

Graphs and Reflection Groups

J.-B. Zuber

CEA, Service de Physique Théorique de Saclay, F-91191 Gif sur Yvette Cedex, France

Received: 23 July 1995 / Accepted: 23 October 1995

Abstract: It is shown that graphs that generalize the ADE Dynkin diagrams and have appeared in various contexts of two-dimensional field theory may be regarded in a natural way as encoding the geometry of a root system. After recalling what are the conditions satisfied by these graphs, we define a bilinear form on a root system in terms of the adjacency matrices of these graphs and undertake the study of the group generated by the reflections in the hyperplanes orthogonal to these roots. Some “non-integrally laced” graphs are shown to be associated with subgroups of these reflection groups. The empirical relevance of these graphs in the classification of conformal field theories or in the construction of integrable lattice models is recalled, and the connections with recent developments in the context of $\mathcal{N} = 2$ supersymmetric theories and topological field theories are discussed.

A. Claude

1. Introduction

Similar features have appeared recently in various problems of two-dimensional field theory and statistical mechanics. In the simplest case based on the $su(2)$ algebra, the classification of ordinary conformal field theories (cft’s), of $\mathcal{N} = 2$ superconformal field theories or of the corresponding topological field theories, and the construction of integrable lattice face models have all been found to have their solutions labelled by the simply laced ADE Dynkin diagrams. This is quite remarkable in view of the fact that the setting of the problem and the techniques of analysis are in each case quite different (see [1] or Sect. 6 below for a short review and a list of references). Although our understanding of the same problems in cases of higher rank $su(N)$ is much poorer, there is some evidence that important data are again provided by a set of graphs. It is the purpose of this paper to show that these graphs may be given a geometrical interpretation as encoding the geometry (the scalar products) of a system of vectors – the “roots” – and thus enable one to construct the group generated by the reflections in the hyperplanes orthogonal to these roots.

After recalling some basic facts and introducing notations concerning reflection groups, I shall define the class of graphs that we are interested in (Sect. 2). These graphs are a natural generalization of the situation encountered with the ADE

diagrams on the one hand, and with the fusion graphs of the affine algebra $\widehat{su}(N)$ on the other. In Sect. 3, the adjacency matrix of such a graph is used to define a scalar product on a root system and thus a reflection group. Some general properties of these groups are proved, in particular those cases that lead to finite groups are identified; also an analogue of the Coxeter element is defined and shown to have interesting properties. In Sect. 4, some other identifications and isomorphisms of groups are derived or conjectured on the basis of several explicit cases. Sect. 5 discusses the cases of “non-integrally laced graphs” and their connections with subalgebras of the “Pasquier algebra.” Finally in Sect. 6, I turn to the discussion of the physical relevance of these graphs and groups in the various contexts mentioned above. The case of $\mathcal{N} = 2$ superconformal theories and of topological field theories seems the most natural setting for that interpretation and we shall see that the graphs and groups discussed here are actual realizations of general results of Cecotti and Vafa, and of Dubrovin.

A short account of this work has been presented in [1].

Let us start with some generalities on reflection groups. Let V denote a vector space of dimension n over \mathbb{R} with a given basis $\{\alpha_a\}$. Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form on V . We denote

$$g_{ab} = \langle \alpha_a, \alpha_b \rangle \tag{1.1}$$

and assume that $\langle \alpha_a, \alpha_a \rangle = 2$. This allows one to define the linear transformation S_a

$$S_a : x \mapsto x' = x - \langle \alpha_a, x \rangle \alpha_a \tag{1.2}$$

or in terms of the components $x_b : x = \sum_b x_b \alpha_b$,

$$\begin{aligned} x'_a &= -x_a - \sum_{c \neq a} g_{ac} x_c, \\ x'_b &= x_b \quad \text{if } b \neq a. \end{aligned} \tag{1.3}$$

The following properties are easily established:

(i) S_a is involutive: $S_a^2 = \mathbb{I}$, and preserves the bilinear form: $\langle S_a x, S_a y \rangle = \langle x, y \rangle$. This is in fact the reflection in the $(n - 1)$ -dimensional hyperplane orthogonal to the vector α_a .

(ii) If the ab entry $g_{ab} = 0$, then S_a and S_b commute and the product $S_a S_b$ is of order 2

$$(S_a S_b)^2 = \mathbb{I}.$$

(iii) If the entry $g_{ab} = 1$, then $S_a S_b$ is of order 3; more generally, if $g_{ab} = 2 \cos \frac{\pi p}{q}$, with p and q coprime integers, the restriction of g to the 2-plane spanned by α_a, α_b endows it with a structure of Euclidean space: the two unit vectors α_a and α_b make an angle $\frac{\pi p}{q}$, hence the product $S_a S_b$ is a rotation of angle $\frac{2\pi p}{q}$, and $S_a S_b$ is of order q .

Let Γ be the group generated by the reflections $S_a, a = 1, \dots, n$. We call *roots* the vectors α_a and *root system* the set of images of the roots under the action of Γ .

An important issue is to know if the group Γ is of finite or infinite order. One proves [2] that the group is finite if and only if the bilinear form $\langle \cdot, \cdot \rangle$ is positive definite. At the end of the discussion, one finds that finite reflection groups are classified [3]: beside the Weyl groups of the simple Lie algebras, A_p, B_p, C_p , (the two

latter groups being isomorphic), D_p, E_6, E_7, E_8, F_4 and G_2 , there are the symmetry groups H_3 and H_4 of the regular icosahedron and of a regular 4-dimensional polytope, and the infinite series $I_2(k)$ of the symmetry groups of the regular k -gons in the plane. If one uses as a basis a system of simple roots, then $\langle \alpha_a, \alpha_b \rangle \leq 0$ if $a \neq b$ [2], and the bilinear form g_{ab} of (1.1) is the so-called Coxeter matrix Γ_{ab} ,

$$g_{ab} = \langle \alpha_a, \alpha_b \rangle = -2 \cos \frac{\pi}{m_{ab}} =: \Gamma_{ab} , \tag{1.4}$$

where the integer $m_{ab} = m_{ba}$ is the order of the product $S_a S_b$, $(S_a S_b)^{m_{ab}} = \mathbb{I}$, $m_{aa} = 1$. Note that for the groups of Weyl type, one usually takes roots $\hat{\alpha}$ normalized differently, with the longest having $(\hat{\alpha}, \hat{\alpha}) = 2$ and the Cartan matrix with integral entries

$$\hat{C}_{ab} = 2 \frac{\langle \hat{\alpha}_a, \hat{\alpha}_b \rangle}{\langle \hat{\alpha}_b, \hat{\alpha}_b \rangle} \in \mathbb{Z} ; \tag{1.5}$$

then Γ_{ab} is the *symmetrized* form of the Cartan matrix

$$\hat{C}_{ab} = \frac{|\hat{\alpha}_a|}{|\hat{\alpha}_b|} \Gamma_{ab} , \tag{1.6}$$

with $|\hat{\alpha}| := (\langle \hat{\alpha}, \hat{\alpha} \rangle)^{\frac{1}{2}}$. For future reference, also note that the adjacency matrix \hat{G} of the Coxeter–Dynkin diagram is related to the Cartan matrix by

$$\hat{C}_{ab} = 2\delta_{ab} - \hat{G}_{ab} = -2 \frac{|\hat{\alpha}_a|}{|\hat{\alpha}_b|} \cos \frac{\pi}{m_{ab}} . \tag{1.7}$$

2. Graphs

The construction of graphs proceeds through generalization of two cases under control, namely the ordinary ADE Dynkin diagrams on the one hand and the fusion graphs of $\widehat{su}(N)_k$ on the other. I briefly review well known facts about these two cases.

2.1. The ADE Dynkin Diagrams. The ADE Dynkin diagrams are unoriented graphs that have the property that their adjacency matrix has a spectrum of eigenvalues satisfying

$$|\gamma| < 2 . \tag{2.1}$$

In fact they are, together with the orbifolds A_{2n}/\mathbb{Z}_2 , the only unoriented connected graphs with that property [4]. (The latter orbifold graphs will be discarded in the following: from the point of view of lattice models, they are uninteresting as they produce no new model; from a different viewpoint, more relevant here, they are not 2-colourable, see below.) Furthermore their eigenvalues take the form

$$\gamma^{(\lambda)} = 2 \cos \frac{\lambda \pi}{h} , \tag{2.2}$$

where h is the Coxeter number, and the integer λ takes r ($=$ rank of the ADE algebra) values between (and including) 1 and $h - 1$, with possible multiplicities.

Note that the ADE graphs are tree graphs and may therefore be bi-coloured.

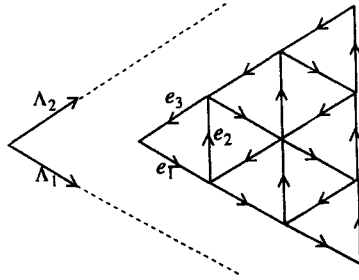


Fig. 1. The graph $\mathcal{A}^{(6)}$

One may also relax the condition that the entries of the matrix under consideration are integers and consider the class of symmetric matrices whose elements are of the form

$$G_{aa} = 0, \tag{2.3}$$

$$G_{ab} = 2 \cos \frac{\pi}{m_{ab}} \quad a \neq b$$

for a set of integers $m_{ab} = m_{ba} \geq 2$. (The case considered above was thus $m = 2$ or 3 .) One may represent such a matrix by a graph with the edge $a - b$ decorated by the integer m_{ab} . The matrix is called indecomposable if the graph is connected. One proves [2] that the only indecomposable matrices (2.3) that satisfy (2.1) are the symmetrized forms of the adjacency matrices \hat{G} of the Coxeter–Dynkin diagrams of all the finite groups $A-I$ discussed in the introduction,

$$G_{ab} = \frac{|\hat{\alpha}_b|}{|\hat{\alpha}_a|} \hat{G}_{ab}. \tag{2.4}$$

Hence they are related to the Coxeter matrices of (1.4) by

$$\Gamma_{ab} = 2\delta_{ab} - G_{ab}. \tag{2.5}$$

2.2. *The $\widehat{su}(N)_k$ Fusion Graphs.* Let $\Lambda_1, \dots, \Lambda_{N-1}$ be the fundamental weights of $su(N)$. Let $\rho = \Lambda_1 + \dots + \Lambda_{N-1}$ be the sum of these fundamental weights. I recall that the set of integrable weights (shifted by ρ) of the affine algebra $\widehat{su}(N)_k$ is

$$\mathcal{P}_{++}^{(k+N)} = \{ \lambda = \lambda_1 \Lambda_1 + \dots + \lambda_{N-1} \Lambda_{N-1} \mid \lambda_i \geq 1, \lambda_1 + \dots + \lambda_{N-1} \leq k + N - 1 \}. \tag{2.6}$$

This set of $\binom{k+N-1}{k}$ weights is a finite subset, the so-called Weyl alcove, of the $(N - 1)$ -dimensional weight lattice. We may represent it by the graph obtained by drawing edges between neighbouring points on this lattice, and orienting them along the weights of the standard N -dimensional representation of $su(N)$, i.e. along the N (linearly dependent) vectors e_i

$$e_1 = \Lambda_1, \quad e_i = \Lambda_i - \Lambda_{i-1}, \quad i = 2, \dots, N - 1, \quad e_N = -\Lambda_{N-1}. \tag{2.7}$$

This graph will be denoted by $\mathcal{A}_N^{(k+N)}$ and the subscript N will be omitted whenever it causes no ambiguity. It is exemplified in Fig. 1 by the case of $\widehat{su}(3)$ at level 3. Recall that the Killing bilinear form is such that $(e_i, e_j) = \delta_{ij} - \frac{1}{N}$.

