Commun. Math. Phys. 138, 409–436 (1991)

Communications in
Mathematical
Physics

On the Complete Integrability of Completely Integrable Systems

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Received July 2, 1990; in revised form October 4, 1990

Abstract. The question of complete integrability of evolution equations associated to $n \times n$ first order isospectral operators is investigated using the inverse scattering method. It is shown that for $n>2$, e.g. for the three-wave interaction, additional (nonlinear) pointwise flows are necessary for the assertion of complete integrability. Their existence is demonstrated by constructing action-angle variables. This construction depends on the analysis of a natural 2-form and symplectic foliation for the groups $GL(n)$ and $SU(n)$.

1. Introduction

A classical Hamiltonian flow with $2N$ degrees of freedom is said to be completely integrable if it has N independent integrals of the motion which are in involution. More generally, k independent commuting Hamiltonian flows in a $2N$-dimensional manifold are said to be a completely integrable family if there are $N-k$ independent integrals of the motions which are in involution, or equivalently the $N-k$ flows may be enlarged to a set of N independent commuting flows. By a theorem of Jacobi and Liouville, there then exist (at least locally in phase space) a new set of canonical variables, called action-angle variables, in which the flows are particularly simple; see [A] for a precise global version due to Arnold.

In recent years a number of nonlinear evolution equations, beginning with the KdV equation, have been shown to have Hamiltonian form on appropriate infinite-dimensional manifolds and to have an infinite family of integrals of the motion which are in involution. Such equations are commonly referred to as "completely integrable," although it no longer makes sense to count half the number of dimensions. Nevertheless the inverse scattering method makes it possible to give a precise form to the question of complete integrability and, indeed, to reduce it to a question in a finite dimensional space.

* Research supported by NSF grants DMS-8916968 and DMS 8901607

Such results are known for the KdV hierarchy and the nonlinear Schrödinger hierarchy; [Ga, ZF]. The scattering map both linearizes and decouples these flows, and action angle variables have been obtained [ZF, ZM]. In both these cases the pointwise dimension of the scattering data is 2, and no new pointwise flows are needed to get to half the dimension. Manakov [Ma] obtained action-angle variables for the 3-wave interaction equation.

The inverse scattering method is based on analysis of an associated linear spectral problem, and the associated hierarchy of flows are isospectral for the linear operator. For KdV the linear operator is the 1-dimensional Schrödinger operator and for NLS it is the 2×2 AKNS-ZS operator

$$\frac{d}{dx} - zJ - q(x); \tag{1.1}$$

here z is the spectral parameter, and J is a given constant matrix. In this paper we consider the $n\times n$ version of (1.1), under the assumption that J is semisimple with distinct eigenvalues and that q takes values in the range of ad J and vanishes rapidly at ∞. The three-wave interaction is an example of an associated isospectral flow, when $n=3$; [ZM, Ma, Ka]. The scattering and inverse scattering theory for (1.1) has been considered by a number of authors: see [ZS1, AKNS] for $n=2$ and [Ma, Ka, Sh, Ne, Ge, BY, BC1, Ca].

Each traceless matrix μ which commutes with J generates a hierarchy of isospectral flows of (1.1). On the scattering side these form an $(n-1)$-parameter family of commuting pointwise flows. The pointwise dimension of the space of scattering data for (1.1) is n^2-n. We show that there is an appropriate Hamiltonian structure on this space of pointwise data, and that the $(n-1)$-parameter family is completely integrable in the classical sense: it is part of an $(n^2-n)/2$-parameter family of commuting Hamiltonian flows. Both the existence of an appropriate pointwise Hamiltonian structure and complete integrability follow from a construction of Darboux coordinates (coordinates which diagonalize the 2-form) which are action-angle variables for the flows of the hierarchies. It should be noted that the additional commuting point-wise flows needed when n is greater than 2 are not linear on scattering data.

The results just described are obtained in the category of complex manifolds and Hamiltonian structures. We are also interested in the real case. The three-wave interaction, for example, is associated to the operator (1.1) with $J+J^*=0$ and the constraint $q+q^*=0$; on the scattering side the appropriate group is $SU(3)$ rather than $SL(3,C)$. We show that one can find real Darboux coordinates for scattering data to provide action-angle variables for the 3-wave interaction and the other flows of the hierarchies. The canonical transformation to action-angle variables is not algebraic in this case: it requires the Liouville method and elliptic functions. Manakov [Ma] used a different method to obtain action-angle variables for the 3-wave interaction which have a simple form but which are nonlocal functions of the entries of the scattering matrix s of Sect. 3; the associated flows are also nonlocal in s.

Our analysis of the Hamiltonian structures (symplectic form, Poisson brackets) leads to a natural closed 2-form of rank n^2-n on $GL(n)$, and a natural symplectic foliation of $GL(n)$. The reduction $J+J^*=0$, $q+q^*=0$ leads to consideration of $SU(n)$ in place of $GL(n)$. The induced Poisson bracket is not a Poisson-Lie structure

[Dr], since it is not degenerate at the identity element. However it was pointed out to us by Lu [Lu] that our structure is the translate by a Weyl element of a Poisson-Lie structure which is the classical limit of a quantum group structure described by Drinfeld [Dr].

The plan of the paper is the following. In Sect. 2 we review the Hamiltonian structure and hierarchy of flows associated to the operator (1.1). The scattering theory for the case $J+J^*=0$ is reviewed in Sect. 3. We then compute the Poisson bracket for scattering data and state the main results on existence of Darboux coordinates and complete integrability. In Sect. 4 we introduce and analyze the 2-form on $GL(n)$ and obtain Darboux coordinates. A symplectic foliation of $GL(n)$ is introduced in Sect. 4, and we calculate the associated Poisson bracket and the Hamiltonians for a family of linear flows.

The algebraic results of Sects. 4 and 5 are used in Sect. 6 to prove the results on Darboux coordinates and complete integrability for scattering data which were stated in Sect. 3. The case of $SU(3)$ is taken up in Sect. 7; complete integrability of the three-wave interaction is a consequence. In Sect. 8 we show that the results stated in Sect. 2 remain valid without the restriction $J+J^*=0$.

2. Symplectic Structure of Hamiltonian Hierarchies

We consider Hamiltonian hierarchies of flows associated to the first order differential operator

$$\frac{d}{dx} - zJ - q(x), \quad z \in \mathbf{C}, \tag{2.1}$$

where J is a constant $n \times n$ semisimple matrix; $q(x)$ is an $n \times n$ matrix whose entries q_{jk} belong to the Schwartz class $\mathscr{S}(\mathbf{R})$; and, for each x, $q(x)$ lies in the range of $\operatorname{ad} J$. We denote by P the linear space of all such q; thus $P = \mathscr{S}(\mathbf{R}; \operatorname{ad} J(M_n))$, where $M_n = M_n(\mathbf{C})$ is the space of $n \times n$ matrices, with the Schwartz topology. We use the following inner product on P:

$$\langle q, p \rangle = \int_{\mathbf{R}} \operatorname{tr}[q(x)p(x)]dx. \tag{2.2}$$

Since P is a linear space we may identify it with its tangent space. We denote tangent vectors (at a given point q) by $\dot{q}$. Associated with (2.1) is the closed 2-form

$$\Omega_P = \frac{1}{2}\int_{\mathbf{R}} \operatorname{tr}[\delta q(x) \wedge [\operatorname{ad} J]^{-1}\, \delta q(x)]dx, \tag{2.3}$$

where $\delta q(x)$ denotes the linear functional taking $\dot{q}$ to $\dot{q}(x)$ and $[\operatorname{ad} J]^{-1}$ maps to the range of $\operatorname{ad} J$, on which $\operatorname{ad} J$ is injective. Thus

$$\Omega_P(\dot{q}_1, \dot{q}_2) = \frac{1}{2}\int_{\mathbf{R}} \operatorname{tr}[\dot{q}_1(x)[\operatorname{ad} J]^{-1}\dot{q}_2(x) - \dot{q}_2(x)[\operatorname{ad} J]^{-1}\dot{q}_1(x)]dx. \tag{2.3'}$$

Since the inner product is non-singular, Ω_P is symplectic. Note that when $J + J^* = 0$, and we restrict to the set $\{q \in P : q + q^* = 0\}$, then the form Ω_P is real.

We shall work with the case in which J is diagonal with distinct eigenvalues:

$J = \operatorname{diag}(i\lambda_1, i\lambda_2, \ldots, i\lambda_n)$. In this case

$$\Omega_P = \int_{\mathbf{R}} \sum_{j<k} \frac{1}{i(\lambda_k - \lambda_j)} \delta q_{jk}(x) \wedge \delta q_{kj}(x) dx. \tag{2.4}$$

A Poisson bracket is associated to the symplectic form Ω_P in the standard way. If F is a functional on P which is Frechet differentiable, and $\dot{q}$ is a tangent vector, we write

$$[\dot{q}F](q) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F(q + \varepsilon\dot{q}) = \left\langle \frac{\delta F}{\delta q}(q), \dot{q} \right\rangle, \tag{2.5}$$

i.e. we identify $\delta F/\delta q$ with the gradient of F. The Hamiltonian vector field associated to F, denoted H_F, is then defined by

$$\Omega_P(H_F, \dot{q}) = -\dot{q}F. \tag{2.6}$$

This definition and (2.3′) imply that $H_F = [J, \delta F/dq]$. The Poisson bracket [Ne] is then given by

$$\{F, G\}_P = H_F G = -\Omega_P(H_G, H_F) = \int_{\mathbf{R}} \operatorname{tr}\left(\left[J, \frac{\delta F}{\delta q}\right]\frac{\delta G}{\delta q}\right)dx. \tag{2.7}$$

There is an $(n-1)$-parameter family of hierarchies of commuting Hamiltonian flows in P, defined as follows. Let μ be a constant matrix with

$$\operatorname{tr}\mu = 0, \quad [J, \mu] = 0,$$

and associate to q in P a sequence of matrix-valued functions F_k defined recursively by

$$F_0(x) = \mu; \quad [J, F_{k+1}] = \frac{dF_k}{dx} + [q, F_k]; \quad \lim_{x \to -\infty} F_{k+1}(x) = 0.$$

The F_k depend nonlinearly on q for $k > 1$ ($k > 2$, if $n = 2$). Various formal and rigorous versions of the following are well-known.

Theorem 2.1. [Sa, BC2, BC3]. *Each $F_k(q)$ is a polynomial in q and its derivatives of order less than k. The hierarchy of flows defined by*

$$\dot{q} = [J, F_{k+1}(q)] \tag{2.8}$$

are Hamiltonian with respect to Ω_P and the Hamiltonians are in involution with respect to the Poisson bracket $\{\ ,\ \}_P$.

We shall discuss the Hamiltonians for these flows later.

It is also well-known that the scattering transform linearizes the flows (2.8). We discuss this in the next section.

3. The Scattering Transform; Symplectic Structure on Scattering Data

We summarize here the basic results of scattering theory for the operator (2.1); cf. [BC1]. In this section we assume

$$J = \operatorname{diag}(i\lambda_1, \ldots, i\lambda_n), \quad \lambda_j \in \mathbf{R}, \quad \lambda_1 > \lambda_2 > \cdots \lambda_n. \tag{3.1}$$

For a given q in P, we seek a matrix-valued solution of the spectral problem

$$\frac{\partial}{\partial x}\psi(x,z) = zJ\psi(x,z) + q(x)\psi(x,z), z\in\mathbf{C}, \tag{3.2}$$

which is normalized by the asymptotic conditions

$$\lim_{x\to-\infty}\psi(x,z)\exp(-xzJ) = 1, \quad \limsup_{x\to+\infty}\|\psi(x,z)\exp(-xzJ)\| < \infty. \tag{3.3}$$

If $\int\|q(x)\|dx < 1$, then there is a unique solution to (3.2), (3.3), and it has a limit

$$\lim_{z\to+\infty}\exp(-x\xi J)\psi(x,\xi) = s(\xi), \quad \xi\in\mathbf{R}. \tag{3.4}$$

The transformation $q\mapsto s = s(\cdot\,;q)$ is one of two versions of the *scattering transform*, and s is called the *scattering matrix*.

