Commun. Math. Phys. 163, 415–431 (1994)

Communications in Mathematical Physics

Kinetic Formulation of the Isentropic Gas Dynamics and p-Systems

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Received: 6 April 1993/in revised form: 14 September 1993

Abstract: We consider the 2×2 hyperbolic system of isentropic gas dynamics, in both Eulerian or Lagrangian variables (also called the p-system). We show that they can be reformulated as a kinetic equation, using an additional kinetic variable. Such a formulation was first obtained by the authors in the case of multidimensional scalar conservation laws. A new phenomenon occurs here, namely that the advection velocity is now a combination of the macroscopic and kinetic velocities. Various applications are given: we recover the invariant regions, deduce new L^∞ estimates using moments lemma and prove $L^\infty - w*$ stability for $\gamma \geq 3$.

Introduction

We consider the equations of isentropic gas dynamics. In the Eulerian coordinates these equations form a 2×2 hyperbolic system of nonlinear conservation laws

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0\,, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p(\varrho)) = 0,\ t \geq 0,\ x \in \mathbb{R}\,, \\ p(\varrho) = \kappa p^\gamma,\ \gamma > 1,\ \kappa = \dfrac{(\gamma-1)^2}{4\gamma}\,, \end{cases} \tag{1}$$

where the unknowns $\varrho(t,x)$ and $q := \varrho u(t,x)$ are respectively the density and the momentum of the gas. They are given at time $t = 0$ by the initial data $\varrho^0(x)$ and $q^0 = \varrho^0 u^0(x)$. And of course, $\varrho \geq 0$ on $\mathbb{R}^+ \times \mathbb{R}$.

We will also consider another 2×2 system,

$$\begin{cases} \partial_t v - \partial_x w = 0\,, \\ \partial_t w + \partial_x p(v) = 0,\ t \geq 0,\ x \in \mathbb{R}\,, \end{cases} \tag{2a}$$

endowed with the pressure law

$$p(v) = \kappa v^{-\gamma}\,, \qquad \gamma > 0\,, \qquad \kappa = \frac{(\gamma-1)^2}{4\gamma}\,. \tag{2b}$$

The system (2a)–(2b) governs the isentropic gas dynamics written in Lagrangian coordinates. In general Eqs. (2a)–(2b) will be referred to as the p-system (see Lax [7], Smoller [14]...). They are also complemented by initial conditions $v^0(x)$, $w^0(x)$.

In this paper we construct and analyze a kinetic formulation of these systems. By this we mean a formulation which is based on an appropriate transport equation such that

- it involves an additional variable, ξ, the so-called kinetic velocity variable;
- its ξ-moments recover the original equations and their augmenting entropy conditions.

This approach was already used by the authors for scalar conservation laws in [9, 10]. The scalar conservation law and all of its associated entropy inequalities were formulated in terms of a single BGK-type kinetic transport equation. Here, we provide kinetic formulations for the isentropic system (1) and the p-system (2). These kinetic formulations are not of BGK-type [1], but instead they involve a limit collision term. This enables us to represent the corresponding 2×2 systems and their associated family of so-called weak entropies. (Strong entropies could be handled by a different kinetic formulation.)

It is evident in both cases of a scalar equation or the present 2×2 systems, that the kinetic formulation is in fact a way to represent a "rich" enough family of entropy inequalities. This seems to give the limits of the method, but also explains its power and why many properties of the system can be proved or recovered so easily using this formulation: these properties are in fact obtained with the use of certain particular entropies which here can be handled easily in the context of our kinetic formulation.

As a first illustration of this advantage, we recover immediately the invariant regions. This provides, as it is well-known, see for instance Dafermos [2], Serre [13], a maximum principle on the Riemann invariants $u \pm \varrho^{\frac{\gamma-1}{2}}$ for weak solutions.

A second illustration is the derivation of a new estimate for weak solutions, which, in the Eulerian case gives for some $C > 0$,

$$\int_0^{+\infty} (\varrho|u|^3 + \varrho^{\frac{3\gamma-1}{2}})(x,t)\,dt \leq C \int_{\mathbb{R}} (\varrho^0|u^0|^2 + (\varrho^0)^\gamma)(y)\,dy\,, \quad \forall x \in \mathbb{R}\,. \tag{3}$$

The proof relies on the moments lemma for transport equations (see Perthame [12]) in the form in which they were set recently by Lions and Perthame [6]. Again it can be interpreted here as a choice of appropriate entropies. A (relatively!) surprising feature is that, with this method, (3) appears very close to dispersive effects for the Schrödinger equation.

Our last illustration is a proof of the strong convergence of families of solutions corresponding to initial data ϱ_ε^0, u_ε^0 bounded in $L^\infty(\mathbb{R})$, in the Eulerian case (1) for $\gamma \geq 3$. Let us recall that this result was proved by DiPerna [3, 4] for values of γ given by $\gamma = (N+2)/N$, $N > 3$ and extended by Chen [1] to $1 < \gamma \leq 5/3$. The main difficulty lies with the degeneracy of the system close to the vacuum ($\varrho = 0$). This restricts the allowed entropies to the so-called weak family, and it is precisely this family of weak entropies that is represented by our kinetic formulation.

[1] It is possible to formulate the isentropic system (1) as a BGK-type kinetic equation but this formulation recovers just one entropy – the mechanical energy

We note that unlike the scalar case our current formulations are not "purely" kinetic, in the sense that the advection velocity of the underlying transport equation involves a combination of the kinetic velocity, ξ, as well as the macroscopic velocities, u and w; consult (18) (with $\gamma \neq 3$) and (44) below. Kinetic formulations of such "non-local" type are familiar from kinetic modeling in other contexts. This particular formulation is compatible with moments lemma, but we do not seem to be able to use here the averaging lemmas (see Golse, Lions, Perthame, Sentis [6]; DiPerna, Lions, Meyer [5]). Instead, our convergence proof employs compensated compactness arguments, as in [3, 4, 1] (see Murat [11], Tartar [15]), and its relative simplicity is due to some algebraic properties of the kinetic formulation.