This graph may also be regarded as the fusion graph of the representation of weight Λ_1 , i.e. the graph whose adjacency matrix is $N_{\Lambda_1} \equiv N_{\square}$. By the Verlinde formula [5], we know how to express its eigenvalues in terms of the unitary matrix S of modular transformations of the characters of the affine algebra $\widehat{su}(N)_k$, and these expressions may be recast as

$$\gamma^{(\lambda)} = \sum_{i=1}^N \exp \frac{-2i\pi}{k+N} (e_i, \lambda), \tag{2.8}$$

with λ running over the same set $\mathcal{P}_{++}^{(k+N)}$. (Since $(e_i, \lambda) \in \frac{1}{N}\mathbb{Z}$, this reduces to (2.2) for $N = 2$.) From now on, we shall write

$$h = k + N. \tag{2.9}$$

Note that the weights of $su(N)$ come naturally with a \mathbb{Z}_N grading, $\tau(\cdot)$, the “ N -ality,”

$$\tau(\lambda) = \sum_{j=1}^{N-1} j\lambda_j \pmod N, \tag{2.10}$$

and that the only non-vanishing entries of the adjacency matrix G_1 are between points of successive N -alities

$$(G_1)_{ab} \neq 0 \quad \text{only if } \tau(b) = \tau(a) + 1 \pmod N. \tag{2.11}$$

The reason why the adjacency matrix has been labelled with a 1 is that, when working with $su(N)$, we may, and in fact we must, generalize this setting and consider the graphs associated with the fusion by all the fundamental representations:

$$N_{\square}, N_{\square}, \dots, N_{\square} \Big\}^{N-1}.$$

The matrix considered above is thus $G_1 = N_{\square}$, and more generally, for the ease of notation, we write

$$G_p = N_{\square} \Big\}^p \quad 1 \leq p \leq N - 1. \tag{2.12}$$

Again by the Verlinde formula, we know that all these matrices are simultaneously diagonalized by the unitary matrix S . The expressions of their eigenvalues read

$$\begin{aligned} \gamma_1^{(\lambda)} &= \sum_{i=1}^N \exp -\frac{2i\pi}{h} (e_i, \lambda) &&= \chi_1(M) \\ \gamma_2^{(\lambda)} &= \sum_{1 \leq i < j \leq N} \exp -\frac{2i\pi}{h} ((e_i + e_j), \lambda) &&= \chi_2(M) \\ &\vdots &&\vdots \\ \gamma_{N-1}^{(\lambda)} &= \sum_{1 \leq i_1 < \dots < i_{N-1} \leq N} \exp -\frac{2i\pi}{h} ((e_{i_1} + \dots + e_{i_{N-1}}), \lambda) \\ &= \left(\gamma_1^{(\lambda)} \right)^* &&= \chi_{N-1}(M). \end{aligned} \tag{2.13}$$

These eigenvalues may be expressed in terms of ordinary $SU(N)$ characters χ_p for the p^{th} fundamental representation evaluated at the $SU(N)$ matrix $M = \text{diag}(\exp -2i\pi(e_i, \lambda)/h)^1$. The γ are pairwise complex conjugate, $\gamma_p = (\gamma_{N-p})^*$, since $\sum_i e_i = 0$, which reflects the fact that the matrices G_p are pairwise transposed of one another:

$$G_p^t = G_{N-p} . \tag{2.14}$$

For example, in the case of $su(3)$, the graphs pertaining to the two fundamental representations are obtained from one another by reversing all the orientations of edges.

Finally, we recall the property that the set of $\gamma_p^{(\lambda)}$, for all $p = 1, \dots, N - 1$, characterizes the weight λ in $\mathcal{P}_{++}^{(h)}$:

$$\text{if for all } p = 1, \dots, N - 1 \quad \gamma_p^{(\lambda)} = \gamma_p^{(\mu)} \quad \text{then } \lambda = \mu . \tag{2.15}$$

This follows from the fact that the fusion ring of $\widehat{su}(N)$ is polynomially generated by the fundamental representations [7]. Thus under the conditions of (2.15), the fusion matrices of λ and μ are identical, which suffices to identify λ and μ .

2.3. Generalized Graphs. Building upon the two particular cases discussed above, it seems natural to introduce a class of generalized graphs satisfying the following properties:

1) We are given a finite set \mathcal{V} of n vertices, that are denoted a, b, \dots . In the set \mathcal{V} acts an involution $a \mapsto \bar{a}$ (a generalization of the conjugation of representations). When working with $su(N)$, we assume that a \mathbb{Z}_N valued grading τ is assigned to these vertices, such that $\tau(\bar{a}) = -\tau(a) \text{ mod } N$.

2) We are also given a set of $N - 1$ commuting $n \times n$ matrices, labelled by the fundamental representations of $su(N)$ and, like in (2.12) above, denoted G_p , $1 \leq p \leq N - 1$. These matrices have entries that are non-negative integers and may thus be regarded as the adjacency matrices of $N - 1$ graphs \mathcal{G}_p . Contrary to the cases encountered above, some of the edges (ab) may be multiple, i.e. have $(G_p)_{ab} > 1$. For the sake of irreducibility, we have also to assume some property of connectivity of the set \mathcal{V} . There is no partition $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$ such that

$$\forall a \in \mathcal{V}', \forall b \in \mathcal{V}'', \forall p = 1, \dots, N - 1, \quad (G_p)_{ab} = 0 .$$

Remark. In the same way as in (2.3), it is natural to extend slightly the previous condition and to allow non-integer G . As we shall see in Sect. 5 below, $(G_p)_{ab}$ of the specific form

$$(G_p)_{ab} = 0 \quad \text{or} \quad \mathcal{P}_p^{(N)} \left(2 \cos \frac{\pi}{h} \right) \tag{2.16a}$$

or

$$(G_p)_{ab} = 2 \cos \frac{\pi}{m_{ab}} \tag{2.16b}$$

seem to be natural choices, with h and m_{ab} integers and $\mathcal{P}_p^{(N)}(x)$ a certain polynomial of x (Eqs. (5.4–5.5)). By a slight abuse of language, we still call G the adjacency matrix of a graph, whose edges are decorated by the integers m_{ab} or h .

¹ An intriguing observation is that these special matrices M are in one-to-one correspondence with the conjugacy classes of elements of finite order in the group $SU(N)$ [6]. I am indebted to J. Patera for this remark.

The considerations that follow (until Sect. 5) do not depend on the integrality of the matrix elements.

3) The edges of the graphs \mathcal{G}_p are compatible with the \mathbb{Z}_N grading τ in the sense that

$$(G_p)_{ab} \neq 0 \quad \text{only if } \tau(b) = \tau(a) + p \pmod N . \tag{2.17}$$

Thus for $p \neq \frac{N}{2}$, the edges are oriented: $(G_p)_{ab} \neq 0 \Rightarrow (G_p)_{ba} = 0$. Also, for a given pair (a, b) there is at most one matrix G_p with a non-vanishing entry $(G_p)_{ab}$.

4) The matrices G_p are pairwise transposed of one another:

$$G_p^t = G_{N-p} . \tag{2.18}$$

Moreover each graph is invariant under the involution in the sense that

$$(G_p)_{\bar{a}\bar{b}} = (G_p)_{ba} . \tag{2.19}$$

5) Since the matrices G commute among themselves, they commute with their transpose, (“normal matrices”), hence they are diagonalizable in a common orthonormal basis $\psi^{(\lambda)}$; we assume that these eigenvectors are labelled by integrable weights λ of $\bar{su}(N)$ and that the corresponding eigenvalues of G_1, G_2, \dots, G_{N-1} read as in (2.14), for some h and $\lambda \in \mathcal{P}_{++}^{(h)}$; some of these λ may occur with multiplicities larger than one.

6) We assume that $\rho = (1, 1, \dots, 1)$ is among these λ , with multiplicity 1: it corresponds to the eigenvector of largest eigenvalue, the so-called Perron–Frobenius eigenvector.

By extension of the ADE case, we call “exponents” these weights λ (with their multiplicities) and denote their set by Exp . Let σ be the automorphism of the Weyl alcove $\mathcal{P}_{++}^{(h)}$,

$$\lambda = (\lambda_1, \dots, \lambda_{N-1}) \mapsto \sigma(\lambda) = (h - \lambda_1 - \dots - \lambda_{N-1}, \lambda_1, \lambda_2, \dots, \lambda_{N-2}) . \tag{2.20}$$

One checks that

$$\exp -\frac{2i\pi}{h}(e_i, \sigma(\lambda)) = e^{\frac{2i\pi}{N}} \exp -\frac{2i\pi}{h}(e_{i-1}, \lambda) , \tag{2.21}$$

and thus, using (2.13),

$$\gamma_p^{(\sigma(\lambda))} = e^{2i\pi \frac{p}{N}} \gamma_p^{(\lambda)} . \tag{2.22}$$

There is of course another automorphism acting on $\mathcal{P}_{++}^{(h)}$: the conjugation \mathcal{C} of representations

$$\text{under } \mathcal{C} \quad \lambda \mapsto \bar{\lambda} \quad \gamma_p^{(\bar{\lambda})} = (\gamma_p^{(\lambda)})^* . \tag{2.23}$$

Now consider $\lambda \in \text{Exp}$, $\psi^{(\lambda)}$ a corresponding eigenvector. Define $\tilde{\psi}_a^{(\lambda)} = e^{2i\pi \frac{\tau(a)}{N}} \psi_a^{(\lambda)}$. Then property 3) implies that $\tilde{\psi}^{(\lambda)}$ is an eigenvector of G_p of eigenvalue $\tilde{\gamma}_p^{(\lambda)} = e^{2i\pi \frac{p}{N}} \gamma_p^{(\lambda)} = \gamma_p^{(\sigma(\lambda))}$. Thus (using (2.15)), we conclude that if λ is an exponent, so is $\sigma(\lambda)$. On the other hand, the real matrices G_p have eigenvalues that come in complex conjugate pairs; thus the exponents also come in complex conjugate pairs.

Proposition 1. *The set Exp is invariant under the action of σ and \mathcal{C} . Moreover one may choose*

$$\begin{aligned} \psi_a^{(\bar{\lambda})} &= \psi_a^{(\lambda)} = (\psi_a^{(\lambda)})^* , \\ \psi_a^{(\sigma(\lambda))} &= e^{\frac{2i\pi}{N}\tau(a)} \psi_a^{(\lambda)} . \end{aligned} \tag{2.24}$$

In [8] we presented lists of solutions to these requirements in the case $N = 3$.

Remarks.

1) For some purposes, in particular for the construction of lattice integrable models based on the graphs, it seems necessary to impose the further constraint that the graph of G_1 , say, has an “extremal point,” i.e. a vertex on which only one edge is ending and from which only one edge is starting. This constraint will not play any role in the following and may thus be omitted.

2) It may be more economic to consider a single matrix G

$$G = G_1 + G_2 + \dots + G_{N-1} . \tag{2.25}$$

The edges of the graph \mathcal{G} it encodes connect only vertices of different τ . Conversely if the graph \mathcal{G} is given, and if the grading of the vertices is known, each matrix G_p may be identified as the adjacency matrix of the subgraph joining pairs of vertices of τ differing by p .