Still assuming $\int\|q(x)\|dx < 1$, the solutions $\psi(x,z)$ for non-real z are holomorphic and have limits on $\mathbf{R}$ which are related by

$$\psi(x,\xi + i0) = \psi(x,\xi - i0)v(\xi), \quad \xi\in\mathbf{R}. \tag{3.5}$$

To describe the target spaces for the maps $q\mapsto s$ and $q\mapsto v$, we define the spaces

$$SL^{\pm} = \{a\in SL(n,C): a_{jk} = 0 \quad \text{if} \quad \pm(j-k) > 0\};$$
$$SL_0^{\pm} = \{a\in SL^{\pm}: \operatorname{diag}(a) = 1\};$$
$$SL_* = (SL^+\cdot SL^-)\cap(SL^-\cdot SL^+). \tag{3.6}$$

This means that SL_* consists precisely of those s in $SL(n) = SL(n,\mathbf{C})$ which have two (unique) triangular factorizations

$$s = s_+v_+^{-1} = s_-v_-^{-1}, \quad s_\pm\in SL^\pm, \quad v_\pm\in SL_0^\mp. \tag{3.7}$$

(3.8) Definition. *SD is the set of matrix-valued functions $s:\mathbf{R}\to SL(n)$ with the properties*

s is smooth and bounded; each derivative has an asymptotic expansion in powers of ξ^{-1} as $|\xi|\to\infty$; (3.8a)

s takes values in SL_*, so it factors as $s(\xi) = s_\pm(\xi)v_\pm(\xi)^{-1}$; (3.8b)

the diagonal-matrix-valued functions $\delta_\pm(\xi) = \operatorname{diag} s_\pm(\xi)$ are the boundary values of a diagonal-matrix-valued function which is bounded, holomorphic, and invertible in $\mathbf{C}\backslash\mathbf{R}$. (3.8c)

(3.9) Definition. *SD′ is the set of pairs of matrix-valued functions (v_+,v_-), $v_\pm$: $\mathbf{R}\to SL^\pm$, with the properties*

each entry of $v_\pm - 1$ belongs to $\mathscr{S}(\mathbf{R})$; (3.9a)

the upper principal minors of the matrix-valued function $v = v_-^{-1}v_+$ are non-zero and have winding number zero. (3.9b)

Condition (3.9b) includes discrete scattering data (bound states); throughout this paper we consider only potentials with purely continuous scattering data.

We equip SD and SD' with the Schwartz topologies.

Theorem 3.1. ([Sh], [BY], [BC1]). *The map $q\mapsto s$ is a diffeomorphism from a* *neighborhood of* 0 *in P onto a neighborhood of* 1 *in SD, and it extends to map an open set in P bijectively to a dense open set in SD.*

The matrix function v in (3.5) is related to the matrix function s by the factorizations (3.7); in fact

$$v = v_-^{-1} v_+ = s_-^{-1} s_+, \tag{3.10}$$

and this equation uniquely determines $v_\pm$ from v. The map $q \mapsto (v_+, v_-)$ is a diffeomorphism from a neighborhood of the origin in P onto a neighborhood of (1, 1) *in SD′, and it extends to map an open set in P bijectively onto a dense open set in SD′.*

(3.11) Remark. It is nearly implicit that SD and SD' are diffeomorphic. The factorizations (3.7) and (3.10) determine v from s. Conversely, write $s_\pm = \delta_\pm t_\pm$ with $\delta_\pm$ diagonal and $t_\pm$ in $SL_0^\pm$. The factorization (3.10) gives $v_-^{-1} v_+ = t_-^{-1}(\delta_-^{-1}\delta_+)t_+$, showing that $t_\pm$ and $\delta_-^{-1}\delta_+$ are determined algebraically from v. The holomorphy properties (3.9) shows that the factors δ_-, δ_+ can be obtained from $\delta_-^{-1}\delta_+$ by solving a Riemann–Hilbert factorization problem, and thus $s_\pm$ and s itself can be obtained from v or from the pair (v_+, v_-).

Proposition 3.2. ([G, BC1]). *The pull-back of the 2-form Ω_P of (2.3) under the inverse of the scattering transform is*

$$\Omega_S = \frac{1}{4\pi i} \int_{\mathbf{R}} \operatorname{tr}\left[v_-(\delta v) v_+^{-1} \wedge s^{-1} \delta s\right]. \tag{3.12}$$

It will be convenient to have a somewhat different formulation.

Proposition 3.3. *The 2-form Ω_S can be written*

$$\Omega_S = \frac{1}{4\pi i} \int_{\mathbf{R}} \operatorname{tr}\left[v_+^{-1} \delta v_+ \wedge s_+^{-1} \delta s_+ - v_-^{-1} \delta v_- \wedge s_-^{-1} \delta s_-\right]. \tag{3.13}$$

Proof. Since $v = v_-^{-1} v_+$ and $s = s_\pm v_\pm^{-1}$, it follows that

$$v_-(\delta v) v_+^{-1} = (\delta v_+) v_+^{-1} - (\delta v_-) v_-^{-1};$$

$$s^{-1}\delta s = s^{-1}[\delta s_\pm v_\pm^{-1} - s_\pm v_\pm^{-1} \delta v_\pm v_\pm^{-1}] = v_\pm s_\pm^{-1}(\delta s_\pm) v_\pm^{-1} - (\delta v_\pm) v_\pm^{-1}.$$

Thus

$$\operatorname{tr}\left[(\delta v_\pm) v_\pm^{-1} \wedge s^{-1} \delta s\right] = \operatorname{tr}\left[v_\pm^{-1} \delta v_\pm \wedge s_\pm^{-1} \delta s_\pm\right],$$

since $(\delta v_\pm) v_\pm^{-1}$ is strictly upper or lower triangular.

Next we consider the image under the scattering transformation of the Poisson bracket $\{\ ,\ \}_P$ of (2.7); equivalently, this is the Poisson bracket associated with the 2-form Ω_S on scattering data. As usual, we may consider the entries of the scattering matrix $s(\xi) = s(\xi; q)$ to be functionals on P and compute the corresponding bracket

$$\{s_{jk}(\xi), s_{lm}(\eta)\}_s(s) = \{s_{jk}(\xi), s_{lm}(\eta)\}_P(q), \quad \xi, \eta \in \mathbf{R}.$$

There are two problems here. First, the gradients $\delta s_{jk}(\xi)/\delta q$ do not decay, so the formula (2.7) does not have an absolutely convergent integrand and it is necessary to use a regularization such as

$$\lim_{N \to \infty} \int_{-N}^{N} \operatorname{tr}\left(\left[J, \frac{\delta s_{jk}}{\delta q}(\xi)\right] \frac{\delta s_{lm}}{\delta q}(\eta)\right) dx.$$

Second, even this limit exists only in the sense of distributions in the two variables ξ, η. Thus the precise meaning of the calculation is this: for any pair of test functions u, w in $C^\infty(\mathbf{R})$ one considers the pair of functionals

$$F(q) = \int_{\mathbf{R}} s_{jk}(\xi)u(\xi)d\xi. \quad G(q) = \int_{\mathbf{R}} s_{lm}(\xi)w(\xi)d\xi, \quad s = s(\cdot\,; q).$$

Then formally one has

$$\begin{aligned}\{F, G\} &= \lim_{N\to\infty} \int_{-N}^{N} \operatorname{tr}\left(\left[J, \frac{\delta F}{\delta q}\right]\frac{\delta G}{\delta q}\right)dx \\ &= \lim_{N\to\infty} \iint \int_{-N}^{N} \operatorname{tr}\left(\left[J, \frac{\delta s_{jk}(\xi)}{\delta q}\right]\frac{\delta s_{lm}(\xi)}{\delta q}\right)u(\xi)w(\eta)dx\,d\xi\,d\eta \\ &= \iint \{s_{jk}(\xi), s_{lm}(\eta)\}_S u(\xi)w(\eta)d\xi\,d\eta \qquad (3.14)\end{aligned}$$

as the defining equation for the distribution $\{s_{jk}(\xi), s_{lm}(\eta)\} \in \mathscr{D}'(\mathbf{R}\times\mathbf{R})$. The following calculation is standard; see [Ma] for the 3×3 case and [Sk, KD] for R-matrix formulations.

Proposition 3.4. *The distribution defined by* (3.14) *is given explicitly by*

$$\begin{aligned}\{s_{jk}(\xi), s_{lm}(\eta)\} &= \pi i s_{jm}(\xi)s_{lk}(\eta)[\operatorname{sgn}(l-j) - \operatorname{sgn}(m-k)]\delta(\xi-\eta) \\ &\quad + s_{jk}(\xi)s_{lm}(\eta)[\delta_{jl} - \delta_{km}]\,\text{p.v.}\frac{1}{\xi-\eta}, \qquad (3.15)\end{aligned}$$

where we take $\operatorname{sgn}(0) = 0$ *and* p.v. *denotes the principal value.*

Proof. The variation of s with respect to q is

$$\dot{s}(\xi) = \int_{\mathbf{R}} s(\xi)\psi(x,\xi)^{-1}\dot{q}(x)\psi(x,\xi)dx; \qquad (3.16)$$

[BC3, (2.45)]. Here the ψ are the eigenfunctions (3.2), normalized at $x = -\infty$. We write $\tilde{\psi}(x,\xi) = \psi(x,\xi)s(\xi)^{-1}$, which is normalized at $x = +\infty$. With F as above, an easy calculation using (2.2), (2.5), and (3.16) shows that

$$\frac{\delta F}{\delta q}(x) = \int_{\mathbf{R}} \psi(x,\xi)e_{kj}\tilde{\psi}(x,\xi)^{-1}u(\xi)d\xi.$$

A similar formula holds for G, so (3.14) becomes

$$\begin{aligned}\{F, G\}_P &= \lim_{N\to\infty} \int_{-N}^{N} \operatorname{tr}\left(\left[J, \frac{\delta F}{\delta q}\right]\frac{\delta G}{\delta q}\right)dx \\ &= \lim_{N\to\infty} \iint \int_{-N}^{N} \operatorname{tr}([J, \psi(x,\xi)e_{kj}\tilde{\psi}(x,\xi)^{-1}] \\ &\quad \cdot \psi(x,\eta)e_{ml}\tilde{\psi}(x,\eta)^{-1})u(\xi)w(\eta)dx\,d\xi\,d\eta. \qquad (3.17)\end{aligned}$$

We use the identity

$$\frac{1}{\xi-\eta}\frac{d}{dx}[\tilde{\psi}(x,\eta)^{-1}\psi(x,\xi)] = \tilde{\psi}(x,\eta)^{-1}J\psi(x,\xi),$$

and the properties of the trace to conclude from (3.17) that

$$\{s_{jk}(\xi), s_{lm}(\eta)\}_S = \lim_{N\to\infty} \frac{1}{\xi - \eta} \operatorname{tr}[e_{kj} g(\xi, \eta, N) e_{ml} g(\eta, \xi, N) - e_{kj} g(\xi, \eta, -N) e_{ml} g(\xi, \eta, -N)] \tag{3.18}$$

in the sense of distributions, where $g(\xi, \eta, x) = \tilde{\psi}(x, \xi)^{-1} \psi(x, \eta)$. Now

$$g(\xi, \eta, x) \approx \exp[x(\eta - \xi)J] s(\eta) \quad \text{as} \quad x \to +\infty;$$
$$\approx s(\xi) \exp[x(\eta - \xi)J] \quad \text{as} \quad x \to -\infty.$$

Thus the right side of (3.18) is

$$\lim_{N\to\infty} \frac{1}{\xi - \eta} [s_{jm}(\eta) s_{lk}(\xi) e^{iN(\xi-\eta)(\lambda_l - \lambda_j)} - s_{jm}(\xi) s_{lk}(\eta) e^{iN(\xi-\eta)(\lambda_m - \lambda_k)}]. \tag{3.19}$$

There is no singularity in (3.19) since the term in brackets vanishes at $\xi = \eta$; therefore we may replace the expression in (3.19) by the principal value integral, i.e. letting the distribution act as the limit as $\varepsilon \downarrow 0$ of the integral over the region $|\xi - \eta| > \varepsilon$. This allows us to decouple the two terms in (3.19) and use the identity

$$\lim_{N\to\infty} \text{p.v.} \frac{1}{\xi - \eta} e^{iaN(\xi - \eta)} = \pi i \operatorname{sgn}(a) \delta(\xi - \eta) \quad \text{if} \quad a \in \mathbf{R} \setminus 0$$

to deduce (3.15) from (3.19).

