon'(r))^{(i)}|\,. \tag{B.17}$$

We remark that for $j \ge 0$ and $\alpha > 0$,

$$\frac{d^j}{dr^j}\left(\frac{1}{1-\varepsilon^2 r^2}\right)^\alpha = \varepsilon^j \frac{d^j}{dy^j}\left(\frac{1}{1-y^2}\right)^\alpha,$$

where $y = \varepsilon x$ with $|y| \le A\varepsilon_A \le \frac{1}{2}$ and

$$\left|\frac{d^j}{dr^j}\left(\frac{1}{1-\varepsilon^2 r^2}\right)^\alpha\right|_{L^\infty(B_A)} \le \varepsilon^j \left|\frac{d^j}{dy^j}\left(\frac{1}{1-y^2}\right)^\alpha\right|_{L^\infty(B_{1/2})} \le c. \tag{B.18}$$

From (B.17)–(B.18) and again Leibniz formula,

$$|N_\varepsilon^{(k)}(r)| \le c\left(1 + \sum_{i=0}^{k-1} |(P_\varepsilon(r)P_\varepsilon'(r))^{(i)}|\right) \le c\left(1 + \sum_{p+q\le k} |P_\varepsilon^{(p)}|\,|P_\varepsilon^{(q)}|\right).$$

Therefore

$$|N_\varepsilon^{(k)}|_{L^2(B_A)} \le c\left(1 + \sum_{p+q\le k} |P_\varepsilon^{(p)}|_{L^2(B_A)}|P_\varepsilon^{(q)}|_{L^2(B_A)}\right) \le c\,.$$

This concludes the recurrence and the proof of Part i).

ii) Let $A > 0$ and $k \ge 1$. Let us prove by compactness arguments

$$(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2) \quad \text{in} \quad H^k(B_A) \times H^k(B_A)\,.$$

We already know that

$$(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2) \quad \text{in} \quad L^2(B_A) \times L^2(B_A)\,.$$

From Part i), there is a $c > 0$ such that for $0 < \varepsilon \le \varepsilon_A$,

$$|P_\varepsilon|_{H^{k+1}(B_A)} + |N_\varepsilon|_{H^{k+1}(B_A)} \le c\,.$$

Therefore by compactness arguments for each sequence $\varepsilon_n \to 0$ as $n \to +\infty$ there is a subsequence (denoted ε_n) such that

$$(P_n, N_n) = (P_{\varepsilon_n}, N_{\varepsilon_n}) \to (\hat{P}, \hat{N}) \quad \text{in} \quad H^k(B_A) \times H^k(B_A) \quad \text{as} \quad n \to +\infty .$$

Thus $(P_n, N_n) \to (\hat{P}, \hat{N})$ in $L^2(B_A) \times L^2(B_A)$. From the uniqueness of the limit, we have $(\hat{P}, \hat{N}) = (Q, -Q^2)$. We conclude that

$$(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2) \quad \text{in} \quad H^k(B_A) \times H^k(B_A),$$

and the proof of Proposition B.1 follows.

Step 4. Uniform estimates of $(P_\varepsilon, N_\varepsilon)$ at infinity in $\mathbb{R}^2$ as $\varepsilon \to 0$.

In Part I, we obtain some estimates on $(P_\varepsilon, N_\varepsilon)$ at infinity for a fixed $\varepsilon > 0$. We prove in this step these estimates uniformly for ε small.

Proposition B.2. *There exist constants $\delta > 0$, $\varepsilon_1 > 0$ and c_k for each $k \leq 1$ such that* $\forall 0 < \varepsilon < \varepsilon_1$, $\forall k \geq 1$, $\forall r > 0$,

$$|P_\varepsilon^{(k)}(r)| \leq c_k e^{-\delta r},$$
$$|N_\varepsilon^{(k)}(r)| \leq \frac{c_k}{1 + r^{k+3}} .$$

Proof. We prove in fact by recurrence on k the property:

$$(\mathscr{P}_k) \qquad \begin{aligned} &|P_\varepsilon^{(l)}(r)| \leq c_k e^{-\delta r} \qquad \text{for} \quad 0 \leq l \leq k+2, \\ &|N_\varepsilon^{(l)}(r)| \leq \frac{c_k}{1 + r^{l+3}} \quad \text{for} \quad 0 \leq l \leq k . \end{aligned}$$

a) We prove $(\mathscr{P}_0)$. We begin by estimates on P_ε.