For later use, I now introduce an explicit parametrization of the matrices G_p . I assume that the vertices of \mathcal{V} have been ordered according to increasing τ : first the vertices with $\tau = 0$, then $\tau = 1$, etc. Then the matrices G_p are $N \times N$ block-matrices of the form

$$\begin{aligned} G_1 &= \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & A_{N-1N} \\ A_{N1} & 0 & \dots & \dots & 0 \end{pmatrix} , \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ G_p &= \begin{pmatrix} 0 & \dots & A_{1p+1} & \dots & 0 \\ \vdots & 0 & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & A_{N-pN} \\ A_{N-p+11} & 0 & & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & A_{Np} & & 0 \end{pmatrix} , \dots \end{aligned} \tag{2.26}$$

with the matrices A_{ij} satisfying

$$A_{ij}^t = A_{ji} \tag{2.27}$$

as a consequence of (2.18). (The matrices A_{ij} are of course subject to further constraints expressing the commutation of the matrices G , etc.) Later, we shall also

encounter the matrix ²

$$T = \begin{pmatrix} \mathbf{1} & -A_{12} & A_{13} & \cdots & -\varepsilon A_{1N} \\ & \mathbf{1} & -A_{23} & \cdots & \varepsilon A_{2N} \\ & 0 & \ddots & & \\ & & & 0 & \mathbf{1} & -A_{N-1N} \\ 0 & 0 & \cdots & 0 & \mathbf{1} \end{pmatrix}, \tag{2.28}$$

where $\varepsilon = (-1)^N$. It may be written as a product of upper triangular matrices in the two following ways:

$$\begin{aligned} T &= \begin{pmatrix} \mathbf{1} & & & & \\ & \mathbf{1} & 0 & & \\ & & \ddots & & \\ & 0 & & \mathbf{1} & -A_{N-1N} \\ & & & & \mathbf{1} \end{pmatrix} \cdots \begin{pmatrix} \mathbf{1} & -A_{12} & A_{13} & \cdots & -\varepsilon A_{1N} \\ & \mathbf{1} & 0 & & 0 \\ & & \ddots & & 0 \\ & 0 & & \mathbf{1} & 0 \\ & & & & \mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & & & -\varepsilon A_{1N} \\ & \mathbf{1} & 0 & \varepsilon A_{2N} \\ & & \ddots & \vdots \\ & 0 & & \mathbf{1} & -A_{N-1N} \\ & & & & \mathbf{1} \end{pmatrix} \cdots \begin{pmatrix} \mathbf{1} & -A_{12} & 0 & \cdots & 0 \\ & \mathbf{1} & & & 0 \\ & & \ddots & & \\ & 0 & & \mathbf{1} & \\ & & & & \mathbf{1} \end{pmatrix}, \tag{2.29} \end{aligned}$$

which allows to write its inverse and its transpose as

$$T^{-1} = \begin{pmatrix} \mathbf{1} & A_{12} & -A_{13} & \cdots & \varepsilon A_{1N} \\ & \mathbf{1} & 0 & & 0 \\ & & \ddots & & 0 \\ & 0 & & \mathbf{1} & \\ & & & & \mathbf{1} \end{pmatrix} \cdots \begin{pmatrix} \mathbf{1} & & 0 \\ & \ddots & \\ & & \mathbf{1} & A_{N-1N} \\ & & & \mathbf{1} \end{pmatrix} \tag{2.30}$$

and

$$T^t = \begin{pmatrix} \mathbf{1} & & 0 & \cdots & 0 \\ -A_{21} & \mathbf{1} & & & 0 \\ & & \ddots & & \\ & 0 & & \mathbf{1} & \\ & & & & \mathbf{1} \end{pmatrix} \cdots \begin{pmatrix} \mathbf{1} & & & & \\ & \mathbf{1} & & & 0 \\ & & \ddots & & \\ & & & 0 & \mathbf{1} \\ -\varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & -A_{N,N-1} & \mathbf{1} \end{pmatrix}. \tag{2.31}$$

Let us finally introduce the matrix J

$$J = \text{diag}(\mathbf{1}, \varepsilon \mathbf{1}, \mathbf{1}, \varepsilon \mathbf{1}, \dots, \mathbf{1}). \tag{2.32}$$

These expressions will be useful soon.

² Here and in the following, by a small abuse of notations, $\mathbf{1}$ denotes a unit matrix, whose dimension is fixed by the context.

3. New Reflection Groups

3.1. *Definition and First Properties.* I now show that with these data, one may associate a reflection group in a natural way.

Let as above V be a n -dimensional space over \mathbb{R} , with a basis $\{\alpha_a\}$ labelled by the points a of the set \mathcal{V} . We then introduce a bilinear form defined by the following expression, that depends on whether N is even or odd

$$g_{ab} = \langle \alpha_a, \alpha_b \rangle = 2\delta_{ab} + ((-1)^{N-1}G_1 + G_2 + (-1)^{N-1}G_3 + \dots + (-1)^{N-1}G_{N-1})_{ab} , \tag{3.1}$$

(or $g_{ab} = 2\delta_{ab} + (-1)^{(N-1)(\tau(a)-\tau(b))}G_{ab}$ in terms of the single matrix G of (2.25)). An alternative expression is

$$g = J(T + T^t)J^{-1} \tag{3.2}$$

in terms of the matrices introduced at the end of Sect. 2.3. We then consider the group Γ generated by the reflections S_a . In the case $N = 2$, one recovers the expressions (1.4)–(2.3). We shall see below what are the virtues of the choice of signs in (3.1).

Note that the groups are generated by the reflections S_a and that these generators satisfy the relations $S_a^2 = \mathbb{I}$, as well as $(S_a S_b)^q = \mathbb{I}$ under the conditions mentioned in Sect. 1. Generically, they satisfy also other relations (see examples below in Sect. 4.1), and thus the group cannot be called a ‘‘Coxeter group.’’ We shall rather use the denomination ‘‘reflection group.’’

Since we know by hypothesis all the eigenvalues of the G_p , those of the metric g read

$$g^{(\lambda)} = \sum_{p=0}^N (-\varepsilon)^p \gamma_p^{(\lambda)} , \quad \lambda \in \text{Exp} , \tag{3.3}$$

where we have extended the formulae (2.13) to $\gamma_0^{(\lambda)} = \gamma_N^{(\lambda)} = 1$ and as before, $\varepsilon = (-1)^N$. It will be very useful to use a multiplicative form of this eigenvalue

$$g^{(\lambda)} = \prod_{i=1}^N (1 - \varepsilon e^{-\frac{2i\pi}{h}(e_i, \lambda)}) \tag{3.4}$$

that follows from (2.13).

For $N = 2$, as recalled above, all the graphs that satisfy the previous constraints lead to a finite reflection group. For $N \geq 3$, on the contrary, the group is generically of infinite order. More precisely,

Proposition 2. *The form \langle , \rangle of Eq. (3.1) is definite positive if and only if*

$$\begin{aligned} N = 2 \quad \forall h \geq 3 , \\ N = 3 \quad h = 4, 5 , \\ N > 3 \quad h = N + 1 . \end{aligned} \tag{3.5}$$

To prove this, we shall exhibit a non-positive eigenvalue of the matrix g whenever the conditions of (3.5) are not fulfilled. According to Proposition 1, all the images of ρ under the action of σ are always exponents. For the exponent $\lambda_l := \sigma^l(\rho)$, $l = 0, \dots, N - 1$, one may compute the eigenvalue of g using (3.4) and (2.21). One

finds, with $\xi = \exp 2i\pi/N$ and $q = \exp i\pi/h$,

$$g^{(\lambda_l)} = (1 + (-1)^{N-1} \xi^l q^{1-N})(1 + (-1)^{N-1} \xi^l q^{3-N}) \dots (1 + (-1)^{N-1} \xi^l q^{N-1}). \tag{3.6}$$

We now choose to look at the eigenvalue corresponding to $l = 1$ for N odd, and to $l = N/2 - 1$ for N even. In both cases, the resulting value of $g^{(\lambda_l)}$ reads

$$g^{(\lambda_l)} = - \prod_{j=0}^{N-1} 2 \cos \left(\frac{1}{N} + \frac{1 - N + 2j}{2h} \right) \pi. \tag{3.7}$$

It is finally a simple matter to check that all the arguments of the cosine are between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and thus $g^{(\lambda_l)} \leq 0$, for $N = 3, h \geq 6$ or for $N \geq 4, h \geq N + 2$.

It remains to examine the cases of (3.5). Leaving aside the case $N = 2$ which is well known, let us consider first the cases $N \geq 3, h = N + 1$. Then the exponents take their values among the integrable weights (shifted by ρ) of $\widehat{su}(N)_1$, and again by Proposition 1, these values are all reached. For these exponents ρ and $\sigma^l(\rho) = \rho + \Lambda_l, l = 1, \dots, N - 1$, the direct calculation shows that the possible eigenvalues of the metric g are either $(N + 1)$ or 1, all positive. Finally, for the last case of (3.5), $N = 3, h = 5$, the possible exponents are among the six integrable weights of $\widehat{su}(3)_2$ and one checks the positivity of the eigenvalues of g for each of them. This establishes (3.5).

What are the graphs satisfying (3.5)? We leave again aside the case $N = 2$, which has already been discussed in the Introduction. In the case $N \geq 3, h = N + 1$, we have seen that all the weights of level 1 appear in the spectrum of exponents. The $\widehat{su}(N)_1$ fusion graphs are solutions, and it is not difficult to prove that there is no isospectral graph satisfying properties 1)–6) of Sect. 2.3. Finally, for $N = 3, h = 5$, if one takes only ρ and its orbit under σ as exponents, the only graph with 3 vertices and that spectrum is the oriented triangle graph $\mathcal{H}^{(5)}$ of Fig. 2 with $G_{01} = G_{12} = G_{20} = 2 \cos \frac{\pi}{5}$; if one takes as exponents all the six integrable weights of $\widehat{su}(3)_2$, i.e. the σ -orbit of ρ and the σ -orbit of 2ρ , the only graph with a matrix of entries integral or of the form (2.16) is the fusion graph $\mathcal{A}^{(5)}$ of $\widehat{su}(3)_2$ (Fig. 2). It is very likely that one cannot take the second triplet of exponents (the orbit of 2ρ) with a multiplicity different from zero or one, as will follow from a conjecture discussed below in Sect. 4.3. If so, this completes the list of graphs that lead to a positive definite form g .

For all these cases, the groups are of finite order, thus in the list discussed in the Introduction. We shall identify them below.

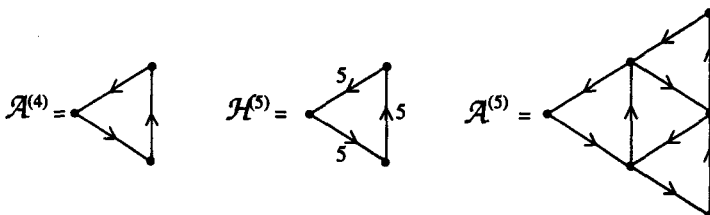


Fig. 2. Three $SU(3)$ graphs yielding a finite group

3.2. *The Coxeter Element.* We now come to a non-trivial property of the groups generated by this procedure, that depends crucially on the assumptions made in Sect. 2 and on the choice of signs in (3.1). I first recall that in the case of finite reflection groups, the product of all generators pertaining to a set of simple roots

$$R = \prod_a \dot{S}_a ,$$

called the Coxeter element, has two remarkable properties:

- i) it is independent, up to conjugation, of the simple set and of the order of the factors;
- ii) its spectrum of eigenvalues is given again by the exponents λ in the form

$$\text{eigenvalues of } R = \left\{ \exp \frac{-2i\pi}{h} \lambda \right\} . \tag{3.8}$$

A weaker version of that property is still true for the groups introduced in this paper.

Let R stand for

$$R = \prod_{\tau(a)=0} S_a \prod_{\tau(b)=1} S_b \cdots \prod_{\tau(f)=N-1} S_f , \tag{3.9}$$

i.e. the product of the blocks of reflections of given N -ality. Then

Proposition 3. *The element R is independent of the order of the S within each block; it is conjugate in the linear group $GL(n)$ to the product $-T^{-1}T^t$ of the matrices defined in (2.30), (2.31), and its spectrum is of the form*

$$(-1)^N \exp N \frac{-2i\pi}{h} (e_j, \lambda), \quad \lambda \in \text{Exp}, \quad j \text{ fixed} : 1 \leq j \leq N . \tag{3.10}$$

In particular this set of eigenvalues is independent of $j = 1, \dots, N$.

That the S may be permuted within each block follows from the fact that with the above assumptions, if $\tau(a) = \tau(b)$, then $g_{ab} = 0$, hence S_a and S_b commute.