As is well-known, the flows of Theorem 2.1 become linear on the scattering side.

Proposition 3.5. *The potential $q(\cdot, t)$ evolves according to (2.8) if and only if the scattering data evolve according to*

$$\frac{\partial}{\partial t} s(\xi, t) = \xi^k [\mu, s(\xi, t)], \quad \frac{\partial}{\partial t} v_\pm(\xi, t) = \xi^k [\mu, v_\pm(\xi, t)]. \tag{3.20}$$

For a proof, see for example [BC2, BC3]. These flows are Hamiltonian (with the same Hamiltonian functions as in the original variables) with respect to the symplectic form Ω_S on scattering data since the structure has simply been transformed from P to SD or SD'. We wish to emphasize that on the scattering side the flows (2.8) are not only linearized but decoupled for different values of ξ, η; equivalently, the Hamiltonian vector fields act in a pointwise fashion on the entries of s or of (v_+, v_-). This allows us to reduce the question of complete integrability of the family of flows (2.8)/(3.20) to a finite-dimensional problem.

This integrability question is related to certain problems and questions concerning the symplectic and Poisson structures on scattering data. Observe that the 2-form Ω_S lifts to the loop space

$$L(SL_*) = \{a : \mathbf{R} \to SL_* : \text{entries of } a - 1 \text{ belong to } \mathscr{S}(\mathbf{R})\}.$$

More precisely Ω_S is the pullback to SD of the 2-form defined by (3.12) or (3.13) on $L(SL_*)$. Moreover, these are pointwise formulas, in the obvious sense: they express the form as the direct integral of forms computed pointwise from entries of a in $L(SL_*)$. However, the form is not symplectic on $L(SL_*)$. (This can be seen from the fact that the dimension of SL is $n^2 - 1$ but the rank of the pointwise form in (3.13), as we show in the next section, is $n^2 - n$.)

The alternative space of scattering data, SD', is itself a loop space, whose pointwise dimension is $n^2 - n$, and the form Ω_S is symplectic on SD'; however the computation of s from (v_+, v_-) involves the solution of a Riemann–Hilbert problem, so that (3.12) does not express Ω_S as a direct integral of pointwise 2-forms on the fiber of the loop space; in fact it has no such pointwise expression when n is larger than 2.

The Poisson bracket $\{\ ,\ \}_S$ lifts to the whole loop group $L(SL(n))$ using the same formula (3.15), but the nonlocal term involving p.v. $(\xi - \eta)^{-1}$ shows that $\{\ ,\ \}_S$ is also not a direct integral of a pointwise defined bracket, unlike the bracket $\{\ ,\ \}_P$ on the space P of potentials. This corresponds to the fact that submanifold $SD \subset L(SL(n))$ is determined in part by the nonlocal constraint (3.8c). The nonlocal term does not vanish when $\{\ ,\ \}_S$ is considered as a bracket on SD' if $n > 2$, which proves the earlier assertion that Ω_S has no pointwise expression on SD'.

The following summarizes these observations.

Proposition 3.6. *Ω_S is a closed, pointwise 2-form but not a symplectic 2-form on the loop space $L(SL_*)$. Ω_S is a symplectic 2-form but not a pointwise 2-form on SD'.*

$\{\ ,\ \}_S$ is not a pointwise Poisson bracket on $L(SL(n))$, nor on SD, nor on SD' (when $n > 2$).

Our main positive results involve choosing new coordinates to overcome the limitations in Proposition 3.6. To state them we must extend the notion of a distribution-valued Poisson bracket beyond the coordinate functions themselves. Suppose f, g are in $C^\infty(SL(n))$ and u, w are test functions. We define functionals on SD, or on P, by

$$f_\xi(s) = f(s(\xi)), \quad F(s) = \int f(s(\xi))u(\xi)d\xi,$$

$$g_\xi(s) = g(s(\xi)), \quad G(s) = \int g(s(\xi))w(\xi)d\xi.$$

Thus the coordinate functions $a_{jk}(s) = s_{jk}$, $a_{lm}(s) = s_{lm}$ give rise to the functionals $s_{jk}(\xi)$, $s_{lm}(\eta)$, F, and G considered above. Then as before we obtain the distribution $\{f_\xi, g_\eta\} \in \mathscr{D}'(\mathbf{R} \times \mathbf{R})$ by the formal calculation

$$(F, G)_P = \iint \{f_\xi, g_\eta\}_s u(\xi)w(\eta)d\xi d\eta.$$

This leads also to the expression

$$\{f_\xi, g_\eta\}_S = \sum \frac{\partial f}{\partial s_{jk}} \frac{\partial g}{\partial s_{lm}} \{s_{jk}(\xi), s_{lm}(\eta)\}_S.$$

Theorem 3.7. *There are functions p_v, q_v, $1 \leqq v \leqq \frac{1}{2}(n^2 - n)$ which are defined and holomorphic on a dense open algebraic subset of $SL(n)$ and which have the following properties:*

The map $s \mapsto (p_1 \circ s, p_2 \circ s, \ldots, q_1 \circ s, q_2 \circ s, \ldots)$ is an injection from a dense open set in SD into $C^\infty(\mathbf{R}; \mathbf{C}^{n^2-n})$, (3.21)

$$\Omega_S = \sum_v \delta(p_v \circ s) \wedge \delta(q_v \circ s), \tag{3.22}$$

$$\{p_{\mu,\xi}, p_{v,\eta}\}_S = \{q_{\mu,\xi}, q_{v,\eta}\}_S = 0; \quad \{p_{\mu,\xi}, q_{v,\eta}\}_S = \delta_{\mu v}\delta(\xi - \eta), \tag{3.23}$$

For any f in $C^*(SL(n))$, the distributions $\{p_{v,\xi}, f_\eta\}$ have support at $\xi = \eta$, all v. (3.24)

In other words, the functions $p_\nu \circ s$, $q_\nu \circ s$ are global Darboux coordinates on the manifold SD of scattering data. The additional fact (3.24) implies that Hamiltonians which are functions of the p_ν give pointwise vector fields on SD. These same functions p_ν provide a strong positive answer to our question about complete integrability in the $SL(n)$ case, as follows.

Theorem 3.8. *The functions p_ν of Theorem 3.2 may be chosen so that for each traceless diagonal matrix μ, the Hamiltonian for the flow (2.8), (3.20) is a linear combination of the functionals*

$$\int_{\mathbf{R}} \xi^k p_\nu(s(\xi))\,d\xi. \tag{3.25}$$

Thus the family of flows (3.20), which is determined pointwise by the $(n-1)$-parameter family of traceless diagonal matrices, is imbedded in the $\frac{1}{2}(n^2-n)$-parameter family of flows generated by the p_ν's. In fact the functions p_ν, q_ν of Theorem 3.2 provide action-angle variables for the flows (3.20).

Theorems 3.7 and 3.8 are proved in Sect. 6.

In this section we have made two restrictive assumptions about J: that the eigenvalues are distinct and that they lie on a line through the origin in $\mathbf{C}$. The same results hold without the second assumption, as we show in Sect. 8.

Another situation arises with *reduction*, i.e. the imposition of restrictions on the potentials q. The most interesting example in the present context is the restriction $q+q^*=0$, still assuming (3.1). For such q, the scattering data satisfies the corresponding constraints

$$s(\xi) \text{ belongs to } SU(n); \quad v_+(\xi)^* v_-(\xi) = 1.$$

The pullbacks of the 2-forms are still symplectic. When $n=3$ the simplest associated nonlinear evolution equation is the 3-wave interaction. (Note that one needs $\mu+\mu^*=0$ in (2.8), (3.20) to preserve the constraints.) Here our manifolds are real, and we need the functions p_ν, q_ν of Theorem 3.2 to be real in order to preserve the structure.

Theorem 3.9. *The functions p_ν, q_ν of Theorems 3.7, 3.8 can be chosen to be real on $SU(3)$ in the case $n=3$. In particular, the three-wave interaction is a completely integrable Hamiltonian evolution in the strong sense.*

This result is proved in Sect. 7.

4. A 2-Form in $GL(n)$

Let $GL(n)$ denote either $GL(n,\mathbf{R})$ or $GL(n,\mathbf{C})$; in the latter case the functions and forms to be considered are complex-valued. The key steps in deriving Theorems 3.2, 3.3, and 3.4 involve an analysis of a 2-form in $GL(n)$: the integrand in (3.13). As in Sect. 3 we introduce the matrix spaces

$$GL^{\pm} = \{a \in GL(n): a_{jk} = 0 \text{ if } \pm(j-k) > 0\};$$
$$GL_0^{\pm} = \{a \in GL^{\pm}: \operatorname{diag}(a) = 1\};$$
$$GL_* = (GL^+ \cdot GL^-) \cap (GL^- \cdot GL^+).$$

Again, GL_* consists of those elements of $GL(n)$ with factorizations

$$a = a_+ v_+^{-1} = a_- v_-^{-1}, \quad a_\pm \in GL^\pm, \quad v_\pm \in GL_0^\mp. \tag{4.1}$$

We consider $a_\pm, v_\pm$ here as functions of a in $GL(n)$; then $da_\pm$ and $dv_\pm$ are matrices of 1-forms on $GL(n)$ and we may define a 2-form by

$$\Omega = \operatorname{tr}[v_+^{-1} dv_+ \wedge a_+^{-1} da_+ - v_-^{-1} dv_- \wedge a_-^{-1} da_-]. \tag{4.2}$$

At the identity, $\Omega = 2 \sum_{j<k} da_{jk} \wedge da_{kj}$, so Ω has rank $\geqq n^2 - n$ on an open set.

Theorem 4.1. *On a dense open algebraic subset of GL_* the 2-form Ω has a representation*

$$\Omega = \sum_{\nu=1}^{N} dp_\nu \wedge dq_\nu, \quad N = \frac{1}{2}(n^2 - n), \tag{4.3}$$

where p_ν and q_ν are analytic (holomorphic) and the 1-forms dp_ν, dq_ν are independent. In particular, Ω is closed and generically has rank $n^2 - n$.

The proof of this theorem is given after Theorem 4.4. The strategy is to obtain the general result by a reduction to the case $n = 2$.

Lemma 4.2. *For $n = 2$,*

$$\Omega = d \log\left[\frac{a_{11} a_{22}}{\Delta}\right] \wedge d \log\left[\frac{a_{21}}{a_{12}}\right], \quad \Delta = \det a. \tag{4.4}$$

Proof. If a is in $GL_*(2)$ then

$$v_+ = \begin{bmatrix} 1 & 0 \\ -a_{21}/a_{22} & 1 \end{bmatrix}, \quad a_+ = \begin{bmatrix} \Delta/a_{22} & a_{12} \\ 0 & a_{22} \end{bmatrix},$$

$$v_- = \begin{bmatrix} 1 & -a_{12}/a_{11} \\ 0 & 1 \end{bmatrix}, \quad v_- = \begin{bmatrix} a_{11} & 0 \\ a_{21} & \Delta/a_{11} \end{bmatrix}.$$

A direct calculation gives

$$\begin{aligned} \Omega = {} & d\left[\frac{a_{21}}{a_{22}}\right] \wedge \left[\frac{a_{12} da_{22}}{\Delta} - \frac{a_{22} da_{12}}{\Delta}\right] \\ & - d\left[\frac{a_{12}}{a_{11}}\right] \wedge \left[\frac{a_{21} da_{11}}{\Delta} - \frac{a_{11} da_{21}}{\Delta}\right], \end{aligned} \tag{4.5}$$

and some further manipulation leads to (4.4).

(4.5) Remark. Given $1 \leqq j < k \leqq n$, consider the subset of GL_* consisting of those a whose only non-vanishing off-diagonal entries occur in the (j,k) and (k,j) places. The pullback of Ω to this subset has, by an analogous computation, the form

$$\Omega = d \log\left[\frac{a_{jj} a_{kk}}{\Delta}\right] \wedge d \log\left[\frac{a_{kj}}{a_{jk}}\right], \quad \Delta = a_{jj} a_{kk} - a_{jk} a_{kj}. \tag{4.6}$$

We now proceed to reduce the general case to a sum of cases as described in the preceding remark.

