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On the Chern Character of θ Summable Fredholm Modules

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Abstract. We show that the entire cyclic cohomology class given by the Jaffe-Lesniewski-Osterwalder formula is the same as the class we had constructed earlier as the Chern character of θ-summable Fredholm modules.

1. Introduction

Cyclic cohomology replaces de Rham homology in the set up of non-commutative differential geometry ([1, 2]). In particular it is a natural receptacle for the Chern character in K-homology ([1]) so that to each K homology cycle of finite dimension, on an algebra A, there corresponds a stable cyclic cohomology class. This class reduces to the index class ([1, 2]) for the K-homology cycle associated to an elliptic differential operator on a manifold M, (where $A = C^\infty(M)$ is the algebra of smooth functions on M). One of the distinctive features of cyclic cohomology is that it fits naturally not only with the non-commutative case but also with the infinite dimensional situation. Indeed, stable (or periodised) cyclic cohomology is the cohomology of cochains with finite support in the (b, B) bicomplex of the algebra A ([1]) and by imposing a suitable growth condition on cochains with infinite support, we introduced in [3] the cohomology of A, which is relevant for the infinite dimensional situation.

In particular it allows to extend the Chern character in K-homology to K-homology cycles $(\mathscr{h}, D)$ on the algebra A (cf. [3]), where the operator D is no longer **finitely summable** (i.e. $\mathrm{Tr}(D^{-p}) < \infty$ for some $p < \infty$) but is only θ-**summable**: $\mathrm{Tr}(e^{-\beta D^2}) < \infty$. Our original construction ([3]) of this Chern character was based on the correspondence between cocycles with infinite support and traces on the algebras QA, εA of Cuntz and Zekri [5, 9]. The algebra εA is an essential ideal in the free product $A * \mathbb{C}(\mathbb{Z}/2)$ of A by the group ring of the group $\mathbb{Z}/2\mathbb{Z}$. The growth condition of **entire** cocycles corresponds to the **vanishing** of the spectral radius of all elements of εA for the trace given by the cocycle. Thus any homomorphism

$\pi: \varepsilon A \to B$ from εA to a quasinilpotent algebra B with a trace τ, gives rise to an entire cocycle φ on A, by the formula: ([3])

$$\varphi_{2n}(a^0, \dots, a^{2n}) = \lambda_n \tau \circ \pi(F, a^0[F, a^1] \dots [F, a^{2n}])$$

(where $a^i \in A$, F is the canonical generator of $\mathbb{C}(\mathbb{Z}/2)$, $F^2 = 1$, and λ_n is a numerical normalisation, $\lambda_n = 2^{-2n}(n!)^{-1}$ (we use the (b, B) bicomplex)).

In the original construction ([3]) we took, for the quasi-nilpotent algebra B, an extension $\tilde{\mathscr{L}}$ of the algebra $\mathscr{L}$ of convolution of operator valued distributions on the interval $[0, +\infty[\subset \mathbb{R}$. Elements T of $\mathscr{L}$ are distributions with value operators in the Hilbert space $\mathscr{h}$ and are assumed to be holomorphic in the parameter $s > 0$ and such that $T(s)$ is an operator of trace class for $s > 0$. The algebra $B = \tilde{\mathscr{L}}$ is obtained by formally adjoining to $\mathscr{L}$ a square root of the distribution δ'_0, the derivative of the dirac mass at the origin (times the identity operator in $\mathscr{h}$). The trace τ was essentially $T \to \mathrm{Trace}(T(1))$, the usual trace of the operator $T(1)$.

Our first point in this paper will be to clarify the nature of this algebra $\tilde{\mathscr{L}}$, using the Hopf algebra of the supergroup $\mathbb{R}^{(1,1)}$.

Our second point will be to show that the later formula [6] of Jaffe, Lesniewski, and Osterwalder (in the context of "Quantum algebras") gives in fact the same cohomology class:

$$Ch(\mathscr{h}, D) \in HC_\varepsilon(A)$$

as our previous formula.

The main advantage of the J.L.O. formula is that it is simpler than ours, and has a clear conceptual meaning in the algebra of cochains introduced by Quillen ([8]). The advantage of our formula is that it yields a normalized cocycle so that the algebraic machinery of εA, QA and traces is available. It is thus relevant that the two formulae in fact are cohomologous.

2. The Algebra $\tilde{\mathscr{L}}$ and the Supergroup $\mathbb{R}^{(1,1)}$

In this section we shall relate the quasinilpotent algebra $\tilde{\mathscr{L}}$ used for technical reasons in [3] with the Hopf algebra of the supergroup $\mathbb{R}^{(1,1)}$.