We will also give details about the case $\gamma = 3$ for the Eulerian case, announced by the authors in [9]. In this case, the kinetic formulation is particularly simple (since it becomes purely local), and we can prove regularizing effects in Sobolev spaces, similar to the scalar case (see [10]).

The rest of this paper is organized as follows. We first give the kinetic formulation in the Eulerian case, from which we derive in a second section the a priori estimate (3). The third section is devoted to the strong convergence of bounded families of solutions for $\gamma \geq 3$, a result that yields existence theorems by the vanishing viscosity method. Such existence results will be presented elsewhere. Treatment of the case $\gamma = 3$ concludes the third section. Finally, in Sect. IV we give the kinetic formulation for the Lagrangian case, and as before we recover invariant regions and the L^∞ estimate of averages in time similar to (3).

Let us finally mention that the case $1 < \gamma < 3$ can also be studied, as far as strong convergence and existence results are concerned, and we shall come back on this in a future publication.

I. Kinetic Formulation in the Eulerian Case

In this section, we consider weak solutions of the system (1) and we will give its kinetic formulation. This requires the knowledge of a complete family of "supplementary conservation laws" or, more precisely, the weak entropy inequalities. This is achieved in Subsect. 1. Then, we present our kinetic formulation and we conclude this section with several remarks in order to connect this formulation with classical notions like the eigenvalues of the system or Riemann invariants.

I.1. Entropy Inequalities

Smooth solutions of (1) satisfy the additional conservation laws

$$\partial_t \eta(\varrho, u) + \partial_x H(\varrho, u) = 0\,, \tag{4}$$

if and only if (η, H) satisfies

$$H_\varrho = u\eta_\varrho + \frac{p'(\varrho)}{\varrho}\,\eta_u\,, \qquad H_u = \varrho\eta_\varrho + u\eta_u\,, \tag{5}$$

or equivalently,

$$\eta_{\varrho\varrho} = \frac{p'(\varrho)}{\varrho^2}\,\eta_{uu}\,. \tag{6}$$

This is easily proved because the existence of such a η reduces to the equality $H_{\varrho u} = H_{u\varrho}$, which yields precisely (6). A function η satisfying (6) is called an entropy.

In the following, we are going to consider the so-called weak entropies, i.e. those functions η which satisfy (6) and are subject to given initial conditions $(0, g(u))$,

$$\begin{aligned} \eta(\varrho = 0, u) &= 0\,, \\ \eta_\varrho(\varrho = 0, u) &= g(u)\,. \end{aligned} \tag{7}$$

Observe that (6) is a wave equation with variable coefficients independent of u, a fact that reflects the Galilean invariance of (1). Thus, the solution of (6), (7) is obtained through a convolution as stated in the

Lemma 1. *For $\varrho \geq 0$, u, $w \in \mathbb{R}$,*

(i) *the fundamental solution of* (6)–(7) i.e. *the solution corresponding to $\eta_\varrho(\varrho = 0, w) = \delta(w)$ (Dirac mass at 0) is given by*

$$\begin{aligned} \chi(\varrho; w) &:= (\varrho^{\gamma-1} - w^2)_+^\lambda\,, \\ \lambda &= \frac{3-\gamma}{2(\gamma-1)}\,, \end{aligned} \tag{8}$$

(ii) *the solution of* (6), (7) *is given by*

$$\eta(\varrho, u) = \int_{\mathbb{R}} g(\xi)\,\chi(\varrho; \xi - u)\,d\xi\,, \tag{9}$$

(iii) *η (given by* (9)*) is convex in $(\varrho, \varrho u)$ for all ϱ, u if and only if g is convex,*

(iv) *the entropy flux H associated with η is*

$$H(\varrho, u) = \int_{\mathbb{R}} g(\xi)\,[\theta\xi + (1-\theta)u]\,\chi(\varrho; \xi - u)\,d\xi\,, \tag{10}$$

where $\theta = \dfrac{\gamma - 1}{2}$.

Remark. In this lemma and in everything that follows we are using the notation $(x)_+^\lambda = \sup(0, |x|^{\lambda-1}x)$.

Proof of Lemma 1. (i) and (ii) are well-known and can be found in [3, 4] for instance. To prove the convexity statement (iii), we notice that

$$\eta(\varrho, u) = \varrho \int_{-1}^{+1} g\left(\frac{q}{\varrho} + z\varrho^{\frac{\gamma-1}{2}}\right)(1 - z^2)_+^\lambda\,dz\,, \tag{11}$$

with $q = \varrho u$ and we still denote by $\eta(\varrho, q)$ this function. We can compute the Hessian matrix of η

$$\begin{aligned}
\eta_{\varrho\varrho} &= \int \frac{\gamma^2-1}{4}\,\varrho^{\frac{\gamma-3}{2}} g'\left(\frac{q}{\varrho} + z\varrho^{\frac{\gamma-1}{2}}\right) z(1-z^2)_+^\lambda\,dz \\
&\quad + \int \left(-\frac{q}{\varrho^2} + \frac{\gamma-1}{2}\,z\varrho^{\frac{\gamma-3}{2}}\right)^2 g''(\ldots)(1-z^2)_+^\lambda\,dz \\
&= \int \frac{\gamma^2-1}{8(\lambda+1)}\,\varrho^{\gamma-2} g''(\ldots)(1-z^2)_+^{\lambda+1}\,dz \\
&\quad + \varrho\int \left(-\frac{q}{\varrho^2} + \frac{\gamma-1}{2}\,z\varrho^{\frac{\gamma-3}{2}}\right)^2 g''(\ldots)(1-z^2)_+^\lambda\,dz \\
\eta_{qq} &= \frac{1}{\varrho}\int g''(\ldots)(1-z^2)_+^\lambda\,dz\,, \\
\eta_{q\varrho} &= \int \left(-\frac{q}{\varrho^2} + \frac{\gamma-1}{2}\,z\varrho^{\frac{\gamma-3}{2}}\right) g''(\ldots)(1-z^2)_+^\lambda\,dz\,.
\end{aligned}$$