Suppose that $\pi_1, \pi_2, \ldots, \pi_N$, $N = \frac{1}{2}(n^2 - n)$, are permutation matrices with the two properties (written using the standard matrix units)

$$\pi_\nu \text{ is the matrix of the transposition } (k, k+1),\ k = k_\nu; \tag{4.7a}$$

$$\text{the product } \pi_1 \pi_2 \cdots \pi_N \text{ is the antidiagonal matrix } r = \sum e_{j,n+1-j}. \tag{4.7b}$$

There are various such decompositions of the antidiagonal matrix r. One such decomposition corresponds to permuting $(1, 2, \ldots, n)$ by moving 1 to the extreme right in $n-1$ steps, then moving 2 to the position left of 1 in $n-2$ steps, and so on.

(4.8) Definition. *Given π_ν satisfying (4.7), set*

$$r_0 = 1, \quad r_\nu = \pi_1 \pi_2 \cdots \pi_\nu, \quad 1 \leqq \nu \leqq N;$$
$$U_\nu = r_\nu G L_0^+ r_\nu^{-1}; \quad L_\nu = r_\nu G L^- r_\nu^{-1}; \quad D_\nu = r_\nu B_\nu r_\nu^{-1},$$

where $B_\nu = \{b \in GL(n); b_{jk} = 0 \text{ for } j \neq k \text{ unless } \{j, k\} = \{k_\nu, k_\nu + 1)\}$. Thus the matrices in D_ν are block diagonal in the sense that after conjugation by r_ν^{-1} the nonzero entries lie in a single 2×2 block along the diagonal.

Note that $U_\nu, L_\nu, D_\nu, U_\nu + D_\nu$, and $L_\nu + D_\nu$ are subalgebras of the matrix algebra M_n. Let

$$P_\nu : M_n \to D_\nu \tag{4.9}$$

be the projection which commutes with left and right multiplication by diagonal matrices. Then

$$(D_\nu + U_\nu) \cap (D_\nu + L_\nu) = D_\nu; \tag{4.10}$$

$$P_\nu \text{ is an algebra homomorphism on } D_\nu + U_\nu \quad \text{and on} \quad D_\nu + L_\nu. \tag{4.11}$$

Note also that because $r_{\nu-1}$ and r_ν differ by a single transposition, we have the identities

$$D_\nu + L_{\nu-1} = D_\nu + L_\nu, \quad D_\nu + U_{\nu-1} = D_\nu + U_\nu. \tag{4.12}$$

(4.13) Definition. *Suppose the permutation matrices π_ν satisfy (4.7), and suppose $r_\nu^{-1} a r_\nu$ is in GL_*, $0 \leqq \nu \leqq N$. Then we may factor the $r_\nu^{-1} a r_\nu$ as in (4.1) to obtain matrices u_ν, l_ν such that*

$$a u_\nu = l_\nu, \quad u_\nu \in U_\nu, \quad l_\nu \in L_\nu. \tag{4.13a}$$

Now define $v_\nu^+, v_\nu^-, a_\nu^+, a_\nu^-$ and 2-forms Ω_ν as follows:

$$v_\nu^- = P_\nu u_{\nu-1}, \quad v_\nu^+ = P_\nu u_\nu, \quad a_\nu^- = P_\nu l_{\nu-1}, \quad a_\nu^+ = P_\nu l_\nu; \tag{4.13b}$$

$$\Omega_\nu = \operatorname{tr}\left[(v_\nu^+)^{-1} dv_\nu^+ \wedge (a_\nu^+)^{-1} da_\nu^+ - (v_\nu^-)^{-1} dv_\nu^- \wedge (a_\nu^-)^{-1} \wedge da_\nu^-\right]. \tag{4.13c}$$

Lemma 4.3. *With a as in (4.13),*

$$v_- = u_0, \quad a_- = l_0, \quad v_+ = u_N, \quad a_+ = l_N, \tag{4.14}$$

$$u_{\nu-1}^{-1} u_\nu = l_{\nu-1}^{-1} l_\nu = (v_\nu^-)^{-1} v_\nu^+ = (a_\nu^-)^{-1} a_\nu^+. \tag{4.15}$$

In particular, $a_\nu^+ (v_\nu^+)^{-1}$ and $a_\nu^- (v_\nu^-)^{-1}$ have a common value a_ν in D_ν, and Ω_ν is the pullback at a_ν of Ω under the map $b \mapsto r_\nu^{-1} b r_\nu$ from D_ν to $B_\nu \subset GL(n)$.

Proof. That (4.14) holds is clear from the definitions, together with the assumption

(4.7), which gives $r_N = r$. The factorizations (4.13a) imply the first equality in (4.15), since $l_{\nu-1}u_{\nu-1}^{-1} = a = l_\nu u_\nu^{-1}$. Because of this first identity and (4.12), the common value belongs to $(D_\nu + L_\nu) \cap (D_\nu + U_\nu) = D_\nu$. Therefore we can project and use the property (4.11) to obtain the remaining identities in (4.15). The final statement is immediate from conjugation of Ω_ν by r_ν.

Note that $a_\nu v_\nu^\pm = a_\nu^\pm$, and that Ω_ν can be expressed directly in terms of the entries of the a_ν via (4.6). By virtue of the following decomposition theorem, the computation of Ω is reduced to a sum of 2×2 problems, as in Remark 4.5. An algorithm for computing a_ν in terms of a will be given below.

Theorem 4.4. *Under the assumption* (4.7), Ω *is the sum*

$$\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_N. \tag{4.16}$$

Proof. We begin by reversing the reasoning in the proof of Proposition 3.3 to write Ω in the alternative form

$$\Omega = \operatorname{tr}[v_-(dv)v_+^{-1} \wedge a^{-1}da].$$

From (4.14), (4.15) it follows that

$$v = v_-^{-1}v_+ = u_0^{-1}u_N = v_1 v_2 \cdots v_N,$$

where v_ν is the common value of the matrices in (4.15). Then

$$v_-(dv)v_+ = v_-\{\textstyle\sum(v_1\cdots v_{\nu-1})dv_\nu(v_{\nu+1}\cdots v_N)\}v_+^{-1} = \textstyle\sum u_{\nu-1}(dv_\nu)u_\nu^{-1}.$$

Note the identity

$$u_{\nu-1}dv_\nu u_\nu^{-1} = u_\nu(v_\nu^+)^{-1}dv_\nu^+ u_\nu^{-1} - u_{\nu-1}(v_\nu^-)^{-1}dv_\nu^- u_{\nu-1}^{-1}.$$

From $a = l_\nu u_\nu^{-1}$ we find

$$\operatorname{tr}\{u_\nu(v_\nu^+)^{-1}dv_\nu^+ u_\nu^{-1} \wedge a^{-1}da\} = \operatorname{tr}\{(v_\nu^+)^{-1}dv_\nu^+ \wedge [(du_\nu^{-1})u_\nu + l_\nu^{-1}dl_\nu]\}.$$

The first term vanishes since v_ν^+ and u_ν both belong to U_ν. For the second term we have

$$\operatorname{tr}\{(v_\nu^+)^{-1}dv_\nu^+ \wedge P_\nu(l_\nu^{-1}dl_\nu)\} = \operatorname{tr}\{(v_\nu^+)^{-1}dv_\nu^+ \wedge (a_\nu^+)^{-1}da_\nu^+\},$$

since P_ν is multiplicative on L_ν. The second term in $\operatorname{tr}(u_{\nu-1}dv_\nu u_\nu^{-1} \wedge a^{-1}da)$ is treated in the same way, using $a = l_{\nu-1}u_{\nu-1}^{-1}$, and (4.13) follows immediately.

Proof of Theorem 4.1. Choose permutation matrices which lead to a decomposition (4.16) of Ω. According to Lemma 4.2 and Remark 4.5, each Ω_ν has the form $dp_\nu \wedge dq_\nu$, so we obtain the desired representation (4.3) on the dense set where the p_ν, q_ν are defined. It follows immediately that Ω is closed and that it has rank at most n^2-n everywhere. The rank is n^2-n at the identity, and therefore is n^2-n on a dense algebraic open set, so the dp_ν, dq_ν are generically independent.

To compute Ω_ν, and hence p_ν, q_ν, we need to find a_ν. To obtain a_ν from a, note that

$$au_\nu^\# = l_\nu^\#, \tag{4.17}$$

where

$$u_\nu^\# = u_{\nu-1}(v_\nu^-)^{-1}, \quad l_\nu^\# = l_{\nu-1}(v_\nu^-)^{-1}, \tag{4.18}$$

and

$$P_\nu l_\nu^\# = P_\nu l_{\nu-1}(v_\nu^-)^{-1} = a_\nu^-(v_\nu^-)^{-1} = a_\nu, \quad P_\nu u_\nu^\# = 1.$$

The matrix $r_{\nu-1}^{-1} a_\nu r_{\nu-1}$ is block diagonal; its nontrivial part is the 2×2 block with entries from rows and columns $k_\nu, k_{\nu+1}$. It follows from (4.17), (4.18), and a simple computation that this block is $D - CA^{-1}B$, where A is $(k_\nu - 1)\times(k_\nu - 1)$, D is 2×2, and

$$[r_{\nu-1}^{-1} a r_{\nu-1}]_{j,k \le k_\nu+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The following notational convention will be useful.

(4.19) Definition. *If J and K are two subsets of $\{1,2,\dots,n\}$ having the same cardinality and a is in M_n, then $m(J;K) = m(J;K;a)$ denotes the determinant of the corresponding submatrix*

$$m(J;K;a) = \det(a_{jk})_{j\in J, k\in K}.$$

We set $m(\varnothing, \varnothing; a) = 1$.

Direct calculation leads to

$$D - CA^{-1}B = \frac{1}{m(J;J)}\begin{bmatrix} m(J,k_\nu;J,k_\nu) & m(J,k_\nu;J,k_\nu+1) \\ m(J,k_\nu+1;J,k_\nu) & m(J,k_\nu+1;j,k_\nu+1) \end{bmatrix},$$

where $J = (1,2,\dots,k_\nu - 1)$ and $m(K;L) = m(K;L;r_{\nu-1}^{-1} a r_{\nu-1})$. Thus

$$\begin{aligned} \Omega_\nu &= dp_\nu \wedge dq_\nu, \\ p_\nu &= \log\left[\frac{m(J,k_\nu;J,k_\nu)m(J,k_\nu+1;J,k_\nu+1)}{m(J;J)m(J,k_\nu,k_\nu+1;J,k_\nu,k_\nu+1)}\right], \\ q_\nu &= \log\left[\frac{m(J,k_\nu+1;J,k_\nu)}{m(J,k_\nu;J,k_\nu+1)}\right]. \end{aligned} \tag{4.20}$$

(4.21) Example. For $n=3$, we consider the decomposition of r corresponding to $(123)\to(312)\to(231)\to(321)$, i.e.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let A be the cofactor matrix $A = \Delta(a^{-1})^t$, where $\Delta = \det a$. Then the corresponding decomposition of Ω is

$$\begin{aligned} \Omega = {} & d\log\left(\frac{a_{11}a_{22}}{A_{33}}\right) \wedge d\log\left(\frac{a_{21}}{a_{12}}\right) + d\log\left(\frac{A_{11}A_{33}}{\Delta a_{22}}\right) \wedge d\log\left(\frac{A_{13}}{A_{31}}\right) \\ & + d\log\left(\frac{a_{22}a_{33}}{A_{11}}\right) \wedge d\log\left(\frac{a_{32}}{a_{23}}\right). \end{aligned}$$

We conclude with some symmetry properties of Ω which will be important later.

Proposition 4.7. *The 2-form Ω is odd under the automorphisms of $GL(n)$,*

$$\Phi_1(a)=(a^{-1})^t, \quad \Phi_2(a)=rar,$$

(where r is again the antidiagonal matrix $\sum e_{j,n+1-j}$), i.e.

$$\Phi_j^*\Omega=-\Omega, \quad j=1,2.$$

Proof. $\Phi_j(GL^{\pm})=GL^{\mp}$. Therefore if $a_j=\Phi_j(a)$, the $\pm$ factors in (4.1) are $\Phi_j(a_{\mp})$, $\Phi_j(v_{\mp})$. It is immediate from this that $\Phi_2^*\Omega=-\Omega$. The result for Φ_1 makes use also of the identity $\Phi_1^*(b^{-1}db)=-(b^{-1}db)^t$, together with the identity $\operatorname{tr}(\alpha\wedge\beta)=\operatorname{tr}(\alpha^t\wedge\beta^t)$ for matrix-valued 1-forms.