Recall from [3] that, given an infinite dimensional Hilbert space $\mathscr{h}$, we defined an algebra $\mathscr{L}$ for the convolution product:

$$(T_1 * T_2)(s) = \int_0^s T_1(u) T_2(s-u) du,$$

and whose elements $T \in \mathscr{L}$ are distributions on $\mathbb{R}$, (with values in the Banach space $\mathscr{L}(\mathscr{h})$ of operators in $\mathscr{h}$) which satisfy the following two conditions:

(1) Support $T \subset \mathbb{R}^+ = [0, +\infty[$.

(2) There exists $r > 0$ and an analytic operator valued function $t(z)$, $z \in C = \bigcup_{s>0} sD_r$, where $D_r = \{z \in \mathbb{C}, |z-1| < r\}$, with

(a) $t(s) = T(s)$ on $]0, +\infty[$,

(b) the function $h(p) = \sup_{z \in 1/pD_r} \|t(z)\|_p$, $p \in]1, +\infty[$ is majorised by a polynomial in p for $p \to \infty$.

The condition (2) essentially means that T takes its values in operators of suitable Schatten class so that the quantity $\mathrm{Trace}\, T(1)$ is well defined.

All operator valued distributions on $\mathbb{R}$ with support $\{0\}$ belong to $\mathscr{L}$ and so do the products $\delta_0 \times \mathrm{id}$, $\delta'_0 \times \mathrm{id}$ of the Dirac mass at 0, or of its derivative, by the identity operator in $\mathscr{h}$. To lighten the notation we shall simply write δ_0, δ'_0.

The algebra $\tilde{\mathscr{L}}$ is obtained from $\mathscr{L}$ by formally adjoining a square root $\lambda^{1/2}$ of $\lambda=\delta'_0$. Thus, elements of $\tilde{\mathscr{L}}$ are given by pairs: (T_0, T_1) of elements of $\mathscr{L}$ with the product:

$$(T_0, T_1)*(S_0, S_1)=(T_0*S_0+\delta'_0*T_1*S_1, T_0*S_1+T_1*S_0), \tag{3}$$

where $*$ denotes the convolution product, which gives $\mathscr{L}$ its algebraic structure.

On the other hand, let us recall that the Hopf algebra H of smooth functions on the super group $\mathbb{R}^{(1,1)}$ is given as follows: as an algebra one has:

$$H=C^\infty(\mathbb{R}^{1,1})=C^\infty(\mathbb{R})\otimes \wedge(\mathbb{R}),$$

the tensor product of the algebra of smooth functions on $\mathbb{R}$ by the exterior algebra $\wedge(\mathbb{R})$ of a one dimensional vector space. Thus every element of H is given by a sum $f+g\xi$, where $f,g\in C^\infty(\mathbb{R})$, $\xi^2=0$. The interesting structure comes from the coproduct $\Delta: H\to H\otimes H$ which corresponds to the super group structure; being an algebra morphism it is fully specified by its value on $C^\infty(\mathbb{R})\subset H$ and by $\Delta(\xi)=\xi\otimes 1+1\otimes\xi$; one has:

$(\Delta f)=\Delta_0(f)+\Delta_0(f')\xi\otimes\xi$, where $f'=\frac{\partial}{\partial s}f(s)$ and,

$$\Delta_0: C^\infty(\mathbb{R})\to C^\infty(\mathbb{R})\otimes C^\infty(\mathbb{R})$$

is the usual coproduct,

$$\Delta_0(f)(s,t)=f(s+t). \tag{4}$$

Equivalently, the (topological) dual H^* of H is endowed with a product which we can now describe. Every element of H^* is uniquely of the form (T_0, T_1), where $T_0, T_1\in C_0^{-\infty}(\mathbb{R})$ are distributions with compact support on $\mathbb{R}$, and:

$$\langle f+g\xi, (T_0, T_1)\rangle = T_0(f)+T_1(g). \tag{5}$$

The product $*$ on H^* dual to the coproduct Δ is given by:

$$\langle (T_0, T_1)*(S_0, S_1), f+g\xi\rangle=\langle (T_0, T_1)\otimes(S_0, S_1), \Delta(f+g\xi)\rangle. \tag{6}$$

Lemma 1. *The product $*$ on H^* is given by:*

$$(T_0, T_1)*(S_0, S_1)=(T_0*S_0+\delta'_0*T_1*S_1,\ T_0*S_1+T_1*S_0).$$

Using $\xi^2=0$ this follows from formula (4). Thus we see that the algebra $\tilde{\mathscr{L}}$ is really a convolution algebra of operator valued distributions on the supergroup $\mathbb{R}^{(1,1)}$, thus clarifying the relations between our formulae ([3]) and supersymmetry.