The proof of (3.10) relies on a simple extension of the original proof by Coxeter of the analogous statement for finite reflection groups [9]. We make use of the notations introduced in (2.26), (2.28) to write the successive blocks of (3.9) as

$$\begin{aligned} S_{[0]} &:= \prod_{\tau(a)=0} S_a = \begin{pmatrix} -1 & \varepsilon A_{12} & -A_{13} & \cdots & \varepsilon A_{1N} \\ & \mathbf{1} & 0 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & & \cdots & & \mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & \varepsilon A_{12} & -A_{13} & \cdots & \varepsilon A_{1N} \\ & \mathbf{1} & 0 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & & \cdots & & \mathbf{1} \end{pmatrix} \begin{pmatrix} -1 & & & & \\ & \mathbf{1} & 0 & & \\ & & \ddots & & \\ & & & & 0 \\ & & & & \mathbf{1} \end{pmatrix} =: B_0 C_0 \end{aligned}$$

$$\begin{aligned}
 S_{[1]} &:= \prod_{\tau(b)=1} S_b = \begin{pmatrix} \mathbf{1} & & & & \\ & \mathbf{1} & \varepsilon A_{23} & -A_{24} & \cdots \\ & & \mathbf{1} & & \\ 0 & & & \ddots & \\ & & & & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & & & & \\ \varepsilon A_{21} & -\mathbf{1} & & & \\ & & \mathbf{1} & & \\ & 0 & & \ddots & \\ & & & & \mathbf{1} \end{pmatrix} =: B_1 C_1 \\
 \dots & \dots \\
 S_{[N-1]} &:= \prod_{\tau(f)=N-1} S_f = \begin{pmatrix} \mathbf{1} & & & & \\ & \mathbf{1} & & & \\ & & \ddots & & \\ \varepsilon A_{N1} & -A_{N2} & \cdots & \varepsilon A_{NN-1} & -\mathbf{1} \end{pmatrix} =: C_{N-1} . \quad (3.11)
 \end{aligned}$$

The matrices B_p and C_q just introduced are such that B_p commutes with all C_q , for $q < p$. This allows one to rewrite

$$\begin{aligned}
 R &= S_{[0]} S_{[1]} \cdots S_{[N-1]} = B_0 C_0 B_1 C_1 \cdots C_{N-1} \\
 &= B_0 B_1 \cdots B_{N-2} C_0 C_1 \cdots C_{N-1} . \quad (3.12)
 \end{aligned}$$

Then it is readily seen that the product $B_0 B_1 \cdots B_{N-2}$ is conjugate to the matrix T^{-1} written in (2.30), and likewise, that $C_0 C_1 \cdots C_{N-1}$ is conjugate to $-T^t$ of (2.31),

$$\begin{aligned}
 J B_0 B_1 \cdots B_{N-2} J^{-1} &= T^{-1} , \\
 J C_0 C_1 \cdots C_{N-1} J^{-1} &= -T^t , \quad (3.13)
 \end{aligned}$$

where the matrix J has been introduced in (2.32). Thus we have shown that our putative ‘‘Coxeter element’’ R is conjugate to $-T^{-1} T^t$.

On the other hand, let us form the polynomial $\Delta(z)$ that admits the roots $\varepsilon_j^{(\lambda)} := \exp -\frac{2i\pi}{h}(e_j, \lambda)$,

$$\begin{aligned}
 \Delta(z) &= \prod_{j=1}^N \prod_{\lambda \in \text{Exp}} (z - \varepsilon_j^{(\lambda)}) \\
 &= \det(z^N \mathbf{1} - z^{N-1} G_1 + z^{N-2} G_2 + \cdots + (-1)^{N-1} z G_{N-1} + (-1)^N \mathbf{1}) \\
 &= \det \begin{pmatrix} (z^N + \varepsilon) \mathbf{1} & -z^{N-1} A_{12} & z^{N-2} A_{13} & \cdots & -\varepsilon z A_{1N} \\ -z \varepsilon A_{21} & (z^N + \varepsilon) \mathbf{1} & -z^{N-1} A_{23} & & \\ \vdots & & & \ddots & \\ -z^{N-1} A_{N1} & z^{N-2} A_{N2} & \cdots & & (z^N + \varepsilon) \mathbf{1} \end{pmatrix} , \quad (3.14)
 \end{aligned}$$

where the second expression is obtained using the formulae (2.13); in the third line, we have used the block decomposition (2.26) of the G_p matrices. If we now multiply the i th row of blocks by z^{-i+1} and the j th column by z^{j-1} , which does not affect the determinant, the result depends only on z^N and is expressed in terms of the matrix T defined in (2.28),

$$\Delta(z) = \det(z^N T + (-1)^N T^t) = \det(z^N \mathbf{1} + (-1)^N T^{-1} T^t) . \quad (3.15)$$

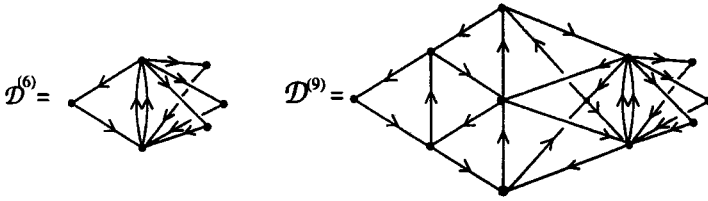


Fig. 3. Two $SU(3)$ orbifold graphs

Then the ‘‘Coxeter element’’ R of (3.9) which is conjugate to $-T^{-1}T^t$ has by the previous discussion the spectrum $\{(-e^{(\lambda)})^N\}$, which completes the proof of the proposition. In this last assertion, we have dropped the index j on $(-e^{(\lambda)})^N$ to emphasize the fact that this set of N^{th} powers of the $\varepsilon_j^{(\lambda)}$ does not depend on j . This follows from the consistency of the argument, but is also a direct consequence of (2.21).

Remarks. Note that if we order the product in (3.9) in the reverse way, as each block has square one, one gets the inverse of R . The latter, however, is also conjugate to R , since any $\varepsilon^{(\lambda)}$ comes along with its complex conjugate $\varepsilon^{(\lambda)*}$ in the spectrum. In the special case $N = 3$, one can do a little more and prove the independence of S (up to conjugacy in the group) with respect to the order of the three blocks. This follows once again from the fact that the three blocks in (3.9) have square one.

Note also that the ‘‘Coxeter element’’ R is of finite order, equal *at most* to h if Nh is even and to $2h$ if Nh is odd. (The order may be smaller, e.g. for the graphs $\mathcal{D}^{(6)}$ and $\mathcal{D}^{(9)}$ of Fig. 3 it is 2 resp. 6, while the value of h is given by the superscript.)

4. Identification of Some Groups

In this section, we shall identify some of the groups introduced above, and establish a certain number of isomorphisms between pairs of such groups. Given two groups Γ and Γ' generated by the reflections relative to two root bases $\{\alpha\}$ and $\{\alpha'\}$, the strategy for establishing the isomorphism $\Gamma \cong \Gamma'$ is to prove that the basis $\{\alpha'_a\}$ is found within the root system of the $\{\alpha_b\}$, i.e. that each α'_a is obtained by a finite number of reflections of Γ acting on some α_b . This will be referred to as a ‘‘change of basis within the root system.’’

4.1. Finite Groups. According to the discussion of Sect. 2.1, the group Γ generated by the S_α is finite for $N = 3$ and $h < 6$, and must therefore identify with one of the well known finite reflection groups. The groups associated with the graphs $\mathcal{A}^{(4)}$, $\mathcal{A}^{(5)}$ and $\mathcal{H}^{(5)}$ of Fig. 2 coincide indeed respectively with the finite groups A_3 , D_6 and H_3 of orders 24, $2^5 6!$ and 120. This is proved by finding a different basis $\{\beta_i\}$ of the space V within the root system α such that the β are simple positive roots of the finite Coxeter group and that their scalar product is thus encoded in the conventional Dynkin diagram. The change of basis in the last two cases is as

follows (Fig. 4)

$$\begin{aligned} \mathcal{A}^{(5)} \cong D_6 \quad \beta_1 = \alpha_1, \quad \beta_2 = S_1\alpha_2, \quad \beta_3 = S_2\alpha_3, \quad \beta_4 = S_2\alpha_4, \\ \beta_5 = S_3S_4\alpha_5, \quad \beta_6 = S_3S_2S_5\alpha_6, \\ \mathcal{H}^{(5)} \cong H_3 \quad \beta_1 = \alpha_1, \quad \beta_2 = -\alpha_2, \quad \beta_3 = S_2S_1\alpha_3. \end{aligned}$$

The discussion of the case of $\mathcal{A}^{(4)}$ may be extended to that of the group associated with the graph of weights of $\widehat{su}(N)_1$, which is nothing else than the finite Coxeter–Weyl group A_N . For N odd, $\langle \alpha_a, \alpha_b \rangle = (\delta_{ab} + 1)$, and the β defined by

$$\beta_1 = \alpha_1, \quad \beta_a = S_{a-1}\alpha_a = \alpha_a - \alpha_{a-1}, \quad a = 2, \dots, N \tag{4.1}$$

satisfy $\langle \beta_a, \beta_a \rangle = 2$, $\langle \beta_a, \beta_{a+1} \rangle = -1$, and all the other scalar products vanish. The β are thus identified with the simple positive roots of A_N . For N even, $\langle \alpha_a, \alpha_b \rangle = (-1)^{a-b}(1 + \delta_{ab})$. Then if one takes

$$\beta_1 = \alpha_1 \quad \beta_2 = \alpha_2 \quad \beta_a = S_{a-2}\alpha_a = \alpha_a - \alpha_{a-2}, \quad a = 3, \dots, N, \tag{4.2}$$

one finds again that the β are the simple positive roots of A_N with a peculiar labelling



As we shall see below, the groups have a natural interpretation in the context of $\mathcal{N} = 2$ superconformal field theories. The first two identifications could thus have

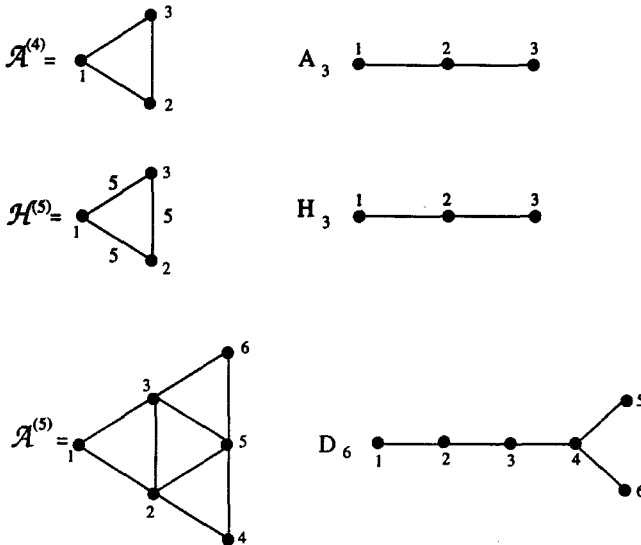


Fig. 4. Labelling of vertices of two pairs of graphs leading to isomorphic groups. Beware that the left one is regarded as a $su(3)$ graph (in which the orientations have been removed) whereas the right one is a $su(2)$ one; the prescription for the scalar products of roots varies according to Eq. (3.1)

been anticipated from identifications between coset realizations of $\mathcal{N} = 2$ superconformal theories [10]. Indeed (see for instance [11])

$$\begin{aligned} \frac{SU(3)_1}{SU(2)_2 \times U(1)} &\equiv \frac{SU(2)_2}{U(1)} , \\ \frac{SU(3)_2}{SU(2)_3 \times U(1)} &\equiv \left[\frac{SU(2)_8}{U(1)} \right]_{\text{“}D_6\text{”}} . \end{aligned} \tag{4.3}$$

The identification of the group associated with the graph of weights of $\widehat{su}(N)_1$ with the finite reflection group A_N reflects also an identification of $\mathcal{N} = 2$ coset superconformal theories, namely

$$\frac{SU(2)_{N-1}}{U(1)} \equiv \frac{SU(N)_1}{SU(N-1)_1 \times U(1)} . \tag{4.4}$$

An alternative description of these groups is by a *presentation* by generators and relations [12]. One may check for example that the group associated with the graph $\mathcal{A}^{(4)}$ is generated by the three generators S_1, S_2, S_3 subject to

$$\begin{aligned} S_1^2 &= S_2^2 = S_3^2 = \mathbb{I} , \\ (S_1 S_2)^3 &= (S_2 S_3)^3 = (S_3 S_1)^3 = \mathbb{I} , \\ S_1 S_2 S_3 S_2 S_1 &= S_3 S_2 S_3 . \end{aligned} \tag{4.5}$$

The last relation (or any permutation thereof) is an example of these non-trivial relations satisfied generically by the reflections S_a of our root systems. Likewise, the group associated with the graph $\mathcal{A}^{(5)}$ above (Fig. 2) is generated by the three generators S_1, S_2, S_3 subject to

$$\begin{aligned} S_1^2 &= S_2^2 = S_3^2 = \mathbb{I} , \\ (S_a S_b)^2 &= S_c S_a S_c S_b S_c , \\ (S_a S_b S_c)^2 &= S_b S_c S_a S_b , \end{aligned} \tag{4.6}$$

with $a \neq b \neq c \neq a$ in the last two relations. Equations (4.6) imply $(S_a S_b)^5 = \mathbb{I}$ as expected, and $(S_a S_b S_c)^5 = \mathbb{I}$.