5. A Symplectic Foliation and Poisson Bracket on $GL(n)$; Flows

We introduce now a foliation of GL_* which is naturally associated to the factorizations (4.1). As in Remark (3.11) we define diagonal matrices $\delta_{\pm}=\operatorname{diag}(a_{\pm})$ and $\delta=\delta_-^{-1}\delta_+$, and set

$$a_{\pm}=\delta_{\pm}b_{\pm}, \quad b_{\pm}\in GL_0^{\pm}, \tag{5.1}$$

so that

$$v_-^{-1}v_+=a_-^{-1}a_+=b_-^{-1}(\delta_-^{-1}\delta_+)b_+=b_-^{-1}\delta b_+. \tag{5.2}$$

In the notation of (4.20) the principal minors of $a\in GL(n)$ are $m(J;J;a)$; we abbreviate this to $m(J;a)$. If a is understood, we may write $m(J)$. In particular the upper and lower principal minors are

$$d_j^+=d_j^+(a)=m(1,\dots,j); \quad d_j^-=m(j,\dots,n); \quad d_0=1=d_{n+1}. \tag{5.3}$$

It follows from (4.1) that $\delta_+(\delta_-)$ has the same lower (upper) minors as a, so

$$(\delta_+)_{jj}=\frac{d_j^-}{d_{j+1}^-}; \quad (\delta_-)_{jj}=\frac{d_j^+}{d_{j-1}^+}, \quad 1\leqq j\leqq n. \tag{5.4}$$

Therefore δ_+ and δ_- are determined uniquely by $\delta=\delta_-^{-1}\delta_+$, together with the quotients

$$\varphi_j(a)=d_j^+(a)/d_{j+1}^-(a), \quad 1\leqq j\leqq n. \tag{5.5}$$

Note that $\varphi_n(a)=\det a$, and that the φ_j determine the product $\delta_+\delta_-$.

Theorem 5.1. *The foliation of GL_* by the functions φ_j is symplectic for Ω. Each leaf $\{a:\varphi_j(a)=c_j, 1\leqq j\leqq n\}$ is parametrized by $V_*=\{(v_+,v_-)\in GL_0^-\times GL_0^+ : v_-^{-1}v_+\in GL\}$; the pullback of Ω to a leaf is generically of rank n^2-n; the pullback from a leaf to V_* is independent of the choice of leaf and is given by*

$$\operatorname{tr}[v_+^{-1}dv_+\wedge\{b_+^{-1}db_++b_+^{-1}(\delta^{-1}d\delta)b\}-v_-^{-1}dv_-\wedge\{b_-^{-1}db_--b_-^{-1}(\delta^{-1}d\delta)b_-)\}], \tag{5.6}$$

where $b_{\pm}$ and δ are determined from (v_+,v_-) by the factorization

$$v_-^{-1}v_+=b_-^{-1}\delta b_+, \quad b_{\pm}\in GL_0^{\pm}, \quad \delta=\operatorname{diag}(\delta). \tag{5.7}$$

Proof. Starting with a in GL_*, we define $b_\pm, \delta_\pm$, as above and again set $\delta = \delta_-^{-1}\delta_+$. Let $\eta = \delta_-\delta_+$. Then

$$\delta_\pm^{-1} d\delta_\pm = \eta^{-1} d\eta \pm \delta^{-1} d\delta; \quad a_\pm^{-1} da_\pm = b_\pm^{-1} db_\pm + b_\pm^{-1}(\delta_\pm^{-1} d\delta_\pm) b_\pm.$$

Consequently Ω is given by the sum of (5.6) and

$$\operatorname{tr}[v_+^{-1} dv_+ \wedge b_+^{-1}(\eta^{-1} d\eta) b_+ - v_-^{-1} dv_- \wedge b_-^{-1}(\eta^{-1} d\eta) b_-]. \tag{5.8}$$

The pullback of $d\eta$ to the leaves of the foliation determined by the functions φ_j vanishes, since these functions determine η, so the pullback of Ω to the leaf is given by (5.6). Now $b_\pm$ and δ are determined from (v_+, v_-) in V_* by (5.2), so the pullback of Ω to V_* is leaf-independent and given also by (5.6). These pullbacks have rank $\leqq \dim(V_*) = n^2 - n$ everywhere and rank $n^2 - n$ at the unique diagonal element in a given leaf, so they have rank $n^2 - n$ generically.

The symplectic foliation gives rise to a Poisson structure on GL_*. In fact there is a Poisson bracket $(,)_L$ on each leaf L corresponding to the pullback of Ω to L, and this may be extended to a (degenerate) Poisson bracket for functions on GL_*, characterized by

$$(f,g)|_L = (f|_L, g|_L)_L; \quad (f, \varphi_j) = 0, \quad 1 \leqq j \leqq n. \tag{5.9}$$

Equivalently, if $\Omega = \Sigma dp_\nu \wedge dq_\nu$ as in Theorem 4.1, then

$$(p_\mu, q_\nu) = \delta_{\mu\nu}, \quad (p_\mu, p_\nu) = 0 = (q_\mu, q_\nu), \quad 1 \leqq \mu,\ \nu \leqq (n^2 - n); \quad (f, \varphi_j) = 0, \quad 1 \leqq j \leqq n. \tag{5.10}$$

Functions such as the φ_j which Poisson commute with all functions are sometimes called *Casimirs.*

This Poisson bracket was computed for the standard coordinates of $GL(n)$ by Lu [Lu] in the cases $n=2, n=3$; Lu also conjectured the general form below and pointed out the connection with the classical limit of a quantum version due to Drinfeld [Dr], as noted in Sect. 1.

Proposition 5.2. *The Poisson bracket* (5.9) *is odd under the automorphisms* Φ_1, Φ_2 *of Proposition* 4.7, i.e.

$$(f \circ \Phi_j, g \circ \Phi_j) = -(f,g) \circ \Phi_j, \quad j = 1, 2.$$

Proof. The Casimirs φ_k satisfy $\varphi_k \circ \Phi_j = \varphi_{n+1-k}$, so Φ_j maps leaves to leaves. Proposition 4.7 implies that the pullback under Φ_j to a leaf L of Ω on $\Phi_j(L)$ is $-\Omega$. Therefore the pushforward of the Poisson bracket $(,)_L$ is $-(,)_{\Phi_i(L)}$, and (5.9) implies the desired result.

Theorem 5.3. *The Poisson bracket on* GL_* *induced by the 2-form* Ω *and the foliation by functions* (φ_j) *extends to the full matrix space* M_n *and is given by the following bracket relations between the coordinate functions* a_{jk}:

$$(a_{jk}, a_{lm}) = \tfrac{1}{4}[\operatorname{sgn}(l - j) - \operatorname{sgn}(m - k)] a_{jm} a_{lk}, \tag{5.11}$$

where again $\operatorname{sgn}(0) = 0$.

Proof. We show first that the calculation can be reduced to the cases $n < 4$. For a fixed κ, $1 \leqq \kappa \leqq n$, consider the map $M_n \to M_{n-1}$ obtained by omitting the κ^{th} row and column. We claim that the pushforward to M_{n-1} of the Poisson structure on

M_n coincides with the structure on M_{n-1} itself, i.e. the Poisson bracket of coordinate functions a_{jk}, a_{lm} on $M_n, i, j, k, l \neq \kappa$, is the same as that obtained by considering them as functions on M_{n-1}. To verify this claim we take the decomposition (4.7) of the antidiagonal matrix $r \in M_n$ obtained from the following three sets of permutations: the first set takes $(1, 2, \ldots, n)$ to $(1, \ldots, \kappa-1, \kappa+1, \ldots, n, k)$ in $n-k$-steps; the second set takes us to $(n, n-1, \ldots, 1, \kappa)$ in $\frac{1}{2}(n-1)(n-2)$ steps; the third set takes us to $(n, n-1, \ldots, 1)$ in $(\kappa-1)$ steps. The corresponding additive decomposition of Ω then takes the form

$$\Omega = \Omega' + \Omega'' + \Omega''' \tag{5.12}$$

with the obvious notational convention. According to the prescription after Lemma 4.5, $\Omega'' = \Sigma\, dp_\nu \wedge dq_\nu$, where the p_ν, q_ν are functions of the matrix b in M_{n-1} which corresponds to a in M_n by the map above; moreover Ω'' has exactly the same form as Ω_{n-1}. To complete the verification of our claim it is, therefore, sufficient to show that the foliation functions for b Poisson commute with all entries of b when these are considered as functions of a. We know that the p_ν which correspond to Ω' and to Ω'' in the decomposition (5.12) commute with all p_ν, q_ν from Ω'' and with each other, so it is enough to show that the foliation functions of b in M_{n-1} are computable from the p_ν in Ω' and Ω''', together with the foliation functions on M_n. The p_ν corresponding to Ω' are

$$\log(g_{k+1}/g_k), \quad \kappa \leqq k < n; \quad g_k = m(1, \ldots, \hat{\kappa}, \ldots, k)/m(1, \ldots, k),$$

where again $m(\cdots)$ denotes the principal minor of a based on the indicated rows and columns. Similarly, the p_ν associated to Ω''' are

$$\log(h_{k+1}/h_k), \quad 1 \leqq k < \kappa; \quad h_k = m(k, \ldots, n)/m(k, \ldots, \hat{\kappa}, \ldots, n).$$

Modulo the foliation functions on M_n, the g_k and h_k can be determined from the p_ν. Again modulo the foliation functions on M_n, the g_k's and h_k's are equivalent to the set of functions

$$\begin{aligned} &m(1, \ldots, \hat{\kappa}, \ldots, k)/m(k+1, \ldots, n), \quad \kappa \leqq k \leqq n; \\ &m(1, \ldots, k-1)/m(k, \ldots, \hat{\kappa}, \ldots, n), \quad 1 \leqq k < \kappa. \end{aligned}$$

These are precisely the foliation functions of b as an element of M_{n-1}. This completes the proof that the two Poisson structures coincide on b.

Suppose now that a_{ij}, a_{kl} are any two coordinate functions on M_n. Repeated use of the preceding argument shows that their Poisson bracket can be computed by taking them to be functions on M_p, with p the cardinality of $\{i, j, k, l\}$. Thus the computation is reduced to the cases M_1 (trivial), M_2, M_3, M_4. The complete computation is tedious, so we merely indicate a few representative cases. Recall that a Poisson bracket is a derivation for each of its arguments.