3. The Normalised Cocycle Associated to a θ-Summable Fredholm Module

We recall in this section the construction of the Chern character of θ-summable Fredholm modules.

Let A be a unital Banach algebra over $\mathbb{C}$, the (b,B) bicomplex of cyclic cohomology ([1]) is given by the two differentials $b: C^n\to C^{n+1}$; $B: C^n\to C^{n-1}$, where $C^n = C^n(A, A^*)$ is the space of continuous $n+1$ linear forms on A and:

$$(b\varphi)(a^0, \dots, a^{n+1}) = \sum_0^n (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, \dots, a^n), \quad (7)$$

$$(B\varphi) = AB_0\varphi, \quad \text{where} \quad (B_0\varphi)(a^0, \dots, a^{n-1}) = \varphi(1, a^0, \dots, a^{n-1}) - (-1)^n, \quad (8)$$

$\varphi(a^0, \dots, a^{n-1}, 1)$ and A is the cyclic antisymmetrisation.

An even (respectively odd) cocycle is given by a sequence $\varphi = (\varphi_{2n})$ (respectively $(\varphi_{2n+1})_{n\in\mathbb{N}}$) such that:

$$b\varphi_{2n} + B\varphi_{2n+2} = 0 \quad (\text{respectively } b\varphi_{2n-1} + B\varphi_{2n+1} = 0) \quad \forall n \in \mathbb{N}. \quad (9)$$

Such a cocycle is *normalized* when for any $n \in \mathbb{N}$, the functional $B_0\varphi_{2n}$ (respectively $B_0\varphi_{2n+1}$) is already cyclic:

$$B_0\varphi_{2n} = \frac{1}{2n} AB_0\varphi_{2n} \quad \left(\text{respectively } B_0\varphi_{2n+1} = \frac{1}{2n+1} AB_0\varphi_{2n+1}\right).$$

It is called *entire* when the radius of convergence of the series $\sum n! z^n \|\varphi_{2n}\|$ is infinity (respectively of $\sum n! z^n \|\varphi_{2n+1}\|$). (We took here the (b, B) differentials instead of (d_1, d_2) of [3]). By [3] Proposition 3, normalized even cocycles on A correspond to traces on the algebra $\mathscr{E}A$, odd cocycles to traces on QA. Here QA, (cf. [5]) is the free product of A by itself, and $\mathscr{E}A$ is the free product of A by the group ring $\mathbb{C}(\mathbb{Z}/2)$ of the group with two elements; $1, F$ with $F^2 = 1$. By [9], $\mathscr{E}A$ is the crossed product algebra $QA \times_\sigma \mathbb{Z}/2$ of QA by the involution $\sigma \in \operatorname{Aut}(QA)$ which exchanges the two copies of A in the free product. Thus by duality for crossed products we see that $QA \otimes M_2(\mathbb{C})$ is the crossed product $\tilde{\mathscr{E}}A = \mathscr{E}A \times_{\hat{\sigma}} \mathbb{Z}/2$ of $\mathscr{E}A$ by the involution $\hat{\sigma}$ dual to σ.

By construction $\tilde{\mathscr{E}}A$ is generated by a subalgebra isomorphic to A, and a pair of elements F, γ such that:

$$F^2 = \gamma^2 = 1, \quad F\gamma = -\gamma F, \quad \gamma a = a\gamma \quad \forall a \in A. \quad (10)$$

Thus a homomorphism $\pi: \tilde{\mathscr{E}}A \to B$ from $\tilde{\mathscr{E}}A$ to an algebra B is given by a homomorphism from A to B and a pair of elements $F, \gamma \in B$ verifying the conditions (10). Since traces on $M_2(QA)$ correspond bijectively to traces on QA, we get:

Lemma 2. *Let B be an algebra, $\pi: A \to B$ a homomorphism and $F, \gamma \in B$ be such that $F^2 = \gamma^2 = 1$, $F\gamma = -\gamma F$ and $\gamma\pi(a) = \pi(a)\gamma$ for any $a \in A$. Then the following functionals (φ_{2n+1}) are the components of an odd cocycle on A, given any trace τ on B:*

$$\varphi_{2n+1}(a^0, \dots, a^{2n+1}) = \lambda_n \tau(\gamma F a^0 [F, a^1] \dots [F, a^{2n+1}]) \quad \forall a^i \in A,$$

where

$$\lambda_n = i \left(\frac{1}{2}\right)^{n+1} \frac{1}{(2n+1)(2n-1)\dots 3 \cdot 1}.$$

We used this lemma in [3] for the even case to associate an entire cyclic cocycle on A to any θ-summable Fredholm module over A. Thus for a change we shall here give the details in the odd case.