From these formulae, we see that whenever g is convex, i.e. g'' is nonnegative we have $\eta_{\varrho\varrho} \geq 0$, $\eta_{qq} \geq 0$ and also

$$\begin{aligned}
\eta_{\varrho\varrho}\eta_{qq} - (\eta_{q\varrho})^2 &\geq \int \left(-\frac{q}{\varrho^2} + \frac{\gamma-1}{2}\,z\varrho^{\frac{\gamma-3}{2}}\right)^2 g''(\ldots)(1-z^2)_+^\lambda\,dz \\
&\quad \times \int g''(\ldots)(1-z^2)_+^\lambda\,dz \\
&\quad - \left[\int \left(-\frac{q}{\varrho^2} + \frac{\gamma-1}{2}\,z\varrho^{\frac{\gamma-3}{2}}\right) g''(\ldots)(1-z^2)_+^\lambda\,dz\right]^2,
\end{aligned}$$

which is again nonnegative using Cauchy-Schwarz inequality. This proves the convexity of η. On the other hand, if $g'' < 0$ on some interval, we can take ϱ small enough and u the center of this interval so that for $|z| \leq 1$, $g''(u + z\varrho^{\frac{\gamma-1}{2}}) < 0$. Thus η_{qq} is negative for these values of (ϱ, u) and η is not convex. This concludes the proof of (iii).

Finally, to recover the entropy flux in (iv), we just need to do it for the fundamental solution and we look for a representation of the following form:

$$H(\varrho, u) = \int_{\mathbb{R}} g(\xi)\, h(\varrho, u; \xi)\, d\xi\,,$$

with

$$\begin{aligned}
h_\varrho(\varrho, u; \xi) &= u\chi_\varrho + \frac{p'(\varrho)}{\varrho}\,\chi_u\,, \\
h_u(\varrho, u; \xi) &= \varrho\chi_\varrho - \chi\,.
\end{aligned}$$

This gives

$$h_\varrho(\varrho,u;\xi)=\lambda(\gamma-1)\varrho^{\gamma-2}(\theta\xi+(1-\theta)u)(\varrho^{\gamma-1}-(\xi-u)^2)_+^{\lambda-1},$$
$$h_u(\varrho=0,u;\xi)=0,$$

and thus, h is given by the formula

$$h(\varrho,u;\xi)=(\theta\xi+(1-\theta)u)\chi(\varrho;\xi-u), \tag{12}$$

and Lemma 1 is proved. □

Remark. The "strong" entropies, $\chi_s(\varrho,w)$, are obtained using the fundamental solution corresponding to $\eta_\varrho(\varrho=0,w)=0$ and given by

$$\chi_s(\varrho;w)=\varrho(\varrho^{\gamma-1}-w^2)_+^\mu,\qquad \mu=-1-\lambda,$$

which belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_w)$ for $\gamma>3$ (since $\lambda>0$). For $\gamma\le 3$ this formula has to be interpreted using finite parts distributions. Then, the entropy flux is given by a more complicated formula than (10). If

$$\eta_s=\int_{\mathbb{R}} g(\xi)\chi_s(\varrho;\xi-u)\,d\xi, \tag{13}$$

then, formally

$$H_s=\int_{\mathbb{R}} g(\xi)\{[\theta\xi+(1-\theta)u]\chi_s+\zeta_s(\varrho;\xi-u)\}\,d\xi, \tag{14}$$

with

$$\frac{\partial}{\partial w}\zeta_s(\varrho;w)=\chi_s(\varrho;w).$$

This can be obtained using the relation

$$\eta_s(\varrho,u)=\int g(u+z\varrho^\theta)(1-z^2)_+^\lambda\,dz, \tag{15}$$

and computing, H_s from the relation on H_u in (5). These entropies are only useful far from the vacuum since they are singular for $\varrho=0$.

I.2. Kinetic Formulation

Following the general theory developed by Lax [7] we are going to restrict our attention to the solutions of (1) which satisfy entropy inequalities. We thus begin with the

Definition 2. *A couple $(\varrho,\varrho u)(t,x)$ is called an entropy solution of* (1) *if it satisfies*

$$\partial_t\eta+\partial_x H\le 0, \tag{16}$$

for all (globally) convex entropies $\eta=\eta(\varrho,q)$ that vanish at $\varrho=0$ for all q, (i.e. *the entropies given by the representation formula* (9) *with g convex).*

This definition needs to be made a bit more precise since we may need to restrict the growth of η (or of g) depending on the integrability properties of (ϱ,q).

Taking $g(\xi) = 1$ or ξ in (9) we just recover $\eta = \varrho$ or ϱu and then (16) is nothing but the original system (1). A more interesting choice is to take $g(\xi) = \xi^2/2$. Then, the corresponding entropy represents the energy

$$\eta_E = \frac{1}{2}\varrho u^2 + \frac{\kappa}{\gamma - 1}\varrho^\gamma .$$

Therefore, entropy solutions will satisfy

$$\varrho u^2 + \varrho^\gamma \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R})) . \tag{17}$$

In fact, in the following only subquadratic g will be used and we will say that a couple $(\varrho, \varrho u)(t, x)$ has finite energy if it satisfies (17). In that case, (16) makes sense and Definition 2 is now precise.

We are now ready to give the kinetic formulation of (1). Observe first that if ϱ and $q = \varrho u$ belong to $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}))$, the term ϱu^2 in (17) should be interpreted as q^2/ϱ.

Theorem 3. *Let $(\varrho, \varrho u) \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}))$ have finite energy and $\varrho \geq 0$, then it is an entropy solution of* (1) *if and only if there exists a non-positive bounded measure m on $\mathbb{R}^+ \times \mathbb{R}^2$ such that the function $\chi(\varrho; \xi - u)$ satisfies*

$$\partial_t \chi + \partial_x[(\theta\xi + (1-\theta)u)\chi] = \partial_{\xi\xi} m(t, x, \xi) . \tag{18}$$

Moreover, if ϱ, ϱu are C^1 in the open subset $\mathcal{O}$ of $\mathbb{R}_+ \times \mathbb{R}$, then $m = 0$ for $(t, x) \in \mathcal{O}$, $\xi \in \mathbb{R}$.