Lemma B.3. *There exist constants $\delta > 0$, $\varepsilon_1 > 0$ and c such that for* $0 < \varepsilon \leq \varepsilon_1$, $\forall r \geq 0$,

i) $|P_\varepsilon(r)| \leq ce^{-\delta r}$.

ii) $|P_\varepsilon'(r)| + |P_\varepsilon''(r)| \leq ce^{-\delta r}$.

Proof. i) We need a crucial estimates on N_ε proved in Part I (see Proposition 4.12): There exists constants $\varepsilon_2 > 0$ and $A > 0$ such that for $0 < \varepsilon \leq \varepsilon_2$,

$$|N_\varepsilon|_{L^\infty(\{|x|>A\})} \leq \tfrac{1}{2} .$$

From Proposition B.1, there exist constants $\varepsilon_1 > 0$, c such that for $0 < \varepsilon \leq \varepsilon_1$,

$$|P_\varepsilon|_{L^\infty(B_A)} \leq c . \tag{B.19}$$

Therefore we only have to estimate $P_\varepsilon(r)$ for $r > A$.

We consider the elliptic problem on $(A, +\infty)$,

$$P_\varepsilon'' + \frac{1}{r} P_\varepsilon' = (N_\varepsilon + 1) P_\varepsilon .$$

We have for $0 < \varepsilon \leq \varepsilon_1$ and $r \geq A$,

$$|P_\varepsilon(A)| \leq c , \; P_\varepsilon(+\infty) = 0 \quad \text{and} \quad \tfrac{1}{2} \leq (N_\varepsilon(r) + 1) \leq \tfrac{3}{2} .$$

Thus by usual techniques of maximum principle, there exists a constant c and $\delta > 0$ which does not depend on ε such that for $0 < \varepsilon \leq \varepsilon_1$,

$$\forall r \in [A, +\infty), \qquad |P_\varepsilon(r)| \leq ce^{-\delta r} \tag{B.20}$$

and Part i) follows.
ii) Let us prove the same estimate for P'_ε and P''_ε.
Writing $(rP'_\varepsilon(r))' = r(N_\varepsilon(r)+1)P_\varepsilon(r)$ and integrating on $(r, +\infty)$, we obtain (by decay of P'_ε for a fixed ε proved in Part I)

$$rP'_\varepsilon(r) = -\int_r^{+\infty} (N_\varepsilon(s)+1)P_\varepsilon(s)s\,ds\,. \tag{B.21}$$

It follows from (B.20) and (B.21) that

$$\forall r \geq A, \qquad |rP'_\varepsilon(r)| \leq cre^{-\delta r}\,. \tag{B.22}$$

We conclude from (B.22)

$$\forall 0 < \varepsilon \leq \varepsilon_2, \forall r \geq 0, \qquad |P'_\varepsilon(r)| \leq ce^{-\delta r}\,.$$

The estimate on P''_ε follows.
– on one hand, from the uniform bound of P''_ε on $[0, A]$,
– on the other hand, from the relation on $[A, +\infty)$,

$$P''_\varepsilon = \frac{-P'_\varepsilon}{r} + (N_\varepsilon + 1)P_\varepsilon$$

and estimates on P_ε and P'_ε. This concludes the proof of Lemma B.3.
We now estimate N_ε.