An odd θ-summable Fredholm module over the Banach algebra A is given by a pair of:

a) A representation ϱ of A in a Hilbert space $\mathfrak{h}$,
b) An unbounded selfadjoint operator D in $\mathfrak{h}$,

such that $[D, \varrho(a)]$ is bounded (by $C\|a\|$) for any $a \in A$ and that $e^{-\beta D^2}$ is a trace class operator for any $\beta > 0$. Let $\tilde{\mathscr{L}}$ be the algebra of operator valued distributions defined in Sect. 3 for the Hilbert space $\mathfrak{h}$.

We take for B the algebra $M_2(\tilde{\mathscr{L}})$ of 2×2 matrices of elements of $\tilde{\mathscr{L}}$ and define the homomorphism π by:

$$\pi(a) = \begin{bmatrix} \varrho(a) & 0 \\ 0 & \varrho(a) \end{bmatrix} \delta_0 \quad \forall a \in A. \tag{11}$$

We define similarly the element $\gamma \in B$ by $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \delta_0$. If we follow exactly what we did in [3], Theorem 2, p. 543 for the odd case, we should take for the operator F, $F^2 = 1$, $F \in B$ the formula:

$$F_0 = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}, \quad U = \frac{D + i\lambda^{1/2}}{\sqrt{D^2 + \lambda}}, \tag{12}$$

where $\lambda^{1/2}$ is the adjoined square root of $\lambda = \delta_0'$. However, to get simpler formulae (I am indebted to A. Jaffe for this point) one should replace F_0 by its *double*

$$F = \begin{bmatrix} 0 & U^2 \\ U^{*2} & 0 \end{bmatrix}, \quad U^2 = \frac{D + i\lambda^{1/2}}{D - i\lambda^{1/2}}. \tag{13}$$

The homotopy invariance formula ([3], Proposition 3, p. 545) and the natural homotopy between the matrices

$$\begin{bmatrix} U^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$$

show that the entire cocycle on A given by F is homotopic to *twice* the entire cocycle associated to F_0. In the next section we shall show that the entire cocycle on A associated to F is cohomologous to twice the J.L.O. cocycle; this computation is more tricky than what would appear at first sight and is the main content of this paper.

4. The Two Chern Character Cocycles are Cohomologous

As above, we let A be a Banach algebra and $(\mathfrak{h}, D)$ an odd θ-summable Fredholm module over A. We now compute our cocycle, obtained with the operator F given by formula (13), and with the trace τ on the algebra $B = M_2(\tilde{\mathscr{L}})$ given by:

$$\tau\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\right) = \tau_1(T_{11}) + \tau_1(T_{22}) \quad \text{for} \quad T_{ij} \in \tilde{\mathscr{L}},$$

with:

$$\tau_1((T, S)) = \operatorname{Trace}(S(1)) \quad \text{for} \quad (T, S) = T + \lambda^{1/2} S \in \tilde{\mathscr{L}}. \tag{14}$$

We then have (Lemma 2):

$$\varphi_{2n+1}(a^0,\dots,a^{2n+1})=\lambda_n\tau(\gamma F a^0[F,a^1]\dots[F,a^{2n+1}])\quad \forall a^i\in A. \tag{15}$$

On the other hand the J.L.O. cocycle (ψ_{2n+1}) is given by the following formula[1] ([6])

$$\psi_{2n+1}(a^0,\dots,a^{2n+1})=\int_{\Sigma s_i=1,\, s_i\geqq 0} ds_0\dots ds_{2n}\,\mathrm{Trace}(a^0 e^{-s_0D^2}[D,a^1]$$
$$\times e^{-s_1D^2}[D,a^2]\dots e^{-s_{2n}D^2}[D,a^{2n+1}]e^{-s_{2n+1}D^2}). \tag{16}$$

Our aim is to show that φ is cohomologous to 2ψ. Since formula 16 is the evaluation at $s=1$ of a convolution of operator valued distributions $T_i\in\mathscr{L}$ we can easily rewrite it in our language as follows:

$$\psi_{2n+1}(a^0,\dots,a^{2n+1})=\tau_0\left(a^0\frac{1}{D^2+\lambda}[D,a^1]\frac{1}{D^2+\lambda}\dots[D,a^{2n+1}]\frac{1}{D^2+\lambda}\right), \tag{17}$$

where $\tau_0(T)$ for $T\in\mathscr{L}$ is the trace of $T(1)$. In this formula λ is the element δ_0' of $\mathscr{L}$ but it is convenient to think of it as the free variable of Laplace transforms, which converts the convolution product of $\mathscr{L}$ into the ordinary pointwise product of operator valued functions of the real positive variable λ.[2]