4.2. Generalities on the Infinite Cases. When the conditions (3.5) are not fulfilled, the bilinear form g is non-definite positive, but one may still assert that the numbers of negative eigenvalues and of zeros are even. Consider an eigenvalue $g^{(\lambda)}$ of g associated with an exponent λ . Proposition 1 tells us that the conjugate $\bar{\lambda}$ is also an exponent. If $\lambda \neq \bar{\lambda}$, as $g^{\bar{\lambda}} = (g^\lambda)^*$, (see (2.23)), and are both real as eigenvalues of a real symmetric form, they give equal contributions to the signature of g . If $\bar{\lambda} = \lambda$, a close look at the expression (3.4) of $g^{(\lambda)}$ shows that its factors come in complex conjugate pairs, that it is thus non-negative and that in fact it cannot vanish. We thus conclude that the signature of g contains an even number of zeros and an even number of minus signs. Note that the form is degenerate only at specific values of h (for example $h = 6, 8, 10, \dots$ for $N = 3$) whereas the existence of negative eigenvalues is the generic situation according to Proposition 2.

4.3. *Identification of Some Infinite Cases.* In this section, we shall identify some of the groups of infinite order associated with graphs and propose some conjectures.

The identity (4.4) is a particular case of a more general one that states that the $\mathcal{N} = 2$ theory based on

$$\frac{SU(n+m)_k}{SU(n)_{k+m} \times SU(m)_{k+n} \times U(1)} \tag{4.7}$$

is independent of permutations of the three integers m, n, k ; in particular, taking $m = 1$,

$$\frac{SU(n+1)_k}{SU(n)_{k+1} \times U(1)} \equiv \frac{SU(k+1)_n}{SU(k)_{n+1} \times U(1)}, \tag{4.8}$$

which suggests the following

Conjecture 1. *The reflection groups associated with the graphs of integrable weights of $\widehat{su}(n+1)_k$ and $\widehat{su}(k+1)_n$ are isomorphic.*

Both graphs have $\binom{k+n}{n}$ vertices, and there is a one-to-one bijection between these vertices (i.e. weights) provided by the reflection of the corresponding Young tableaux along their diagonal.³

This conjecture may be verified in the case of $\widehat{su}(4)_2 \equiv \widehat{su}(3)_3$, for which the following change of basis within the root system maps the graphs on one another (the β 's refer to $\widehat{su}(4)_2$, the α 's to $\widehat{su}(3)_3$):

$$\begin{aligned} \beta_1 &= \alpha_1 & \beta_2 &= \alpha_2 - \alpha_1 & \beta_3 &= \alpha_4 - \alpha_2 + \alpha_1 & \beta_4 &= \alpha_3 - \alpha_1 & \beta_5 &= \alpha_6 - \alpha_5, \\ \beta_6 &= \alpha_5 - \alpha_4 - \alpha_3 + \alpha_2 & \beta_7 &= \alpha_8 - \alpha_7 + \alpha_6 - 2\alpha_5 + \alpha_4 & \beta_8 &= \alpha_7, \\ \beta_9 &= \alpha_{10} + \alpha_9 - 3\alpha_8 + \alpha_7 - 3\alpha_6 + 4\alpha_5 + \alpha_3 - 3\alpha_2 + \alpha_1, \\ \beta_{10} &= \alpha_9 + \alpha_8 - \alpha_7 - 3\alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 - \alpha_1. \end{aligned} \tag{4.9}$$

We shall see below that the conjecture is also consistent with further data and conjectures.

Since this duality maps the representations of $\widehat{su}(n+1)_k$ onto those of $\widehat{su}(k+1)_n$ by just reversing the Young tableaux it is clear that the number of representations with a given number of boxes is the same in both:

$$\begin{aligned} &\text{card} \left\{ \lambda \in \mathcal{P}_{++}^{(k+n+1)}(su(n+1)), \sum_{i=1}^n (\lambda_i - 1)i = l \right\} \\ &= \text{card} \left\{ \mu \in \mathcal{P}_{++}^{(k+n+1)}(su(k+1)), \sum_{i=1}^k (\mu_i - 1)i = l \right\}. \end{aligned} \tag{4.10}$$

This leads to the observation that the spectra of their ‘‘Coxeter elements’’ computed according to (3.10) coincide. Indeed the spectrum of R for $\widehat{su}(n+1)_k$ is

$$\exp -i\pi \frac{(n+1)(k+1)}{n+k+1} \exp \frac{2i\pi}{n+k+1} \sum_i i(\lambda_i - 1), \tag{4.11}$$

³ Note that the relations between these two situations is *not* what is referred to as level-rank duality in the literature [13], which compares $\widehat{su}(n)_k$ and $\widehat{su}(k)_n$.

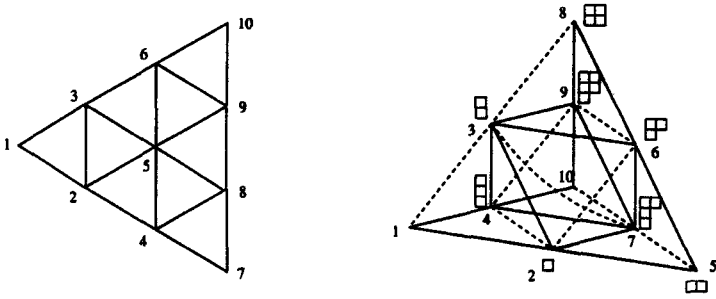


Fig. 5. Labelling of vertices (hence of roots) of the two graphs of $\widehat{su}(3)_3$ and $\widehat{su}(4)_2$. In the former, edges code scalar products equal to +1. The latter graph is the graph of matrix (2.25), with the edges of G_2 that give rise to scalar products = +1 indicated by broken lines, those of $G_1 + G_3$ (solid lines) coding scalar products = -1. (For clarity, the Young tableau corresponding to vertex 10 has not been depicted: it is made of three rows of two boxes)

and is by (4.10) the same as that of $\widehat{su}(k + 1)_n$. The same property may be checked in the other cases where we have established the isomorphism of two reflection groups Γ and Γ' . Moreover, in the cases $H_3 \cong \mathcal{H}^{(5)}$, $A_N \cong \mathcal{A}_N^{(N+1)}$, N odd or $N = 4$, one checks explicitly that the two elements $R(\Gamma)$ and $R(\Gamma')$ are conjugate in the group. This leads to the conjecture that the ‘‘Coxeter element’’ R has (up to conjugation) a more intrinsic nature than suggested by the special presentation (3.9). More precisely

Conjecture 2. For two groups Γ and Γ' associated by the previous construction with two graphs \mathcal{G} and \mathcal{G}' of $su(N)$, resp. $su(N')$, the isomorphism $\Gamma \cong \Gamma'$ implies that the Coxeter elements R and R' are conjugate in the group.

Remark. Returning to the discussion at the end of Sect. 3.1, we thus see that a graph with a spectrum given by the σ -orbit of ρ , and the σ -orbit of 2ρ with a multiplicity larger than 1 could not match the spectrum of the Coxeter element of any of the finite Coxeter groups.

In the rest of this section, we shall see that some groups of infinite order associated with graphs may be identified with groups encountered in singularity theory.

This is the case of the group associated with the graph corresponding to the weight lattice of $\widehat{su}(3)$ at level 3. At level 3, (i.e. $h = 6$), the graph depicted on Fig. 1 has 10 vertices. By a suitable change of basis of the α 's within the root system, it is seen that the group is isomorphic to the monodromy group of the singularity $X^6 + Y^3 + aX^2Y^2$ tabulated as J_{10} in [14]. This may not be obvious on the appearance of the generalized Dynkin diagram describing the intersection form of the vanishing cycles of the J_{10} singularity (Fig. 6).

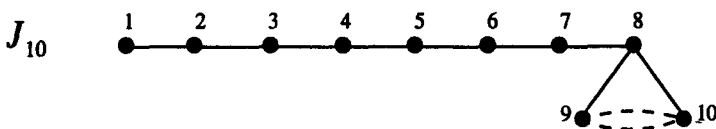


Fig. 6. The Dynkin diagram of the J_{10} singularity. Here the double broken line is coding a scalar product equal to +2, the solid ones a scalar product equal to -1

Note that this singularity is also often associated with the Dynkin diagram of the affine algebra \widehat{E}_8 . (The apparent mismatch between the rank 9 of \widehat{E}_8 and the number 10 of vertices of J_{10} reflects the extension of the Cartan algebra of the former by an additional independent generator dual to the central element.)

That this graph has to do with this singularity is no surprise, as the polynomial $X^6 + Y^3 + aX^2Y^2$ is (for a specific value of the coefficient a) the homogeneous part of the fusion polynomial of $\widehat{su}(3)$ at level 3 [7]. This leads to the natural

Conjecture 3. *The group associated with the graph of weights of $\widehat{su}(N)_k$ is the monodromy group of the singularity described by the homogeneous part of the fusion polynomial, i.e. the terms of degree $k + N$ in t in the expansion of $\ln(1 - tX_1 + t^2X_2 + \dots + (-t)^{N-1}X_{N-1})$.*

This conjecture may be established for $N = 3$, because for polynomials in only two variables, one can make use of the method of A’Campo [15] which simplifies greatly the determination of the monodromy group whenever one has a resolution of the singularity such that: i) all critical points are real, ii) all the saddle point values vanish. As shown by Warner [16], one may find such a resolution of the homogeneous part of the fusion potential of $\widehat{su}(3)$ at all levels. Then the method of A’Campo provides us with a description of the monodromy group by a generalized Dynkin diagram which upon a change of basis within the root system may be recast in the form of the weight lattice of $\widehat{su}(3)_k$ [16] (Fig. 7).

This conjecture may be shown to be consistent with the previous one on the rank-level duality. Consider the pair

$$\widehat{su}(N)_{N-2}, \widehat{su}(N - 1)_{N-1} .$$

Although the fusion potential of $\widehat{su}(N)_{N-2}$ has one more variable than the one of $\widehat{su}(N - 1)$, the extra variable X_{N-1} appears at most quadratically in it and does not affect the singularity theory. The two monodromy groups must thus be isomorphic, which agrees with Conjecture 1.