Lemma B.4. *There is $c > 0$ such that for $\varepsilon > 0$ small,*

$$|N_\varepsilon(r)| \leq \frac{c}{1+r^3}\,.$$

Proof. We use for this estimate the integral formula (B.16) of N_ε,

$$N_\varepsilon(r) = \frac{1}{(\varepsilon^2 r^2 - 1)^{3/2}} K_\varepsilon(r)\,,$$

where

$$K_\varepsilon(r) = \int_{1/\varepsilon}^{r} P_\varepsilon(s)P'_\varepsilon(s)\,(\varepsilon^2 s^2 - 1)^{1/2}\,ds\,. \tag{B.23}$$

We remark that $\forall r$,

$$\begin{aligned} |K_\varepsilon(r)| &\leq c\int_{1/\varepsilon}^{r} e^{-2\delta s}(\varepsilon^2 s^2 - 1)^{1/2}\,ds \leq c\int_{1/\varepsilon}^{r} e^{-2\delta s}(\varepsilon s + 1)\,ds \\ &\leq c\int_{1/\varepsilon}^{r} e^{-\delta s}\,ds \leq c(e^{\frac{-\delta}{\varepsilon}} + \varepsilon^{-\delta r})\,. \end{aligned} \tag{B.24}$$

Therefore

– for $0 \le r \le \dfrac{1}{2\varepsilon}$,

$$|N_\varepsilon(r)| \le c|K_\varepsilon(r)| \le c(e^{\frac{-\delta}{\varepsilon}} + e^{-\delta r}) \le \frac{c}{1+r^3}\,.$$

– for $r \ge \dfrac{2}{\varepsilon}$,

$$|N_\varepsilon(r)| \le c\,\frac{|K_\varepsilon(r)|}{(\varepsilon^2 r^2-1)^{3/2}} \le c\,\frac{e^{-\delta r}}{(\varepsilon^2 r^2)^{3/2}} \le \frac{c}{1+r^3}\,.$$

– for $\dfrac{1}{2\varepsilon} \le r \le \dfrac{2}{\varepsilon}$ and $\varepsilon > 0$ small

$$\begin{aligned}
|N_\varepsilon(r)| &\le \frac{1}{(\varepsilon^2 r^2-1)^{3/2}} \int_{1/\varepsilon}^{r} e^{-2\delta s}(\varepsilon^2 s^2-1)^{1/2}\,ds \\
&\le \frac{e^{\frac{-\delta}{\varepsilon}}}{(\varepsilon r-1)^{3/2}(1+\varepsilon r)^{3/2}} \int_{1/\varepsilon}^{r} (\varepsilon s-1)^{1/2}\,ds \\
&\le c\,\frac{e^{\frac{-\delta}{\varepsilon}}}{\varepsilon}\,\frac{(\varepsilon r-1)^{3/2}}{(\varepsilon r-1)^{3/2}} \le c e^{\frac{-\delta}{2\varepsilon}} \le \frac{c}{1+\left(\dfrac{1}{2\varepsilon}\right)^3} \le \frac{c}{1+r^3}\,.
\end{aligned}$$

This concludes the proof of the lemma.

b) Property $(\mathscr{P}_k)$ implies property $(\mathscr{P}_{k+1})$.

We first prove the estimates on N_ε and then on P_ε.

Lemma B.5. *There is a constant c_k such that*

$$\forall 0 < \varepsilon \le \varepsilon_1,\ \forall r\,, \qquad |N_\varepsilon^{(k+1)}(r)| \le \frac{c_k}{1+r^{k+4}}\,.$$

Proof. i) Estimates for $0 \le r \le \dfrac{1}{2\varepsilon}$ and $r \ge \dfrac{2}{\varepsilon}$.

Writing again $N_\varepsilon(r) = b_\varepsilon(r)K_\varepsilon(r)$, where

$$\begin{aligned}
K_\varepsilon(r) &= \int_{1/\varepsilon}^{r} P_\varepsilon(s)P_\varepsilon'(s)\,(\varepsilon^2 r^2-1)^{1/2}\,ds\,, \\
b_\varepsilon(r) &= \frac{1}{(\varepsilon^2 r^2-1)^{3/2}}\,.
\end{aligned} \tag{B.25}$$