The cocycle property of ψ: $b\psi_{2n-1}+B\psi_{2n+1}=0$ (cf. [6]) can be checked directly using the following straightforward equalities:

$$(b\psi_{2n-1})(a^0,\dots,a^{2n})$$
$$=-\tau_0\left([D,a^0]\frac{1}{D^2+\lambda}[D,a^1]\dots\frac{1}{D^2+\lambda}[D,a^{2n}]\frac{1}{D^2+\lambda}\right), \tag{18}$$

$$(B_0\psi_{2n-1})(a^0,\dots,a^{2n})$$
$$=\tau_0\left(\frac{1}{(D^2+\lambda)^2}[D,a^0]\frac{1}{D^2+\lambda}[D,a^1]\dots\frac{1}{D^2+\lambda}[D,a^{2n}]\right). \tag{19}$$

One gets indeed that:

$$(B\psi_{2n-1})(a^0,\dots,a^{2n})=\tau_0\left(\frac{\partial}{\partial\lambda}T\right),\qquad b\psi_{2n-1}=\tau_0(T)$$

for the element $T=-[D,a^0]\frac{1}{D^2+\lambda}\dots[D,a^{2n}]\frac{1}{D^2+\lambda}$ of the algebra $\mathscr{L}$, so that the cocycle property follows from:

$$\tau_0\left(\frac{\partial}{\partial\lambda}T\right)=-\tau_0(T)\quad \forall T\in\mathscr{L}. \tag{20}$$

Let us now compute the cocycle φ.

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[1] In fact the set of Quantum Algebras is more restrictive than ours since it requires that multiple commutators $[D,[D,\dots[D,a]\dots]$ be bounded, which we do not want to assume

[2] One cannot however permute the Laplace transform with the trace, since an operator like $a^0\frac{1}{D^2+\lambda}[D,a^1]\dots[D,a^{2n+1}]\frac{1}{D^2+\lambda}$ is in general *not* of trace class for λ a scalar, when D is only θ-summable

Lemma 3. *One has, for any* $a^0, \ldots, a^{2n+1} \in A$,

$$\varphi_{2n+1}(a^0, \ldots, a^{2n+1}) = -i\lambda_n \tau_0((4\lambda)^{n+1} H_{2n+1} + 4^{n+1}\lambda^n R_{2n+1}),$$

where

$$H_{2n+1} = a^0 \frac{1}{D^2+\lambda}[D, a^1]\frac{1}{D^2+\lambda} \ldots [D, a^{2n+1}]\frac{1}{D^2+\lambda} \in \mathscr{L}$$

and

$$R_{2n+1} = Da^0 D\frac{1}{D^2+\lambda}[D, a^1]\frac{1}{D^2+\lambda} \ldots [D, a^{2n+1}]\frac{1}{D^2+\lambda} \in \mathscr{L}.$$

Proof. Computing in the algebra $M_2(\tilde{\mathscr{L}})$ one gets, for $a \in A$,

$$[F, a] = \begin{bmatrix} 0, 2i\lambda^{1/2}(D - i\lambda^{1/2})^{-1}[D, a](D - i\lambda^{1/2})^{-1} \\ -2i\lambda^{1/2}(D + i\lambda^{1/2})^{-1}[D, a](D + i\lambda^{1/2})^{-1}, 0 \end{bmatrix},$$

$$\begin{aligned} &[F, a^k][F, a^{k+1}] \\ &= 4\lambda \begin{bmatrix} (D - i\lambda^{1/2})^{-1}[D, a^k](D^2+\lambda)^{-1}[D, a^{k+1}](D + i\lambda^{1/2})^{-1}, 0 \\ 0, (D + i\lambda^{1/2})^{-1}[D, a^k](D^2+\lambda)^{-1}[D, a^{k+1}](D - i\lambda^{1/2})^{-1} \end{bmatrix}, \end{aligned}$$

$$[F, a^{2n+1}]\gamma F = -2i\lambda^{1/2} \begin{bmatrix} (D - i\lambda^{1/2})^{-1}[D, a^{2n+1}](D + i\lambda^{1/2})^{-1}, 0 \\ 0, (D + i\lambda^{1/2})^{-1}[D, a^{2n+1}](D - i\lambda^{1/2})^{-1} \end{bmatrix}.$$

