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On the Classification of Simple Vertex Operator Algebras

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Abstract: Inspired by a recent work of Frenkel–Zhu, we study a class of (pre-)vertex operator algebras (voa) associated to the self-dual Lie algebras. Based on a few elementary structural results we propose that $\mathscr{V}$, the category of $\mathbf{Z}_+$-graded prevoas V in which $V[0]$ is one-dimensional, is a proper setting in which to study and classify simple objects. The category $\mathscr{V}$ is organized into what we call the minimal k^{th} types. We introduce a functor Γ – which we call the Frenkel–Lepowsky–Meurman functor – that attaches to each object in $\mathscr{V}$ a Lie algebra. This is a key idea which leads us to a (relative) classification of the *simple minimal first type*. We then study the set of all Virasoro structures on a fixed minimal first type V, and show that they are in turn classified by the orbits of the automorphism group $\mathrm{Aut}(\Gamma(V))$ in $\mathrm{cent}(\Gamma(V))$. Many new examples of voas are given. Finally, we introduce a generalized Kac–Casimir operator and give a simple proof of the irreducibility of the prolongation modules over the affine Lie algebras.

Contents

1. Introduction

The theory of *vertex operator algebras* (voas) since its inception, has undergone many developments [1, 2, 7, 8, 9, 11, 15, 16, 24–27, 31] – both at the level of structural theory and at the level of new examples (see the introduction in [9] for a historical review).

This work is an attempt – partly inspired by a recent paper of Frenkel–Zhu – to understand certain basic aspects of the structure of voas. As a result, we discover a large class of new voas. Recall that Frenkel–Zhu attach to every finite dimensional *simple* Lie algebra g, a family of voas $I(g, \mathbf{C}_\chi)$ (among other things). One of the main results of the present work is the observation that the family of voas attached to g fits naturally into a much larger class of voas, each of which is associated to a finite-dimensional *self-dual* Lie algebra. We propose studying this new family of voas as a first step toward classifying the simple voas.

We begin with an outline of our motivations and goals.

1.1. *Conformal Field Theory, Quantum Groups and Chern–Simons–Witten Theory.* Voa theory is deeply rooted, aside from the theory of the Monster, in string theory and conformal field theory (CFT). This work is partly motivated by two classes of CFTs.

One of the richest classes of CFTs is the Wess–Zumino–Witten (WZW) models[28, 13, 18]. These models were discussed originally within the framework of *semisimple* groups G – partly inspired by similar models associated with abelian groups. When G is compact, the corresponding WZW models are relatively well-understood. For example, it can be shown that the genus-zero and genus-one properties of these models are substantially captured by the properties of the corresponding voas (WZW voas) [18, 26, 11]. For non-compact semisimple G, some partial results on WZW models are known (see [21] and references therein). No example is known beyond the case of a reductive G. However, there are hints that non-reductive WZW-type models might exist.

In the case of a compact G, the genus-zero correlation functions of WZW models are known to be governed by the Knizhnik–Zamolodchikov (KZ) differential equations. These equations also bear some connections with the theory of quantum groups. In particular, the monodromy of the KZ equations is related to the tensor product structure of the representations of a quantized enveloping algebra [4]. On the other hand, Drinfel'd has recently pointed out that the KZ equations can in fact be formulated for *any* finite dimensional Lie algebra with an invariant bilinear form. But it was not known whether such equations actually *arise* in CFT (see [10] for a related discussion). Still this is a first hint that there might exist non-reductive WZW-type models and their voas. This is one of the motivations of the present work.

Another interesting class of CFTs are the super unitary CFTs. It has been shown [14] that the super chiral algebra of such a CFT is a tensor product of "spin half fermion" theories and a $\mathbf{Z}_+$-graded chiral algebra V with one-dimensional V [0]. The chiral algebra V lies in what we call the category $\mathscr{V}$. This category is the starting point of the present work (Sect. 3). Thus this paper may be viewed as an extension of the work of Goddard–Schwimmer.

In one of his papers, Witten has uncovered a deep connection between a certain gauge theory and $2+1$-dimensional gravity [29]. The gauge theory he studied was the Chern–Simons actions with gauge group $ISO(2,1)$. This group may be viewed as the semi-direct product $SL(2,\mathbf{R}|\!\!\times sl(2,\mathbf{R}')$ where $SL(2,\mathbf{R})$ acts on the vector subgroup sl(2,$\mathbf{R}$)$'$ via the coadjoint action. The complexified Lie algebra of this group is $g = sl_2|\!\!\times sl_2'$. One of the key ingredients that were used to construct the Chern–Simons action was the non-degenerate symmetric g-invariant pairing between Sl_2 and sl_2' [29].

There is in fact a host of other examples of non-compact groups similar to $SL(2,\mathbf{R}|\!\!\times sl(2,\mathbf{R})'$, i.e. the ones of the form H $|\!\!\times h'$, where H is any Lie group, and h' is the dual of the Lie algebra of H. Its Lie algebra g clearly has a non-degenerate form, just as one does in the case above. Thus one can at least write down the Chern–Simons action for each of these "self-dual" groups – groups with invariant metrics. We have no idea whether it makes sense to quantize such an action.

Now in [30], it is shown that Chern–Simons theory based on the group $SU(N)$ gives rise to the WZW models based on the same group. More precisely, the conformal blocks of the two-dimensional models arise as the physical Hilbert spaces of the three-dimensional theory. Thus it is plausible that the conformal blocks of some other WZW-type models (perhaps one that might arise from self-dual group $ISO(2,1)$ may be related, in a similar fashion, to the quantization of the corresponding Chern–Simons theory. This suggests that we should look for non-reductive WZW-type models.

Where do we look? We know that the current algebra of $SU(N)$ leads to a WZW voa and to a conformally invariant quantum field theory. The local conformal transformations are effected by the action of the Virasoro algebra. Is there a voa associated to a general self-dual group? If so, what is the Virasoro action in the case?

1.2. *The Virasoro Action.* Motivated by quantum conformal field theory, one requires that a voa be equipped with an action of the Virasoro algebra, satisfying a number of "physical" conditions. This action corresponds to local conformal transformations on a conformally invariant physical system. This requirement of having a Virasoro action is clearly independent of the Jacobi identity, for there are in fact many interesting objects – similar to voas – which satisfy the Jacobi identity, and yet carry no action of the Virasoro algebra. The space $I(g,\mathbf{C}_{\chi})$ mentioned above with χ equal to the negative of the dual Coxeter number of g, is one such example [11, 5].

From an abstract point of view and the point of view of classification, it seems advantageous to begin with only the fundamental axiom and with no auxiliary assumption. The Jacobi identity is clearly fundamental. A Virasoro action on what we call a *pre-vertex operator algebra* should be viewed as an auxiliary structure. (To see how prevoa is related to existing notion, see Sect. 2.) Therefore it is important for us to start with the notion of prevoas. Once an interesting category of such objects is identified, we can then ask for the existence of a Virasoro structure.

Closely related to the question of the existence of Virasoro action is the following problem in representation theory. Let g be any Lie algebra with a symmetric-bilinear invariant form $(|)$. To the pair $(g, (|))$, we can attach the loop algebra $g \otimes \mathbf{C}[t,t^{-1}]$ and its canonical central extension $\tilde{g}$. The Virasoro algebra Vir acts on $\tilde{g}$ canonically with zero central charge. Thus we can form the semi-direct product $\mathbf{Z}$-graded Lie algebra Vir $\ltimes \tilde{g}$. Given a positive energy $\tilde{g}$-module M, when can we extend it to a Vir $\ltimes \tilde{g}$ -module?

We should point out that this problem has well-known solutions in a number of special cases:

(i) When g is a finite dimensional simple Lie algebra with its Cartan–Killing form $\langle|\rangle$, then a $\tilde{g}$-module M under suitable conditions, admits a Virasoro action. This action is given by the Suguwara–Sommerfield formula of Laurent series

$$\sum_n L_n z^{-n-2} = \frac{1}{2(\chi + h^v)}\sum_i : u_i(z)u^i(z) : . \tag{1.1}$$

Here h^v is the dual Coxeter number of g, χ is the central character of $\tilde{g}$ with $\chi \neq -h^v$, and $u_i(z), u^i(z)$ are the vertex operators associated to the dual bases vectors u_i, u^i of g,

(ii) Similarly, when g is finite dimensional abelian with a non-degenerate invariant form $(|)$, then a $\tilde{g}$-module M admits the Virasoro action

$$\sum_n L_n z^{-n-2} = \frac{1}{2\chi}\sum_i : u_i(z)u^i(z) : . \tag{1.2}$$

This is the original Virasoro formula.

(iii) In the abelian case, there is actually a more general action of the Virasoro algebra which subsumes (ii) above. It is the Chodos–Thron–Feigin–Fuchs action:

$$\sum_n L_n z^{-n-2} = \frac{1}{2\chi}\sum_i (:\ u_i(z)u^i(z)\ : + \alpha_i \frac{d}{dz}u^i(z))\ , \tag{1.3}$$

where the α_i are arbitrary scalar constants. This action is known to be of fundamental importance, both in physics and in mathematics (see [20, 6] and references therein).

All three cases are strongly related to voa theory. In view of the formal resemblance that they have, one should wonder whether there might be a natural common root to all three in the context of voa theory.

We now summarize the problems, motivated by the above discussion, to be studied in this paper.

1.3. Problem Statements. 1. What is structurally special about the objects in category $\mathscr{V}$, i.e. prevoas V with a one-dimensional $V[0]$?

2. Given a Lie group with an invariant metric, construct an analogue of the WZW (pre-)vertex operator algebra.

3. A WZW prevoa is generated by its weight one elements. Describe all the prevoas V generated by $V[1]$ (minimal type ones), in category $\mathscr{V}$.

4. Classify all *simple* minimal type ones.

5. Classify in Virasoro elements on each minimal type one.

1.4 Organization. In the following outline, the main results of this paper are highlighted with **bold-faced** characters.

In Sect. 2, we discuss the notion of a pre-vertex operator algebra and the associated enveloping algebra. We show that there is a distinguished derivation in every prevoa. This derivation plays an important role throughout this work. We also make a short excursion to the notions of normal ordering and operator product expansion.

In Sect. 3, we study the category $\mathscr{V}$ consisting of the prevoas with one-dimensional $V[0]$. We use some elementary structural results to demonstrate the importance of restricting to $\mathscr{V}$ **(Proposition 3.1, Corollary 3.3, Theorem 3.7)**. We organize this category by the notion of the minimal k^{th} types.

In Sect. 4, we begin with the construction of a prevoa structure on $I(g, \mathbf{C})$. The main results in this section are **Theorems 4.7** and **4.11**. Theorem 4.7 generalizes Theorem 2.3.3, Theorem 2.3.4 and the second part of Theorem 2.4.1 of Frenkel–Zhu [11], while Theorem 4.11 describes the type ones in category $\mathscr{V}$.

In Sect. 5, we classify the simple type ones in terms of the self-dual Lie algebras (**Theorem 5.4**). We discuss many examples of self-dual Lie algebras – all of which correspond to new simple prevoas.

In Sect. 6, we classify the Virasoro elements of a type one prevoa **(Theorem 6.4, Corollary 6.7, Theorem 6.11, Theorem 6.14).** We discuss the Virasoro elements in those new examples we give in the last section. As an application, we use the Virasoro elements to study the reducibility problem of the prolongation modules in the generic case **(Theorem 6.19).** We also use the action of the automorphism group on a self-dual Lie algebra to classify the vertex operator algebra structures on $I(g, \mathbf{C}_\chi)$ (**Theorem 6.22**). We then conclude with a few remarks.

1.5 Notations. The bold-faced characters: $\mathbf{C}, \mathbf{R}, \mathbf{Z}, \mathbf{Z}_+$ denote respectively the complex numbers, the reals, the integers and the non-negative integers. Let V be a

Z-graded vector space over **C**. We denote the n^{th} graded piece as $V[n]$. Thus we have $V = \bigoplus_n V[n]$. If f is a linear map of **Z**-graded vector spaces, i.e. f preserves the grading, the $f[n]$ denotes the restriction of f to the n^{th} graded piece. If a $\in V$ is homogeneous element of weight n, we write $|a| = n$. If A is a linear operator on V such that $AV[n] \subset V[n+m]$, then we write $|A| = m$. The restricted dual of V is $V' = \bigoplus_n V[n]'$, where $V[n]'$ is linear dual of $V[n]$.

$V[[z, z^{-1}]]$ denotes the linear space of formal Laurent series with coefficients in V. The subspace of elements with at most finitely many powers of z^{-1} is denoted $V((z))$.

Given a rational function $f(z,w)$ of two variables on the Riemann sphere, we denote the Laurent series expansions in the domains $|z| > |w|, |w| > |z|$, and $|w| > |z-w|$ respectively as $\iota_{z,w} f(z,w), \iota_{w,z} f(z,w)$, and $\iota_{w,z-w} f(z,w)$. For example,

$$\iota_{w,z-w} z^n = \sum_{i \geqq 0} \binom{n}{i} w^{n-1} (z-w)^i \,. \tag{1.4}$$

Other notations used in this paper will be defined locally as we go along.

2. Preparations

We begin with some basic definitions. We review the notion of the universal enveloping algebra associated to a voa, introduced in [11]. We then discussed the notions of normal ordering and operator product expansion. To every prevoa, we attach a canonical derivation which plays an important role throughout our discussion. Some of the results in this section are straightforward generalizations of those in [8].

2.1 Pre-Vertex Operator Algebras.

Definition 2.1 [2,9]. *A prevoa is a pair $(V, Y(-,z))$ where V is a **Z**-graded vector space, $Y(-,z)$ is a linear map $V \longrightarrow (\mathrm{End}V)[[z, z^{-1}]]$ with $a \mapsto \sum_n a_n z^{-n-\Delta}$ for each $a \in V[\Delta]$. In addition, for homogeneous $a, b \in V$, we have*

V1. $Y(a,z) = 0$ iff $a = 0$;

V2. *There is a distinguished* $\mathbf{1}_V \in V$ *with* $Y(\mathbf{1}_V, z) = \mathrm{id}_V$;

V3. $a_n V[m] \in V[m-n]$ *for all* m, n; $a_n b = 0$ *for* $n \gg 0$;

V4. *The Jacobi identity: for all m, n,*

$$\begin{aligned} &\mathrm{Res}_{z-w}(Y(Y(a,z-w)b,w)(z-w)^m \iota_{w,z-w}(w+(z-w))^n) \\ &\quad = \mathrm{Res}_z(Y(a,z)Y(b,w)\iota_{z,w}(z-w)^m z^n) \\ &\quad - \mathrm{Res}_z(Y(b,w)Y(a,z)\iota_{w,z}(z-w)^m z^n) \,. \end{aligned} \tag{2.1}$$

The formal Laurent series $Y(a,z)$ is called the vertex operator associated with a, and $\mathbf{1}_V$ is called the vacuum vector.