Leibniz formula yields

$$\begin{aligned}
|N_\varepsilon^{(k+1)}(r)| \le c\sum_{i=0}^{k} |b_\varepsilon^{(i)}(r)|\,|((1-\varepsilon^2 r^2)^{1/2} P_\varepsilon(r)P_\varepsilon'(r))^{(k-i)}| \\
+ |b_\varepsilon^{(k+1)}(r)|\,|K_\varepsilon(r)|\,.
\end{aligned} \tag{B.26}$$

– From (B.24) and direct calculations, we have for $0 \le r \le \dfrac{1}{2\varepsilon}$ or $r \ge \dfrac{2}{\varepsilon}$,

$$|b_\varepsilon^{(i)}(r)| \le c\,, \tag{B.27}$$

$$|K_\varepsilon(r)| \le c(e^{\frac{-\delta}{\varepsilon}} + e^{-\delta r})\,. \tag{B.28}$$

– From estimates on $P_\varepsilon, \dots, P_\varepsilon^{(k+1)}$, and again Leibniz formula we have $\forall i = 0, \dots, k$,

$$|((1-\varepsilon^2 r^2)^{1/2} P_\varepsilon(r) P_\varepsilon'(r))^{(k-i)}| \le c e^{-\delta r}\,. \tag{B.29}$$

– We claim that

$$\text{for} \quad |r\varepsilon - 1| \ge \frac{1}{2}\,, \qquad |b_\varepsilon^{(k+1)}(r)| \le \frac{c}{(1+r^{k+4})\varepsilon^3}\,. \tag{B.30}$$

Indeed,

$$b_\varepsilon^{(k+1)} = \varepsilon^{k+1} \frac{d^{k+1} b(y)}{dy^{k+1}}\,,$$

where $b(y) = \dfrac{1}{(y^2-1)^{3/2}}$ and $y = \varepsilon x$. From direct computations, for $|y-1| \ge \frac{1}{2}$,

$$\left| \frac{d^{k+1} b(y)}{dy^{k+1}} \right| \le \frac{1}{1+|y|^{k+4}}\,,$$

and therefore

$$|b_\varepsilon^{(k+1)}(r)| \le \frac{c\varepsilon^{k+1}}{(1+|\varepsilon r|^{k+4})} \le \frac{c}{(1+r)^{k+4}\varepsilon^3}$$

which proves (B.30).

We then deduce from (B.27)–(B.30) that for $|\varepsilon r - 1| \ge \frac{1}{2}$,

$$|N_\varepsilon^{(k+1)}(r)| \le \frac{c}{1+r^{k+4}} \left(1 + \frac{e^{\frac{-\delta}{\varepsilon}}}{\varepsilon^3} \right) \le \frac{c}{1+r^{k+4}}\,.$$

ii) Estimates for $r \in \left[\dfrac{1}{2\varepsilon}, \dfrac{2}{\varepsilon} \right]$.

We write $N_\varepsilon(r) = a_\varepsilon(r) Y_\varepsilon(r)$ where

$$a_\varepsilon(r) = \frac{1}{\varepsilon(\varepsilon+1)^{3/2}}\,,$$

$$Y_\varepsilon(r) = \frac{1}{\left(r - \frac{1}{\varepsilon}\right)^{3/2}} \int_{1/\varepsilon}^{r} Z_\varepsilon(s) \left(s - \frac{1}{\varepsilon}\right)^{1/2} ds\,,$$

$$Z_\varepsilon(r) = 2 P_\varepsilon(r) P_\varepsilon'(r)\,(\varepsilon r + 1)^{1/2}\,.$$

Leibniz formula yields

$$|N_\varepsilon^{(k+1)}(r)| \le c \Big(\sum_{p+q=k+1} |a_\varepsilon^{(p)}(r)|\, |Y_\varepsilon^{(q)}(r)| \Big)\,.$$

On one hand

$$|a_\varepsilon^{(p)}(r)| = \frac{c_p \varepsilon^p}{\varepsilon(\varepsilon r + 1)^{\frac{3}{2}+p}} . \tag{B.31}$$