We thus get:

$$\begin{aligned} &\tau(\gamma F a^0 [F, a^1] \ldots)[F, a^{2n+1}]) \\ &= -2i\tau_0((4\lambda)^n((D - i\lambda^{1/2})a^0(D + i\lambda^{1/2}) + (D + i\lambda^{1/2})a^0(D - i\lambda^{1/2})) \\ &\quad \times (D^2+\lambda)^{-1}[D, a^1](D^2+\lambda)^{-1} \ldots [D, a^{2n+1}](D^2+\lambda)^{-1}) \\ &= -i\tau_0((4\lambda)^{n+1} H_{2n+1} + 4^{n+1}\lambda^n R_{2n+1}). \quad \square \end{aligned}$$

With the notations of Lemma 3 one can rewrite Eq. (17) in the form:

$$\psi_{2n+1}(a^0, \ldots, a^{2n+1}) = \tau_0(H_{2n+1}). \tag{17'}$$

It is then easy to express the term $\tau_0((4\lambda)^{n+1}(H_{2n+1})$ of Lemma 3 as a function of the cocycle ψ_{2n+1}; for this we let ψ^{β}_{2n+1} be the J.L.O. cocycle corresponding to the operator $\beta^{1/2}D$, (β real and positive), instead of the original D.

Lemma 4. *One has: for any* $a^0, \ldots, a^{2n+1} \in A$,

$$\tau_0((4\lambda)^{n+1}(H_{2n+1})) = \left(4\frac{\partial}{\partial\beta}\right)^{n+1}(\beta^{n+1/2}\psi^{\beta}_{2n+1}) \quad at \quad \beta = 1.$$

Proof. The element H_{2n+1} of $\mathscr{L}$ is given by the convolution

$$H_{2n+1}(s) = \int_{\Sigma s_i = s, s_i \geq 0} \prod_0^{2n} ds_i a^0 e^{-s_0 D^2}[D, a^1]e^{-s_1 D^2} \ldots [D, a^{2n+1}]e^{-s_{2n+1} D^2}.$$

Thus we get:

$$\psi^{\beta}_{2n+1}(a^0, \ldots, a^{2n+1}) = \beta^{-n-1/2}\,\mathrm{Trace}(H_{2n+1}(\beta)). \tag{21}$$

Hence Lemma 4 follows from the equality $\lambda = \delta'_0$ in $\mathscr{L}$. $\square$

Now one has:

$$\left(4\frac{\partial}{\partial\beta}\right)^{n+1}(\beta^{n+1/2}\psi^\beta_{2n+1})=4^{n+1}\sum_0^{n+1}C^k_{n+1}\left(n+\frac{1}{2}\right)\dots\left(k+\frac{1}{2}\right)\beta^{k-1/2}\left(\frac{\partial}{\partial\beta}\right)^k\psi^\beta_{2n+1}.$$

Combining this with Lemma 4 we get:

$$-i\lambda_n\tau_0((4\lambda)^{n+1}H_{2n+1})=\sum_0^{n+1}C^k_{n+1}\frac{\beta^{k-\frac{1}{2}}}{(k-\frac{1}{2})\dots\frac{3}{2}\frac{1}{2}}\left(\frac{\partial}{\partial\beta}\right)^k\psi^\beta_{2n+1}\quad\text{at}\quad\beta=1. \tag{22}$$

The first term in the sum of the right-hand side is just ψ_{2n+1}, moreover one knows ([7]) that $\left(\frac{\partial}{\partial\beta}\psi^\beta_{2n+1}\right)_{\beta=1}$ is a coboundary.

At this point it would be natural to guess that (22) is the main part of the proof, and that the other term $\tau_0((4^{n+1}\lambda^nR_{2n+1})$ in Lemma 3 does not contribute to the cohomology class of φ. This is however wrong, the next lemma shows that the two terms: $(4\lambda)^{n+1}H_{2n+1}$ and $4^{n+1}\lambda^nR_{2n+1}$ contribute equally to the cohomology class of φ, thus accounting for the coefficient 2 in the relation $\varphi\sim 2\psi$.

Lemma 5. *Let for $\beta>0$, θ^β_{2n} be the following cochain:*

$$\theta^\beta_{2n}(a^0,\dots,a^{2n})=\beta^{-(n+1/2)}\int_{\Sigma s_i=s,\,s_i\geqq 0}\Pi ds_i\,\mathrm{Tr}(a^0De^{-s_0D^2}[D,a^1]$$
$$\times e^{-s_1D^2}[D,a^2]\dots e^{-s_{2n-1}D^2}[D,a^{2n}]e^{-s_{2n}D^2}),\qquad\forall a^i\in A.$$

Then the cochain $-i\lambda_n\tau_0((4\lambda)^{n+1}H_{2n+1}-4^{n+1}\lambda^nR_{2n+1})$ is equal to $b(P_n\theta^\beta_{2n})$, $\beta=1$, where P_n is the differential operator:

$$P_n=\sum_0^n C^k_n\frac{\beta^{k+\frac{1}{2}}}{(k+\frac{1}{2})\dots\frac{3}{2}\frac{1}{2}}\left(\frac{\partial}{\partial\beta}\right)^k$$

and b is the Hochschild coboundary.