Remark. On the set $\{\varrho(t, x) = 0\}$, the function $u(t, x)$ is not defined, however it is never used on this set since $\chi(0, w) = 0$. In all that follows we just define, for convenience, $u = 0$ whenever $\varrho = 0$.

This formulation is clearly an extension of the one obtained by the authors in [9, 10] for scalar conservation laws. The new point here is that the advection velocity in (18) is no longer purely kinetic. This fact creates difficulties in the application of the known methods for kinetic equations. However, a possible application is given in next section.

Proof of Theorem 3. Let $(\varrho, \varrho u)$ have finite energy and let us define the distribution m by

$$\partial_t \chi + \partial_x\{[\theta\xi + (1-\theta)u]\chi\} = \partial_{\xi\xi} m , \tag{19}$$

(or equivalently m is constructed as

$$m = \partial_t \bar{\chi} + \partial_x \bar{h} ,$$

where $\bar{\chi}$ and $\bar{h}$ are second primitives in ξ of χ and $h = [\theta\xi + (1-\theta)u]\chi)$. In other words, (19) means

$$\partial_t \eta + \partial_x H = \langle g''(\xi), m \rangle$$

for any couple (η, H) given by the formulae of Lemma 1. Now, to write the entropy inequalities for g convex is equivalent to the following inequalities:

$$\langle k(\xi), m \rangle \leq 0 , \quad \forall k(\xi) \geq 0 ;$$

which in turn is equivalent to $m \leq 0$. The equivalence of the formulation is completed by noticing that the choice $g(\xi) = \xi^2/2$ gives the bound

$$-\int_{[0,T]\times\mathbb{R}^2} dm(t,x,\xi) = \int_{\mathbb{R}} (\eta_E(x,T) - \eta_E(x,0))\,dx$$

which is finite for any T by the assumption (17).

Finally, if $(\varrho, \varrho u)$ is smooth on $\mathscr{O}$ then the entropy inequality turns out to be an exact equality by the very construction of entropies and thus $m = 0$ on $\mathscr{O} \times \mathbb{R}$. Another way to prove it, is to compute exactly the left-hand side of (19) using Eqs. (1) and the chain rule allowed for smooth functions ϱ, ϱu.

I.3. Remarks

In order to clarify this formulation let us first notice that (1) is a hyperbolic system (actually strictly hyperbolic as long as $\varrho > 0$). The corresponding eigenvalues are $u \pm c$ with $c = p'(\varrho)^{1/2}$ the speed of sound. We recover the same speeds of propagation for the advection term in (18). Indeed, when $\xi \in \operatorname{supp}\chi = [u - \varrho^{\frac{\gamma-1}{2}}, u + \varrho^{\frac{\gamma-1}{2}}]$, then

$$\theta\xi + (1-\theta)u \in [u-c, u+c]\,.$$

Finally, using the convex function g given by

$$g(\xi) = (\xi - \xi_0)_+ \quad (\text{resp. } (\xi - \xi_0)_-)\,,$$

we recover the classical maximum principle on the two Riemann invariants

$$\max_x (u + \varrho^\theta)(x,t) \leq \max_x (u^0 + (\varrho^0)^\theta)(x)\,,$$
$$\min_x (u - \varrho^\theta)(x,t) \geq \min_x (u^0 - (\varrho^0)^\theta)(x)\,.$$

From this follows the existence of a constant M such that

$$\|u(t,\cdot)\|_\infty + \|\varrho(t,\cdot)\|_\infty \leq M(\|\varrho^0\|_\infty, \|u^0\|_\infty)\,. \tag{20}$$

Let us also give another possible derivation of (18). We consider the family of convex entropies indexed by $k \in \mathbb{R}$ corresponding to $g_k(\xi) = (k - \xi)_+$. Then, one also has

$$\eta_k(\varrho, u) = \int_{-\infty}^{k} \int_{-\infty}^{\eta} \chi(\varrho, \xi - u)\,d\xi\,d\eta\,,$$

or in other words

$$\chi(\varrho, k-u) = \partial_{kk}\eta_k(\varrho, u)\,.$$

If we write the Lax entropy inequalities for η_k as

$$\partial_t \eta_k(\varrho, u) + \partial_x H_k(\varrho, u) = \bar{m}(t,x,k) \leq 0\,,$$

for some measure $\bar{m}$, and we differentiate twice in k, we just recover (18) with k in place of ξ and $\bar{m} = m$. This family of entropies is very similar to the Kruzkov entropies for scalar conservation laws.

II. An L^∞ Estimate

We now give a first application of the kinetic formulation (18), namely the L^∞ estimate for integrals in time given in the introduction by formula (3). It is in fact a variant of the higher moments Lemma [12].

To prove (3), we follow the method of [8]; we multiply Eq. (18) by $\xi|\xi|\,\mathrm{sgn}(x-y)$ and integrate on $(0,T)\times\mathbb{R}^2_{x,\xi}$. This gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \xi|\xi|\,(\chi(T,x,\xi)-\chi(0,x,\xi))\,\mathrm{sgn}(x-y)\,dx\,d\xi \\
&\qquad -2\int_0^T\int_{\mathbb{R}} |\xi|\xi(\theta\xi+(1-\theta)u(t,y))\chi(t,y,\xi)\,dt\,d\xi \\
&\quad = \int_0^T\int_{\mathbb{R}^2} 2\,\mathrm{sgn}(x-y)\,\mathrm{sign}\,\xi\,dm(t,x,\xi) \\
&\quad \geq 2\int_0^T\int_{\mathbb{R}^2} dm(t,x,\xi) \\
&\quad \geq -2E_0\,,
\end{aligned}$$

where E_0 denotes the inital energy

$$E_0 = \int_{\mathbb{R}} \left(\frac{1}{2}\varrho^0 u^{0^2} + \frac{\kappa}{\gamma-1}(\varrho^0)^\gamma\right) dx\,.$$