For $n=2$ the foliation functions are a_{11}/a_{22}, $\Delta = \det a$. From this fact and (4.4) we deduce

$$\begin{aligned} &(a_{11}, a_{22}) = (a_{11}, a_{11}(a_{22}/a_{11})) = (a_{11}, a_{11})a_{22}/a_{11} = 0; \\ &(a_{11}, a_{12}a_{21}) = (a_{11}, a_{11}a_{22} - \Delta) = 0. \end{aligned}$$

Also, $p = \log(a_{11}a_{22}/\Delta)$, $q = \log(a_{21}/a_{12})$ and $(p, q) = 1$, so

$$(a_{11}a_{22}, a_{21}/a_{12}) = a_{11}a_{22}a_{21}/a_{12};$$

$$(a_{11}, a_{21}/a_{12}) = (a_{11}^2, a_{21}/a_{12})/2a_{11} = (a_{11}a_{22}a_{11}/a_{22}, a_{21}/a_{12})/2a_{11}$$
$$= (a_{11}a_{22}, a_{21}/a_{12})/2a_{22} = a_{11}a_{21}/2a_{12}.$$

Therefore

$$(a_{11}, a_{21}) = \frac{1}{2a_{21}}(a_{11}, a_{21}^2) = a_{12}(a_{11}, a_{21}/a_{12})/2 = \tfrac{1}{4}a_{11}a_{21}.$$

Because of the symmetries in Proposition 4.7, $(a_{22}, a_{21}) = -(a_{11}, a_{21})$ and so on. Also

$$(a_{21}, a_{12}) = (a_{21}, a_{12}a_{21})/a_{21} = (a_{21}, -a_{11}a_{22})/a_{21}$$
$$= (a_{11}a_{22}a_{11}/a_{22}, a_{21})a_{21}/a_{11}a_{12} = 2(a_{11}, a_{21})a_{22}/a_{21} = a_{11}a_{22}/2.$$

For $n=3$, reduction to $n=2$ gives all brackets such as $(a_{22}, a_{12}), (a_{23}, a_{33})$. Let $A = (\det a)(a^{-1})^t$ be the cofactor matrix. The foliation functions are

$$\det a, \quad a_{11}/A_{11}, \quad a_{33}/A_{33}.$$

The decomposition (4.21) implies, therefore, that

$$0 = (a_{22}^2 a_{11}a_{13}/A_{11}A_{33}, A_{13}/A_{31}) = 2a_{22}a_{11}a_{33}(a_{22}, A_{13}/A_{31})/A_{11}A_{33};$$
$$(a_{22}, A_{13}A_{31}) = (a_{22}, A_{11}A_{33} - a_{22}\det a) = A_{11}A_{33}(a_{22}, a_{11}a_{33})/a_{11}a_{33} = 0.$$

Therefore $(a_{22}, A_{12}) = 0 = (a_{22}, A_{31})$, and the 2×2 results allow one to calculate (a_{22}, a_{31}) and (a_{22}, a_{13}) from these identities. Similar computations yield all the 3×3 brackets, though for some it is convenient to replace the decomposition in (4.21) with the decomposition obtained from factoring r by means of $(123)\to(132)\to(312)\to(321)$.

The case $n=4$ is similar, and again brackets like (a_{12}, a_{24}) are known from the 3×3 computation. This completes our sketch of the proof of Theorem 5.3.

We consider now the $(n-1)$-parameter family of flows in M_n:

$$a(t) = \exp(t\mu)a(0)\exp(-t\mu), \quad \mu \text{ diagonal}, \quad \operatorname{tr}\mu = 0. \tag{5.13}$$

This conjugation preserves the principal minors of a, so the flow preserves $GL(n), GL_*$, and the leaves of the foliation. The factorization (4.1) is also preserved, so the flow preserves the 2-form Ω. Therefore these flows are Hamiltonian.

Theorem 5.4. *The Hamiltonian function for the flow* (5.13) *is* $\operatorname{tr}[\mu\log\delta]$, $\delta = \delta_-^{-1}\delta_+$, *and it is a linear combination of the functions* p_ν *of* (4.20).

Proof. We use the additive decomposition of Theorem 4.4, with the factorization of r described after (4.7). If π_ν is associated with the interchange of j and k, then (4.20) implies that under the flow (5.13), $\dot q_\nu = \mu_k - \mu_j$. Therefore the Hamiltonian for (5.13) is a linear combination of the p_ν. The p_ν themselves are logarithms of quotients of principal minors of a:

$$m_{jk} = m(j, j+1, \ldots, k), \quad j \leqq k.$$

The term m_{jk} occurs in the *numerator* of $\exp(p_\nu)$ when π_ν is associated to the interchange of $j-1$ with k or of j with $k+1$; the term m_{jk} occurs in the *denominator* when π_ν is associated to the interchange of $j-1$ with $k+1$ or of j with k. Thus the total weight attached to $\log m_{jk}$ in the Hamiltonian for the flow (5.13) is

$$(\mu_k-\mu_{j-1})+(\mu_{k+1}-\mu_j)-(\mu_{k+1}-\mu_{j-1})-(\mu_k-\mu_j),$$

with the convention that if $j-1=0$ or $k+1=n$, the corresponding term in parentheses is omitted. Thus $\log m_{jk}$ has weight zero unless $j=1$ or $k=n$, and the Hamiltonian for (5.13) is

$$\sum_{k=1}^{n-1}(\mu_{k+1}-\mu_k)\log m_{1k}+\sum_{j=1}^{n-1}(\mu_{j+1}-\mu_j)\log m_{jn}$$

$$=\sum_{j=1}^{n}\mu_j\log[d_j^-(a)d_{j-1}^+(a)/d_{j+1}^-(a)d_j^+(a)]=\sum_{j=1}^{n}\mu_j\log\delta_{jj}.$$

(5.14) Remark. Theorem 5.2 and its proof show that the flows (5.13) are completely integrable in the classical sense: they are an $(n-1)$-parameter family of commuting Hamiltonian flows in each n^2-n dimensional symplectic leaf of the foliation, and are part of the $\frac{1}{2}(n^2-n)$-parameter family of commuting flows generated by the p_ν. Note that the flows (5.13) are the only members of the larger family which are linear as flows on the full matrix algebra M_n; in fact the generator of a linear flow which commutes with all the flows (5.13) must have each matrix unit e_{jk}, $j\neq k$, as an eigenvector and if such a flow leaves all the p_ν invariant it can be shown to be included among the flows (5.13).

6. Proofs of Theorems 3.7 and 3.8: Darboux Coordinates for Scattering Data

Up to a trivial normalization, the functions p_ν, q_ν of Theorem 4.1 are the functions of Theorems 3.7 and 3.8. To see this, we return to the notation introduced before Theorem 3.7. Observe that if f, g belong to $C^\infty(SL(n))$ then in view of Proposition 3.4 the distribution-valued Poisson bracket can be decomposed as

$$\{f_\xi, g_\eta\}_S=[f,g](s(\xi))\delta(\xi-\eta)+\langle f,g\rangle(s(\xi),s(\eta))p.v.\frac{1}{\xi-\eta}, \tag{6.1}$$

where $[\,,]$ and $\langle\,,\rangle$ have the following properties.

The map $f,g\mapsto[f,g]$ is an alternating bilinear map from $C^\infty(SL(n))\times C^\infty(SL(n))$ to $C^\infty(SL(n))$ which is a derivation in each variable:

$$[fg,h]=f[g,h]+g[f,h],\quad [f,gh]=g[f,h]+h[f,g]. \tag{6.2}$$

$$[a_{jk},a_{lm}]=\pi i a_{jm}a_{lk}[\operatorname{sgn}(l-j)-\operatorname{sgn}(m-k)]. \tag{6.3}$$

The map $f,g\to\langle f,g\rangle$ is a symmetric bilinear map from $C^\infty(SL(n))\times C^\infty(SL(n))$ to $C^\infty(SL(n)\times SL(n))$ such that

$$\langle fg,h\rangle(s,s')=f(s)\langle g,h\rangle(s,s')+g(s)\langle f,g\rangle(s,s'),$$
$$\langle f,gh\rangle(s,s')=g(s')\langle f,h\rangle(s,s')+h(s')\langle f,g\rangle(s,s'). \tag{6.4}$$

$$\langle a_{jk},a_{lm}\rangle(s,s')=a_{jk}(s)a_{lm}(s')[\delta_{jl}-\delta_{km}]. \tag{6.5}$$

Here the a_{jk} are the coordinate functions and s, s' are points of $SL(n)$.

The properties (6.2), (6.3) imply that the bracket $[\,,]$ is precisely $4\pi i(\,,)$, where $(\,,)$ is the Poisson bracket (5.11); this corresponds to the fact that Ω_S is the direct integral of $4\pi i\Omega$. Consequently we may replace the functions p_ν and q_ν of Sect. 4 by the renormalized versions

$$\frac{1}{2}p_\nu, \quad \frac{1}{2\pi i}q_\nu \tag{6.6}$$

to obtain

$$\{p_{\mu,\xi}, q_{\nu,\eta}\}_S = \delta_{\mu\nu}\delta(\xi-\eta) + \langle p_\mu, q_\nu\rangle(s(\xi), s(\eta))\,\mathrm{p.v.}\frac{1}{\xi-\eta}. \tag{6.7}$$

To complete the proof of Theorem 3.7 we must show

$$\langle p_\nu, f\rangle = 0 \quad \text{all} \quad f \quad \text{in} \quad C^\infty(SL(n)), \quad \text{all} \quad \nu; \tag{6.8}$$

$$\langle q_\mu, q_\nu\rangle = 0, \quad \text{all} \quad \mu, \nu. \tag{6.9}$$

(6.10) Definition. *Given subsets J, J', K, K' of $(1, 2, \ldots, n\}$, set*

$$\varepsilon(J, K; J', K') = \mathrm{card}\,(J \cap J') - \mathrm{card}\,(K \cap K').$$

Lemma 6.1. *The bracket $\langle\ ,\ \rangle$ between minors of s satisfies*

$$\langle m(J;K), m(J';K')\rangle(s, s') = \varepsilon(J, K; J', K')m(J;K;s)m(J';K';s'). \tag{6.11}$$

Proof. The case when J, K, J', K' all have cardinality 1 is immediate from (6.5). The general case follows by expanding the determinants and using the derivation property (6.4).

We can now prove (6.8) and (6.9), using (4.20). Each p_ν is the logarithm of a term $m(J)m(J')/m(K)m(K')$, where each element j of $(1, 2, \ldots, n)$ occurs with the same frequency in the pair of sets J, J' as in the pair K, K'. From the derivation property (6.5) we deduce that (6.11) is equivalent to

$$\langle \log m(J;K), \log m(J';K')\rangle = \varepsilon(J, K; J', K'). \tag{6.12}$$

Each coordinate function a_{jk} is itself a minor, and therefore (6.12) implies

$$\left\langle \log\frac{m(J)m(J')}{m(K)m(K')}, \log a_{jk}\right\rangle = \varepsilon(J, J; j, k) + \varepsilon(J', J'; j, k) - \varepsilon(K, K; j, k) - \varepsilon(K', K'; j, k)$$
$$= 0.$$

This proves (6.8). To prove (6.9) we note that according to (4.20), each q_ν has the form $\log m(J;K)/m(K;J)$, which we abbreviate slightly as $\log m(JK)/m(KJ)$. Again we deduce from (6.12) that

$$\left\langle \log\frac{m(JK)}{m(KJ)}, \log\frac{m(J'K')}{m(K'J')}\right\rangle$$
$$= \varepsilon(J, J; J', K') - \varepsilon(J, K; K', J) - \varepsilon(K, J; J', K') + \varepsilon(K, J; K', J')$$
$$= 0$$

because $\varepsilon(J, K; J'K') = -\varepsilon(K, J; K', J')$, and $\varepsilon(J, K; K', J') = -\varepsilon(K, J; J', K')$. This proves (6.9). For the injectivity property (3.21) we need the foliation functions φ_j in addition to p_ν, q_ν. By Theorem 5.4, entries of $\delta_-^{-1}\delta_+$ are linear combinations of the p_ν As in Remark 3.11, δ_- and δ_+ are Riemann–Hilbert factors of $\delta_-^{-1}\delta_+$. Finally, (5.4) and (5.5) determine the φ_j from δ_+, δ_-.

Proof of Theorem 3.8. It is a well-known fact that the Hamiltonian for the flow (2.8) is the negative of the coefficient of z^{-k-1} in the asymptotic expansion of $\operatorname{tr}\mu\log\delta(z)$ as $z\to\infty$, where

$$\delta(z)=\lim_{x\to+\infty}\psi(x,z)\exp(-xzJ),\quad z\in\mathbf{C}\backslash\mathbf{R}.$$

See [S, BC3]. Now δ is piecewise holomorphic with limits $\delta_\pm$ on $\mathbf{R}$, so the Hamiltonian can be expressed in the form

$$\frac{1}{2\pi i}\int_{\mathbf{R}}\xi^k[\operatorname{tr}\mu\log\delta_+(\xi)-\operatorname{tr}\mu\log\delta_-(\xi)]d\xi=\frac{1}{2\pi i}\int_{\mathbf{R}}\xi^k\operatorname{tr}\mu\log[\delta_-^{-1}(\xi)\delta_+(\xi)]d\xi.$$

According to Theorem 5.4, this integral is a linear combination of the integrals (3.25).