Observe the minus sign in the expression

$$\tau_0((4\lambda)^{n+1}-4^{n+1}\lambda^nR_{2n+1})$$

instead of the plus sign in the similar expression of Lemma 3.

Proof. As in (21) we get:

$$\theta^\beta_{2n}(a^0,\dots,a^{2n})=\beta^{-(n+1/2)}\mathrm{Trace}(X_{2n}(\beta)), \tag{23}$$

where X_{2n} is the element of $\mathscr{L}$ given by:

$$X_{2n}(a^0,\dots,a^{2n})=a^0\frac{D}{D^2+\lambda}[D,a^1]\frac{1}{D^2+\lambda}[D,a^2]\dots\frac{1}{D^2+\lambda}[D,a^{2n}]\frac{1}{D^2+\lambda}.$$

Let then $a^0,\dots,a^{2n+1}\in A$, and define $bX_{2n}(a^0,\dots,a^{2n+1})$ as

$$\sum_0^{2n}(-1)^jX_{2n}(a^0,\dots,a^ja^{j+1},\dots,a^{2n+1})-X_{2n}(a^{2n+1}a^0,a^1,\dots,a^{2n}).$$

A direct calculation shows that:

$$bX_{2n}(a^0,\dots,a^{2n+1}) = -\lambda a^0 \frac{1}{D^2+\lambda}[D,a^1]$$
$$\times \frac{1}{D^2+\lambda}[D,a^2]\dots\frac{1}{D^2+\lambda}[D,a^{2n+1}]\frac{1}{D^2+\lambda}$$
$$-a^0\frac{D}{D^2+\lambda}[D,a^1]\dots\frac{1}{D^2+\lambda}[D,a^{2n}]D\frac{1}{D^2+\lambda}[D,a^{2n+1}]\frac{1}{D^2+\lambda}$$
$$+a^0\frac{D}{D^2+\lambda}[D,a^1]\dots\frac{1}{D^2+\lambda}[D,a^{2n}]a^{2n+1}\frac{1}{D^2+\lambda}$$
$$-a^{2n+1}a^0\frac{D}{D^2+\lambda}[D,a^1]\dots\frac{1}{D^2+\lambda}[D,a^{2n}]\frac{1}{D^2+\lambda}.$$

Thus using the identity

$$\left[a^{2n+1},\frac{1}{D^2+\lambda}\right] = \frac{1}{D^2+\lambda}D[D,a^{2n+1}]\frac{1}{D^2+\lambda} + \frac{1}{D^2+\lambda}[D,a^{2n+1}]D\frac{1}{D^2+\lambda}$$

we see that modulo commutators:

$$bX_{2n} = R_{2n+1} - \lambda H_{2n+1}.$$

Hence for any $\beta > 0$ we get that:

$$\mathrm{Trace}((\lambda^n bX_{2n})(\beta)) = \mathrm{Trace}((\lambda^n R_{2n+1} - \lambda^{n+1}H_{2n+1})(\beta)).$$

Multiplying both terms by $-i\lambda_n 4^{n+1}$ and using (20) we get the desired equality. □

We are now ready to prove:

Theorem 6. *The two Chern character cocycles of* [3] *and* [6] *define the same entire cyclic cohomology class.*

Proof. With the above notations, it follows from Sect. 3 that our Chern character (in the odd case) is cohomologous to $\frac{1}{2}\varphi$. Thus it is enough to show that φ is cohomologous to 2ψ. We shall first define a cochain (α_{2n}) on A such that for any n one has:

$$b\alpha_{2n} + B\alpha_{2n+2} = \varphi_{2n+1} - 2\psi_{2n+1}, \tag{24}$$

and then check that it is indeed an entire cochain.