When the context is clear we often denote $(V, Y(-,z))$ simply as V, and $\mathbf{1}_V$ as $\mathbf{1}$. For convenience, we often write $Y(a,z) = \sum a(n) z^{-n-1}$. Thus the a_n and the $a(n)$ are related by $a_n = a(n - \mathbf{1} + \Delta)$ whenever $a \in V[\Delta]$. The advantage of the notation $a(n)$ is that it is meaningful even when a itself is not homogeneous. We often denote $Y(a,z)$ simply as $a(z)$.

A prevoa here is what's called a vertex algebra with $\mathbf{Z}$-grading in [2]. If a prevoa is equipped with a distinguished element $\omega \in V[2]$, known as the Virasoro element satisfying certain conditions, it is called a vertex operator algebra (voa) [9]. Not every prevoa admits a Virasoro element. If a prevoa admits an sl_2-action satisfying some additional conditions, it is called a quasi-voa [8].

Definition 2.2 *An ideal I of the prevoa $(V, Y(-,z))$ is a graded subspace of V such that $Y(I,z)V \in I[[z,z^{-1}]] and Y(V,z)I \in I[[z,z^{-1}]]$. A subprevoa V' of $(V, Y(-z))$ is a graded subspace of V such that $(V', Y(-,z)|v')$ is a prevoa is simple if the only ideals are the prevoa itself and* (0).

For a discussion of this and other categorical notions (homomorphisms, modules, etc.) in voa theory, see [8]. It is easy to check that if I is an ideal of the prevoa V, then the quotient space V/I is also a prevoa in a natural way.

The following is an elementary but useful result [8]. It tells us how to recover a from the corresponding vertex operator $Y(a,z)$.

Lemma 2.3. *Let $a, b \in V$. Then we have*
(i) $(a(-1)\mathbf{1})(n) = a(n)$ *for all n,* i.e. $a = a(-1)1$;
(ii) $a(n)1 = 0$ *for all $n \geq 0$.*

Proof.
(i) The first part of (i) is a direct consequence of the Jacobi identity V4 in the case $b = 1, m = -1, n = 0$. Now we have $Y(a(-1)\mathbf{1} - a, z) = 0$. From V1, it follows that $a = a(-1)\mathbf{1}$.
(ii) This part is a consequence of V4 in the case $b = \mathbf{1}, m \geq 0, n = 0$ □

2.2. Universal Enveloping Algebra. Following Frenkel–Zhu [11], we can define the universal enveloping algebra of a prevoa. Again, we weaken their definition by lifting the existence of a Virasoro element. Much of the detail is the same as in Sect. 1.3 of [11]. We will describe briefly the construction.

Let A be any $\mathbf{Z}$-graded associative algebra. A formal Laurent series $b(z) = \sum_i b(i)z^{-i-1}$ in $A[[z,z^{-1}]]$ is called *regular of weight Δ* if $|b(i)| = \Delta - i - 1$ for all i. The subspace spanned by the regular series in $A[[z,z^{-1}]]$ is denoted as $A\langle z\rangle$.

Let $(V, Y(-,z))$ be a prevoa, $S(V)$ be the $\mathbf{Z}$-graded free algebra generated by the symbols $a_s(i)$, where $a \in V$ and $i \in \mathbf{Z}$, subject only to the conditions that

(i) the $a_s(i)$ are linear in a;
(ii) $\mathbf{1}_s(i) = \delta_{i,-1}1$;
(iii) $|a_s(i)| = |a| - i - 1$ whenever a is homogeneous.

Let $\tilde{S}(V)$ be a suitable completion of $S(V)$ (see [11] for details). Let $Y_s(a,z)$ denote the Laurent series of formal symbols: $\sum_i a_s(i)z^{-i-1}$. The completion above is defined so that it is a $\mathbf{Z}$-graded associative algebra containing the following elements:

$$\begin{aligned}&\mathrm{Res}_w \mathrm{Res}_{z-w}(Y_S(Y(a,z-w)b,w)(z-w)^m \iota_{w,z-w}(w+(z-w))^n w^l)\\&\quad - \mathrm{Res}_w \mathrm{Res}_z(Y_S(a,z)Y_S(b,w)\iota_{z,w}(z-w))^m z^n w^l)\\&\quad + \mathrm{Res}_w \mathrm{Res}_z(Y_S(b,w)Y_S(a,z)\iota_{z,w}(z-w))^m z^n w^l)\end{aligned} \tag{2.2}$$

where m, n, l are integers, and a, b are homogeneous elements in V. the universal enveloping algebra $U(V)$ of the prevoa V defined to be quotient of $\tilde{S}(V)$ by the two-sided ideal generated by the above elements (2.2).

By an abuse of notation, we will denote the image of the $a_s(i)$ *in* $U(V)$ *simply as* $a(i)$. *Similarly,* $Y(a,z)$ *will denote the formal series* $\sum_i a(i)z^{-i-1} \in U(V)[[z,z^{-1}]]$. *We will call* $Y(a,z)$ *the vertex series attached to a.* When viewed as a formal series of linear operators acting on some V-module $M, Y(a,z)$ will be called the vertex operator attached to a.

Since the elements (2.2) are themselves homogeneous elements of $\tilde{S}(V)$, it follows that $U(V)$ becomes $\mathbf{Z}$-graded. Note that if a homogeneous element of V, then the vertex series $Y(a,z)$ is regular of weight $|a|$. Thus we have the linear map

$$Y(-,z): \ V \to U(V)\langle z\rangle \ . \tag{2.3}$$

Note that by construction, $Y(-,z)$ also satisfies the Jacobi identity V4, but with $Y(a,z)$ viewed as a vertex series in $U(V)\langle z\rangle$.

2.3. Normal Ordering. Given two vertex operators $Y(a,z), Y(b,z)$ acting on V, it is general meaningless to speak of their product "$Y(a,z)Y(b,z)$" because the coefficients of the z^n in this formal product are in general not a well-defined operator on V. Let

$$Y(a,z)^+ = \sum_{n\geqq 0} a(n)z^{-n-1} \ , \tag{2.4}$$

$$Y(a,z)^- = \sum_{n<0} a(n)z^{-n-1} \ . \tag{2.5}$$

By V3, $Y(b,z)c \in V((z))$ and $Y(a,z)^+c \in V[z,z^{-1}]$. Thus both $Y(a,z)^-Y(b,z)c$ and $Y(b,z)Y(a,z)^+c$ are elements of $V((z))$. Note that both $Y(a,z)^-Y(b,z)$ and $Y(b,z)Y(a,z)^+$ also make sense as elements of $U(V)\langle z\rangle$.

Definition 2.4. *For a, b in V, we define the normal ordered product of the two vertex series* $Y(a,z), Y(b,w)$ *by the following series*:

$$: Y(a,z)Y(b,w) := Y(a,z)^-Y(b,w) + Y(b,w)Y(a,z)^+ \ .$$

Part (ii) of the following lemma shows that: $Y(a,z)Y(b,w)$: is nothing but the non-singular part of the ordinary product.

Lemma 2.5 *For* a, b *is* V, *we have*

(i) $[a(n), Y(b,z)] = \sum_{i\geqq 0} \binom{n}{i} z^{n-i} Y(a(i)b,z)$.

(ii) $Y(a,z)Y(b,w) = \sum_{i\geqq 0} Y(a(i)b,w)\iota_{z,w}(z-w)^{-i-1} + : Y(a,z)Y(b,w):$.

(iii) $Y(a(-2)\mathbf{1},z) = \frac{d}{dz}Y(a,z)$.

(iv) $Y(a(-i-1)b,z) = \frac{1}{i!} : \left(\frac{d}{dz}\right)^i Y(a,z)(b,z) :$.

Proof.

(i) This is the Jacobi identity for vertex series in the case $m = 0$.

(ii) Using the definition of the normal ordered product, we get

$$Y(a,z)Y(b,w) - : Y(a,z)Y(b,w) := [Y(a,z)^+, Y(b,w)] \ . \tag{2.6}$$

Applying part (i) to compute the right-hand side, we get the desired result.

(iii) This is the Jacobi identity in the case $b = 1, m = -2$ and $n = 0$.

(iv) This is the Jacobi identity in the case $m = -i - 1, n = 0$. □

Let's define the normal ordered product of n vertex series $A_1(z), \ldots, A_n(z)$ inductively by

$$: A_1(z) \ldots A_n(z) : \stackrel{\text{def}}{=} A_1(z)^- : A_2(z) \ldots A_n(z) : + : A_2(z) \ldots A_n(z) : A_1(z)^+ . \qquad (2.7)$$

Now applying Lemma 2.5(iv) repeatedly, we get

Corollary 2.6. *For $a_1, \ldots, a_n \in V$ and non-negative integers $i_1, \ldots, i_n$, we have*

$$Y(a_1(-i_1 - 1) \ldots a_n(-i_n - 1)\mathbf{1}, z) = \frac{1}{i_1!} \cdots \frac{1}{i_n!} : \left(\frac{d}{dz}\right)^{i_n} a_1(z) \ldots \left(\frac{d}{dz}\right)^{i_n} a_n(z) : . \qquad (2.8)$$

Lemma 2.5(i), (ii) show that the singular part of $Y(a,z)Y(b,w)$ completely determines the commutators $[a(n), b(m)]$. In fact there is a quick way to *compute* such commutators. Given a V-module M and its restricted dual M', for $v' \in M', v \in M$ and $a, b \in V$ it can be shown using the Jacobi identity [9] that $\langle v', Y(a,z)Y(b,w)v \rangle$ converges to a rational function $R_{v,v'}(a,z;b,w)$ with poles at $z = 0, w = 0, z = w$, in the region $|z| > |w|$. Thus we can define the rational function

$$\text{sing}_{v',v}(a,z;b,w) = R_{v',v}(a,z;b,w) - \langle v', : Y(a,z)Y(b,w) : v \rangle . \qquad (2.9)$$

Note that the last term in Eq. (2.9) is a Laurent *polynomial* in z, w. By Lemma 2.5(ii), we have

$$\text{sing}_{v',v}(a,z;b,w) = \sum_{i \geqq 0} \langle v', Y(a(i)b, w)v \rangle (z - w)^{-i-1} . \qquad (2.10)$$

This will be called the singular part of the operator product expansion of $Y(a,z)$ and $Y(b,w)$. For convenience, we often write

$$Y(a,z)Y(b,w) \sim \sum_{i \geqq 0} Y(a(i)b, w)(z - w)^{-i-1} \qquad (2.11)$$

to mean Eq. (2.10).

Lemma 2.7. *Let C_0 be a contour around* 0, *and C_w around w but not* 0. *Then we have*

$$\langle v', [a(n), b(m)]v \rangle = \frac{1}{(2\pi i)^2} \int_{C_0} \int_{C_w} \text{sing}_{v',v}(a,z;b,w) z^n w^m dz dw .$$

Conversely, we have

$$\iota_{z,w} \text{sing}_{v',v}(a,z;b,w) = \sum_{n \geqq 0} \langle v', [a(n), Y(b,w)]v \rangle z^{-n-1} .$$

Proof. Eq. (2.10), we have

$$\frac{1}{(2\pi i)^2}\int_{C_0}\int_{C_w} \mathrm{sing}_{v',v}(a,z;b,w)z^n w^m dz dw$$
$$=\frac{1}{(2\pi i)}\int_{C_0}\sum_{i\geq 0}\binom{n}{i}\langle v',Y(a,(i)b,w)v\rangle w^{m+n-i}dw$$
$$=\frac{1}{2\pi i}\int_{C_0}\langle v',\ [a(n),Y(b,w)]v\rangle w^m dw$$
$$=\langle v',[a(n),b(m)]v\rangle\ . \tag{2.12}$$

The converse follows immediately from Eqs. (2.6) and (2.9). □

2.4. Derivations. It is interesting to consider the Jacobi identity V4 in the case $m = n = 0$. One gets

$$Y(a(0)b,w) = a(0)Y(b,w) - Y(b,w)a(0)\ . \tag{2.13}$$

We call a linear map $d: V \to V$ a derivation of the prevoa $(V, Y(-,z))$ if for every $a, b \in V$, we have $d(Y(a,z)b) = Y(da,z)b + Y(a,z)db$. A derivation is called inner if $d = a(0)$ for some $a \in V$. In particular, $a(0)$ is an inner derivation for every $a \in V$.

It is trivial to show that the linear space Der V of derivations of the prevoa, V, is a Lie algebra whose bracket is given by the commutator. The inner derivations Inn V form a Lie subalgebra of Der V. Among the derivations, there is one which will play an important role. This derivation is in effect acting as a formal differentiation operator on the prevoa.

Theorem 2.8. *Every prevoa V has a distinguished derivation L_{-1} satisfying the following: for a,b in V,*

(i) $|L_{-1}| = 1$;

(ii) $[L_{-1}, a(i)] = -ia(i-1)$ *for all* i, i.e, $Y(L_{-1}a,z) = \frac{d}{dz}Y(a,z)$. *In particular, L_{-1} centralizes* Inn V *in* Der V;

(iii) $(L_{-1})^i a = i!a(-i-1)\mathbf{1}$ *for* $i \geq 0$;

(iv) $Y(a,z)\mathbf{1} = e^{zL_{-1}}a$;

(v) $Y(a,z)b = e^{zL_{-1}}Y(b,-z)a$;

(vi) L_{-1} *stabilizes every ideal of V*;

(vii) *A submodule M of the adjoint module is an ideal* iff *M is stabilized by L_{-1}.*

Proof.
(i) Define L_{-1} by $L_{-1}a = a(-2)\mathbf{1}$. Thus if a is homogeneous, we have

$$|L_{-1}a| = |a(-2)| = |a(-1)| + 1 = |a| + 1\ . \tag{2.14}$$

The last equality follows from Lemma 2.3(i).
(ii) Let b be a fixed but arbitrary element of V. Fix i and let

$$v = (L_{-1}a(i) - a(i)L_{-1})b + ia(i-1)b$$
$$= (a(i)b)(-2)\mathbf{1} - a(i)b(-2)\mathbf{1} + ia(i-1)b\ . \tag{2.15}$$

We will show that $v = 0$ for all i. For $i < 0$, using Lemma 2.5(iii) and (iv), we have

$$\begin{aligned}Y(v,z) &= \frac{1}{(-i-1)!}\frac{d}{dz}\left(:\left(\left(\frac{d}{dz}\right)^{-i-1}a(z)\right)b(z):\right)\\ &\quad -\frac{1}{(-i-1)!}:\left(\left(\frac{d}{dz}\right)^{-i-1}a\right)\left(\frac{d}{dz}b\right):\\ &\quad +\frac{i}{(-i)!}:\left(\left(\frac{d}{dz}\right)^{-i}a\right)b:\\ &= 0 \end{aligned} \tag{2.16}$$

By V1, $v = 0$. This proves part (ii) for $i < 0$. We now do induction on $i \geqq 0$. First observe that by Lemma 2.5(iii), for every a we have

$$Y(L_{-1}a,z) = Y(a(-2)\mathbf{1},z) = \frac{d}{dz}Y(a,z) \tag{2.17}$$

Also, recall that $a(0)$ is a derivation, for every a in V. Thus for $i = 0$, we have

$$\begin{aligned}Y(v,z) &= Y(L_{-1}a(0)b - a(0)L_{-1}b,z)\\ &= \frac{d}{dz}[a(0),Y(b,z)] - [a(0),\frac{d}{dz}Y(b,z)]\\ &= 0\end{aligned} \tag{2.18}$$