On the other hand, Lemma 2.9 in Part I and estimates for $r \in \left[\frac{1}{2\varepsilon}, \frac{2}{\varepsilon}\right]$,

$$|P_\varepsilon(r)| + \ldots + |P_\varepsilon^{(k+2)}(r)| \le c e^{\frac{-\delta}{2\varepsilon}} ,$$

we deduce

$$|Y_\varepsilon^{(q)}(r)| \le c |Z_\varepsilon|_{W^{q,\infty}\left(\frac{1}{2\varepsilon}, \frac{2}{\varepsilon}\right)} \le c e^{\frac{-\delta}{2\varepsilon}} . \tag{B.32}$$

From (B.31) and (B.32), $\forall r \in \left[\frac{1}{2\varepsilon}, \frac{2}{\varepsilon}\right]$,

$$|N_\varepsilon^{(k+1)}(r)| \le c \sum_{0 \le p \le k+1} \varepsilon^{p-1} e^{\frac{-\delta}{2\varepsilon}} \le \frac{c}{1 + r^{k+4}} ,$$

which concludes the proof of the lemma.

We now estimate $P_\varepsilon^{(k+3)}$.

Lemma B.6. *There exists c such that for all $0 < \varepsilon \le \varepsilon_1$,*

$$|P_\varepsilon^{(k+3)}(r)| \le c e^{-\delta r} .$$

Proof. From Proposition B.1, there exists a constant c such that for $0 < \varepsilon \le \varepsilon_1$,

$$|P_\varepsilon^{(k+3)}|_{L^\infty(B_1)} \le c .$$

Let us estimate $P_\varepsilon^{(k+3)}(r)$ for $r \ge 1$. We derive $(k+1)$ the following relation:

$$P_\varepsilon'' = P_\varepsilon - \frac{1}{r} P_\varepsilon' + N_\varepsilon P_\varepsilon ,$$

and we obtain

$$|P_\varepsilon^{(k+3)}(r)| \le |P_\varepsilon^{(k+1)}(r)| + c \sum_{p+q=k+1} \left| \left(\frac{1}{r}\right)^{(p)} P_\varepsilon^{(q)}(r) \right| + c \sum_{p+q=k+1} |N_\varepsilon^{(p)}(r) P_\varepsilon^{(q)}(r)| .$$

From the estimates on $P_\varepsilon, \ldots, P_\varepsilon^{(k+2)}$ and $N_\varepsilon, \ldots, N_\varepsilon^{(k+1)}$ we deduce

$$\forall r \ge 1 , \quad |P_\varepsilon^{(k+3)}(r)| \le c e^{-\delta r} .$$

Thus the lemma and property $(\mathscr{P}_{k+1})$ is proved.

This concludes the proof of Proposition B.2.

Step 5. We are now able to prove $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (Q, -Q^2, 0)$ in H^k, $\forall k \ge 1$, which with Step 2 concludes the proof of Theorem 3.

Proposition B.7. *Let $p, q \ge 0$. We then have as $\varepsilon \to 0$:*

i) $x^p P_\varepsilon^{(q)}(x) \to x^p Q^{(q)}(x)$ *in* L^2,

ii) $N_\varepsilon^{(q)}(x) \to -(Q^2)^{(q)}(x)$ *in* L^2,

iii) $x N_\varepsilon^{(q)}(x) \to -x(Q^2)^{(q)}(x)$ *in* L^2.

Proof. i) Let $p, q \geq 0$.

On the one hand, from Proposition B.2 there is a constant c such that

$$\forall 0 < \varepsilon \leq \varepsilon_1, \forall x \in \mathbb{R}^2, \qquad |x^p P_\varepsilon^{(q)}(x)| \leq c|x|^p e^{-\delta|x|}$$

which belong to L^2.

On the other hand from step 3, $P_\varepsilon \to Q$ in H^k_{loc} for $k \geq 0$. Therefore $P_\varepsilon^{(q)}(x) \to Q^{(q)}(x)$ on compact sets and

$$\forall x, \quad P_\varepsilon^{(q)}(x) \to Q^{(q)}(x).$$

The convergence dominated theorem allows us to conclude to the proof.

Proofs of Parts ii) and iii) are similar.

Let us now conclude the proof of Theorem 3.