Using again the energy inequality for the first term, we obtain, for any $y\in\mathbb{R}$,

$$\int_0^T\int_{\mathbb{R}} |\xi|\,\xi(\theta\xi+(1-\theta)u(t,y))\chi(t,y,\xi)\,dt\,d\xi \leq 4E_0\,. \tag{22}$$

And we will recover (3) if we prove

Lemma 4. *There exists a constant $\delta>0$ (depending only on γ) such that*

$$u\int_{\mathbb{R}} |\xi|\,\xi\chi(\varrho;\xi-u)\,d\xi \geq \delta\varrho|u|^2(\varrho^\theta+|u|)\,, \tag{23}$$

$$\int_{\mathbb{R}} |\xi|^3\chi(\varrho;\xi-u)\,d\xi \geq \delta\varrho(|u|^3+\varrho^{3\theta})\,, \tag{24}$$

$$\int_{\mathbb{R}} \xi(\xi-u)\,|\xi|\,\chi(\varrho;\xi-u)\,d\xi \geq \delta\varrho(\varrho^{3\theta}+\varrho^{\gamma-1}|u|)\,. \tag{25}$$

Indeed, a combination of (22), (23) and (25) yields (3) – notice that θ is positive but $1-\theta$ can be negative.

Proof of Lemma 4. The proofs are similar for the three estimates and we just prove (25). We set $\sigma = u/\varrho^\theta$, then the left-hand side of (25) can also be written as

$$\varrho^{1+3\theta} \int_{\mathbb{R}} |z+\sigma|\,(z+\sigma)z(1-z^2)_+^\lambda\,dz\,,$$

and it is enough to prove that

$$\Psi(\sigma) = \int_{\mathbb{R}} |z+\sigma|\,(z+\sigma)z(1-z^2)_+^\lambda\,dz \geq \delta(1+|\sigma|)\,. \tag{26}$$

But Ψ is even, and thus we can only consider positive values of σ.

We have, for some $\alpha > 0$,

$$\Psi''(\sigma) = \int_{-1}^{+1} \operatorname{sgn}(z+\sigma)z(1-z^2)_+^\lambda\,dz$$

$$= 2\int_{\sigma}^{1} z(1-z^2)_+^\lambda\,dz \geq \alpha\left(\frac{1}{2}-\sigma\right)_+ .$$

Therefore, for some $\beta > 0$,

$$\Psi'(\sigma) \geq \beta(\sigma \wedge 1)\,,$$

and, integrating again, we obtain (26) and thus (25).

Remarks. 1. Again the proof of Lemma 4 consists in using the appropriate family of (non-convex) entropies, which is very intuitive at the level of the transport equation, but which could be done directly at the macroscopic level.

2. The form of the estimate (3) is very similar to the regularizing effects known for Schrödinger equations (regularity in x of averages in time). The relationship can be understood by the moments lemma and the Wigner transform as in [8].

III. Strong Convergence of Bounded Solutions for $\gamma \geq 3$

III.1. The Case $\gamma \geq 3$

In this section we consider a bounded family $(\varrho_n(t,x), \varrho_n u_n(t,x))$ of uniformly bounded entropy solutions to the isentropic gas dynamics equations in Eulerian coordinates (1), with finite energy (uniformly in n). From the estimate (20) and the energy estimate, this amounts to say that the initial data $\varrho_n^0(x)$, $u_n^0(x)$ are bounded in L^∞ with finite energy.

In such a situation, we may assume that, as n tends to $+\infty$,

$$\varrho_n(t,x) \rightharpoonup \varrho(t,x)\,, \qquad u_n(t,x) \rightharpoonup u(t,x)\,, \tag{27}$$

in $L^\infty((0,T)\times\mathbb{R})$ weak $*$, for all $T \in (0,\infty)$.

The question, solved by DiPerna [3, 4] for some values of γ and extended by Chen [1] to the range $\gamma \in]1, \frac{5}{3}]$, is to prove that this convergence is strong, which allows to show that $(\varrho, \varrho u)$ is an entropy solution of (1) for the limiting (in L^∞ weak $*$)

initial data. This is achieved in [3, 4, 1] for $1 < \gamma \leq 5/3$ and we will show below that a very simple argument allows to treat the case $\gamma \geq 3$.

The main idea, as in [1, 3, 4], is to use compensated compactness on all the family of weak entropies (in order to deal with a possible vacuum). The advantage of the kinetic formulation is that it is equivalent to do it ξ by ξ, thus giving a functional relation in ξ that is easy to use.

Our precise result is

Theorem 5. *Let $\gamma \geq 3$ and ϱ_n, u_n as above, then a subsequence of ϱ_n (still denoted by ϱ_n) converges pointwise to ϱ, and (a subsequence of) u_n converges pointwise to u on the set $\{\varrho(x,t) > 0\}$. In particular, $\varrho_n u_n$ converges pointwise to ϱu.*

Remark. It is possible to deduce from this "strong convergence" result a global existence result for general initial conditions. This can be done by a viscosity type approximation and requires some careful estimates that will be presented in a future publication.