Thus we have $v = 0$. Suppose that $v = 0$ for $i = 0,1,\ldots,k-1$. We want to show that it holds for $i = k$. This requires a lengthy calculation for

$$Y(v,z) = Y(L_{-1}a(k)b,z) - Y(a(k)L_{-1}b,z) + kY(a(k-1)b,z)\,. \tag{2.19}$$

First by Lemma 2.5(i), we have

$$[a(k),Y(b,z)] = Y(a(k)b,z) + \sum_{i=0}^{k-1}\binom{k}{i}z^{k-i}Y(a(i)b,z)\,. \tag{2.20}$$

Differentiating this and applying Eq. (2.17), we get

$$\begin{aligned}[a(k),Y(L_{-1}b,z)] =& Y(L_{-1}a(k)b,z) + \sum_{i=0}^{k-1}\binom{k}{i}(k-i)z^{k-i-1}Y(a(i)b,z)\\ &+ \sum_{i=0}^{k-1}\binom{k}{i}z^{k-i}Y(L_{-1}a(i)b,z)\,.\end{aligned} \tag{2.21}$$

Applying Eq. (2.20) again, but with b replaced by $L_{-1}b$, we can replace the left-hand side of Eq. (2.21) by a new expression, i.e. Eq. (2.21) becomes

$$\begin{aligned}&Y(a(k)L_{-1}b,z) + \sum_{i=0}^{k-1}\binom{k}{i}z^{k-i}Y(a(i)L_{-1}b,z)\\ &= Y(L_{-1}a(k)b,z) + \sum_{i=0}^{k-1}\binom{k}{i}(k-i)z^{k-i-1}Y(a(i)b,z)\\ &+ \sum_{i=0}^{k-1}\binom{k}{i}z^{k-i}Y(L_{-1}a(i)b,z)\end{aligned} \tag{2.22}$$

Isolate the term $Y(L_{-1}a(k)b,z) - Y(a(k)L_{-1}b,z)$ and substitute it into Eq. (2.19). Then using the inductive hypothesis

$$Y(L_{-1}a(i)b,z) - Y(a(i)L_{-1}b,z) = -iY(a(i-1)b,z), \quad i = 0,1,\ldots,k-1 \tag{2.23}$$

to simplify the resulting equation, we get

$$\begin{aligned} Y(v,z) =& \sum_{i=0}^{k-1} \binom{k}{i} z^{k-i} iY(a(i-1)b,z) - \sum_{i=0}^{k-1} \binom{k}{i}(k-i)z^{k-i-1}Y(a(i)b,z) \\ &+ kY(a(k-1)b,z) \\ =& \sum_{i=0}^{k-2} \binom{k}{i+1} z^{k-i-1}(i+1)Y(a(i)b,z) - \sum_{i=0}^{k-1} \binom{k}{i}(k-i)z^{k-i-1}Y(a(i)\,b,z) \\ &+ kY(a(k-1)b,z) \\ =& \sum_{i=0}^{k-1} \left[\binom{k}{i+1}(i+1) - \binom{k}{i}(k-i)\right] z^{k-i-1}Y(a(i)b,z) \\ =& 0\,. \end{aligned} \tag{2.24}$$

The last equality follows from the fact that the term $[\ldots]$ in the summand is identically zero. Now by V1, we conclude that $v = 0$. Thus we have proved that $[L_{-1}, a(i)] = -ia(i-1)$. Combining this with Eq. (2.17), we get

$$Y(L_{-1}a,z) = \frac{d}{dz}Y(a,z) = [L_{-1}, Y(a,z)]\,. \tag{2.25}$$

This completes the proof of part (ii).

(iii) $Y(L_{-1}a,z) = \frac{d}{dz}Y(a,z)$ implies that

$$Y((L_{-1})^i a,z) = \left(\frac{d}{dz}\right)^i Y(a,z)\,. \tag{2.26}$$

This in turn implies that

$$((L_{-1})^i a)(-1) = i!a(-i-1)\,. \tag{2.27}$$

Thus by Lemma 2.3(i), we have

$$(L_{-1})^i a = i!a(-i-1)\mathbf{1} \tag{2.28}$$

(iv) follows immediately from (iii).
(v) This part is also a lengthy calculation:

$$\begin{aligned} Y(a,z)b =& Y(a,z)b(-1)\mathbf{1} \\ =& -[b(-1), Y(a,z)]\mathbf{1} + b(-1)Y(a,z)\mathbf{1} \\ =& -\sum_{i\geq 0} \binom{-1}{i} z^{-1-i} Y(b(i)a,z)\mathbf{1} + b(-1)Y(a,z)\mathbf{1} \quad (\textit{Lemma } 2.5) \\ =& -\sum_{i\geq 0} \binom{-1}{i} z^{-1-i} e^{zL_{-1}} b(i)a + b(-1)e^{zL_{-1}}a \quad (\textit{part } (\mathrm{iv})) \end{aligned}$$

$$=e^{zL-1}\left[\sum_{i\geqq 0} b(i)(-z)^{-1-i} + e^{zL-1}b(-1)e^{zL-1}\right] a$$

$$=e^{zL-1}\left[\sum_{i\geqq 0} b(i)(-z)^{-1-i}\right.$$

$$\left. + \sum_{i\geqq 0}\frac{1}{i!}\left[L_{-1},[L_{-1},..i\ times..[L_{-1},b(-1)]..i\ times..]\right](-z)^i\right] a$$

$$=e^{zL-1}\left[\sum_{i\geqq 0} b(i)(-z)^{-i-1} + \sum_{i\geqq 0}\frac{1}{i!}b(-i-1)(-z)^i\right] a \quad (part\ (ii))$$

$$=e^{zL-1}Y(b,-z)a\,. \tag{2.29}$$

(vi) Let I be an ideal of V, and a in I. Then by definition, $Y(a,z)\mathbf{1} = \sum_i a(i)\mathbf{1}z^{-i-1}$ is in $I[[z,z^{-1}]]$. In particular, $L_{-1}a = a(-2)\mathbf{1}$ is in I.

(vii) Let M be a submodule of the adjoint module V. This means that $Y(V,z)M \subset M[[z,z^{-1}]]$. Suppose M is stabilized by L_{-1}. We need to show that $Y(M,z)V \subset M[[z,z^{-1}]]$. So let a be in M, b in V. Thus $Y(b,-z)a$ is in $M[[z,z^{-1}]]$. By assumption, $e^{zL-1}Y(b,-z)a$ is also in $M[[z,z^{-1}]]$. It follows from part (v) that $Y(a,z)b$ is also in $M[[z,z^{-1}]]$. Part (vi) provides converse. □

3. Prevoas With One-Dimensional V[0]

From now on, we assume that a prevoa V is $\mathbf{Z}_+$-graded. In this section, we present three structural results (Proposition 3.1, Corollary 3.3, Theorem 3.7), for the category $\mathscr{V}$ consisting of the prevoas with one-dimensional $V[0]$. We use some new information concerning the structure of prevoas, to demonstrate the importance of restriction to $\mathscr{V}$.

3.1. A Commutative Algebra Associated to a Prevoa. In this section, we attach a commutative associative algebra to every prevoa V. The point of this exercise is to indicate that in order to get some kind of classification of prevoas (or voas), we must somehow restrict the level zero $V[0]$. This is why we will later begin with the case in which $V[0]$ is one-dimensional.

Let $(V,Y(-,z))$ be a prevoa. Define a bilinear operation $*$ on $V[0]$ by (cf. $*$ operation in [31]).

$$a * b = a(-1)b\,. \tag{3.1}$$

Proposition 3.1. *The space $V[0]$ is a commutative associative algebra with the product* $*$

Proof.

<u>The unit.</u> Obviously, $\mathbf{1}$ is in $V[0]$. Also by Lemma 2.3, we have

$$\begin{aligned} a*\mathbf{1} &= a(-1)\mathbf{1} = a\,,\\ \mathbf{1}*a &= \mathbf{1}(-1)a = a\,. \end{aligned} \tag{3.2}$$

<u>Commutativity.</u> By Lemma 2.5(i), we have

$$[a(n),b(m)] = \sum_{i\geqq 0}\binom{n}{i}(a(i)b)(m+n-i) \tag{3.3}$$

for all integers m,n, and a,b in $V[0]$. By definition

$$|a(i)b| = |a| - i - 1 + |b| - i - 1 \ . \tag{3.4}$$

This means that $a(i)b = 0$ for all $i \geqq 0$. Thus

$$[a(n),b(m)] = 0 \ . \tag{3.5}$$

In particular, we have

$$a * b = a(-1)b(-1)\mathbf{1} = b(-1)a(-1)\mathbf{1} = b(-1)a = b * a \ . \tag{3.6}$$

Associativity. Consider

$$\begin{aligned}(a*b)*c &= (a(-1)b)(-1)c \\ &= \mathrm{Res}_w \mathrm{Res}_{z-w} Y(Y(a,z-w)b,w)c(z-w)^{-1}w^{-1} \\ &= \mathrm{Res}_w \mathrm{Res}_z Y(a,z)Y(b,w)c\iota_{z,w}(z-w)^{-1}w^{-1} \\ &\quad - \mathrm{Res}_w \mathrm{Res}_z Y(b,w)Y(a,z)c\iota_{w,z}(z-w)^{-1}w^{-1} \quad (\textit{Jacobi identity}) \\ &= \sum_{i\geqq 0}(-i-1)b(i-1)c + \sum_{i\geqq 0} b(-i-2)a(i)c \\ &= a(-1)b(-1)c \ \ (|b(n)| = -n-1 \textit{ and } V[m] = 0 \textit{ for } m < 0) \\ &= a*(b*c) \ . \end{aligned} \tag{3.7}$$

This completes our proof. □

It is easy to show that if V, V' are two prevoas, the commutative algebra attached to the prevoa $V \otimes V'$ is the tensor product $V[0] \otimes V'[0]$. Using this, we can show that there is a prevoa V with $V[0] = A$ for *any* commutative algebra A. This is the first reason for us to restrict our consideration to the case in which $V[0]$ is one-dimensional. We now discuss the second reason.

3.2. The Maximal Ideal. We will denote by $\mathscr{V}$, the category of prevoas V with one-dimensional $V[0]$.

Let a,b be elements of the prevoa V. According to Lemma 2.5(i), the coefficients $a(n), b(m)$ of the vertex series satisfy [2, 9]

$$[a(n),b(m)] = \sum_{i\geqq 0}\binom{n}{i}(a(i)b)(n+m-i) \ . \tag{3.8}$$

If a,b are homogeneous, we can write

$$[a_n,b_m] = \sum_{i\geqq 0}\binom{n-1+|a|}{i}(a(i)b)_{n+m} \ . \tag{3.9}$$

Note that $|a_n| = -n$ with respect to the grading of $U(V)$. Thus the subspace of $U(V)$ spanned by the coefficients, a_n, form a $\mathbf{Z}$-graded Lie algebra whose bracket is given by the commutator (3.9). We denote by $\Xi(V)$ the Lie algebra spanned by the a_n's.

The prevoa V is clearly a module, cyclically generated by $\mathbf{1}$, over the graded Lie algebra $\Xi(V)$. Let J_V be the sum of all proper (graded) $\Xi(V)$-submodules in V. Since every proper submodule M has $M[0] = 0$, we have $J_V[0] = 0$. Thus J_V is the unique maximal submodule over the Lie algebra $\Xi(V)$. It is clear that J_V is also the maximal submodule over the prevoa V.

Proposition 3.2. *The derivation L_{-1} of V stabilizes J_V.*

Proof. Suppose $L_{-1}a$ does not belong to J_V, for some $a \in J_V$. We will obtain a contradiction. Since J_V is graded, we may assume that a is homogeneous. We know that $J_V = \oplus_{n \geq n_0} J_V[n]$ for some $n_0 > 0$. So let a be an element of *lowest* weight in J_V for which $L_{-1}a \notin J_V$. Then by the maximality of J_V as $\Xi(V)$-submodule, we see that $L_{-1}a$ generates V. In particular, we have

$$\mathbf{1} \in U\Xi(V)L_{-1}a\,, \tag{3.10}$$

where $U\Xi(V)$ is enveloping algebra $\Xi(V)$. By PBW theorem, we have the decomposition

$$U\Xi(V) = UE(V)_+ U\Xi(V)[0] U\Xi(V)_-\,, \tag{3.11}$$

where $\Xi(V)_\pm = \bigoplus_{\pm n > 0} \Xi(V)[n]$

Now Eq. (3.10) means that there is an element $P \in U\Xi(V)$, homogeneous of weight $-|a| - 1$, such that $PL_{-1}a = \mathbf{1}$. By Eq. (3.11), we may choose P to be in $U\Xi(V)\Xi(V)_-$. So there must be some $b \in V$ and $m > 0$ such that $b_m L_{-1}a \notin J_V$. For otherwise it would mean that $\Xi(V)_- L_{-1}a \subset J_V$, and hence $PL_{-1}a \in J_V$ (J_V *is* $U\Xi(V)$ invariant).

Now by Theorem 2.8(ii), we have

$$b_m L_{-1}a = L_{-1}b_m a + (m - 1 + |b|)b_{m-1}a\,. \tag{3.12}$$

The last term is in J_V because a is. Since the first term is not the J_V, neither is the second. But then $b_m a \in J_V, |b_m a| = |a| - m < |a|$ and yet $L_{-1}b_m a \notin J_V$. This contradicts the minimality of $|a|$. □

Corollary 3.3. *The V-submodule J_V is also the unique maximal ideal of V. In particular, V/J_V is a simple prevoa.*

Proof. Since every ideal of V is, in particular, a submodule of the adjoint module, it follows that J_V contains all the ideals. But J_V itself is also an ideal, by Theorem 2.8(vii) and Proposition 3.2. □

This suggests that there is an abundance of simple objects in $\mathscr{V}$, and that the number of simple quotients of each object is well under control. There is a third reason for this category to be interesting to look at. It turns out to be intimately connected with Lie algebras of a certain type.

3.3. The Functor Γ. For every object V in $\mathscr{V}$, we identify the one-dimensional commutative algebra $V[0]$ with $\mathbf{C}$, by $\mathbf{1} \equiv 1$.

Lemma 3.4. *Let V be an object $\mathscr{V}$. Let a, b be elements of $V[1]$. Then the components $a(n)$, $b(m)$ of the formal series $Y(a,z), Y(b,z)$ in $U(V)\langle z\rangle$ satisfy*

$$[a(n), b(m)] = (a(0)b)(n+m) + na(1)b\,\delta_{n+m,0}1\,.$$

Proof. By Lemma 2.5(i), we have

$$[a(n), Y(b,z)] = \sum_{i \geqq 0} \binom{n}{i} z^{n-i} Y(a(i)b, z)$$
$$= z^n Y(a(0)b, z) + nz^{n-1} Y(a(1)b, z) \quad (b \in V[1]) \tag{3.13}$$

Extracting the coefficients of the z^{-m-1} on both sides, we obtain

$$[a(n), b(m)] = (a(0)b)(n+m) + n(a(1)b)(n+m-1)\,. \tag{3.14}$$

But then $a(1)b \in V[0] = \mathbf{C}1 = \mathbf{C}$. This means that $(a(1)b(n+m-1) = \delta_{n+m,0}1$. □

Lemma 3.5. *Let V be an object in $\mathscr{V}$. For a, b in V,*

(i) $a(0)b = -b(0)a$;
(ii) $a(1)b = b(1)a$.