Proof of Theorem 3. We recall that

$$\phi_{0\varepsilon}(x) = \frac{\omega_\varepsilon}{T_\varepsilon} e^{i\frac{|x|^2}{4}c_0\varepsilon} P_\varepsilon(x),$$

$$n_{0\varepsilon}(x) = N_\varepsilon(x), \qquad n_{1\varepsilon}(x) = c_0\varepsilon(|x|N'_\varepsilon(x) + 2N_\varepsilon(x)),$$

and the proof of Theorem 3 is reduced to the proof of the convergence of $(\phi_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$ to $(Q, -Q^2, 0)$ in H^k for $k \geq 1$, that is

i) $\phi_{0\varepsilon} \to 0$ in H^k,

ii) $n_{0\varepsilon} \to -Q^2$ in H^{k-1},

iii) $n_{1\varepsilon} \to 0$ in H^{k-2} if $k \geq 2$, in $\hat{H}^{-1}$ if $k = 1$.

i) Let us prove that $\phi_{0\varepsilon} \to Q$ in H^k for all $k \geq 0$: $\forall k \geq 0$, $\phi_{0\varepsilon}^{(k)} \to Q^{(k)}$ in L^2. By Leibniz formula and from Proposition B.2,

$$|\phi_{0\varepsilon}^{(k)} - e^{\frac{-i|x|^2 c_0\varepsilon}{4}} P_\varepsilon^{(k)}|_{L^2} \leq c \sum_{p+q=k} |\varepsilon^p |x|^p P_\varepsilon^{(q)}|_{L^2} \leq c\varepsilon.$$

Furthermore from Proposition B.7 and the dominated convergence theorem,

$$e^{\frac{-i|x|^2 c_0\varepsilon}{4}} P_\varepsilon^{(k)} \to Q^{(k)} \quad \text{in} \quad L^2 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore $\phi_{0\varepsilon}^{(k)} \to Q^{(k)}$ in L^2 and this concludes the proof of Part i).

ii) $n_{0\varepsilon} = N_\varepsilon \to -Q^2$ in H^k by Proposition B.7.

iii) Case $k = 1$. We have by definition

$$|n_{1\varepsilon}(x)|_{\hat{H}^{-1}} = |v_\varepsilon|_{L^2},$$

where $\nabla \cdot v_\varepsilon = n_{1\varepsilon}$. By direct computations (see Part I),

$$v_\varepsilon(x) = \varepsilon x N_\varepsilon(x).$$

Therefore $|n_{1\varepsilon}(x)|_{\hat{H}^{-1}} = \varepsilon|xN_\varepsilon|_{L^2} \leq c\varepsilon \to 0$ as $\varepsilon \to 0$.

Case $k \geq 2$. $|n_{1\varepsilon}(x)|_{H^{k-2}} \leq c_0\varepsilon| \, |x|N'_\varepsilon(x) + 2N_\varepsilon(x)|_{H^{k-2}} \leq c\varepsilon$ by Proposition B.7, that is $|n_{1\varepsilon}(x)|_{H^{k-2}} \to 0$ as $\varepsilon \to 0$.

B.2 Case of Multiple Periodic Solutions. We consider as in Theorem 3 a real radial solution of

$$(\mathrm{V}_\omega) \qquad \omega V = \Delta V + |V|^2 V \quad \text{in} \quad \mathbb{R}^2$$

and $(u(t), n(t)) = (e^{i\omega t}V(x), -|V(x)|^2)$ associated periodic solution of (I_{c_0}).

We assume that V is a nondegenerated in the following sense: the operator

$$L_V : W \to \Delta W - \omega W + 3V^2 W$$

is a continuous one to one application from H_r^2 to L_r^2 with continuous inverse.