Proof of Theorem 5. Let us denote $d\nu_{x,t}(\varrho, u)$ or $d\nu$ the Young measure (see [11, 15]) associated to the weak limit (27). Take two smooth functions with compact support $g(\xi_1), h(\xi_2)$ and use the compensated compactness weak limit in the determinant built with the equations

$$\begin{aligned}
&\partial_t \int g(\xi_1)\chi(\varrho_n; u_n - \xi_1)d\xi_1 + \partial_x \int g(\xi_1)[\theta\xi_1 + (1-\theta)u_n] \\
&\quad \times \chi(\varrho_n; u_n - \xi_1)d\xi_1 = \langle g'', m\rangle\,, \\
&\partial_t \int h(\xi_2)\chi(\varrho_n; u_n - \xi_2)d\xi_2 + \partial_x \int h(\xi_2)[\theta\xi_2 + (1-\theta)u_n] \\
&\quad \times \chi(\varrho_n; u_n - \xi_2)d\xi_2 = \langle h'', m\rangle\,.
\end{aligned}$$

Observe that all the quantities appearing on the LHS are in $W^{-1,\infty}_{x,t}$ and, since $\langle g'', m\rangle$ and $\langle h'', m\rangle$ on the RHS are bounded measures in x, t, we can use Murat's Lemma to conclude as in [1, 3, 4],

$$\begin{aligned}
&\int g(\xi_1)\overline{\chi(\xi_1)}d\xi_1 \int h(\xi_2)\overline{[\theta\xi_2 + (1-\theta)u]\chi(\xi_2)}d\xi_2 \\
&\qquad - \int h(\xi_2)\overline{\chi(\xi_2)}d\xi_2 \int g(\xi_1)\overline{[\theta\xi_1 + (1-\theta)u]\chi(\xi_1)}d\xi_1 \\
&\quad = \int g(\xi_1)h(\xi_2)\overline{\chi(\xi_1)[\theta\xi_2 + (1-\theta)u]\chi(\xi_2)}d\xi_1 d\xi_2 \\
&\qquad - \int h(\xi_2)g(\xi_1)\overline{\chi(\xi_1)[\theta\xi_1 + (1-\theta)u]\chi(\xi_2)}d\xi_1 d\xi_2\,.
\end{aligned}$$

The last equality holds for arbitrary functions, g, h, and this yields

$$\begin{aligned}
&\overline{\chi(\xi_1)[\theta\xi_2 + 1(1-\theta)u]\chi(\xi_2)} - \overline{\chi(\xi_2)[\theta\xi_1 + 1(1-\theta)u]\chi(\xi_2)} \\
&\quad = \overline{\chi(\xi_1)[\theta\xi_2 + (1-\theta)u]\chi(\xi_2)} - \overline{\chi(\xi_2)[\theta\xi_1 + (1-\theta)u]\chi(\xi_1)} \\
&\quad = \theta(\xi_2 - \xi_1)\overline{\chi(\xi_1)\chi(\xi_2)}\,. \qquad (28)
\end{aligned}$$

Here and below we use the overbar to indicate the usual integration with respect to the Young measure; for instance

$$\bar{\chi}(\xi) := \int \chi(\varrho; u - \xi)d\nu_{x,t}(\varrho, u)\,.$$

We may rewrite (28) as [2]

$$\frac{\theta}{1-\theta}\left[\frac{\overline{\chi(\xi_1)\chi(\xi_2)}}{\overline{\chi(\xi_1)}\,\overline{\chi(\xi_2)}}-1\right]=\frac{1}{\xi_2-\xi_1}\left(\frac{\overline{u\chi(\xi_2)}}{\overline{\chi(\xi_2)}}-\frac{\overline{u\chi(\xi_1)}}{\overline{\chi(\xi_1)}}\right), \tag{29}$$

for $\xi_1,\xi_2\in\mathscr{C}$, where $\mathscr{C}$ is any open connected component in the union of the sets $]u-\varrho^\theta,u+\varrho^\theta[$, for which $(\varrho,u)\in\operatorname{supp}\nu$.

The first step of the proof is to show that for $\gamma\geq 3$,

$$\frac{\overline{u\chi(\xi)}}{\overline{\chi(\xi)}}\ \text{is a nonincreasing function of}\ \xi\in\mathscr{C},\ (\gamma\geq 3). \tag{30}$$

To this end, we denote by $f_0(\xi):=\dfrac{\chi(\xi)-\bar{\chi}(\xi)}{\bar{\chi}(\xi)}$, so that (29) takes the equivalent form

$$\frac{\theta}{1-\theta}\,\overline{f_0(\xi_1)f_0(\xi_2)}=\frac{1}{\xi_2-\xi_1}\left(\frac{\overline{u\chi(\xi_2)}}{\overline{\chi(\xi_2)}}-\frac{\overline{u\chi(\xi_1)}}{\overline{\chi(\xi_1)}}\right). \tag{31}$$

Sending ξ_2 to ξ_1 in (31), we should end up with

$$\frac{\theta}{1-\theta}\,\overline{f_0^2(\xi)}=\frac{\partial}{\partial\xi}\left(\frac{\overline{u\chi(\xi)}}{\overline{\chi(\xi)}}\right),\quad\forall\xi\in\mathscr{C},$$

and (30) follows, since $1-\theta$ and hence the left-hand side are negative for $\gamma>3$. Notice that there is no difficulty to pass to the limit on the right-hand side of (31) (in distributions sense) since $\overline{\chi(\xi)}$ does not vanish on $\mathscr{C}$. In order to pass to the limit on the left-hand side (in $L^2_{\mathrm{loc}}(\mathscr{C})$), we require $f_0(\xi)$ and hence $\chi(\xi)\in L^2(\mathbb{R}_\xi)$; but since $\|\chi(\xi)\|^2_{L^2(\mathbb{R}_\xi)}=\varrho^{(5-\gamma)/2}\int_{-1}^{1}(1-\zeta^2)^{2\lambda}\,d\zeta$, this requirement of $L^2(\mathbb{R}_\xi)$-integrability restricts the range of admissible γ's with $\gamma<5$. To extend the statement of (30) for all $\gamma\geq 3$, we first regularize by mollifying both sides of (31) against a unit mass mollifier, $\psi_\alpha(\xi)\geq 0$, arriving at $f_\alpha:=f_0*\psi_\alpha$ which satisfies

$$\frac{\theta}{1-\theta}\,\overline{f_\alpha(\xi_1)f_\alpha(\xi_2)}=\frac{1}{\xi_2-\xi_1}\left(\frac{\overline{u\chi(\xi_2)}}{\overline{\chi(\xi_2)}}-\frac{\overline{u\chi(\xi_1)}}{\overline{\chi(\xi_1)}}\right)\underset{\xi_1}{*}\psi_\alpha(\xi_1)\underset{\xi_2}{*}\psi_\alpha(\xi_2).$$