7. Coordinates and Flows on $SU(3)$

Formula (4.2) defines a complex 2-form on the intersection of $GL=GL(n,\mathbf{C})$ with the real submanifold $GL_*\cap U(n)$. The automorphism $a\mapsto(a^{-1})^*$ takes $GL^\pm$ to $GL^\mp$. Therefore, in the factorizations (4.1), we have

$$a_\pm=(a_\mp^{-1})^*,\quad v_\pm=(v_\mp^{-1})^*,\quad a\in U(n). \tag{7.1}$$

As in the proof of Proposition 4.7 we can conclude that $\Omega=-\bar{\Omega}$ on $GL_*\cap U(n)$.

Theorem 7.1. (a) *$i\Omega$ is a closed real 2-form on $GL_*\cap U(n)$.*

(b) *The foliation of $GL_*\cap U(n)$ induced by the foliation of GL_* in Sect. 5 has leaves with real dimension n^2-n. The 2-form generically has rank n^2-n on each leaf, so the foliation is again symplectic.*

(c) *If μ is a real diagonal matrix with $\operatorname{tr}(\mu)=0$, the flow*

$$a(t)=e^{it\mu}a(0)e^{-it\mu} \tag{7.2}$$

is Hamiltonian in $GL_*\cap U(n)$ with real Hamiltonian function $\operatorname{tr}(\mu\log\delta)$, *where again* $\delta=\delta_-^{-1}\delta_+$, $\delta_\pm=\operatorname{diag}(a_\pm)$.

Proof. Part (a) follows from the preceding remarks. For part (b), note that (7.1) implies

$$\delta_\pm(a)=[\delta_\mp(a)^{-1}]^*,\quad a\in U(\eta). \tag{7.3}$$

Because of (5.3) and (5.4), (7.3) implies that the foliation functions (5.5) take values in $\{|z|=1\}$, so the induced foliation is defined by the n independent real functions $\arg\varphi_j$, The 2-form $i\Omega$ has (real) rank n^2-n at the unique diagonal element in each leaf,, so it has rank n^2-n generically. Finally, the Poisson bracket associated to this symplectic foliation is $-i(\ ,\ )$, where $(\ ,\ )$ is the Poisson bracket of Sect. 5 restricted to $U(n)$. According to Theorem 5.3, therefore, the Hamiltonian for the flow (7.2) is $\operatorname{tr}(\mu\log\delta)$, and according to (7.3) $\delta=\delta_-^{-1}\delta_+=\delta_\mp^*\delta_+$ is real.

The Darboux coordinates p_v,q_v constructed in Sect. 4 are not real when specialized to $U(n)$ (and suitably normalized) except for $n=2$. We show in this section that real Darboux coordinates can be chosen in $SL(3,\mathbf{C})$ in such a way that: (a) the restrictions to $SU(3)$ are real; (b) linear combinations of the p_v still include the Hamiltonians for the flows (7.1). As we shall see, the third Hamiltonian flow which commutes with the 2-parameter family of linear flows (7.2) is not linear on M_3.

We begin our discussion by recalling the Darboux coordinates for $SL(3)$ in example (4.21). Corresponding to the new normalization $i\Omega$, we take these to be

$$\tilde{p}_1 = \log(a_{11}a_{22}/A_{33}), \quad \tilde{p}_2 = \log(A_{11}A_{33}/a_{22}), \quad \tilde{p}_3 = \log(a_{22}a_{33}/A_{11});$$
$$\tilde{q}_1 = i\log(a_{21}/a_{12}) \qquad \tilde{q}_2 = i\log(A_{13}/A_{31}) \qquad \tilde{q}_3 = i\log(a_{32}/a_{23}). \tag{7.4}$$

Here again the A_{jk} are the entries of the cofactor matrix $(\det a)(a^{-1})^t$. Therefore

$$A_{jk} = \bar{a}_{jk} \quad \text{for} \quad a \quad \text{in} \quad SU(n). \tag{7.5}$$

It is convenient to make a preliminary linear canonical transformation to new Darboux coordinates

$$p_1 = \tilde{p}_1 + \tilde{p}_2 = \log(a_{11}A_{11}),$$
$$p_2 = \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = \log(a_{11}a_{22}a_{33}),$$
$$p_3 = \tilde{p}_2 + \tilde{p}_3 = \log(a_{33}A_{33}),$$
$$q_1 = \tilde{q}_2 - \tilde{q}_3 = i\log(a_{23}A_{13}/a_{32}A_{31}),$$
$$q_2 = \tilde{q}_1 - \tilde{q}_2 + \tilde{q}_3 = i\log(a_{21}A_{31}a_{32}/a_{12}A_{13}a_{23}),$$
$$q_3 = -\tilde{q}_1 + \tilde{q}_2 = i\log(A_{13}a_{12}/A_{31}a_{21}). \tag{7.6}$$

Then p_1 and p_3 are real on $SU(3)$, but p_2 is not. It follows either from direct calculation using the Poisson bracket (5.11) or from the first symmetry in Proposition 5.2 that $\log(A_{11}A_{22}A_{33})$ is also in involution with p_1 and p_2, so we may take

$$I_j = a_{jj}A_{jj}, \quad j = 1, 2, 3, \tag{7.7}$$

as the action variables for a new set of Darboux coordinates. The corresponding angle variables $\Theta_1, \Theta_2, \Theta_3$ are then obtained by the classical Liouville method; cf. [W]. We briefly recall the method: in principle one solves the equations

$$p_j = f_j(I, q) = f_j(I_1, I_2, I_3, q_1, q_2, q_3), \quad j = 1, 2, 3. \tag{7.8}$$

Because the I_j are independent and in involution, it follows that $\partial f_j/\partial p_k = \partial f_k/\partial q_j$ on the level surfaces $\{I_j = c_j, j = 1, 2, 3\}$. Therefore there is a generating function $S(q, I)$ such that

$$\partial S/\partial q_j = f_j(I, q) = p_j, \quad j = 1, 2, 3. \tag{7.9}$$

Putting $\Theta_j = \partial S/\partial I_j$, it follows that

$$dS = \Sigma p_j dq_j + \Sigma \Theta_j dI_j,$$

and therefore

$$i\Omega = \Sigma dp_j \wedge dq_j = \Sigma dI_j \wedge d\Theta_j. \tag{7.10}$$

By (7.5) the functions I_j are real on $SU(3)$ and we may replace the Θ_j in (7.10) by their real parts (if necessary) to obtain real Darboux coordinates. In the remainder of this section we show that the angle variables Θ_j are elliptic functions.

We turn to Eq. (7.8). We already have $p_j = \log I_j$, $j = 1, 3$. The remaining equation can be obtained, in principle, once we have a nontrivial identity involving p_2, I_j, q_j.

Proposition 7.2. *Let* $\zeta = e^{p_2} = a_{11}a_{22}a_{33}$, $I_0 = I_2 - I_1 - I_3 + 1$. *The following identity holds on* $SL(3, \mathbf{C})$:

$$(\zeta I_0 + 2I_1, I_3)^2 + 4\cos^2(\tfrac{1}{2}q_2)(\zeta - I_1)(\zeta - I_3)(\zeta - I_1 I_3) = 0. \tag{7.11}$$

Proof. We make extensive use of the identities defining the cofactors: $A_{12} = a_{23}a_{31} - a_{21}a_{33}$ and so on, and of the corresponding identities coming from the inverse matrix A: $a_{12} = A_{23}A_{31} - A_{21}A_{33}$ and so on. In the following computation each term in braces is replaced by its expression obtained from such identities in order to pass to the next in the sequence of identities.

$$\begin{aligned} I_2 &= a_{22}A_{22} = a_{11}a_{22}a_{33} - \{a_{22}a_{13}\}\{a_{22}a_{31}\}/a_{22} \\ &= a_{11}a_{22}a_{33} - [\{A_{13}A_{31}\} + \{a_{12}a_{21}\}\{a_{23}a_{32}\} - (a_{12}A_{13}a_{23} + a_{21}A_{31}a_{32})]/a_{22} \\ &= I_1 + I_3 + 1 - 2A_{11}A_{13}/a_{22} + (a_{12}A_{13}a_{33} + a_{21}A_{31}a_{32})/a_{22} \end{aligned}$$

or

$$\begin{aligned} a_{12}A_{13}a_{23} + a_{21}A_{31}a_{32} &= a_{22}I_0 + 2A_{11}A_{33} \\ &= (\zeta I_0 + 2I_1 I_3)/a_{11}a_{33}. \end{aligned} \tag{7.12}$$

On the other hand

$$\begin{aligned} (a_{12}A_{13}a_{23})(a_{21}A_{31}a_{32}) &= (a_{12}a_{21})(A_{13}A_{31})(a_{23}a_{32}) \\ &= (a_{11}a_{22} - A_{33})(A_{11}A_{33} - a_{22})(a_{22}a_{33} - A_{11}) \\ &= \frac{1}{a_{33}}(\zeta - I_3)\frac{1}{a_{11}a_{33}}(I_1 I_3 - \zeta)\frac{1}{a_{11}}(\zeta - I_1). \end{aligned} \tag{7.13}$$

Let $f = a_{12}A_{13}a_{23}$ and $g = a_{21}A_{31}a_{32}$. Then

$$\begin{aligned} (f+g)^2 &= fg[\sqrt{f/g} + \sqrt{g/f}]^2 = fg[\exp(-\tfrac{1}{2}iq_2) + \exp(\tfrac{1}{2}iq_2)]^2 \\ &= 4fg\cos^2(\tfrac{1}{2}q_2). \end{aligned} \tag{7.14}$$

Combining (7.12), (7.13), and (7.14), we obtain (7.11).

Returning to the generating function S, we have

$$\partial S/\partial q_1 = p_1 = \log I_1, \quad \partial S/\partial q_3 = p_3 = \log I_3, \quad \partial S/\partial q_2 = p_2 = \log \zeta$$

so

$$S(I, q) = q_1 \log I_1 + q_3 \log I_3 + \int^{q_2(\zeta, I)} \log \zeta(I, u)\,du. \tag{7.15}$$

The angle variables corresponding to the action variables I_j are

$$\Theta_j = \frac{\partial S}{\partial I_j} = \frac{q_j}{I_j} + \int^{q_2} \frac{1}{\zeta}\frac{\partial \zeta}{\partial I_j}\,du, \quad j = 1, 3;$$

$$\frac{\partial S}{\partial I_2} = \int^{q_2} \frac{1}{\zeta}\frac{\partial \zeta}{\partial I_2}\,du.$$

We rewrite (7.11) in the form

$$\Phi(\zeta, I, u) = F(\zeta, I) + \cos^2(u/2)G(\zeta, I) = 0 \tag{7.16}$$

to define ζ or u in terms of u or ζ and I. Thus

$$\frac{\partial\zeta}{\partial I_j} = -\frac{\partial\Phi/\partial I_j}{\partial\Phi/\partial\zeta} = \frac{(\partial\Phi/\partial I_j)(\partial\Phi/\partial u)}{\partial\zeta/\partial u}.$$

In particular, on the surface $\Phi = 0$,

$$\frac{\partial\Phi}{\partial I_2} = \frac{\partial F}{\partial I_2} = 2\zeta\sqrt{F}; \qquad \frac{\partial\Phi}{\partial u} = -\sin\left(\frac{1}{2}u\right)\cos\left(\frac{1}{2}u\right)G;$$

$$\cos\left(\frac{1}{2}u\right) = i[F/G]^{1/2}; \quad \sin\left(\frac{1}{2}u\right) = [1 + F/G]^{1/2};$$

so

$$\Theta_2 = \frac{\partial S}{\partial I_2} = i\int^{q_2} \frac{1}{\sqrt{F+G}}\frac{\partial\zeta}{\partial u}du = i\int^{\zeta(I,q_2)} \frac{1}{\sqrt{F+G}}d\zeta.$$

This can be written as a Jacobi elliptic integral; cf. [Co, p. 400]. Set $z^2 = (\zeta - \alpha)/(\zeta - \beta)$, where α, β are the nonzero roots of $F + G$; then the last integral becomes

$$\Theta_2 = \frac{i}{\sqrt{\alpha}}\int^{z(q_2,I)} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad k^2 = \beta/\alpha. \tag{7.17}$$

Note that $\alpha\beta = I_1I_2I_3$ is real and positive on $SU(3)$.