By a similar proof to the one in Part I, we can prove that, for $\varepsilon \in (0, \varepsilon_1)$ small enough, there exists a radial solution $(P_{V,\varepsilon}, N_{V,\varepsilon})$ of

$$(\mathrm{II}_\varepsilon) \qquad \Delta P_{V,\varepsilon} - P_{V,\varepsilon} = N_{V,\varepsilon} P_{V,\varepsilon}\,, \tag{B.33}$$

$$\varepsilon^2\left(r^2 \frac{\partial^2}{\partial r^2} N_{V,\varepsilon} + 6r \frac{\partial}{\partial r} N_{V,\varepsilon} + 6N_{V,\varepsilon}\right) - \Delta N_{V,\varepsilon} = \Delta |P_{V,\varepsilon}|^2\,, \tag{B.34}$$

such that $(P_{V,\varepsilon}, N_{V,\varepsilon}) \to (V, -V^2)$ in $H^1 \times L^2$ as $\varepsilon \to 0$.

In addition,

$$u_\varepsilon(t,x) = \frac{\omega_\varepsilon}{T-t} e^{i\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{\omega_\omega^2}{(-T+t)}\right)} P_{V,\varepsilon}\left(\frac{x\omega_\varepsilon}{T-t}\right),$$

$$n_\varepsilon(t,x) = \left(\frac{\omega_\varepsilon}{T-t}\right)^2 N_{V,\varepsilon}\left(\frac{x\omega_\varepsilon}{T-t}\right),$$

where $\omega_\varepsilon = \dfrac{1}{c_0 \varepsilon}$, $T > 0$, and $\theta \in S^1$, is a blowing up solution in H_1 of equation (I_{c_0}).

Indeed, we only use in Part I

– exponential decay at infinity of V (which is still true – see Berestycki-Lions [4]),

– the nondegeneracy condition,

to be able to prove that the operator

$$T_{V,\varepsilon}(h) = L_V^{-1}((V+h)\mathcal{N}_\varepsilon((V+h)^2) + V^3 + 3V^2 h)$$

has a unique fixed point $h_{V,\varepsilon}$ in a neighbour of 0 in $H_r^2 = H_r^1 \cap H^2$ for $\varepsilon > 0$ small enough.

We remark then $(P_{V,\varepsilon}, N_{V,\varepsilon}) = (V + h_{V,\varepsilon}, \mathcal{N}_\varepsilon((V + h_{V,\varepsilon})^2))$ is a solution of (II_λ). Moreover we have $P_{V,\varepsilon} \to V$ in H^2 as $\varepsilon \to 0$ and there exists constants $\varepsilon_2 > 0$ and $A > 0$ such that for $0 < \varepsilon \le \varepsilon_2$,

$$|N_{V,\varepsilon}|_{L^\infty(\{|x| > A\})} \le \tfrac{1}{2}\,.$$

We now apply the same procedure as the one of B.1 to prove the instability of the periodic solution $(e^{i\omega t}V(x), -|V(x)|^2, 0)$. As in Sect. B.1, we prove that the initial data

$$\phi_{0\varepsilon}(x) = \frac{\omega_\varepsilon}{T_\varepsilon} e^{i\frac{|x|^2}{4} c_0 \varepsilon} P_{V,\varepsilon}(x)\,,$$

$$n_{0,\varepsilon}(x) = N_{V,\varepsilon}(x)\,,$$

$$n_{1\varepsilon}(x) = c_0\varepsilon(|x| N_{V,\varepsilon}'(x) + 2N_{V,\varepsilon}(x))\,,$$

of the associated blowing up solution

$$u_\varepsilon(t,x) = \left(\frac{1}{1-c_0\varepsilon t}\right) e^{i\left(\frac{|x|^2 c_0\varepsilon}{4(c_0\varepsilon t-1)}+\frac{t}{(1-c_0\varepsilon t)}\right)} P_{V,\varepsilon}\left(\frac{x}{1-c_0\varepsilon t}\right),$$

$$n_\varepsilon(t,x) = \left(\frac{1}{1-c_0\varepsilon t}\right)^2 N_{N,\varepsilon}\left(\frac{x}{1-c_0\varepsilon t}\right),$$

converges to $(V, -|V|^2, 0)$ in H_k, $\forall k \geq 1$ as $\varepsilon \to 0$.

This concludes the proof of Theorem 4.

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