Granted the boundedness of the LHS and the smoothness of the RHS, we may now take $\xi_2=\xi_1$, to find out that

$$\frac{\theta}{1-\theta}\,\overline{f_\alpha^2(\xi)}=\frac{1}{\xi_2-\xi_1}\left(\frac{\overline{u\chi(\xi_2)}}{\overline{\chi(\xi_2)}}-\frac{\overline{u\chi(\xi_1)}}{\overline{\chi(\xi_1)}}\right)\underset{\xi_1}{*}\psi_\alpha(\xi_1)\underset{\xi_2}{*}\psi_\alpha(\xi_2)|_{\xi_1=\xi_2=\xi}. \tag{32$_\alpha$}$$

If we now let α tend to zero, then the left-hand side of (32) yields a negative measure (again, since $1-\theta$ is negative for $\gamma\geq 3$) whereas the right-hand side tends to $\dfrac{\partial}{\partial\xi}\left(\dfrac{\overline{u\chi(\xi)}}{\overline{\chi(\xi)}}\right)$, and thus (32$_\alpha$) yields the desired (30) as $a\to 0$.

[2] The case $\theta=1$ corresponding to $\gamma=3$ will be treated in Proposition 7 below

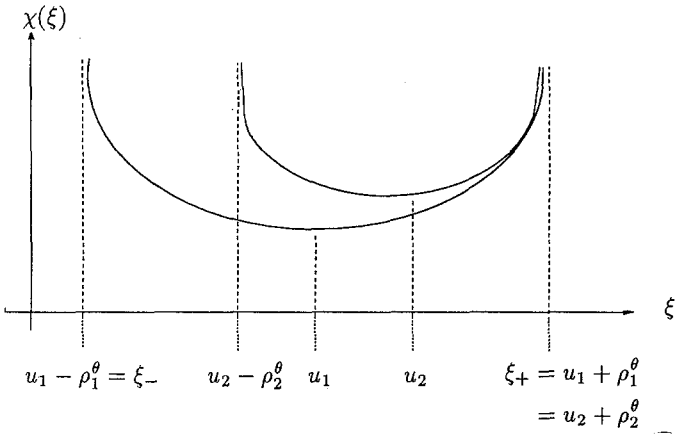

Fig. 1. $\mathscr{C} =]\xi_-, \xi_+[$ and two χ functions for two values (ϱ_1, u_1), (ϱ_2, u_2) in the support of ν

The second and final step in our proof is stated as

Lemma 6. *Let* $\mathscr{C} =]\xi_-, \xi_+[$ *stand for any open connected component as above, and set* $u_0 := (\xi_- + \xi_+)/2$, *then*

$$\lim_{\xi \to \xi_+} \frac{\overline{u\chi(\xi)}}{\overline{\chi(\xi)}} \geq u_0\,, \qquad \lim_{\xi \to \xi_-} \frac{\overline{u\chi(\xi)}}{\overline{\chi(\xi)}} \leq u_0\,. \tag{33}$$

Before proving Lemma 6, let us conclude the proof of Theorem 5. Combining (33) with (30) we obtain that $\overline{u\chi}/\bar{\chi}$ is constant, which in turn tells us, by (32α), that $\overline{f_\alpha^2(\xi)} = 0$. Hence, $f_\alpha(\xi)$ vanishes on the support of ν and in particular, by letting $\alpha \to 0$, so does $f_0(\xi)$,

$$f_0(\xi) = \frac{\chi(\varrho, u - \xi)}{\overline{\chi(\xi)}} - 1 = 0\,, \quad (\varrho, u) \in \operatorname{supp} \nu\,.$$

This shows that on the set $\{\varrho > 0\}$, the Young measure ν is reduced to a Dirac mass and the conclusion holds as usual (see [1, 3, 4]).

Proof of Lemma 6. According to (30), $\overline{u\chi}/\bar{\chi}$ is a monotone function on $\mathscr{C}$, and we turn to consider its one-sided limits as $\xi \to \xi_\pm$. The values of (ϱ, u) such that $\chi(\xi) > 0$ in an interval $]\xi_+ - \varepsilon, \xi_+[$ satisfy

$$u + \varrho^\theta \geq \xi_+ - \varepsilon\,,$$

and therefore, since $\xi_- \leq u - \varrho^\theta$ for these (ϱ, u) values, we have

$$\begin{aligned} \lim_{\xi \to \xi_+} \frac{\overline{u\chi(\xi)}}{\overline{\chi(\xi)}} &\geq \min\{u; (\varrho, u) \in \operatorname{supp} \nu,\ u + \varrho^\theta = \xi_+\} \\ &\geq \frac{\xi_+ + \xi_-}{2}\,. \end{aligned}$$

A similar argument holds for ξ_-, thus concluding the proof of Lemma 6 and of Theorem 5.

Remark. The case $\gamma \geq 3$ is also the case where both families of entropies, weak and strong, are well-defined in L^1.

III.2. The Case $\gamma = 3$

In the case when $\gamma = 3$, we have $\theta = 1$ and the kinetic formulation reads

$$\partial_t \chi + \xi \partial_x \chi = \partial_{\xi\xi} m\,. \tag{34}$$

Thus, we can apply the averaging lemma in the version [5] and obtain

Proposition 7. *Let $(\varrho, \varrho u)$ satisfy (1) with ϱ^0, $\varrho^0 (u^0)^2 + (\varrho^0)^\gamma$ in $L^1(\mathbb{R})$, ϱ^0 and u^0 in $L^\infty(\mathbb{R})$, then*

$$\varrho, \varrho u \in W^{s,p}_{\mathrm{loc}}([0, +\infty[\times\mathbb{R}) \quad \textit{for all } 0 \le s < \frac{1}{7}\,, \tag{35}$$

where $p = 7/4$.

Remark. The regularity given by (35) is probably not optimal and is just an indication of regularizing phenomena taking place in the case when $\gamma = 3$.