5. Non Integrally Laced Graphs

This section is devoted to a closer study of the situation where we allow some of the matrices G_p to have non-integral entries. This is a generalization of what was encountered with the classical finite groups not of ADE type, see the end of Sect. 2.1. It must be stressed that all the considerations of this section are based on empiric observations, a good understanding of which is still missing.

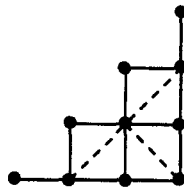


Fig. 7. A generalized Dynkin diagram equivalent to the graphs of Figs. 5 and 6. Same convention as in Fig. 5

We shall first recall an observation made in [17] where it was shown that the non-ADE Coxeter–Dynkin diagrams appear in the discussion of algebras associated with the ADE ones. Let us return to the normal matrices $(G_p)_{ab}$ satisfying the properties listed in Sect. 2.3 and assumed to have integral entries, and let us introduce their orthonormal eigenvectors $\psi_a^{(\lambda)}$. We use these eigenvectors to construct the two following sets of numbers:

$$M_{\lambda\mu}^v = \sum_{a \in \mathcal{V}} \frac{\psi_a^{(\lambda)} \psi_a^{(\mu)} \psi_a^{(v)*}}{\psi_a^{(\rho)}} , \tag{5.1a}$$

$$N_{ab}^c = \sum_{\lambda \in \text{Exp}} \frac{\psi_a^{(\lambda)} \psi_b^{(\lambda)} \psi_c^{(\lambda)*}}{\psi_1^{(\lambda)}} , \tag{5.1b}$$

where we have assumed the existence of a selected vertex denoted 1 such that none of the components $\psi_1^{(\lambda)}$ vanishes. These two sets of numbers may be regarded as structure constants of two commutative associative algebras attached to the graph \mathcal{G} . For the graphs of type \mathcal{A} (the truncated weight lattices), these two algebras are isomorphic and reduce to the fusion algebra. I call the first algebra M the Pasquier algebra [18]. Now, in all cases, we may regard the numbers N_{ab}^c as the entries of matrices $N_{ab}^c = (N_a)_b^c$, and write

$$\forall \lambda \in \text{Exp}, \quad \forall p = 1, \dots, N - 1 \quad \sum_a (G_p)_{1a} \psi_a^{(\lambda)} = \gamma_p^{(\lambda)} \psi_1^{(\lambda)} ,$$

i.e.

$$\gamma_p^{(\lambda)} = \sum_a (G_p)_{1a} \frac{\psi_a^{(\lambda)}}{\psi_1^{(\lambda)}} ,$$

and hence

$$G_p = \sum_a (G_p)_{1a} N_a . \tag{5.2}$$

Thus the matrices G_p are linear combinations of the N_a 's with non-negative integral coefficients. In fact in many cases, the vertex “1” is connected (in the sense of G_1) to a single vertex a and hence $G_1 = N_a$. In those cases, the matrices N of (5.1b) provide an actual realization of an idea of Ocneanu and Pasquier [19, 18] to look at the associative algebra attached to the vertices of the graph and generated by G_1 .

Now, start from a graph \mathcal{G} , compute its M algebra and look for possible subalgebras of the latter consistent with the requirements of Sect. 2.4. In other words, we look for a subset $\widetilde{\text{Exp}}$ of the set Exp of exponents such that

$$\lambda, \mu \in \widetilde{\text{Exp}}, \quad M_{\lambda\mu}^v \neq 0 \Rightarrow v \in \widetilde{\text{Exp}} , \tag{5.3}$$

and we demand that the set $\widetilde{\text{Exp}}$ satisfies the same properties as Exp , namely that it is stable under the action of σ and \mathcal{C} and contains the weight ρ of the identity representation. Call $\widetilde{M}_{\lambda\mu}^v$ the new structure constants obtained by restricting $M_{\lambda\mu}^v$ to $\lambda, \mu, v \in \widetilde{\text{Exp}}$. It is easily seen that they are diagonalized by a set of $\widetilde{\psi}_a$,

$$\widetilde{\psi}_a^{(\lambda)} = \frac{\psi_a^{(\lambda)}}{\mathcal{N}_a}$$

for a subset of $a \in \mathcal{V}$ and a suitable normalization \mathcal{N}_a . Use these $\tilde{\psi}$ to construct the dual algebra of matrices \tilde{N}_α . In general, the entries of the \tilde{N}_α are irrationals. In [17] it was noticed that when this procedure is applied to the ADE Dynkin diagrams, among these matrices \tilde{N} , one at least, call it \tilde{G} , has non-negative entries of the form $\tilde{G}_{\alpha\beta} = 2 \cos \frac{\pi}{m_{\alpha\beta}}$, $m_{\alpha\beta} \in \{2, 3, \dots\}$, and in fact is the symmetrized form (2.4) of the adjacency matrices of a non-ADE Coxeter–Dynkin diagram! In that way, the Dynkin diagram B_n is obtained from A_{2n-1} , C_n from D_{n+1} , F_4 from E_6 , G_2 from D_4 , H_3 from D_6 and H_4 from E_8 . The only exception is the case of $I_2(k)$, where the \tilde{N} algebra is generated by the two-by-two matrices

$$\tilde{N}_1 = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{N}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whereas the Coxeter matrix (2.3) is $(2 \cos \frac{\pi}{k})\tilde{N}_2$.⁴

Remarkably, the same procedure applies to the known solutions with integral $(G_p)_{ab}$ to the conditions of Sect. 2.4, that have a set of non-negative structure constants of the M and N algebras. It produces more solutions with non-integral entries and manufactures subgroups out of the original groups. We may call these new solutions “non-integrally laced” graphs resp. groups, to refer to the generic non-integrality of the entries of the matrices. As in the case $N = 2$, two classes of solutions emerge. In the first class, among the matrices \tilde{N} of the dual subalgebra, one can find at least $N - 1$ matrices that have non-negative entries of the form $2 \cos \frac{\pi}{m_{\alpha\beta}}$, $m_{\alpha\beta} \in \{2, 3, \dots, \infty\}$. Moreover these matrices (or linear combinations with integral coefficients thereof) qualify as possible matrices G_p , i.e. satisfy the conditions of Sect. 2.3 (cp. (5.2)). This is exemplified on three cases on Fig. 8 using three graphs found in [8]. These graphs were shown to be associated with modular invariants coming from conformal embeddings of $\widehat{su}(3)_k$ in a larger algebra (resp. $\widehat{su}(3)_5 \subset \widehat{su}(6)_1$, $\widehat{su}(3)_9 \subset (\hat{e}_6)_1$, and $\widehat{su}(3)_{21} \subset (\hat{e}_7)_1$) [21]; the subalgebra of the Pasquier algebra is labelled by the weights belonging to the block of the identity.

The second class of solutions is a simple generalization of the situation discussed above with the Coxeter graph $I_2(k)$. For any graph relative to $N \geq 2$, the Pasquier M algebra admits a subalgebra whose generators are labelled by the σ -orbit of ρ : $M_\rho, M_{\sigma\rho}, \dots, M_{\sigma^{N-1}\rho}$. This is in fact the smallest subalgebra consistent with our requirements. In that case, the procedure of considering the dual subalgebra does not produce the right adjacency matrices but rather the adjacency matrices G_p of $\widehat{su}(N)_1$. Those have a spectrum of eigenvalues equal to N^{th} roots of unity, ξ^{lp} , $\xi := \exp \frac{2i\pi}{N}$, $l = 0, \dots, N - 1$, instead of the eigenvalues $\gamma_p^{(\rho)} \xi^{lp}$, which follow from (2.13). The correct adjacency matrices are thus of the form

$$(G_p)_{ab} = \delta_{a+p, b \bmod N} \gamma_p^{(\rho)}, \quad a, b = 0, \dots, N - 1, \tag{5.4}$$

where $\gamma_p^{(\rho)}$ may be expressed as a q -deformed binomial coefficient, $q = e^{i\pi/h}$,

$$\gamma_p^{(\rho)} = \binom{N}{p}_q = \frac{\sin \frac{\pi N}{h} \dots \sin \frac{\pi(N-p+1)}{h}}{\sin \frac{\pi}{h} \sin \frac{2\pi}{h} \dots \sin \frac{p\pi}{h}}. \tag{5.5}$$

⁴ As shown by Shcherbak, Moody and Patera [20], there is a simple way to see how the Coxeter groups of type $B \cong C, F, G, H$ or I appear as subgroups of some ADE Coxeter group and how an appropriate folding of the ADE Dynkin diagram gives rise to the Coxeter–Dynkin diagram of the subgroup. The two previous constructions are not independent, as will be explained elsewhere.

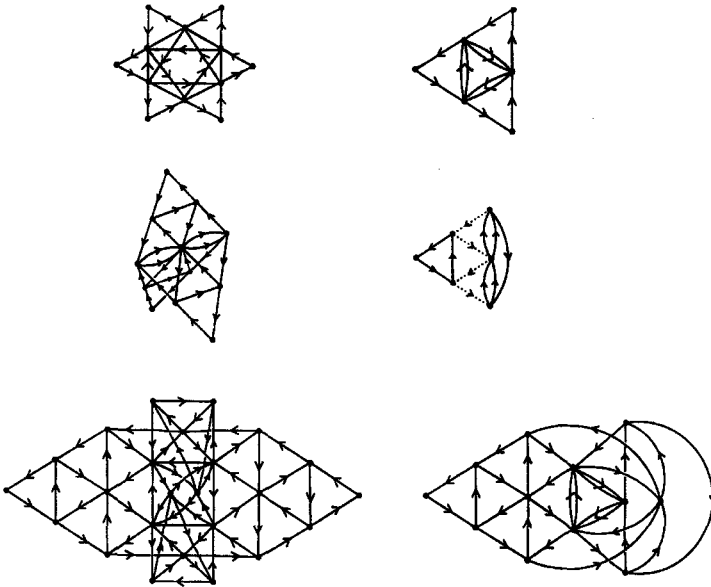


Fig. 8. Three graphs of $su(3)$ (on the left) having a M -subalgebra that leads to new graphs (on the right). The dotted line denotes an edge carrying $\sqrt{3} = 2 \cos \frac{\pi}{6}$, and the double lines are indeed edges with $G_{ab} = 2$

One may prove that the latter expression is a polynomial of degree $p(N - p)$ in $2 \cos \frac{\pi}{h}$. An example of this class is provided by the graph $\mathcal{H}^{(5)}$ exhibited in Fig. 2 for which the non-vanishing entries of the adjacency matrix G_1 are $(G_1)_{01} = (G_1)_{12} = (G_1)_{20} = 1 + 2 \cos \frac{2\pi}{5} = 4 \cos^2 \frac{\pi}{5} - 1 (= 2 \cos \frac{\pi}{5})$.

Clearly a systematic analysis and classification of the possible solutions would be highly desirable. We shall see below how these Pasquier algebra and subalgebras appear in the context of conformal field theories.