Now the homotopy invariance of the J.L.O. cocycle ([7]) gives explicitly $\frac{\partial}{\partial\beta}\psi^\beta$ as a coboundary, one has:

$$\frac{\partial}{\partial\beta}\psi^\beta_{2n+1} = b\varrho^\beta_{2n} + B\varrho^\beta_{2n+2}, \tag{25}$$

where the cochain ϱ^β is given by:

$$\varrho^\beta_{2n}(a^0, \ldots, a^{2n}) = \tfrac{1}{2}\beta^{-n-3/2} \int_{\Sigma s_i = \beta, s_i \geqq 0} \prod_0^{2n} ds_i$$
$$\times \left(\sum_0^{2n} (-1)^j \operatorname{Trace}(a^0 e^{-s_0 D^2}[D, a^1] e^{-s_1 D^2} \ldots e^{-s_{j-2} D^2}[D, a^{j-1}] e^{-s_{j-1} D^2} \right.$$
$$\left. \times D e^{-s_j D^2}[D, a^j] \ldots e^{-s_{2n} D^2}[D, a^{2n}] e^{-s_{2n+1} D^2} \right). \tag{26}$$

In other words one has

$$\varrho^\beta_{2n}(a^0, \ldots, a^{2n}) = \tfrac{1}{2}\beta^{-n-3/2} \operatorname{Trace}(Y_{2n}(\beta)),$$

where Y_{2n} is the following element of the algebra $\mathscr{L}$:

$$Y_{2n} = \sum_0^{2n} (-1)^j a^0 (D^2 + \lambda)^{-1}[D, a^1] \ldots (D^2 + \lambda)^{-1}[D, a^{j-1}]$$
$$\times (D^2 + \lambda)^{-1} D (D^2 + \lambda)^{-1}$$
$$\times [D, a^j] \ldots (D^2 + \lambda)^{-1}[D, a^{2n}] (D^2 + \lambda)^{-1}).$$

Combining this with formula (19) we can write:

$$-i\lambda_n \tau_0((4\lambda)^{n+1} H_{2n+1}) - \psi_{2n+1}$$
$$= \sum_0^n C^{k+1}_{n+1} \frac{\beta^{k+\frac{1}{2}}}{(k+\frac{1}{2}) \ldots \frac{3}{2}\frac{1}{2}} \left(\frac{\partial}{\partial\beta}\right)^k (b\varrho^\beta_{2n} + B\varrho^\beta_{2n+2}) \quad \text{at} \quad \beta = 1. \tag{27}$$

Now, by Lemma 3 one has:

$$\varphi_{2n+1} = -2i\lambda_n \tau_0((4\lambda)^{n+1} H_{2n+1}) + i\lambda_n \tau_0((4\lambda)^{n+1} H_{2n+1} - 4^{n+1}\lambda^n R_{2n+1}).$$

And by Lemma 5 we get

$$\varphi_{2n+1} = -2i\lambda_n \tau_0((4\lambda)^{n+1} H_{2n+1}) - b(P_n \theta^\beta_{2n})_{\beta=1}.$$

Combining this with (27) we thus get:

$$\varphi_{2n+1} - 2\psi_{2n+1} = 2\sum_0^n C^{k+1}_{n+1} \frac{\beta^{k+\frac{1}{2}}}{(k+\frac{1}{2}) \ldots \frac{3}{2}\frac{1}{2}} \left(\frac{\partial}{\partial\beta}\right)^k (b\varrho^\beta_{2n} + B\varrho^\beta_{2n+2})$$
$$- \sum_0^n C^k_n \frac{\beta^{k+\frac{1}{2}}}{(k+\frac{1}{2}) \ldots \frac{3}{2}\frac{1}{2}} \left(\frac{\partial}{\partial\beta}\right)^k b\theta^\beta_{2n} \quad (\text{at } \beta = 1). \tag{28}$$

Thus if we let (α_{2n}) be the cochain:

$$\alpha_{2n} = 2\sum_0^n C^{k+1}_n \frac{\beta^{k+\frac{1}{2}}}{(k+\frac{1}{2}) \ldots \frac{3}{2}\frac{1}{2}} \left(\frac{\partial}{\partial\beta}\right)^k \varrho^\beta_{2n} \quad (\text{at } \beta = 1).$$

We then get, using $C^{k+1}_{n+1} - C^{k+1}_n = C^k_n$, that

$$\varphi_{2n+1} - 2\psi_{2n+1} - b\alpha_{2n} - B\alpha_{2n+2} = bP_n(2\varrho^\beta_{2n} - \theta^\beta_{2n}), \tag{29}$$

where $P_n = \sum_0^n C^k_n \frac{\beta^{k+\frac{1}{2}}}{(k+\frac{1}{2}) \ldots \frac{3}{2}\frac{1}{2}} \left(\frac{\partial}{\partial\beta}\right)^k$ is the differential operator that we used in Lemma 5.

But the right-hand side of (29) is a coboundary since a simple calculation shows that, for $\beta = 1$, $B(2\varrho_{2n}^{\beta} - \theta_{2n}^{\beta}) = 0$. Applying the technique of [3], Lemma 1, p. 532 to control derivatives one checks that the cochains (α_{2n}), $P_n(\varrho_{2n} - \frac{1}{2}\theta_{2n})$ are entire cochains so the conclusion follows. □

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