Proof of Proposition 7. With these assumptions on the initial data we have

$$\chi(t,x,\xi) \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^2))\,,$$
$$m(t,x,\xi) \in M^1(\mathbb{R}_+ \times \mathbb{R}^2) \quad \text{(bounded measures)}\,,$$

and thus (35) follows from the general results of [5] observing that $\operatorname{supp} \chi(t,x,\cdot)$ is uniformly bounded for $x \in \mathbb{R}$, $t \ge 0$.

IV. The *p*-System

To treat the case of Eq. (2), we follow the lines of Sects. I and II. We give first the kinetic formulation, then the invariant regions and finally the "dispersion" estimate analogous to (3).

We only consider $v(t,x) \ge 0$ and thus v plays the role of ϱ in the Eulerian case.

IV.1. Kinetic Formulation

The couples entropy-entropy flux are now given by the relations

$$\eta_{vv} + p'(v)\eta_{ww} = 0\,, \tag{36}$$
$$H_v = \eta_w p'(v)\,, \qquad H_w = -\eta_v\,. \tag{37}$$

Therefore the fundamental solutions to (36) are now

$$\chi_1(v;\xi - w) = v(v^{1-\gamma} - (w-\xi)^2)_+^\lambda\,, \qquad \lambda = \frac{3-\gamma}{2(\lambda - 1)}\,, \tag{38}$$

$$\chi_2(v;\xi - w) = (v^{1-\gamma} - (w-\xi)^2)_+^\mu\,, \qquad \mu = -1 - \lambda\,. \tag{39}$$

The entropy fluxes are obtained integrating $g(\xi)$ against the function h given by

$$h_1(v,w;\xi) = \theta \frac{\xi - w}{v}\, \chi_1(v;\xi - w)\,, \tag{40}$$

$$h_2(v,w;\xi) = \theta \frac{\xi - w}{v}\, \chi_2(v;\xi - w) + \int_0^v (s^{1-\gamma} - (\xi - w)^2)_+^\mu\, ds\,. \tag{41}$$

We still have $\theta = (\gamma - 1)/2$ and these formulae are obtained integrating the relation $H_v = \dots$ in (37). Again, depending upon γ, these functions belong to $L^1(\mathbb{R}_\xi)$ or not and in the following we restrict our attention to those cases corresponding to $\chi_1 \in L^1$, namely

$$1 < \gamma < +\infty\,. \tag{42}$$

The energy is now given by

$$E(t,x) = \frac{w^2}{2} + \frac{\kappa}{\gamma - 1}\, v^{1-\gamma}\,. \tag{43}$$

It is always positive and its L^1 norm in x decreases with t. Other entropies are given by the

Lemma 8. *$\eta(v,w) = \int_{\mathbb{R}} g(\xi)\chi_1(v;\xi - w)\,d\xi$ is a convex function of v, w if and only if g is convex.*

Proof of Lemma 8. Skipping standard calculations we have

$$\eta_{vv} = \frac{\gamma^2 - 1}{4}\, v^{-(\gamma+1)} \left[v^{-\theta} \int_{\mathbb{R}} g'(w + zv^{-\theta})\, z(1 - z^2)_+^\lambda\, dz \right.$$
$$\left. + \int_{\mathbb{R}} g''(\dots)\, z^2 (1 - z^2)_+^\lambda\, dz \right] \geq 0\,,$$
$$\eta_{ww} = \int_{\mathbb{R}} g''(\dots)\,(1 - z^2)_+^\lambda\, dz \geq 0\,,$$
$$\eta_{vw} = \frac{1 - \gamma}{2}\, v^{-\frac{1+\gamma}{2}} \int_{\mathbb{R}} g''(\dots)\, z(1 - z^2)_+^\lambda\, dz\,,$$

and we conclude again using Cauchy-Schwarz inequality and letting $v \to 0$ or $+\infty$.

As far as we are interested by shocks described by the convex entropies in w, v, we obtain that the entropy solutions to (2) with finite energy satisfy (recall that $|\gamma| > 1$)

$$\partial_t \chi_1 + \partial_x \left(\theta\, \frac{\xi - w}{v}\, \chi_1 \right) = \partial_{\xi\xi} m\,, \tag{44}$$

for some non-positive bounded measure m.

Remark. Notice that multiplying (44) by $(\xi, \xi^2/2)$ we recover the conservation of the momentum (second equation of (2)) and the energy inequality. The weight 1 does not give the first conservation law, and we do not know how to recover the conservation law (associated of the conservation of mass) from (44).

IV.2. Invariant Regions

As before if $\chi_1(t=0,x,\xi)=0$ for $\xi \geq \xi_+$, then this has to be true for all times (same for $\xi \leq \xi_-$) thus recovering the maximum principle on the Riemann invariants

$$(w+v^{-\theta})(x,t) \leq \max_x (w^0+(v^0)^{-\theta})(x), \tag{45}$$

$$(w-v^{-\theta})(x,t) \geq \min_x (w^0-(v^0)^{-\theta})(x). \tag{46}$$

Thus v is bounded for $\gamma < -1$ and bounded from below for $\gamma > 1$. In any case, w remains bounded.

IV.3. An L^∞_x Estimate of the Integral in Time

A straightforward application of (25) in Lemma 4 yields as in Sect. II the "dispersion" estimates

$$\int_0^{+\infty} (|w|v^{1-\gamma} + v^{\frac{3(1-\gamma)}{2}})(y,t)\,dt$$

$$\leq C \int ((w^0)^2 + (v^0)^{1-\gamma})(x)\,dx, \quad \forall y \in \mathbb{R}, \tag{47}$$

for entropy solutions with finite energy. Of course, for $\gamma > 1$, those have infinite L^1 norm of v but in that case the physical density is $1/v$.

Acknowledgements. The third author (E. Tadmor) has been partially supported by ONR Contract Nr. N 00014-91-J-1076 and DARPA Contract Nr. N 0014-92-J-1890.

Note added in proof. After this work was achieved, other examples of kinetic formulation for $N \times N$ hyperbolic systems were obtained in [16].

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Communicated by J. L. Lebowitz