6. Physical Interpretation of the Graphs and Reflection Groups

6.1. *In Conformal Field Theories or Lattice Models.* It has been mentioned in the Introduction that there are empirical relations between our graphs and the classification of $su(N)$ integrable lattice models and of $su(N)$ WZW or coset conformal field theories. Recall that in the context of lattice models one looks for solutions of the Yang-Baxter equation based on the quantum group $\mathcal{U}_q(sl(N))$. One finds classes of solutions indexed by graphs of the type discussed above [22, 8, 23]. The rôle of the graph \mathcal{G}_1 is to specify what are the allowed configurations of the degrees of freedom (or “heights”). The Boltzmann weights of the model are obtained by finding a representation of a quotient of the Hecke algebra (a deformation of the algebra of the symmetric group) on the space of paths on the graph \mathcal{G}_1 . In the context of cft’s, the rôle of the graph is more indirect. The partition functions of these theories on a torus are sesquilinear forms in characters χ_λ of the affine algebra $\widehat{su}(N)_k$ indexed by integrable weights λ . The constraint of modular invariance and of non-negativity of the coefficients restricts enormously the possible expressions

$\sum_{\lambda, \lambda'} N_{\lambda\lambda'} \chi_\lambda \chi_{\lambda'}^*$. It turns out that in many instances, diagonal terms in these expressions are labelled by λ running over one of the sets of “exponents” encountered above in the spectrum of integrally laced graphs. This correspondence is known to be one-to-one for $\widehat{su}(2)$ theories, for which all the modular invariants may be labelled by an ADE Dynkin diagram [24]. For $\widehat{su}(N)$ theories, $N > 2$, however, some modular invariants exhibit diagonal terms that do not satisfy the property of invariance of the set Exp under σ that followed from condition 3) imposed in Sect. 2.3 on the graphs [25–26]. It seems, however, that these cases may always be recovered from the others by a twist of the right sector with respect to the left one by an automorphism of the fusion algebra. Another observation is that there is a class of modular invariants relative to the coset theories

$$\frac{SU(N)_{k-1} \times SU(N)_1}{SU(N)_k},$$

the “minimal W_N models,” for which the problem does not seem to occur: see [27] for a further discussion.

Conversely, it has been known for some time that some graphs satisfying the conditions of Sect. 2.3 are irrelevant for the classification of modular invariants and do not seem to support an integrable lattice model. Indeed in [23], some graphs had to be discarded. It thus appears that we are still missing some further restriction on the graphs.

These little discrepancies notwithstanding, it seems that there is a hidden connection between the problems of classification of graphs, of cft’s and of integrable lattice models. At this point, it may be useful to recall that the manifestation of this connection goes beyond the mere coincidence of spectra of “exponents” of graphs with the diagonal terms of modular invariants and involves the Pasquier algebras introduced in Sect. 5. First there is an empirical correspondence between the graphs that have a pair of non-negative M and N algebras and the cft’s whose partition function is a sum of squares of combinations of characters. Moreover there is some evidence that the pattern of algebras and subalgebras of type M is connected with the structure of the operator product algebra (OPA) of conformal field theories and lattice models. It is in fact in that context that these algebras were first introduced [18], and the quantitative role of the M algebra in the determination of the structure constants of $su(2)$ conformal theories has been clarified recently [28]. In a more recent work [27], the extension of these considerations to higher rank cases is discussed: it is shown that the vanishing of the matrix elements of the M algebra implies that of the OPA structure constants. Therefore a subalgebra of the Pasquier algebra signals a subalgebra of the OPA.

A few more facts are known about the association between a cft and a graph. Starting from a graph with non-negative M and N algebras, an empirical reconstruction of the modular invariant using the theory of “ c -algebras” [29] has been developed in [23]. It is also believed that the non-simply (or “non-integrally”) laced graphs do not lead to an acceptable modular invariant partition function, but rather to an invariant under a subgroup of the modular group [17]. Reciprocally, in a variety of cases of cft’s, (typically orbifolds or conformal embeddings), considerations on the OPA that generalize those of [28] enable one to determine an appropriate graph [27].

In yet another approach, these graphs have been used in a recent work to construct invariants of three-manifolds, à la Turaev–Viro [30].

In view of these connections of graphs with cft’s and lattice models, it is natural to wonder whether the reflection groups constructed in this work manifest themselves in those physical contexts. I have unfortunately no definite answer to that question. The previous discussion suggests that the reflection group might be hidden in some features of the OPA.

As already mentioned in [1], this suggests a programme based on the assumption that the previous observations are of general validity. If one could ascertain the connection between the consistent subalgebras of the OPA and reflection groups, one might consider classifying the relevant reflection groups, find their presentation by generators subject to the conditions of Sect. 2, and reconstruct the graph; one would then discard the cases of non-integrally laced graphs, that correspond to theories that are inconsistent at higher genus. This would yield a set of admissible graphs, and with the methods of c -algebras or of counting of “essential paths” [30], one would reconstruct the modular invariant partition functions. It remains to see how realistic this programme is ...

6.2. *In $\mathcal{N} = 2$ Superconformal Theories and Topological Field Theories.* The context of $\mathcal{N} = 2$ superconformal theories and of topological field theories (tft’s) seems the most relevant one for the interpretation of the reflection groups. We first recall that there is a large class of $\mathcal{N} = 2$ theories amenable to a description by an effective Landau–Ginsburg (LG) superpotential [31]. The latter is a quasihomogeneous polynomial W in some chiral superfields X_i , $i = 1, \dots, n$, with an isolated critical point at the origin in field space. In the simplest cases, it is thus to be found in the lists of singularities [14]. This is in particular the case of the so-called minimal $\mathcal{N} = 2$ theories, based on $sl(2)$, that are all described in that way and for which the relevant singularity is a simple one (with no modulus), i.e. falls once again in an ADE classification [14].

In all cases, the elements of the chiral ring \mathcal{R} are in one-to-one correspondence with those of the local ring of the singularity, i.e. the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$ quotiented by the ideal generated by the derivatives $\partial W / \partial X_i$. The $U(1)$ charges q_j of the chiral fields are proportional to the homogeneity degrees of the elements of a basis of the local ring, with the proportionality factor fixed by the requirement that $q(W) = 1$. If c denotes the central charge (of the Virasoro algebra), the $U(1)$ charges of the Ramond ground states are $q_j^{(\text{Ramond})} = q_j^{(\text{chiral})} - \frac{c}{6}$. On the other hand, it is a standard practice in singularity theory to look at deformations of the polynomial that resolve the singularity and to study the intersection form of the vanishing cycles and the monodromy group of these cycles when the deformation parameters are changed along loops. For n , the number of variables, even, the intersection form is encoded in a generalized Dynkin diagram. Cecotti and Vafa [32] have shown that this intersection form counts the (signed) number of solitons A_{ab} interpolating between the vacua of the $\mathcal{N} = 2$ supersymmetric theory obtained by perturbing the original $\mathcal{N} = 2$ superconformal theory. For a special choice of deformation and of labelling, the matrix A may be taken upper triangular. The monodromy operator is the form $H = SS^{-t}$ (with the notations of [32]), with $S = \mathbb{I} - A$, and its eigenvalues are $\exp 2\pi i q_j^{(\text{Ramond})}$.

This applies in particular to the $\mathcal{N} = 2$ theories based on the cosets

$$\frac{SU(N)_k}{SU(N-1)_{k+1} \times U(1)} \tag{6.1}$$

(cp. (4.8)). There the LG potential W is the quasihomogeneous part of the fusion potential of $\widehat{su}(N)_k$; as already discussed in Sect. 4, it is a polynomial of degree $h = k + N$ in X_1, \dots, X_{N-1} of respective degrees $1, \dots, N - 1$. The chiral fields may be labelled by integrable weights $\lambda \in \mathcal{P}_{++}^{(k+N)}$ (cp. (2.6)). Their $U(1)$ charges are thus

$$q_\lambda = \frac{1}{h} \sum_{i=1}^{N-1} i(\lambda_i - 1) \quad \text{with} \quad \sum_i \lambda_i \leq h - 1 \quad \lambda_i \geq 1, \quad (6.2)$$

with $h = k + N$. The central charge c of the $\mathcal{N} = 2$ theories under discussion is

$$c = 3(N - 1) \left(1 - \frac{N}{h} \right), \quad (6.3)$$

thus

$$q_\lambda^{(\text{Ramond})} = q_\lambda - \frac{c}{6} = \frac{1}{h} \sum_{i=1}^{N-1} i(\lambda_i - 1) - \frac{1}{2}(N - 1) + \frac{N(N - 1)}{2h}. \quad (6.4)$$

One thus sees that the eigenvalues of the monodromy operator H coincide with those of the opposite $-R \approx T^{-1}T^t$ of the ‘‘Coxeter’’ operator of the present paper, (Proposition 3), given by (4.11):

$$\begin{aligned} \text{Eigenvalues of } (-R) &= (-1)^{N-1} \exp \left\{ i\pi \frac{N(N - 1)}{h} + \frac{2i\pi}{h} \sum_{i=1}^{N-1} i(\lambda_i - 1) \right\} \\ &= e^{2i\pi q_\lambda^{(\text{Ramond})}}. \end{aligned} \quad (6.5)$$

In fact, the upper triangular S of [32] identifies with the conjugate by J (Eq. (3.11)) of our T of (2.28), $S = J T J^{-1}$, so that our operator $-R$ identifies (up to conjugation by J) with the transpose H^t of their monodromy operator.

The previous discussion has been implicitly dealing with the graph $\mathcal{A}_N^{(k+N)}$ and the corresponding group. To make the connection with the preceding section, we have been considering a $\mathcal{N} = 2$ superconformal theory (and its deformation) whose genus-one partition function is constructed out of the diagonal modular invariant of $\widehat{su}(N)_k$. One may also consider other, non-diagonal, modular invariants, and the resulting $\mathcal{N} = 2$ coset theory. In many cases, however, for $N > 2$, the theory does not possess a LG superpotential. The simplest example is provided by the $SU(3)_3 / (SU(2)_4 \times U(1))$ coset theory in which one chooses the orbifold modular invariant for the numerator. Correspondingly, we take the graph $\mathcal{D}^{(6)}$ of Fig. 3. Then the counting of chiral fields as exposed by the partition function, or the $U(1)$ charges (6.2) computed from the exponents of that graph, are incompatible with the Poincaré polynomial that would follow from a LG superpotential.

Cecotti and Vafa, however, have been able to extend their discussion to cases where the LG picture does not apply. The same results hold true: the operator H is the monodromy matrix around the origin of the solution to a linear system, whose consistency equations are the tt^* equations [33], and S is its Stokes matrix. Also they considered the matrix $B = S + S^t$ and prove (under some assumptions) that the number r (resp. s) of positive (resp. negative) eigenvalues of B is

$$\begin{aligned} r &= \# \left\{ 2p - \frac{1}{2} < q^{(\text{Ramond})} < 2p + \frac{1}{2} \right\} \\ s &= \# \left\{ 2p + \frac{1}{2} < q^{(\text{Ramond})} < 2p + \frac{3}{2} \right\}. \end{aligned} \quad (6.6)$$

In view of the previous identification, their matrix B is nothing else than our metric g (cp. Eq. (3.2)),

$$B = S + S^t = J(T + T^t)J^{-1} = g. \quad (6.7)$$

Thus Eq. (6.6) is a statement on the signature of our metric, which may be verified on the expression on (6.4) for graphs with low values of N and k .

To summarize, we have found that the graphs discussed in this paper yield actual realizations of those discussed by Cecotti and Vafa, in the specific case of the $\mathcal{N} = 2$ theories (6.1). It is most likely that they describe the pattern of solitons that arise when the theory is perturbed by the least relevant operator: this is the Chebishev perturbation, so called because in the simplest case of $SU(2)$ theories of A_n type, it changes the homogeneous superpotential into the Chebishev polynomial $T_{n+1}(X)$ [34]. It is indeed known that for that perturbation, the pattern of solitons reproduces the classical Dynkin diagrams or their generalizations [35, 11].

The situation is quite parallel in the case of the topological field theories. Those may be obtained by the “twisting” of $\mathcal{N} = 2$ theories but may also be defined and studied for their own sake. Then the defining equations in genus zero are the so-called Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [36]. Recently, Dubrovin has been able to reformulate these equations in a coordinate independent way, as expressing the existence of a geometrical structure on the moduli space of these theories. First he proved that he could associate such a structure, i.e. a solution to the WDVV equations with the space of orbits of any finite Coxeter group [37]. This accounts for all the “simple” topological theories, i.e. with a central charge of the corresponding $\mathcal{N} = 2$ theory $c < 3$, or alternatively, such that all $U(1)$ charges of moduli be positive. In that way, he recovered not only the *ADE* solutions once again, but also others associated with the non-simply laced Coxeter matrices. (The consistency of the latter theories at higher genus when coupled to gravity has been questioned recently [38].) Dubrovin also showed the existence of two independent flat metrics on the moduli space of tft’s, and, in a subsequent work, he studied the differential equations that express the flat coordinates for the first metric in terms of those of the second one [39]. He proved that this differential system has a non-trivial monodromy, which under certain assumptions, is generated by reflections.