Proof. (i) Applying Lemma 3.4 in the case $m = 0$, we get

$$(a(0)b)(n) + (b(0)a)(n) = [a(n), b(0)] + [b(0), a(n)] = 0 \tag{3.15}$$

for all n. Thus (i) follows.

(ii) Applying the same lemma again in the case $m = -n = 1$, and using part (i), we get

$$(a(1)b - b(1)a)1 = [a(1), b(-1)] - (a(0)b)(0) + [b(-1), a(1)] - (b(0)a)(0) = 0 \quad \square \tag{3.16}$$

It is important to emphasize that Lemma 3.5 does *not* hold in general if $V[0]$ is not one-dimensional.

Definition 3.6. *Define two bilinear maps*

$$[\cdot,\cdot]: V[1] \times V[1] \longrightarrow V[1]$$
$$(\cdot\,|\cdot): V[1] \times V[1] \longrightarrow \mathbf{C}\,. \tag{3.17}$$

For a, b in $V[1]$, let

$$[a,b] = a(0)b \in V[1]$$
$$(a\,|b) = a(1)b \in V[0] = \mathbf{C}\,. \tag{3.18}$$

This definition has been indicated by Frenkel–Lepowsky–Meurman in the special case, in which $V = V_L$ is the untwisted voa associated to an even positive definite lattice (see [9], Remark 8.9.1).

Theorem 3.7. *The space $V[1]$ is a Lie algebra with the bracket $[\cdot,\cdot]$, and the $V[1]$-invariant symmetric form $(\cdot\,|\cdot)$. We will denote the pair $(V[1], (|))$ by $\Gamma(V)$. If $f: V \longrightarrow V'$ is a homomorphism in $\mathscr{V}$, then it induces a Lie algebra homomorphism $\Gamma(f)$ which preserves the invariant forms. Moreover if f is injective (surjective), then so is $\Gamma(f)$.*

Proof. Let a, b, c be elements of $V[1]$.

Skew-symmetry. By Lemma 3.5(i), we have

$$[a,b] = a(0)b = -b(0)a = -[b,a]\,. \tag{3.19}$$

Lie algebra Jacobi identity. By definition,

$$\begin{aligned}-[a,[c,b]] + [c,[a,b]] + [b,[c,a]] &= -a(0)c(0)b + c(0)a(0)b + b(0)c(0)a \\ &= [c(0),a(0)]b - (c(0)a)(0)b \quad (\textit{Lemma } 3.5(\text{i})) \\ &= 0 \quad (\textit{Lemma } 3.4)\,. \end{aligned} \tag{3.20}$$

Symmetry. Lemma 3.5(ii) implies that

$$(a\,|b) = a(1)b = b(1)a = (b\,|a)\,. \tag{3.21}$$

V[1]-invariant form. By definition,

$$\begin{aligned}([b,a]|c) &= -([a,b]|c) \\ &= -(a(0)b(1)c \\ &= -c(1)(a(0)b) \ (\textit{Lemma } 3.5(\text{ii})) \\ &= [a(0),c(1)]b - a(0)c(1)b \\ &= (a(0)c)(1)b \ (\textit{Lemma } 3.4;\ \ a(0)c(1)b \in a(0)V[0] = 0) \\ &= (a(0)c|b) \\ &= ([a,c]|b) \\ &= (b|[a,c]) \ (\textit{symmetry})\,. \end{aligned} \tag{3.22}$$

Functoriality of Γ. Let $f\colon V \longrightarrow V'$ be a prevoa homomorphism. Let $\Gamma(f)$ be the restriction of f to $V[1]$. Then

$$\begin{aligned}[][fa,fb] &= (fa)(0)fb = f(a(0)b) = f[a,b]\,, \\ (fa|fb) &= (fa)(1)fb = f(a(1)b) = f((a|b)\mathbf{1}_V) = (a|b)\mathbf{1}_{V'}\,. \end{aligned} \tag{3.23}$$

The fact that Γ preserves injectivity and surjectivity of f is clear from the definition. □

We call $\Gamma(V)$ the Frenkel–Lepowsky–Meurman Lie algebra of V.

Proposition 3.8. *If J is a proper ideal of V, then $J[1]$ is a proper ideal of the Lie algebra $\Gamma(V)$. Moreover, $J[1]$ is a subalgebra of the kernel of the invariant form $(|)$. Let's denote $J[1]$ by $\Gamma[J]$. Then there is a natural isomorphism $\Gamma(V/J) \cong \Gamma(V)/\Gamma(J)$.*

Proof. Let $a \in \Gamma(V) = V[1], b \in J[1]$. By definition, $Y(J,z)V$, $Y(V,z)J \subset J[[z,z^{-1}]]$ and $J[0] = 0$. In particular, $[a,b] = a(0)b \in J[1]$ and $(a|b) = a(1)b \in J[0] = 0$. This proves the first two parts.

From the projection p: $V \longrightarrow V/J$ in $\mathscr{V}$, we get a surjective Lie algebra homomorphism in $\mathscr{L}, \Gamma(p)$: $\Gamma(V) = V[1] \longrightarrow \Gamma(V/J) = (V/J)[1]$. The kernel of this map is obviously $J[1] = \Gamma(J)$. □

Lemma 3.4 and Theorem 3.7 clearly suggest that the affinization of a Lie algebra [17] arises naturally in every object V with $V[1] \neq 0$.

3.4. Affinization of $\Gamma(V)$.

Definition 3.9. *Let $\mathscr{L}$ be the following category:* Obj $\mathscr{L}$: *the pairs $(g, (|))$, where g is any Lie algebra with the g-invariant symmetric bilinear form $(|)$.*

Morphisms: homomorphisms of Lie algebras ϕ: $(g,(|)) \longrightarrow (g',(|)')$, which preserve the invariant forms: $(\phi(x)|\phi(y))' = (x|y)$ for all x, y in g.

We define the following functor$^\wedge$: $\mathscr{L} \longrightarrow \mathscr{L}$, called the affinization. Thus given a pair $(g,(|))$, we will define a new pair $(\hat{g},(|))\hat{}$, called the affine Lie algebra associated with the pair $(g,(|))$.

Given a pair $(g,(|))$, define the *loop algebra* associated to g as the Lie algebra $g \otimes \mathbf{C}[t,t^{-1}]$ whose bracket is given in an obvious way. Let $\tilde{g}$ be the one-dimensional central extension:

$$0 \longrightarrow \mathbf{C} \longrightarrow \tilde{g} \longrightarrow g \bigotimes \mathbf{C}[t,t^{-1}] \longrightarrow 0 \,, \tag{3.24}$$

defined by

$$[a \bigotimes t^n, b \bigotimes t^m] = [a,b] \bigotimes t^{n+m} + n(a|b)\delta_{n+m,0}\zeta \,, \tag{3.25}$$

where ζ is the basis element $\mathbf{1} \in \mathbf{C}$ (we use ζ to avoid confusing it with $\mathbf{1}$ of a prevoa). Now $\tilde{g}$ admits a derivation $d = t\dfrac{d}{dt}$. We let $\hat{g}$ be the semi-direct product algebra

$$\hat{g} = \mathbf{C}d| \times \tilde{g} \,. \tag{3.26}$$

We call ζ and d respectively the canonical central element and the canonical derivation of $\hat{g}$.

We now define $(|)\hat{}$. For a,b in g and m,n integers, let

$$\begin{aligned} (a \otimes t^n | b \otimes t^m)\hat{} &= \delta_{m+n,0}(a|b) \,, \\ (a \otimes t^n|\zeta) &= 0 \,, \\ (a \otimes t^n|d) &= 0 \,, \\ (\zeta|d) &= 1 \,, \end{aligned} \tag{3.27}$$

and extended $(|)\hat{}$ by bilinearity and symmetry. It is trivial exercise to check that $(|)\hat{}$ is a well-defined $\hat{g}$-invariant form. This completes the construction of the pair $(\hat{g}),(|)\hat{})$. Note the Lie algebra $\hat{g}$ is naturally $\mathbf{Z}$-graded, with $|a \otimes t^n| = -n$ and $|\zeta| = |d| = 0$. It is also trivial to check that every morphism ϕ: $(g,(|)) \longrightarrow (g'|(|)')$ induces the new morphism $\hat{\phi}$: $(\hat{g},(|)\hat{}) \longrightarrow (\hat{g}'|(|))\hat{}$, defined by

$$\begin{aligned} \hat{\phi}(a \otimes t^n) &= \phi(a) \otimes t^n, \\ \hat{\phi}(\zeta) &= \zeta \,, \\ \hat{\phi}(d) &= d \,. \end{aligned} \tag{3.28}$$

This morphism also respects the new invariant forms:

$$(\hat{\phi}(x)|\hat{\phi}(y))'\hat{} = (x|y) \,. \tag{3.29}$$

Let E be a any g-module. Let p be the subalgebra $\bigoplus_{n \leq 0} \hat{g}_n$. Let χ be a fixed scalar. We extend E to become a $\hat{g}_0 = g \oplus \mathbf{C}\zeta \oplus \mathbf{C}d$-module E_χ, by letting ζ act by scalar χ and d act by zero. Now further extend E_χ to become a p-module by letting $\hat{g}_n$ with $n < 0$, act by zero. We now form the induced module:

$$I(g,E_\chi) = U(\hat{g} \bigotimes_p E_\chi) . \tag{3.30}$$

We call this the *prolongation* of E_χ[19,21,22]. Note that $I(g,E_\chi)$ is $\mathbf{Z}_+$-graded according to the eigenvalues of $-d$. When ζ acts by the scalar 1, we write $I(g,E)$. When E is the trivial g-module $\mathbf{C}$, we will often denote the vector $1 \otimes 1$ as $\mathbf{1}$, and will call this the *vacuum vector* of $I(g,C_\chi)$. In this case, we have $I(g,\mathbf{C}_\chi)[0] = \mathbf{C1}$.

The $\hat{g}$-module $I(g,E_\chi)$ can be characterized by the following universal property. Let E_χ be the p-module as defined above. Then there is a unique pair $(\hat{E}_{\chi^\iota})$, where $\hat{E}_\chi$ is a $\hat{g}$-module and ι: $E_\chi \longrightarrow \hat{E}_\chi$ is a p-module map, satisfying the following property: for every $\hat{g}$-module M and every p-module map f: $E_\chi \longrightarrow M$, there is a unique $\hat{g}$-module map $\hat{f}$: $\hat{E}_\chi \longrightarrow M$ such that $\hat{f} \circ \iota = f$. Clearly we have $\hat{E}_\chi \cong I(g,E_\chi)$.

We now return to the discussion of prevoas. By Theorem 3.7, from every V in $\mathscr{V}$, we obtain an object in $\mathscr{L}$: $\Gamma(V) = (V[1],(|))$. By Lemma 3.4, we have a natural Lie algebra homomorphism $\widehat{\Gamma(V)} \longrightarrow U(V)$, with $a \otimes t^n \mapsto a(n) \in U(V), \zeta \mapsto 1$. Now every V-module M is a $U(V)$-module. This means that M is also a $\widetilde{\Gamma(V)}$-module. Since M is $\mathbf{Z}$-graded, we can let the canonical derivation d of the affine algebra $\widehat{\Gamma(V)}$ act by $d|_{M[n]} = -n$. The space M now becomes a $\widehat{\Gamma(V)}$-module. We summarize this as follows:

Proposition 3.10. *Let V be an object in $\mathscr{V}$. Then every V-module M is a module over the affine Lie algebra $\widehat{\Gamma(V)}$ whose action on M is defined above.*

Let's consider the $\widehat{\Gamma(V)}$-module, $M = V$ itself. Clearly according to the action given by the Proposition 3.10 above, the one-dimensional subspace $V[0]$ satisfies

$$\begin{aligned} (a \otimes t^n) \cdot b &= a(n)b = 0 , \\ \zeta \cdot b &= b , \\ d \cdot b &= 0 \end{aligned} \tag{3.31}$$

for all a in $\Gamma(V), n \geqq 0$ and b in $V[0]$. Here ζ and d are the canonical central element and the canonical derivation of $\widehat{\Gamma(V)}$ respectively. Thus by the universal property of the induced module $I(\Gamma(V),\mathbf{C})$, we have a $\widehat{\Gamma(V)}$-module map

$$I(\Gamma(V),\mathbf{C}) \longrightarrow V, \quad \mathbf{1} \mapsto \mathbf{1}_V . \tag{3.32}$$

This leads us to the following natural questions: *is $I(\Gamma(V),\mathbf{C})$ a prevoa? If so, is the map* (3.32) *a prevoa homomorphism*?

We first organize the category $\mathscr{V}$. We say that a prevoa V is generated by a subset S of V, if V is spanned by the elements [8]

$$a_1(i_1)\dots a_n(i_n)a \tag{3.33}$$

where $a_1,\dots,a_n,a$, range over the S and $i_1,\dots,i_n$ are integers. We say that I is an ideal of V generated by the subset $S \subset V$, if I is the smallest ideal containing S.

Definition 3.11. *Let $\mathscr{V}_k$, where $k \geqq 0$, be the subcategory of $\mathscr{V}$ in which every object V is generated by the subspace $\bigoplus_{0 \leqq i \leqq k} V[i]$. We call an object of $\mathscr{V}_k$ a prevoa of the minimal k^{th} type.*

Thus the only zeroth type is the commutative algebra $\mathbf{C}$. It is obvious that we have $\mathscr{V}_0 \subset \mathscr{V}_1 \subset \ldots$. One of the main tasks of this paper is to describe all the (minimal) type ones.

4. The Prevoa $I(g, \mathbf{C})$

Let $(g, (|))$ be a pair in the category $\mathscr{L}$. We begin with the construction of a prevoa structure on $I(g, \mathbf{C})$. The main results in this section are Theorems 4.7 and 4.11. Theorem 4.7 generalizes Theorem 2.3.3, Theorem 2.3.4 and the second part of Theorem 2.4.1 of Frenkel–Zhu [11], while Theorem 4.11 describes the type ones in category $\mathscr{V}$.

To define the vertex operators on $I(g, \mathbf{C})$, we use many ideas of Frenkel–Zhu[11] where they consider the case in which g if finite dimensional and simple. However we emphasize that to prove Theorem 4.7 here, while following the *general strategy* of [11], we use different ingredients. For example, Frenkel–Zhu begin with an *explicit* formula for the "n-point correlation function":

$$\langle v', a_1(z_1) \cdots a_n(z_n)v)\rangle \ . \tag{4.1}$$

This formula requires the use of multilinear trace form on g. In our general setting, however, where g may be infinite dimensional and $(|)$ may be degenerate, a multilinear trace form may not exist. Thus we must prove the required properties of the n-point function *abstractly*.