The other angle variables are also elliptic integrals. Straightforward calculation gives

$$\Theta_1 = \frac{1}{I_1}q_1 + i\int^{\zeta} \frac{1}{\sqrt{F+G}}\left\{-1 + \frac{I_0 + 2I_3}{\zeta - I_1} + \frac{I_3(I_0+2)}{\zeta - I_1I_3}\right\}d\zeta;$$

$$\Theta_3 = \frac{1}{I_3}q_3 + i\int^{\zeta} \frac{1}{\sqrt{F+G}}\left\{-1 + \frac{I_0 + 2I_1}{\zeta - I_3} + \frac{I_1(I_0+2)}{\zeta - I_1I_3}\right\}.$$

The first of the three integrals in each line are equal to $-\Theta_2$ and the remaining two transform, under the same change of variables as in (7.17), into sums of Jacobi elliptic integrals of the first and third kinds.

To complete our discussion of $SU(3)$ we consider the integration of the flow with Hamiltonian I_2, in the original coordinate system. The Poisson bracket determined by the foliation of the form $i\Omega$ differs from the Poisson bracket (5.11) by a factor $-i$. Thus the flow on $SU(3)$ is given by

$$\dot{f} = -i(\log I_2, f) = -i(\log a_{22}A_{22}, f), \quad f \in C^\infty(SL(n)). \tag{7.18}$$

Theorem 7.3. *On $SL(3, \mathbf{C})$, let*

$$\rho = (I_1I_2I_3)^{1/2}, \quad \omega = \frac{1}{2i}\log(a_{11}a_{22}a_{33}/A_{11}A_{22}A_{33}).$$

These functions and the functions $a_{jk}A_{jk}$ and $\frac{1}{2i}\log(a_{jk}/A_{jk})$ are real on $SU(3)$. Under *the flow* (7.17) ω *evolves according to the pendulum equation*

$$\ddot{\omega} = -2\rho \sin \omega. \tag{7.19}$$

Moreover

each $a_{jk}A_{jk}$ *is an algebraic function of* $\cos\omega$ *and the* I_j; (7.20)

each time derivative of $\left[\frac{1}{2i}\log(a_{jk}/A_{jk})\right]$ *is an algebraic function of* $\cos\omega$ *and the* I_j. (7.21)

Proof. A direct but somewhat tedious calculation using (7.18), (5.11), and various identities for cofactors gives

$$\begin{aligned}\ddot{\omega} &= \frac{1}{2i}\log(a_{22}/A_{22})^{\cdot\cdot} = i(a_{11}a_{22}a_{33} - A_{11}A_{22}A_{33}) \\ &= -2\rho \sin\omega.\end{aligned}$$

To obtain (7.20) we use the identities

$$a_{j1}A_{k1} + a_{j2}A_{k2} + a_{j3}A_{k3} = \delta_{jk} = a_{1j}A_{1k} + a_{2j}A_{2k} + a_{3j}A_{3k},$$

which come from (7.05) to obtain

$$a_{jk}A_{jk} + a_{kj}A_{kj} = 1 - I_j - I_k + I_l,$$

for distinct j, k, l. Also

$$\begin{aligned}a_{jk}A_{jk}a_{kj}A_{kj} &= a_{jk}a_{kj}A_{jk}A_{kj} = (a_{jj}a_{kk} - A_l)(A_{jj}A_{kk} - a_{ll}) \\ &= I_jI_k + I_l - 2\rho\cos\omega.\end{aligned}$$

Therefore $a_{jk}A_{jk}$ and $a_{kj}A_{kj}$ are the roots of a quadratic equation with coefficients which are polynomials in the I_j, $\rho = (I_1I_2I_3)^{1/2}$, and $\cos\omega$.

Finally, we consider $\log(a_{12}/A_{12})$. Another direct calculation gives

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2i}\log\left(\frac{a_{12}}{A_{12}}\right)\right] &= -\frac{1}{4}I_2\left\{\frac{a_{11}A_{21}}{a_{12}A_{22}} + \frac{A_{11}a_{21}}{A_{12}a_{22}}\right\} \\ &= -\frac{1}{4}\{a_{11}a_{22}(A_{11}A_{22} - a_{33}) + A_{11}A_{22}(a_{11}a_{22} - A_{33})\} \\ &= \frac{1}{4}\left\{\frac{A_{11}A_{22}A_{33} + a_{11}a_{22}a_{33} - 2I_1I_2}{a_{12}A_{12}}\right\} \\ &= \frac{1}{2a_{12}A_{21}}(\rho\cos\omega - I_1I_2).\end{aligned}$$

The calculations for other such terms are similar, and (7.21) follows, using (7.20).

Note that on $SU(3)$, the functions considered above are $a_{jk}A_{jk} = |a_{jk}|^2$ and $\frac{1}{2i}\log(a_{jk}/A_{jk}) = \arg a_{jk}$.

8. General Nondegenerate J

In this section we discuss the case of a spectral problem (3.2) whose characteristic matrix J has n distinct eigenvalues but is not otherwise constrained. Thus we may assume

$$J = \operatorname{diag}(i\lambda_1, i\lambda_2, \ldots, i\lambda_n), \quad \lambda_j\text{'s distinct.} \tag{8.1}$$

The corresponding space of potentials P, the 2-form Ω_p, and the associated Poisson bracket $\{\ ,\ \}_p$ are defined as in Sect. 2. The (continuous) scattering data s or (v_+, v_-) which correspond to a potential q in P is a matrix-valued function or pair of such functions, defined on the set

$$\Sigma = \{\xi \in \mathbf{C}: \operatorname{Re}(i\lambda_j \xi) = \operatorname{Re}(i\lambda_k \xi), \text{ some } j \neq k\}. \tag{8.2}$$

This set is a union of lines through the origin; we will consider it as a union of rays from the origin and orient each ray from 0 to ∞. We will describe the spaces $SD = \{s\}$ and $SD' = \{(v_+, v_-)\}$ in more detail below. The analogue of Theorem 3.1 carries over to this more general setting; [BC1]. Therefore the 2-form and Poisson bracket can be carried over to a form Ω_S and Poisson bracket $\{\ ,\ \}_S$ on scattering data. In this section we prove the existence of Darboux coordinates and the complete integrability of the linear flows.

Theorem 8.1. *There are functions $p_\nu q_\nu$, $1 \leqq \nu \leqq (n^2 - n)/2$, holomorphic on a dense open algebraic subset of $SL(n)$, which have the following properties. Let $\Sigma_1, \ldots, \Sigma_m$ be the rays of Σ. There is an assignment of rays $\nu \mapsto \Sigma_{k(\nu)}$ such that*

s in SD is uniquely determined by the values

$$\{p_\nu(s(\xi)), q_\nu(s(\xi)), \quad \xi \in \Sigma_{k(\nu)}, \quad 1 \leqq \nu \leqq (n^2 - n)/2\}. \tag{8.3}$$

$$\Omega_S = \sum_\nu \int_{\Sigma_{k(\nu)}} \delta p_\nu \wedge \delta q_\nu. \tag{8.4}$$

$$\{p_{\mu,\xi}, p_{\nu,\eta}\}_S = \{q_{\mu,\xi}, q_{\nu,\eta}\}_S = 0; \quad \{p_{\nu,\xi}, q_{\nu,\eta}\}_S = \delta_{\mu\nu}\delta(\xi - \eta), \quad \xi, \eta \in \Sigma_{k(\nu)}. \tag{8.5}$$

For any f in $C^\infty(SL(n))$, the distributions $\{f_\xi, p_{\nu,\eta}\}$ have support at $\xi = \eta \in \Sigma_{k(\nu)}$, all ν; as before, $f_\xi(s) = f(s(\xi))$. (8.6)

Theorem 8.2. *The functions p_ν of Theorem 8.1 may be chosen so that for each traceless diagonal matrix μ, the Hamiltonian for the flows (2.8), (3.20) is a linear combination of the functionals*

$$\int_{\Sigma_{k(\nu)}} \xi^k p_\nu(s(\xi)) d\xi.$$

The machinery needed for the proofs of Theorems 8.1 and 8.2 has already been developed in Sects. 3 and 4. To show how it applies, we must describe the space of scattering data. Assume $\int \|q(x)\| dx < 1$ and consider the spectral problem (3.1), (3.2). Again there is a unique solution $\psi(\cdot, z)$, This solution is holomorphic with respect to $z, z \in \mathbf{C} \backslash \Sigma$, and its boundary values satisfy

$$\psi(x, (1 + i0)\xi) = \psi(x, (1 - i0)\xi) v(\xi), \quad \xi \in \Sigma \backslash 0. \tag{8.7}$$

Given ξ in $\Sigma \backslash 0$, let $\Pi_\xi: M_n \to M_n$ be the projection defined by

$$(\Pi_\xi a)_{jk} = \begin{cases} a_{jk} & \text{if } \operatorname{Re}(i\lambda_j \xi) = \operatorname{Re}(i\lambda_k \xi), \\ 0, & \text{otherwise.} \end{cases}$$

Then the limit

$$\lim_{x\to+\infty} \Pi_\xi \psi(x,\xi)e^{-x\xi J} = s(\xi) \tag{8.8}$$

exists on $\Sigma\backslash 0$. There are factorizations

$$v(\xi)=v_-(\xi)^{-1}v_+(\xi)=s_-(\xi)^{-1}s_+(\xi); \quad s_\pm(\xi)=s(\xi)v_\pm(\xi). \tag{8.9}$$

The factors in (8.9) are characterized by the following conditions:

$$\Pi_\xi v_\pm(\xi)=v_\pm(\xi); \quad \Pi_\xi s_\pm(\xi)=s_\pm(\xi); \quad \operatorname{diag} v_\pm = 1;$$
$$(v_\pm(\xi))_{jk}=(s_\mp(\xi))_{jk}=0 \quad \text{if} \quad \operatorname{Re} i(\lambda_j-\lambda_k)w>0 \quad \text{for} \quad w=(1+i\varepsilon)\xi, \quad \text{small} \quad \varepsilon>0. \tag{8.10}$$

There are further conditions on s and v as functions of $\xi\in\Sigma$, which we do not need to cite here; see [BC1] for a complete discussion of these conditions and of the algebraic facts we are using in this section.

Proposition 3.2 carries over, in the following form [BC3]:

$$\Omega_S=\frac{1}{2\pi i}\int_\Sigma \operatorname{tr}(v_-(\delta v)v_+^{-1}\wedge s^{-1}\delta s)dz. \tag{8.11}$$

The strategy for proving Theorems 8.1 and 8.2 is the same as for proving Theorems 3.7 and 3.8: pointwise analysis of the form Ω_S.

Suppose ξ is in $\Sigma\backslash 0$. After conjugation by a permutation matrix (which depends on the ray of Σ containing ξ), we may assume that

$$\begin{aligned}\operatorname{Re} i\lambda_i\xi=\cdots=\operatorname{Re} i\lambda_{d_1}\xi>\operatorname{Re} i\lambda_{d_1+1}\xi=\cdots\\ =\operatorname{Re} i\lambda_{d_2}\xi>\operatorname{Re} i\lambda_{d_2+1}\xi\cdots.\end{aligned} \tag{8.12}$$

Then $v(\xi)$ and $s(\xi)$ have the block diagonal form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0\\ 0 & A_2 & \cdots & 0\\ & \cdots & & \\ 0 & 0 & \cdots & A_s\end{bmatrix}, \quad A_j\in SL(d_j). \tag{8.13}$$

Moreover $v_-(\xi)$ and $s_+(\xi)$ have this form and are upper triangular, while $v_+(\xi)$ and $s_-(\xi)$ have this form and are lower triangular. With this normalization, the pointwise 2-form being integrated over the ray containing ξ is (a constant multiple of) the sum of the forms as in Theorem 4.1 for the matrix groups $GL(d_1), GL(d_2), \dots$. There is an analogous decomposition of the Poisson bracket, which is computed as in Sect. 3.

It follows from these considerations that the (trivial) extension of the results in Sects. 4 and 5 to block diagonal matrix groups yields the desired functions p_v, q_v of Theorems 8.1 and 8.2.

Acknowledgements. We are grateful for discussions with V. Zurkowski, J. H. Lu, G. Moore, and G. Zuckerman. B. Konopelchenko drew our attention to the work of Manakov.

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Communicated by N. Yu. Reshetikhin