In the special cases of tft’s described by a LG potential [40], this monodromy group is the monodromy group of the singularity. We have seen that this is also the case with the groups studied here (Conjecture 3). It is thus a very natural conjecture that the groups studied in the present paper are actual realizations of the considerations of Dubrovin for those tft’s that emanate from $su(N)$ $\mathcal{N} = 2$ theories.

7. Discussion and Conclusion

In this paper I have shown that graphs that have appeared recently in various contexts of mathematical physics have the natural interpretation of encoding the geometry of a root system and allow one to construct reflection groups.

From the mathematical point of view, in addition to all the conjectures that have been proposed, this paper has left many questions unanswered. To quote a few:

- We have found a certain number of isomorphisms of reflection groups. Under which conditions on two graphs does one get two isomorphic groups?

- In the finite cases like the graphs $\mathcal{A}^{(4)}$ or $\mathcal{A}^{(5)}$ of $su(3)$, what is the specific property of the choice of roots α within the root system of the A_3 or D_6 Coxeter groups?
- Given a reflection group, when can we assign it to a $su(N)$ graph? If the Coxeter element has an intrinsic meaning, (as suggested in Conjecture 2), and may be identified in the group, it gives some information about the spectrum Exp of the graph, but it remains to reconstruct the graphs with a given spectrum.
- Can one classify these groups/graphs?
- What can one say about the nature of the possible invariants of these groups? Does the set Exp encode any information about those, as it does in the $su(2)$ cases?

Another problem deserves special mention. For each of the graphs considered in this paper there exists a family of matrices V_{ab}^λ intertwining its adjacency matrices G_p with those – denoted here A_p – of the basic fusion graph of Sect. 2.2 with the same value of h , i.e. satisfying

$$(A_p)_{\lambda\mu} V_{ab}^\mu = V_{ac}^\lambda (G_p)_{cb} \tag{7.1}$$

(with summation over repeated indices). In fact an explicit formula may be given for a class of V :

$$V_{ab}^\lambda = \sum_{\mu \in \text{Exp}} \frac{\phi_\lambda^{(\mu)}}{\phi_\rho^{(\mu)}} \psi_a^{(\mu)} \psi_b^{(\mu)*} . \tag{7.2}$$

Here the ϕ denote the eigenvectors of the A_p matrices. Equation (7.1) may be recast as recursive equations for the V 's which show that these coefficients are integers. The surprise, however, is that they are *non-negative* integers. This was checked case by case in [8], and then Dorey, in the $su(2)$ cases, was able to derive it from properties of the root system of the ADE algebras [41]. It would be interesting to see if these considerations extend to $su(N)$, $N > 2$ using the root systems of the present paper. (For a physical interpretation of these coefficients in terms of boundary conditions see [8, 41].)

From the physical point of view, there are still several missing links.

1) In spite of many hints, we have no general proof that the graphs that satisfy the constraints of Sect. 2.3, possibly supplemented by some additional conditions, encode the data relative to modular invariants.

2) In spite of many hints, we have no general proof either that the graphs that satisfy the constraints of Sect. 2.3, possibly supplemented by some additional conditions, support a representation of the appropriate Hecke algebra and thus yield an integrable lattice model.

3) In both contexts, the rôle of the reflection group has remained elusive, although there is a suggestive matching of these groups and their subgroups with the algebras and subalgebras of Pasquier type, that are known to be related to the structure of the Operator Product Algebra.

4) In the contexts of $\mathcal{N} = 2$ and topological field theories, there are many indications but no general proof that the graphs are a particular case of those introduced by Cecotti and Vafa, and that the groups are those considered by these authors and Dubrovin in the study of monodromy problems.

Another interesting question is to extend what has been done here for $su(N)$ algebras to other simple algebras. On the one hand, it is known that appropriate graphs are again closely related to modular invariants and should permit the construction of lattice models. See for example some graphs relevant for G_2 in [23]. On the other hand, not all simple algebras give rise to $\mathcal{N} = 2$ superconformal field theories [10]. It is thus likely that the interpretation in terms of reflections groups is less developed in those latter cases.

Acknowledgements. This paper is dedicated to the memory of Claude Itzykson, who was my mentor and my friend, and without whom I wouldn't be what I am. Claude was the touchstone of my papers – of the few that I didn't write with him – demanding ever more clarity, more precision. He saw only the beginning of the present work and made as usual several judicious comments and there is no doubt that his criticisms and suggestions would have helped me to improve this paper greatly, both on the matter and on the presentation. The possibility to interpret the graphs of [23] as root diagrams was considered several years ago in conversations with P. Dorey, to whom I express my thanks for his constant interest and encouragement. Several gaps in the present work have been filled thanks to the energetic and friendly help of M. Bauer and N. Warner. I have benefited a lot from conversations with B. Dubrovin. It is also a pleasure to thank D. Bernard, Ph. Di Francesco, R. Dijkgraaf, T. Eguchi, J. Gaiete, T. Gannon, P. Mathieu, H. Ooguri, V. Pasquier, J. Patera and V. Petkova for comments and suggestions.

References

1. Zuber, J.-B.: Conformal, Integrable and Topological Theories, Graphs and Coxeter Groups. In: Proceedings of the XIth International Conference of Mathematical Physics, Paris July 1994, D. Iagolnitzer ed., Boston: International Press, 1995
2. Humphreys, J.E.: Reflection Groups and Coxeter Groups. Cambridge: Cambridge Univ. Pr., 1990; Bourbaki, N.: Groupes et Algèbres de Lie. Chaps. 4–6, Paris: Masson, 1981
3. Coxeter, H.S.M.: Ann. Math. **35**, 588 (1934)
4. Goodman, F.M., de la Harpe, P., Jones, V.F.R.: Coxeter Graphs and Towers of Algebras. Berlin, Heidelberg, New York: Springer, 1989
5. Verlinde, E.: Nucl. Phys. **300** [FS22], 360 (1988)
6. Kac, V.: Funct. Anal. Appl. **3**, 252 (1969); Moody, R.V., Patera, J.: SIAM J. Alg. Disc. Meth. **5**, 359 (1984) and further references therein
7. Gepner, D.: Commun. Math. Phys. **141**, 381 (1991)
8. Di Francesco, P., Zuber, J.-B.: Nucl. Phys. **B338**, 602 (1990)
9. Coxeter, H.S.M.: Duke Math. J. **18**, 765 (1951)
10. Kazama, Y., Suzuki, H.: Phys. Lett. **B216**, 112 (1989); Nucl. Phys. **B321**, 232 (1989)
11. Nemeschansky, D., Warner, N.P.: Nucl. Phys. **B380**, 241 (1992); Di Francesco, P., Lesage, F., Zuber, J.-B.: Nucl. Phys. **B408**, 600 (1993)
12. Coxeter, H.S.M., Moser, W.O.J.: Generators and Relations for Discrete Groups. Berlin, Heidelberg, New York: Springer, 1957
13. Fuchs, J.: Fortschr. Phys. **42**, 1 (1994) and further references therein
14. Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differentiable maps. Basel: Birkhäuser, 1985
15. A'Campo, N.: Math. Ann. **213**, 1 (1975)
16. Warner, N.: Private communication
17. Zuber, J.-B.: Mod. Phys. Lett. **A8**, 749 (1994)
18. Pasquier, V.: J. Phys. **A20**, 5707 (1987)
19. Ocneanu, A.: In: Operator Algebras and Applications. Vol. 2, London Math. Soc. Lecture Notes Series, Vol. 136, London: Cambridge Univ. Press, 1988, pp. 119–172
20. Shcherbak, O.P.: Russ. Math. Surveys **43:3**, 149 (1988); Moody, R.V., Patera, J.: J. Phys. A **26**, 2829 (1993)

21. Christe, P., Ravanini, F.: *Int. J. Mod. Phys.* **A4**, 897 (1989)
22. Pasquier, V.: *Nucl. Phys.* **B285** [FS19], 162 (1987)
23. Di Francesco, P., Zuber, J.-B.: In: *Recent Developments in Conformal Field Theories*. Trieste Conference, 1989, S. Randjbar-Daemi, E. Sezgin, J.-B. Zuber eds., Singapore: World Scientific 1990; Di Francesco, P.: *Int. J. Mod. Phys.* **A7**, 407–500 (1992)
24. Cappelli, A., Itzykson, C., Zuber, J.-B.: *Nucl. Phys.* **B280** [FS], 445 (1987); *Commun. Math. Phys.* **113**, 1 (1987); Kato, A.: *Mod. Phys. Lett.* **A2**, 585 (1987)
25. Itzykson, C.: *Nucl. Phys. Proc. Suppl.* **58**, 150 (1988)
26. Gannon, T.: *Commun. Math. Phys.* **161**, 233 (1994); The classification of $SU(3)$ modular invariants revisited. hep-th 9404185, *Ann. I.H.P.*, to appear
27. Petkova, V., Zuber, J.-B.: From CFT's to graphs. To appear in *Nucl. Phys. B* 463 (1996)
28. Petkova, V., Zuber, J.-B.: *Nucl. Phys. B.* **B438**, 347 (1995)
29. Bannai, E., Ito, T.: *Algebraic Combinatorics I: Association schemes*. Reading, MA: Benjamin–Cummings, 1984
30. Ocneanu, A.: Communication at the Workshop Low Dimensional Topology, Statistical Mechanics and Quantum Field Theory, Fields Institute, Waterloo, Ontario, April 26–30, 1995
31. Martinec, E.: *Phys. Lett.* **B217**, 431 (1989); Criticality, catastrophes and compactifications. In: *Physics and mathematics of strings*. V.G. Knizhnik memorial volume, L. Brink, D. Friedan, A.M. Polyakov eds., Singapore: World Scientific, 1990; Vafa, C., Warner, N.P.: *Phys. Lett.* **B218**, 51 (1989); Lerche, W., Vafa, C., Warner, N.P.: *Nucl. Phys.* **B324**, 427 (1989)
32. Cecotti, S., Vafa, C.: *Commun. Math. Phys.* **158**, 569 (1993)
33. Cecotti, S., Vafa, C.: *Nucl. Phys.* **B367**, 359 (1991); Dubrovin, B.: *Commun. Math. Phys.* **152**, 539 (1993)
34. Fendley, P., Intriligator, K.: *Nucl. Phys.* **B372**, 533 (1992)
35. Lerche, W., Warner, N.P.: In: *Strings and Symmetries*, 1991. N. Berkovits, H. Itoyama et al. (eds.) Singapore: World Scientific 1992; Intriligator, K.: *Mod. Phys. Lett.* **A6**, 3543 (1991)
36. Witten, E.: *Nucl. Phys.* **B340**, 281 (1990); Dijkgraaf, R., Verlinde, E., Verlinde, H.: *Nucl. Phys.* **B352**, 59 (1991); In: *String Theory and Quantum Gravity*. Proceedings of the 1990 Trieste Spring School, M. Green et al. (eds.) Singapore: World Sci. 1991
37. Dubrovin, B.: *Nucl. Phys.* **B379**, 627 (1992); *Differential Geometry of the space of orbits of a reflection group*. hep-th/9303152
38. Eguchi, T., Yamada, Y., Yang, S.-K.: On the genus expansion in topological string theory. *Rev. Math. Phys.* **7**, 279 (1995)
39. Dubrovin, B.: *Geometry of 2D Topological Field Theories*. hep-th/9407018
40. Vafa, C.: *Mod. Phys. Lett.* **A6**, 337 (1991)
41. Dorey, P.: *Int. J. Mod. Phys.* **8A**, 193 (1993)

Communicated by R.H. Dijkgraaf