4.1. The Frenkel–Zhu construction. Let $(g, (|))$ be a pair in the category $\mathscr{L}$. Recall that $\tilde{g}$ is the one-dimensional central extension, as defined in Sect. 3.4, of the loop algebra of g, and that $\hat{g}$ is the affinization of g. Let ζ be the canonical central element of $\tilde{g}$. Let $A^\infty\tilde{g}$ be the completed universal enveloping algebra of $\tilde{g}$ (for definition, see [32, 11]). We denote as $U^\infty\tilde{g}$ the quotient of $A^\infty\tilde{g}$ by the ideal generated by $\zeta - 1$. As before, we denote by $U^\infty\tilde{g}\langle z\rangle$ the subspace spanned by the regular series in $U^\infty\tilde{g}[[z, z^{-1}]]$. For a $\subset$ g, we write $a(n) = a \otimes t^n$ and let $a(z) = \sum_n a(n)z^{-n-1}$, viewed as an element of $U^\infty\tilde{g}\langle z\rangle$.

Definition 4.1. *For every regular series* $b(z) = \sum_n b(n)z^{-n-1}$ *in* $U^\infty\tilde{g}\langle z\rangle$, *and* $a \in g$ *we define the action of* $a(n)$ *on* $b(z)$ *by*

$$a(n) \bullet b(z) = \mathrm{Res}_w(a(w)b(z)\iota_{w,z}(w-z)^n - b(z)a(w)\iota_{z,w,}(z-w)^n) \ . \tag{4.2}$$

Then applying a similar argument as in [11], we show that Eq. (4.2) defines a $\tilde{g}$-module structure on the space $U^\infty\tilde{g}\langle z\rangle$. Moreover if $b(z)$ is a regular series of weight $\Delta - n$ then $a(n) \bullet b(z)$ is a regular series of weight $\Delta - n$. Thus if we let the weights of the regular series define a $\mathbf{Z}$-grading on the space $U^\infty\tilde{g}\langle z\rangle$, then this space becomes a graded module, i.e. it is now a $\hat{g}$-module, where the canonical derivation acts by $d \bullet b(z) = \Delta b(z)$.

Now the regular series 1 satisfies

$$\begin{aligned} d \bullet 1 &= 0 \\ \zeta \bullet 1 &= 1 \\ a(n) \bullet 1 &= 0 \end{aligned} \tag{4.3}$$

for all $n \geqq 0$. Thus by the universal property of $I(g, \mathbf{C})$, we have a $\hat{g}$-module map

$$Y(-,z): I(g,\mathbf{C}) \longrightarrow U^{\infty}\tilde{g}\langle z\rangle\,,$$
$$a_1(i_1)\cdots a_n(i_n)\mathbf{1} \mapsto a_1(i_1)\bullet\cdots\bullet a_n(i_n)\bullet 1\,. \tag{4.4}$$

In particular, this map respects the $\mathbf{Z}$-grading. Thus if $a \in I(g,\mathbf{C})$ is homogeneous, then $Y(a,z) = \sum_n a(n)z^{-n-1}$ is a regular series of weight $|a|$, i.e. $|a(n)| = |a| - n - 1$ for all n.

Since $I(g, \mathbf{C})$ is a $U^{\infty}\tilde{g}$-module, we have a homomorphism

$$U^{\infty}\tilde{g} \longrightarrow \text{End } I(g,\mathbf{C})\,. \tag{4.5}$$

Composing this homomorphism with $Y(-,z)$ (and using the same notation), we get

$$Y(-,z): I(g,\mathbf{C}) \longrightarrow \text{End } I(g,\mathbf{C})\langle z\rangle\,. \tag{4.6}$$

Proposition 4.2. *Let $Y(-,z)$ be given by Eq.* (4.6), *and $V = I(g,\mathbf{C})$ be graded according to the eigenvalues of $-d$. Then $(V, Y(-,z))$ satisfies* V1, V2 *and* V3 *of Definition* 2.1.

The proof is similar to the one for the first part of Theorem 2.4.1 of [11]. We now prepare for the proof of the Jacobi identity for modules. *In the rest of this section, we let M be a $\hat{g}$-module with $M[n] = 0$ for $n \ll 0$, and M' be the restricted dual. Thus we have an algebra homomorphism $\pi_M: U^{\infty}\tilde{g} \longrightarrow$* End$M$.

4.2. The N-point Functions

Proposition 4.3. *For any fixed $v \in M, v' \in M'$, and $a, b \in g$, the Laurent series $\langle v', a(z)b(w)v\rangle$ converges to a rational function $R_{v',v}(a,z;b,w)$ in the domain $|z| > |w|$, with possible poles at $z = 0$, $w = 0$, $z = w$.*

Proof. A simple calculation shows that

$$\begin{aligned}\langle v', a(z)b(w)v\rangle &= \langle v', v\rangle(a|b)\iota_{z,w}(z-w)^{-2}\\ &= +\langle v', [a,b](w)v\rangle\iota_{z,w}(z-w)^{-1} + \langle v', :a(z)b(w):v\rangle\,,\end{aligned} \tag{4.7}$$

where the normal ordered product is given by

$$:a(z)b(w): \,= a(z)^{-}b(w) + b(w)a(z)^{+}\,. \tag{4.8}$$

Since $\langle v', [a,b](n)v\rangle = 0$ for all but finitely many n, we see that $\langle v', [a,b](w)v\rangle$ is a Laurent polynomial in w. Therefore the first two terms on the right-hand side of Eq. (4.7) converge to a rational function.

Since $a(n)v = 0$ for $n \gg 0, a(z)^{+}v$ is a Laurent polynomial. Thus $\langle v', b(w)a(z)^{+}v\rangle$ is a Laurent polynomial in z, w. Similarly for $\langle v', a(z)^{-}b(w)v\rangle$. Thus the last term of Eq. (4.7) is a Laurent polynomial in z, w. □

Proposition 4.4. *The rational function above $R_{v',v}(a,z;b,w)$ satisfies*

$$R_{v',v}(a,z;b,w) = R_{v',v}(b,w;a,z)\,. \tag{4.9}$$

Proof. From the proof of Proposition 4.3, we have

$$\begin{aligned}R_{v',v}(a,z;b,w) =&\langle v', v\rangle(a|b)(z-w)^{-2} + \langle v', [a,b](w)v\rangle(z-w)^{-1}\\ &+ \langle v', :a(z)b(w):v\rangle\,,\end{aligned} \tag{4.10}$$

$$R_{v',v}(b,w;a,z) = \langle v',v\rangle(b|a)(w-z)^{-2} + \langle v',[b,a](z)v\rangle(w-z)^{-1} + \langle v', :b(w)a(z): v\rangle\ . \tag{4.11}$$

One should keep in mind that all the terms of the form $\langle v',\cdots v\rangle$ are Laurent polynomials in z, w.

Now to prove our claim, it is enough to show that $R_{v',v}(b,w;a,z) - R_{v'v}(a,z;b,w)$ is zero is the domain $|z| > |w| > |z-w|$. This will involve a straightforward but rather lengthy calculation:

$$\begin{aligned}
&R_{v',v}(b,w;a,z) - R_{v',v}(a,z;b,w)\\
&= \langle v',[b,a](z)v\rangle(w-z)^{-1} - \langle v',[a,b](w)v\rangle(z-w)^{-1}\\
&\quad + \langle v', :b(w)a(z): v\rangle - \langle v', :a(z)b(w): v\rangle\\
&= \sum_{n>0}\left(\frac{d}{dw}\right)^n \langle v',[a,b](w)v\rangle\frac{(z-w)^{n-1}}{n!}\\
&\quad - \langle v',[a(z)^-,b(w)^-]v\rangle - \langle v',[b(w)^+,a(z)^+]v\rangle\\
&= \sum_{n>0;k<0}\left(\frac{d}{dw}\right)^n w^{-k-1}\langle v',[a,b](k)v\rangle\frac{(z-w)^{n-1}}{n!}\\
&\quad + \sum_{n>0;k\geqq 0}\left(\frac{d}{dw}\right)^n w^{-k-1}\langle v',[a,b](k)v\rangle\frac{(z-w)^{n-1}}{n!}\\
&\quad - \langle v',[a(z)^-,b(w)^-]v\rangle - \langle v',[b(w)^+,a(z)^+]v\rangle\\
&= \sum_{l>n>0}(l-1)(l-2)\cdots(l-n)w^{l-n-1}\langle v',[a,b](-l)v\rangle\frac{(z-w)^{n-1}}{n!}\\
&\quad + \sum_{1>0;k\geqq 0}(-k-1)(-k-2)\cdots(-k-n)w^{-k-n-1}\langle v',[a,b](k)v\rangle\frac{(z-w)^{n-1}}{n!}\\
&\quad - \langle v',[a(z)^-,b(w)^-]v\rangle - \langle v',[b(w)^+,a(z)^+]v\rangle\\
&= \sum_{1>1}\langle v',[a,b](-l)v\rangle(z^{l-2}+z^{l-3}w+\cdots+zw^{l-3}+w^{l-2})\\
&\quad - \sum_{k\geqq 0}\left(\sum_{n>0}\binom{n+k}{k}\left(\frac{w-z}{w}\right)^{n-1}\right)\langle v',[a,b](k)v\rangle w^{-k-2})\\
&\quad - \langle v',[a(z)^-,b(w)^-]v\rangle - \langle v',[b(w)^+,a(z)^+]v\rangle\\
&= \sum_{m,n>0}\langle v',[a,b](-m-n)v\rangle z^{m-1}w^{n-1}\\
&\quad - \sum_{k\geqq 0}\left(\sum_{\geqq n\geqq 0} z^{-n-1}w^{n+1}\right)\langle v',[a,\,b](k)v\rangle w^{-k-2}\\
&\quad - \langle v',[a(z)^-,b(w)^-]v\rangle - \langle v',[b(w)^+,a(z)^+]v\rangle\\
&= +\langle v',[a(z)^-b(w)^-]v\rangle + \langle v',[b(w)^+,a(z)^+]v\rangle\\
&\quad - \langle v',[a(z)^-,b(w)^-]v\rangle - \langle v',[b(w)^+,a(z)^+]v\rangle\\
&= 0 \quad \square
\end{aligned} \tag{4.12}$$

Proposition 4.5. *For $v \in M, v' \in M'$, and $a_1,\ldots a_n \in g$, the Laurent series $\langle v', a_1(z_1)\cdots a_n(z_n)v\rangle$ converges to a rational function $R_{v',v}(a_1,z_1;\cdots;a_n,z_n)$ in the domain*

$|z_1| > \cdots > |z_n|$, *with possible poles at* $z_i = 0$ *and* $z_i = z_j$ *for* $i \neq j$. *We call the rational function an n-point function.*

Proof. When $n = 1$, we get a Laurent polynomial in one variable. When $n = 2$, we have Proposition 4.3. Suppose our claim holds for the n-variable case. Let's consider the $n+1$-variable case:

$$\begin{aligned}
&\langle v', a_0(z_0)a_1(z_1)\cdots a_n(z_n)v\rangle \\
&= \langle v', a_0(z_0)^- a_1(z_1)\cdots a_n(z_n)v\rangle + \langle v', [a_0(z_0)^+, a_1(z_1)\cdots a_n(z_n)]v\rangle \\
&\quad + \langle v', a_1(z_1)\cdots a_n(z_n)a_0(z_0)^+ v\rangle \\
&= \langle v', a_0(z_0)^- a_1(z_1)\cdots a_n(z_n)v\rangle \\
&\quad + \sum_{n\geqq k\geqq 1} \langle v', a_1(z_1)\cdots \widehat{a_k(z_k)}\cdots a_n(z_n)v\rangle (a_0|a_k)\iota_{z_0,z_k}(z_0 - z_k)^{-2} \\
&\quad + \sum_{n\geqq k\geqq 1} \langle v', a_1(z_1)\cdots [a_0, a_k](z_k)\cdots a_n(z_n)v\rangle \iota_{z_0,z_k}(z_0 - z_k)^{-1} \\
&\quad + \langle v', a_1(z_1)\cdots a_n(z_n)a_0(z_0)^+ v\rangle \, . \qquad (4.13)
\end{aligned}$$

Since $a_0(z_0)^+ v$ is a polynomial, the last term has at most finitely many powers of z_0. The coefficient of each power of z_0 converges to an n-point function. Thus by inductive hypothesis, the last term of Eq. (4.13) converges to a $n+1$-variable rational function with the desired properties. Similarly, the second and the third sums also converge to (sums of) rational functions. Thus we only need to check that the first term also has the desired properties. It is enough to show that $\langle v', a_0(m)a_1(z_1)\cdots a_n(z_n)v\rangle = 0$ for all but finitely many $m < 0$. Without loss of generality, suppose that both v', v are homogeneous. If $\langle v', a_0(m)a_1(z_1)\cdots a_n(z_n)v\rangle \neq 0$, then

$$\langle v', a_0(m)a_1(i_1)\cdots a_n(i_n)v\rangle \neq 0 \qquad (4.14)$$

for some $i_1, \cdots, i_n$ with $|v'| = |v| - m - i_1 - \cdots - i_n$ and $|v| - i_1 - \cdots - i_n \geqq N$, because the grading of M is bounded from below by N. This means that $m \geqq N - |v'|$. □

Proposition 4.6. *For the same hypotheses as in Proposition* 4.5, *the n-point function is permutation invariant,* i.e.

$$R_{v',v}(a_1, z_1; \cdots; a_n, z_n) = R_{v',v}(a_{i_1}, z_{i_1}; \cdots; a_{i_n}, z_{i_n}) \qquad (4.15)$$

for any permutation $(i_, \ldots, i_n)$ *of* $(1, \ldots, n)$.

Proof. The $n = 2$ case is given by Proposition 4.4. Assume that the claim holds for n-point functions, and consider the $n+1$-case where $n > 1$. If the permutation $(i_0, i_1, \ldots, i_n)$ of $(0, 1, \ldots, n)$ stabilizes 0, then form the recursion formula (4.13) that defines $R_{v',v}$, it is clear that Eq. (4.15) holds. Therefore, it is enough to show that Eq. (4.15) holds for the permutation $(1, 0, 2\ldots, n)$.

By a similar calculation as in the proof of Proposition 4.5, we get

$$\begin{aligned}
&\langle v', a_0(z_0)a_1(z_1)\cdots a_n(z_n)v\rangle \\
&= \langle v', a_n(z_n)^- a_0(z_0)\cdots a_{n-1}(z_{n-1})v\rangle
\end{aligned}$$

$$+\sum_{n-1\geqq k\geqq 0}\langle v',a_0(z_0)\cdots\widehat{a_k(z_k)}\cdots a_{n-1}(z_{n-1})v\rangle(a_n|a_k)\iota_{z_k,z_n}(z_k-z_n)^{-2}$$
$$+\sum_{n-1\geqq k\geqq 0}\langle v',a_0(z_0)\cdots[a_k,a_n](z_k)\cdots a_{n-1})v\rangle\iota_{z_k,z_n}(z_k-z_n)^{-1}$$
$$+\langle v',a_0(z_0)\cdots a_{n-1}(z_{n-1})a_n(z_n)^{+}v\rangle\,. \tag{4.16}$$

Now we know that this Laurent series converges to the left-hand side of Eq. (4.15). But by inductive hypothesis, Eq. (4.16) shows that this series also converges to a rational function that is invariant under the interchange of (a_0,z_0) and (a_i,z_i) for any $i<n$ – in particular for $i=1$. This completes our proof. □

4.3. The Jacobi Identity

Theorem 4.7. *Fix $v'\in M',v\in M$.*
(i) *Let $b_i,\ldots,b_n$ be elements of $I(g,\mathbf{C})$. Then the series*

$$\langle v',Y(b_1,z_1)\cdots Y(b_n,z_n)v\rangle \tag{4.17}$$

converges to a rational function $R_{v',v}(b_1,z_1;\cdots;b_n,z_n)$ in the domain $|z_1|>\cdots>|z_n|$, with possible poles at $z_i=0$ and $z_i=z_j$ for $i\neq j$.
(ii) *For every permutation $(i_1,\cdots,i_n)$ of $(1,\cdots,n)$, we have*

$$R_{v',v}(b_1,z_1;\cdots;b_n,z_n)=R_{v',v}(b_{i_1},z_{i_1};\cdots;b_{i_n},z_{i_n})\,. \tag{4.18}$$

(iii) *The series*

$$\langle v',Y(Y(b_1,z-w)b_2,w)v\rangle \tag{4.19}$$

converges to the rational function $R_{v',v}(b_1,z;b_2,w)$ in the domain $|w|>|z-w|$.
(iv) *Let $Y_M(-,z)=\pi_M\circ Y(-,z):I(g,\mathbf{C})\longrightarrow \mathrm{End}\ M\langle z\rangle$. Then the Jacobi identity holds for $(M,Y_M(-z))$.*

We note that Theorem 4.7(i)–(iv) here generalize respectively the following four results of Frenkel–Zhu[11]: the two parts of Theorem 2.3.3, Theorem 2.3.4, and the second part of Theorem 2.4.1. In the proofs of their results, their Proposition 2.3.1 was the key ingredient. Propositions 4.5 and 4.6 above are stronger versions of their Proposition 2.3.1. By using the same line of argument as Frenkel–Zhu did for their Theorems 2.3.3 and 2.3.4, but with the key ingredients Propositions 4.5 and 4.6 above replacing their Proposition 2.3.1, one can easily generalize their proofs to the current setting for Theorem 4.7(i)–(iii). Thus we refer the readers to the reference [11].

As for Theorem 4.7(iv), one might first try to imitate Frenkel–Zhu's argument for their Theorem 2.4.1. But the argument, aside from using their Theorems 2.3.3 and 2.3.4, also relies on the following fact: given a finite dimensional simple Lie algebra g, the Killing form $(|)$ of g, and an irreducible g-module E, the $\hat{g}$-module $I(g,E_\chi)$ is irreducible for generic values of χ. It is easy to show that this breaks down in general. In fact, for a general pair $(g,(|))$ in the category $\mathscr{L}$, $I(g,E)$ is always reducible, whenever $(|)$ is degenerate. Even when $(|)$ is non-degenerate, it is not obvious how to prove the irreducibility property.

The way we bypass this difficult in our setting is by first strengthening the key ingredient – replacing their Proposition 2.3.1 by Propositions 4.5 and 4.6 above. This leads to stronger versions – namely Theorems 4.7(i)–(iii) above – of their Theorems

2.3.3 and 2.3.4. We now see that for proving the Jacobi identity, Theorems 4.7(i)–(iii) suffice.

The first part of the following theorem summarizes the results given by Proposition 4.2 and Theorem 4.7(iv).

Theorem 4.8. *For every pair* $(g, (|))$ *in the category* $\mathscr{L}$, $I(g,\mathbf{C})$ *is a prevoa in* $\mathscr{V}$. *Moreover, we have* $\Gamma(I(g,\mathbf{C})) \cong (g,(|))$.

Proof. We only need to prove the second part. By definition, we have

$$I(g,\mathbf{C})[1] = \{a(-1)\mathbf{1} | a \in g\} . \tag{4.20}$$

Using the PBW basis of $I(g,\mathbf{C})$, it is easy to see that $a \mapsto a(-1)\mathbf{1}$ defines a linear isomorphism $g \longrightarrow I(g,\mathbf{C})[1]$. Now by definition, the bracket on $\Gamma(I(g,\mathbf{C})$ is given by

$$[a(-1)\mathbf{1}, b(-1)\mathbf{1}] = a(0)b(-1)\mathbf{1} = [a,b](-1)\mathbf{1} \tag{4.21}$$

for all $a,b \in g$. Similarly, the bilinear form on $\Gamma(I(g,\mathbf{C}))$ is given by

$$(a(-1)\mathbf{1}|b(-1)\mathbf{1}) = a(1)b(-1)\mathbf{1} = a(1)b(-1)\mathbf{1} = (a|b)_g\mathbf{1} \equiv (a|b)_g . \tag{4.22}$$

This completes the proof. □

4.4. Functoriality. Next we show that the construction of the prevoa $I(g,\mathbf{C})$, is functorial.

Let ϕ: $(g,(|)) \longrightarrow (g',(|)')$ be a morphism in the category $\mathscr{L}$. As seen in Sect. 3.4, this induces a morphism $\hat{\phi}: (\hat{g},(|)^\wedge) \longrightarrow (\hat{g}',(|)'^\wedge)$. Let E be any g'-module. The $\hat{g}'$-module $I(g',E)$ becomes a $\hat{g}$-module $\hat{\phi}^* I(g',E)$ by pull back. On the other hand, the g'-module E becomes a g-module ϕ^*E. Now in an obvious way, we get a p-module map $(p = \bigoplus_{n\leqq 0}\hat{g}_n)\phi^*E \longrightarrow \hat{\phi}^* I(g',E)$. Thus by the universal property of the prolongation module, we have a $\hat{g}$-module map

$$\begin{aligned} \hat{\phi}: I(g,\phi^*E) &\longrightarrow \hat{\phi}^* I(g',E) , \\ a_1(n_1)\cdots a_k(n_k) \otimes e &\mapsto (\phi a_1)(n_1)\cdots(\phi a_k)(n_k) \otimes e \end{aligned} \tag{4.23}$$

for $a_1,\ldots,a_k \in g, e \in E$ and integers $n_1,\ldots,n_k$. More generally we have

$$a_1(n_1)\cdots a_k(n_k)\cdot b \mapsto (\phi a_1)(n_1)\cdots(\phi a_k)(n_k)\cdots\hat{\phi}\cdot b . \tag{4.24}$$

Now we restrict to the case where E is the trivial g'-module. By the second part of Theorem 4.8, we can identify $I(g,\mathbf{C})[1]$ with g. Thus by Corollary 2.6, for $a_1,\ldots,a_n \in g = I(g,\mathbf{C})[1]$ and non-negative integers $i_1,\ldots,i_n$, we have

$$Y(a_1(-i_1-1)\cdots a_n(-i_n-1)\mathbf{1},z) = \frac{1}{i_1!}\cdots\frac{1}{i_n!}: \left(\frac{d}{dz}\right)^{i_n} a_1(z)\cdots\left(\frac{d}{dz}\right)^{i_n} a_n(z): . \tag{4.25}$$

Since the prevoa $(I(g,\mathbb{C}),Y(-,z))$ is generated by its level one elements, it follows that the image of $Y(-,z)$ is spanned by those expressions in (4.25). The same holds for g' and the prevoa $(I(g',\mathbf{C}),Y(-,z))$. It is now clear from Eq. (4.24) that

$$\hat{\phi}Y(a,z)b = Y(\hat{\phi}a,z)\hat{\phi}b \tag{4.26}$$

for every a,b in $I(g,\mathbf{C})$. To summarize, we have

Proposition 4.9. *The correspondence* $(g,(|)) \to I(g,\mathbf{C})$ *is a functor from* $\mathscr{L}$ *to* $\mathscr{V}_1$.

We now return to a question which we raised at the end of Sect. 3. Given a prevoa V in the category $\mathscr{V}$, we considered the $\widehat{\Gamma(V)}$-module map

$$\begin{aligned} \widehat{i_V}: I(\Gamma(V),\mathbf{C}) &\longrightarrow V \\ a_1(n_1)\cdots a_k(n_k)\mathbf{1} &\mapsto a_1(n_1)\cdots a_k(n_k)\mathbf{1}_V \end{aligned} \tag{4.27}$$

for $a_1,\ldots,a_k \in \Gamma(V)$ and integers $n_1,\ldots,n_k$. By an argument similar to that of Proposition 4.9, we have

$$\widehat{i_V}Y(a,z)b = Y(\widehat{i_V}a,z)\widehat{i_V}b \tag{4.28}$$

for all $a,b \in I(\Gamma(V),\mathbf{C})$. Thus the map $\hat{i}_V$ above is a prevoa homomorphism.

Let $f: V \longrightarrow V'$ be a morphism in the category $\mathscr{V}$. By functoriality of Γ,f induces a morphism $\Gamma(f): \Gamma(V) \longrightarrow \Gamma(V')$ in the category $\mathscr{L}$.

Proposition 4.10. *Given a commutative diagram in* $\mathscr{L}$:

$$\begin{array}{ccc} g & \longrightarrow & g' \\ \downarrow & & \downarrow \\ \Gamma(V) & \longrightarrow & \Gamma(V') \end{array} \tag{4.29}$$

we have a commutative diagram in $\mathscr{V}$:

$$\begin{array}{ccc} I(g,\mathbf{C}) & \longrightarrow & I(g',\mathbf{C}) \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array} \tag{4.30}$$

Proof. By the functoriality of $I(-,\mathbf{C})$ (Proposition 4.9), we have the following commutative diagram in $\mathscr{V}$:

$$\begin{array}{ccc} I(g,\mathbf{C}) & \longrightarrow & I(g',\mathbf{C}) \\ \downarrow & & \downarrow \\ I(\Gamma(V),\mathbf{C}) & \longrightarrow & I(\Gamma(V'),\mathbf{C}) \end{array} \tag{4.31}$$

Since $\widehat{i_V}$ and $\widehat{i'_V}$ are prevoa homomorphisms, we have yet another diagram in $\mathscr{V}$:

$$\begin{array}{ccc} I(\Gamma(V),\mathbf{C}) & \stackrel{\widehat{\Gamma(f)}}{\longrightarrow} & I(\Gamma(V'),\mathbf{C}) \\ \widehat{i_V}\downarrow & & \widehat{i_{V'}}\downarrow \\ V & \stackrel{f}{\longrightarrow} & V' \end{array} \tag{4.32}$$

Thus we need to show that diagram (4.32) is commutative, for then combining diagrams (4.31) and (4.32) gives the desired result.

Let's denote the vacuum vectors of V, V', $I(g,\mathbf{C})$, $I(g',\mathbf{C})$ respectively as $\mathbf{1}_V$, $\mathbf{1}_{V'}$, $\mathbf{1}$, $\mathbf{1}'$. For any $a,b \in V$, we have

$$fY(a,z)b = Y(fa,z)fb\ . \tag{4.33}$$

In particular, if $a_1, \ldots, a_n \in \Gamma(V) = V[1]$ and $i_1, \ldots, i_n$ are integers, then

$$f(a_1(i_1)\cdots a_n(i_n)\mathbf{1}_V) = (fa_1)(i_1)\cdots(fa_n)(i_n)\mathbf{1}_{V'} \tag{4.34}$$

Now by Eqs. (4.23) and (4.27), we have

$$\begin{aligned}\widehat{i'_V} \circ \widehat{\Gamma(f)}(a_1(i_1)\cdots a_n(i_n)\mathbf{1}) &= \widehat{i'_V}(fa_1)(i_1)\cdots(fa_n)(i_n)\mathbf{1}' \\ &= (fa_1)(i_1)\cdots(fa_n)(i_n)\mathbf{1}_{V'} \\ &= f(a_1(i_1)\cdots(i_n)\mathbf{1}_V) \\ &= f \circ \widehat{i_V}(a_i(i_1)\cdots a_n(i_n)\mathbf{1})\,. \end{aligned} \tag{4.35}$$

This completes our proof. □

Theorem 4.11. *Every minimal type one is a quotient of $I(g,\mathbf{C})$, for some pair $(g,(|))$ in the category $\mathscr{L}$. Conversely, every quotient of $I(g,\mathbf{C})$ by an ideal is a minimal type one.*

Proof. Let V be a minimal type one. By definition, V is spanned by elements of the form $a_1(i_1)\cdots a_n(i_n)a$, where the a's are elements of $V[1]+V[0]$. But $V[0] = \mathbf{C}\mathbf{1}_V$. Thus without loss of generality, we may assume that $a = \mathbf{1}_V$ and $a_1,\cdots,a_n \in V[1]$. This means that the *pre*voa map $\widehat{i_V} : I(\Gamma(V),\mathbf{C}) \to V$ in onto.

Conversely for every quotient V of $I(g,\mathbf{C})$, we have an onto map $I(g,\mathbf{C}) \to V$. Since $I(g,\mathbf{C})$ is generated by $I(g,\mathbf{C})[1]$, it follows that V is generated by $V[1]$. □

4.5. Modules over $I(g,\mathbf{C})$

Proposition 4.12. *Let M be $\mathbf{Z}$-graded space with $M[n] = 0$ for $n \ll 0$. M is a $I(g,\mathbf{C})$-module* iff *it is a $U^\infty\tilde{g}$-module.*

Proof. Let $(M, Y_M(-,z))$ be an $I(g,\mathbf{C})$-module. By Theorem 4.8, $I(g,\mathbf{C})$ is a prevoa in $\mathscr{V}$ and $\hat{\Gamma}(I(g,\mathbf{C})) \cong (\hat{g},(|)\hat{\ })$. Thus by Proposition 3.10, M is a $\hat{g}$-module in which ζ acts by 1. Since $M[n] = 0$ for $n \ll 0$, it is also a $U^\infty\tilde{g}$-module.

Conversely given such an algebra homomorphism $\pi_M : U^\infty\tilde{g} \to \mathrm{End}\, M$, we obtain $Y_M(-,z) = \pi_M \circ zY(-,z)$ which satisfies the Jacobi identity according to Theorem 4.7(iv). Obviously, $Y_M(\mathbf{1},z) = \mathrm{id}_M$. Since both maps π_M and $Y(-,z)$ respects the $\mathbf{Z}$-grading, so does $Y_M(-,z)$. This completes the proof. □

5. The Simple Minimal Type Ones

In this section, we prove that the simple objects in $\mathscr{V}_1$ are in one-to-one correspondence with the self-dual Lie algebras (Theorem 5.4). A pair $(g, (|))$ in the category $\mathscr{L}$ is called *selfdual* if $(|)$ is non-degenerate. We discuss many examples of self-dual Lie algebras all of which correspond to *new* simple prevoas.

5.1. The Radical of V. Let E be an irreducible g-module Let J be the maximal $\hat{g}$-submodule of the $\hat{g}$-module $I(g,E)$, and let $L(g,E)$ be the irreducible quotient. By Proposition 4.12, it is clear that J is also the maximal submodule of the adjoint module over the prevoa $I(g, \mathbf{C})$. Thus by Proposition 3.2, J is the maximal ideal of the prevoa $I(g, \mathbf{C})$. In particular, its unique simple quotient prevoa is $L(g, \mathbf{C})$.

Definition 5.1. *Given a pair $(g, (|))$ in $\mathscr{L}$ we denote as* rad $(|)$ *the radical of the bilinear form* $(|)$. *Given a minimal type one V in $\mathscr{V}$, we let* rad V *be the ideal of V generated by* rad $(|)_{\Gamma(V)}$. *(Recall that $\Gamma(V)$ is identified with $V[1]$.) Whenever possible, we will drop the subscript $\Gamma(V)$ from $(|)$.*

Theorem 5.2. *For any minimal type one V in $\mathscr{V}$, we have*

$$\text{rad } V = \sum_{a \in \text{rad } (|)} a(n)V \tag{5.1}$$

Moreover, rad V *is a proper ideal.*

Proof. Call the right-hand side of Eq. (5.1) K. By definition rad V contains all of rad $(|)$. Since rad V is an ideal, it must also contain the coefficients of the Laurent series $Y(a,z)b$, for all $a \in$ rad $(|)$ and $b \in V$. Thus K is clearly a subspace of rad V. Now K contains $a(-1)\mathbf{1}_V = a$ for all $a \in$ rad $(|)$. Since rad V is the smallest ideal containing rad $(|)$, we need to show that K is an ideal and that K is proper.

First we show that K is a V-submodule of the adjoint module. Consider the action of the affine Lie algebra $\widehat{\Gamma(V)}$. Now rad $(|)$ is an ideal in the Lie algebra $\Gamma(V)$. This means that for $a \in$ rad $(|)$, $b \in \Gamma(V)$, n, $m \in \mathbf{Z}$, we have

$$\begin{aligned} b(m)a(n)V &= (a(n)b(m) + [b,a](m+n) + m(b|a)\delta_{m+n,0})V \\ &\subset K + [b,a](m+n)V\,, \\ &\subset K\,. \end{aligned} \tag{5.2}$$

Thus K is $\widehat{\Gamma(V)}$-stable. Since V is generated by $V[1]$, Corollary 2.6 tells us that K is stable under any $b(m)$, $b \in V$.

Now we show that K is stable under L_{-1}. By Theorem 2.8(vii), this means that K is an ideal. For $a \in$ rad $(|)$, n integer, we have

$$L_{-1}a(n)V \subset [L_{-1}, a(n)]V + a(n)L_{-1}V\,. \tag{5.3}$$

The second term on the right-hand side is clearly contained in K. The first term is too, by Theorem 2.8(ii). Thus K is stable under L_{-1}.

Finally, we want to show that $\mathbf{1} \notin K$. We suppose otherwise and will get a contradiction. Since K is a sum of graded spaces $a(n)V$, $a \in$ rad $(|)$, we may write $\mathbf{1}$ as a sum of homogeneous elements of the form $a(n)b$ having weight zero, i.e.

$$\mathbf{1} = a_1(n_1)b_1 + \cdots + a_k(n_k)b_k\,. \tag{5.4}$$

Since V is in category $\mathscr{V}$, we have $V[0] = \mathbf{C}\mathbf{1}$. Thus each $a_i(n_i)b_i$ is a multiple of $\mathbf{1}$. So we may assume that $\mathbf{1} = a(n)b$ for some $a \in$ rad $(|)$, $b \in V$.

Since V is generated by $V[1]$, b must be a sum of elements of the form $b_1(m_1)\cdots b_k(m_k)\mathbf{1}$, where the b_i's are in $V[1]$. Again because $V[0]$ is one-dimensional, we may assume that

$$\mathbf{1} = a(n)b_1(m_1)\cdots b_k(m_k)\mathbf{1}\,. \tag{5.5}$$

Let $c = c_1(n_1)\cdots c_k(n_k)\mathbf{1}$, where the c_i's are in $V[1]$ and at least one of the c_i's is in rad $(|)$. We will show by induction that if $|c| = 0$, then $c = 0$. This means that Eq. (5.5) is impossible.

For $k = 1$, $|c_1(n_1)\mathbf{1}| = -n_1 = 0$ implies that $c = c_1(0)\mathbf{1} = 0$. Consider the $k+1$-case: suppose $|c| = 0$, i.e. $n_1 + \cdots + n_{k+1} = 0$. If all $n_i = 0$, there is nothing to show. so let $n_j > 0$ for some j. We move $c_j(n_j)$ to annihilate $\mathbf{1}$, by commuting it through $c_{j+1}(n_{j+1}), \ldots, c_{k+1}(n_{k+1})$. Using the fact that one of the c_i's is in rad $(|)$ (which is a Lie algebra ideal in $V[1]$), it is easy to see that c is now expressed as a sum of terms of the form $c_1'(n_1')\cdots c_k'(n_k')\mathbf{1}$ – having weight zero and with at least one of the c_i''s in rad $(|)$. By inductive hypothesis on k, each of these terms must be zero. Thus $c = 0$. This completes the proof. □

Corollary 5.3 *Let V be any minimal type one in $\mathscr{V}$, $(g, (|))$ be any pair in $\mathscr{L}$. Then we have*

(i) Γ (rad V) = rad $(|)_{\Gamma(V)}$.
(ii) $\Gamma(V\text{rad } V) \cong \Gamma(V\text{rad } (|)_{\Gamma(V)}$.
(iii) *If V is simple, then $\Gamma(V)$ is self-dual.*
(iv) *The maximal ideal J_V of V has* $\Gamma(J_V) = $ rad $(|)_{\Gamma(V)}$.
(v) $\Gamma(L(g, \mathbf{C})) \cong g/\text{rad } (|)$.

Proof.
(i) We will drop the subscript of $(|)$. By Proposition 3.8 and the fact that rad V is a proper ideal of V, we have $\Gamma(\text{rad } V) \subset$ rad $(|)$. By definition, $\Gamma(\text{rad } V) = (\text{rad } V)[1]$. By Theorem 5.2, we see that $a = a(-1)\mathbf{1}_V \in$ rad V for every $a \in$ rad $(|)$. This gives rad $(|) \subset (\text{rad } V)[1]$.
(ii) This follows from Proposition 3.8 and part (i).
(iii) If V is simple, then rad $V = 0$. Now use part (ii).
(iv) From the projection in $\mathscr{V}$:

$$p: V \to V/J_V \tag{5.6}$$

we get the surjective morphism (Theorem 3.7, Proposition 3.8)

$$\Gamma(p): \Gamma(V) \to \Gamma(V/J_V) \tag{5.7}$$

with ker $\Gamma(p) = \Gamma(J_V)$. By part (iii), $\Gamma(V/J_V)$ is self-dual. This means that ker $\Gamma(p)$ must coincide with rad $(|)_{\Gamma(V)}$.
(v) By definition, $L(g, \mathbf{C}) = I(g, \mathbf{C})/J$, where J is the maximal ideal of $I(g, \mathbf{C})$. By Theorem 4.8, $\Gamma(I(g, \mathbf{C})) \cong (g, (|))$. Thus the desired result follows from part (iv) and Proposition 3.8. □

5.2. Correspondence between the Simple Objects of $\mathscr{V}_1$ and $\mathscr{L}$. In Theorem 4.8, we saw the "universal objects" $I(g, \mathbf{C})$ in $\mathscr{V}$ are in one-to-one correspondence with the objects in $\mathscr{L}$. To refine the relation between the two categories, we have

Theorem 5.4.
(i) *Every simple minimal type one is isomorphic to $L(g, \mathbf{C})$ for some self-dual pair $(g, (|))$ in $\mathscr{L}$.*
(ii) *For any simple minimal type ones V, V', we have $V \cong V' \Leftrightarrow \Gamma(V) \cong \Gamma(V')$.*
(iii) *The simple objects in $\mathscr{V}_1$ are in one-to-one correspondence with the self-dual objects in $\mathscr{L}$.*

Proof.
(i) Let V be a simple minimal type one and $\Gamma(V) = (g, (|))$. By Corollary 5.3, $(g, (|))$ is self-dual. By the proof of Theorem 4.11, we have an onto map $\widehat{i_V} : I(g, \mathbf{C}) \to V$. Since $L(g, \mathbf{C})$ is the unique simple quotient of $I(g, \mathbf{C})$, the map above induces an isomorphism $L(g, \mathbf{C}) \to V$.
(ii) Obviously, $V \cong V'$ implies that $\Gamma(V) \cong \Gamma(V')$. By part (i), we have $V \cong L(\Gamma(V), \mathbf{C})$ and $V' \cong L(\Gamma(V'), \mathbf{C})$. Thus $\Gamma(V) \cong \Gamma(V')$ implies that $V \cong V'$.
(iii) follows from parts (i) and (ii). □

Let's briefly recapitulate what we have established so far. In Sect. 4, we see how to attach to each pair $(g, (|))$ in the category $\mathscr{L}$, a prevoa $I(g, \mathbf{C})$ of the minimal type one. By taking quotients, we exhaust *all* the minimal type ones. The prevoas of the kind $I(g, \mathbf{C})$ are constructed in an utterly general fashion – with little assumption on $(g, (|))$. In this section, we describe the simple type ones in terms of the *self-dual* Lie algebras. What is still lacking, however, is a description of the class of self-dual Lie algebra themselves. For example, we know that among them are the finite dimensional reductive Lie algebras. But are there any other interesting examples?

5.3. Examples: Self-Dual Lie Algebras. We now describe a number of important classes of examples of self-dual objects in the category $\mathscr{L}$.[1]

5.3.1. Double Extensions. In Sect. 3.4, we saw that affinization is a way of getting new objects out of old ones in the category $\mathscr{L}$. Here we will describe a procedure – known as *double extension* – that generalizes affinization. The original context in which this procedure was discussed is the geometry of real groups with pseudo-Riemannian invariant metric [23]. For completeness, we review the construction here.

Let $(g, (|))$ be an object in $\mathscr{L}$. We denote by $\mathrm{Der}(g, (|))$ the Lie algebra of derivations d which are skew-symmetric on g with respect to $(|)$, i.e. $(da|b) = -(a|db)$ for all $a, b \in g$. Let h be any finite dimensional Lie algebra with a homomorphism $\theta : h \to \mathrm{Der}(g, (|))$. Let h' be the linear dual of h.

Define a bilinear form $\beta : g \times g \to h'$ as follows:

$$\langle \beta(a,b), c \rangle = (a|\theta(c)b) \tag{5.8}$$

for $a, b \in g$ and $c \in h$.

Lemma 5.5. *β is a two-cocycle.*

Proof. Let $a, b, \in g$ and $d \in h$. Then

$$\langle \beta(a,b) + \beta(b,a), d \rangle = (a|\theta(d)b) + (b|\theta(d)a) \,. \tag{5.9}$$

Now the right-hand side vanishes because $\theta(d)$ is skew-symmetric. This means that β is skew-symmetric. Similarly, we have

$$\begin{aligned} &\langle \beta([a,b], c) + \beta([c,a], b) + \beta([b,c], a), d \rangle \\ &= ([a,b] \,|\, \theta(d)c) + ([c,a] \,|\, \theta(d)b) + ([b,c] \,|\, \theta(d)a) \\ &= (a \,|\, [b, \theta(d)c]) + (a \,|\, [\theta(d)b, c]) - (a \,|\, \theta(d)[b,c]) \,. \end{aligned} \tag{5.10}$$

[1] I thank G. Zuckerman for pointing out many examples to me.

This right-hand side vanishes because $\theta(d)$ is a derivation on g. Thus β is a cocycle. □

We now regard h' as an abelian Lie algebra. Then using β, we can form the central extension – which we denote by $g\bigoplus_\beta h'$, θ'.

Lemma 5.6. *θ' defines an action of h on $g\bigoplus_\beta h'$ by Lie algebra derivations.*

Proof. Let $a,b\in g$, and $c,d\in h$. Then we have

$$\begin{aligned}(\beta(a,b)\circ ad_h c)d &= \langle\beta(a,b),[c,d]\rangle\\ &= (a\,|\,\theta([c,d])b)\\ &= (a\,|\,\theta(c)\theta(d)b)-(a\,|\,\theta(d)\theta(c)b)\\ &= -(\theta(c)a|\theta(d)b)+(\theta(d)a|\theta(c)b)\\ &= -\langle\beta(\theta(c)a,b),d\rangle-\langle\beta(a,\theta(c)b),d\rangle\,. \end{aligned}\tag{5.11}$$

This means that

$$\beta(a,b)\circ ad_h c = -\beta(\theta(c)a,b)-\beta(a,\theta(c)b)\,. \tag{5.12}$$

Let $\mu,\nu\in h'$. Consider

$$\begin{aligned}\theta'(c)[(a,\mu),(b,\nu)] &= \theta'(c)([a,b],\beta(a,b))\\ &= (\theta(c)[a,b],-\beta(a,b)\circ ad_h c)\,,\end{aligned}\tag{5.13}$$

$$\begin{aligned}&[\theta'(c)(a,\mu),(b,\nu)]+[(a,\mu),\theta'(c)(b,\nu)]\\ &= [(\theta(c)a,-\mu\circ ad_h c),(b,\nu)]+[(a,\mu),(\theta(c)b,-\nu\circ ad_h c)]\\ &= ([\theta(c)a,b],\beta(\theta(c)a,b))+([a,\theta(c)b],\beta(a,\theta(c)b))\\ &= (\theta(c)[a,b],\beta(\theta(c)a,b)+\beta(a,\theta(c)b))\,.\end{aligned}\tag{5.14}$$

By Eq. (5.12), we see that right-hand sides of Eqs. (5.13), (5.14) agree. This means that $\theta'(c)$ acts on $g\oplus_\beta h'$ by derivation. □

Definition 5.7. *Let $\Delta(g,g;\theta)$ denote the semi-direct product Lie algebra $h\ltimes_{\theta'}(g\bigoplus_\beta h')$. It is called the double extension of $(g,(|))$ by $(h;\theta)$.*

Proposition 5.8. *$\Delta(g,g;\theta)$ has an invariant form $(|)_\Delta$ such that there is an inclusion $(g,(|))\hookrightarrow(\Delta(g,h;\theta),(|)_\Delta)$ in the category $\mathscr{L}$.*

Proof. As vector spaces,

$$\Delta(g,h;\theta) = h\oplus g\oplus h'\,. \tag{5.15}$$

For $a,b\in g$, $c,d\in h$ and $\mu,\nu\in h'$, we let

$$((c,a,\mu)|(d,b,\nu))_\Delta = (a|b)_g+\langle\mu,d\rangle+\langle\nu,c\rangle\,. \tag{5.16}$$

Checking the invariance property of $(|)_\Delta$ is an easy exercise. Now it is obvious that we have the desired inclusion map. □

The following is a direct consequence of the construction above.

Proposition 5.9. *The pair* $(\Delta(g,h;\theta),(|)_\Delta)$ *is self-dual* iff $(g,(|))$ *is. More precisely, we have* rad $(|)_\Delta =$ rad $(|)_g$.

5.3.2. A Few Special Cases. Given a pair $(g,(|))$ from $\mathscr{L}$, the loop algebra $Lg = g\otimes \mathbf{C}[t,t^{-1}]$ has a canonical invariant form defined by

$$(a\otimes t^n|b\otimes t^m)_{Lg} = (a|b)\delta_{n+m,0} \tag{5.17}$$

for $a,b\in g$ and integers m,n.

Let $\mathbf{C}d$ be the one-dimensional abelian Lie algebra which acts on Lg by θ: $\mathbf{C}d\to \mathrm{Der}(Lg,(|)_{Lg})$: $\theta(d)a\otimes t^n = na\otimes t^n$. Then the affinization $(\hat{g},(|)\hat{})$ of $(g,(|))$ is the double extension of $(Lg,(|)_{Lg})$ by $(\mathbf{C}d;\theta)$. Moreover, $(|)\hat{}$ is non-degenerate iff $(|)$ is.

If we take $h=g$ and $\theta = ad$, we get

$$\langle\beta(a,b),c\rangle = -([a,b]|c) \tag{5.18}$$

for all $a,b,c\in g$. Thus the central extension $g\oplus_\beta g'$ is a split extension. Thus we have $\Delta(g,g;ad) = g\ltimes(g\oplus g')$, where g acts on g' by adjoint and coadjoint action respectively.

Clearly, the construction in Sect. 5.3.1 makes sense even when g is the zero algebra. In this case, the double extension of 0 by h is nothing but the semi-direct product $h\ltimes h'$, where h acts on h' by the coadjoint action.

We call $h\ltimes h'$ the *self-dual double* of h, and denote it by $\delta(h)$. Note that the invariant form given by Eq. (5.16) in this case is nothing but the (non-degenerate) pairing – denoted by $(|)_\delta$ – between h and h'. In fact even when g is nonzero, we have an inclusion map $(\delta(h),(|)_\delta)\hookrightarrow(\Delta(g,h;\theta),(|)_\Delta)$ in the category $\mathscr{L}$.

5.3.3. Drinfel'd Lie Bialgebras and Manin Triples. Let g be a Lie algebra which admits a Lie coalgebra structure $g\xrightarrow{\delta}\bigwedge^2 g$, such that δ is a one-cocycle. then g is called a Lie bialgebra. This notion, first introduced in [3], may be used to construct examples of quantum groups.

It has been pointed out by Drinfel'd that Lie bialgebras are in one-to-one correspondence with certain self-dual Lie algebras known as the Manin triples. A Manin triple (h,h_1,h_2) consists of a Lie algebra h with a non-degenerate invariant form, and two isotropic subalgebras h_1, h_2, such that $h=h_1\oplus h_2$ as vector spaces. Thus given a Lie bialgebra, g, we can let $h_1=g$, $h_2=g'$ and $h=g\oplus g'$. The bracket $[,]: g\times g'\to h$ is defined so that the natural pairing between g and g' is h-invariant. Thus corresponding to the Lie bialgebra g is the triple $(g\oplus g',g,g')$.

Let h be any Lie algebra. Then h can be endowed with the trivial coalgebra structure, i.e. let $h\to\bigwedge^2 h$ be the zero map. Hence h becomes a Lie bialgebra. It is easy to show that the Manin triple corresponding to h is nothing but $(h\ltimes h',h,h')$, where $h\ltimes h'$ is the self-dual double of h discussed above.

Drinfel'd has constructed many examples of quantized objects $U_q g$ out of certain Lie bialgebras g. It is an interesting problem to compare these quantized objects with the vertex operator algebras $I(g\oplus g',\mathbf{C})$, corresponding to the Manin triple $(g\oplus g',g,g')$.

6. A Classification of the Virasoro Elements

In this section, we classify the Virasoro elements of a prevoa of minimal type one (Theorem 6.4, Corollary 6.7, Theorem 6.11, Theorem 6.14). We discuss the Virasoro elements in those new examples we give in the last section. As an application, we use the Virasoro elements to study the reducibility problem of the prolongation module in the generic case (Theorem 6.19). We also use the action of the automorphism group on a self-dual Lie algebra to classify the *vertex operator algebra* structures on $I(g, \mathbf{C})$ (Theorem 6.22). We then conclude with a few remarks.

6.1 Uniqueness of the Virasoro Element

Definition 6.1 [9] *Let $(V, Y_V(-,z))$ be a prevoa. An element $\omega \in V$ is called a Virasoro element if the vertex operator $Y_V(\omega,z) = \sum_n L_n z^{-n-2}$ satisfies the following*:

(i) $[L_{-1}, Y_V(a,z)] = \frac{d}{dz} Y_V(a,z)$;
(ii) $L_0|_{V[n]} = n\, id$;
(iii) $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$

for all $a \in V$, integers m, n. The fixed scalar c is called the central charge of ω.

Although (i)–(iii) above are stated in terms of *vertex operators* acting on V, these conditions are equivalent to the following conditions on *vertex series*:

(i)′ $[L_{-1}, Y(a,z)] = \frac{d}{dz} Y(a,z)$;
(ii)′ $[L_0, Y(a,z)] = z\frac{d}{dz} Y(a,z) + |a|\, Y(a,z)$;
(iii)′ $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$

The proof of the equivalence is a straight forward application of Lemma 2.5. Hence we use the two sets of conditions interchangeably without explicitly stating so.

In this subsection, we assume that V is a prevoa in $\mathscr{V}$ with a Virasoro element ω. We use our further Γ, to obtain information about ω.

Proposition 6.2. *Given a Virasoro element ω of V, there is a unique $\Gamma(V)$-character $\lambda \in (\Gamma(V)/[\Gamma(V), \Gamma(V)])'$ such that*

$$[L_n, a(m)] = -ma(n+m) + n(n+1)\langle \lambda, a\rangle \delta_{n+m,0}$$

for all $a \in V[1]$ and integers m, n.

Proof. By Lemma 2.5(i) and the fact that $L_n = \omega(n+1)$, we have

$$\begin{aligned}[L_n, Y(a,z))] &= \sum_{i\geq 0} \binom{n+1}{i} z^{n-i+1} Y(\omega(i)a,z) \\ &= z^{n+1} Y(L_{-1}a,z) + (n+1)z^n Y(L_0 a,z) \\ &\quad + \frac{1}{2}n(n+1)z^{n-1} Y(L_1 a,z)\,. \end{aligned} \tag{6.1}$$

Since $L_1 a \in \mathbf{C1}$ is linear in $a \in V[1] = \Gamma(V)$, we have

$$L_1 a = 2\langle \lambda, a \rangle \tag{6.2}$$

for some $\lambda \in \Gamma(V)'$. Now extracting the coefficients of the powers of z from Eq. (6.1), we get

$$[L_n, a(m)] = -ma(n+m) + n(n+1)\langle \lambda, a \rangle \delta_{n+m,0} \,. \tag{6.3}$$

To show that λ is zero on $[\Gamma(V), \Gamma(V)]$, we compute $[L_n, [a(m), b(l)]]$ for $a, b \in V[1]$, in two ways. First we have

$$\begin{aligned}[L_n, [a(m), b(l)]] &= [L_n, [a,b](m+1)] \\ &= -(m+1)[a,b](n+m+l) \\ &\quad + n(n+1)\langle \lambda, [a,b] \rangle \delta_{n+m+l,0} \,. \end{aligned} \tag{6.4}$$

But we also have

$$\begin{aligned}[L_n, [a(m), b(l)]] &= [[L_n, a(m)], b(l) + [a(m), L_n, b(l)]] \\ &= -m[a(n+m), b(l)] - l[a(m), b(n+l)] \\ &= -(m+l)[a,b](n+m+l) - m(n+m)(a|b)\delta_{n+m+l,0} \\ &\quad - lm(a|b)\delta_{n+m+l,0} \\ &= -(m+l)[a,b](n+m+l) \,. \end{aligned} \tag{6.5}$$

Equations (6.4), (6.5) show that $\langle \lambda, [a,b] \rangle = 0$ for all $a, b \in V[1]$.
Uniqueness of λ is obvious. □

Theorem 6.3. *Suppose V is a minimal type one such that V admits a Virasoro element ω_0, $\Gamma(V)$ is self-dual and finite dimensional. Then there is a unique Virasoro element ω satisfying*

$$[L_n, a(m)] = -ma(n+m) \,, \tag{6.6}$$

where $L_n = \omega(n+1)$, for all $a \in V[1]$ and integers m, n.

Proof. Let λ_0 be the $\Gamma(V)$-character determined by ω_0. By Proposition 6.2, we have

$$[\omega_0(n+1), a(m)] = -ma(n+m) + n(n+1)\langle \lambda_0, a \rangle \delta_{n+m,0} \,. \tag{6.7}$$

By hypotheses, we can pick (uniquely) $\varepsilon \in \Gamma(V)$ such that

$$(\varepsilon | a) = \langle \lambda_0, a \rangle \tag{6.8}$$

for all $a \in V[1]$. We let

$$\omega = \omega_0 + \varepsilon(-2)\mathbf{1} \,. \tag{6.9}$$

We now check that ω has the desired properties. For $a, b \in \Gamma(V)$, we have

$$([\varepsilon, a] | b)_{\Gamma(V)} = (\varepsilon | [a,b]) = \langle \lambda_0, [a,b] \rangle = 0 \,. \tag{6.10}$$

Since $(|)_{\Gamma(V)}$ is non-degenerate, this means that ε is in the center of $\Gamma(V)$. By Lemma 2.5(iii), we have

$$L_n = \omega(n+1) = \omega_0(n+1) - (n+1)\varepsilon(n)\,. \tag{6.11}$$

Using Eq. (6.7), (6.8) and the fact that ε is in the center of $\Gamma(V)$, we get

$$\begin{aligned}[L_n, a(m)] &= -ma(n+m) + n(n+1)\langle\lambda, a\rangle\delta_{n+m,0} \\ &\quad - (n+1)[\varepsilon, a](n+m) - n(n+1)(\varepsilon|a)\delta_{n+m,0} \\ &= -ma(n+m)\,. \end{aligned} \tag{6.12}$$

This gives Eq. (6.6).

By Eq. (6.11), we have $\omega(0) = \omega_0(0)$. Since ω_0 is a Virasoro element, $\omega_0(0)$ satisfies Definition 6.1(i). Hence so does $\omega(0)$. Since ε is in the center of $\Gamma(V)$, it follows that $\varepsilon(0)$ commutes with $a(n)$, for all $a \in V[1]$ and integers n. Now V is generated by $V[1]$. Thus $\varepsilon(0)$ commutes with $Y(b,z)$, for all $b \in V$. In particular, we have $\varepsilon(0)b = \varepsilon(0)b(-1)\mathbf{1} = 0$, i.e. $\varepsilon(0)$ acts by zero in V. Thus by Eq. (6.11), we have $\omega(1)|_{V[n]} = \omega_0(1)|_{V[n]} = n\ id$, i.e. $\omega(1)$ satisfies Definition 6.1(ii). Finally, suppose the Virasoro element ω_0 has central charge c. Then we have

$$\begin{aligned}[L_n, L_m] &= [\omega_0(n+1) - (n+1)\varepsilon(n), \omega_0(m+1) - (m+1)\varepsilon(m)] \\ &= (n-m)\omega(n+m+1) + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ &\quad - n(n+1)\varepsilon(n+m) + m(m+1)(n+1)\langle\lambda_0, \varepsilon\rangle \\ &\quad + m(m+1)\varepsilon(n+m) - n(n+1)(m+1)\langle\lambda_0, \varepsilon\rangle \\ &\quad + (n+1)(m+1)n(\varepsilon|\varepsilon)\delta_{n+m,0} \\ &= (n-m)(\omega_0(n+m+1) - (n+m+1)\varepsilon(n+m)) \\ &\quad + \frac{c}{12}(n^3 - n)\delta_{n+m,0} + m(m+1)(n+1)(\varepsilon|\varepsilon)\delta_{n+m,0} \\ &= (n-m)L_{n+m} + \frac{c + 12(\varepsilon|\varepsilon)}{12}(n^3 - n)\delta_{n+m,0}\,. \end{aligned} \tag{6.13}$$

Thus ω is a Virasoro element of V with central charge $c + 12(\varepsilon|\varepsilon)_{\Gamma(V)}$.

To prove uniqueness, let ω, ω' be two virasoro elements satisfying Eq. (6.6). This means that $[\omega(n) - \omega'(n), a(m)] = 0$ for all $a \in V[1]$ and integers m, n. Since V is generated by $V[1]$, it follows that $\omega(n) - \omega(n)'$ commutes with $Y(b,z)$ for all $b \in V$. In particular for $L_0 = \omega(1)$, we have

$$L_0(\omega - \omega') = [L_0, \omega(-1) - \omega'(-1)]\mathbf{1} = 0\,. \tag{6.14}$$

On the other hand because $\omega, \omega' \in V[2]$, the left-hand side of Eq. (6.14) is equal to $2(\omega - \omega')$. Thus we have $\omega = \omega'$. This completes the proof. □

Theorem 6.4. *Let V be a prevoa satisfying the hypotheses of Theorem* 6.3. *Then the set of Virasoro elements of V is the affine subspace of $V[2]$ given by* $\{\omega_0 + \varepsilon(-2)\mathbf{1} | \varepsilon \in \mathrm{cent}(\Gamma(V))\}$. *Thus if exists, the set of Virasoro elements of V is classified by* cent $(\Gamma(V))$. *Moreover if ω_0 has central charge c, then $\omega_0 + \varepsilon(-2)\mathbf{1}$ has central charge $c + 12(\varepsilon|\varepsilon)_{\Gamma(V)}$.*

Proof. Using a similar argument as in Theorem 6.3, we can show that if ω_0 is a Virasoro element, then so is $\omega_0 + \varepsilon(-2)\mathbf{1}$ for every $\varepsilon \in \mathrm{cent}(\Gamma(V))$. The same proof shows that conversely, every Virasoro element differs from ω_0 by exactly $\varepsilon(-2)\mathbf{1}$ for some $\varepsilon \in \mathrm{cent}(\Gamma(V))$. This gives the first of our claims.

For the second claim, let's assume that $\varepsilon \in \mathrm{cent}(\Gamma(V))$ and $\varepsilon(-2)\mathbf{1} = 0$. We need to show that $\varepsilon = 0$. By Lemma 2.5(iii), we have $0 = Y(\varepsilon(-2)\mathbf{1}, z) = \frac{d}{dz} Y(\varepsilon, z)$. This implies that $\varepsilon(n) = \varepsilon(-1)\delta_{n,-1}$. In particular, $(\varepsilon|a)_{\Gamma(V)} = \varepsilon(1)a = 0$ for all $a \in \Gamma(V)$. By hypothesis, $(|)_{\Gamma(V)}$ is non-degenerate. This means that $\varepsilon = 0$.

The last claim of the theorem follows from the proof of Theorem 6.3. □

In proposition 6.2, we see that every Virasoro element gives rise to a $\Gamma(V)$-character in a natural way. One wonders whether the converse is true. Indeed, under the same hypotheses in Theorem 6.3, this holds. In fact if one considers the map $\mathrm{cent}(\Gamma(V)) \to (\Gamma(V)/[\Gamma(V)\Gamma(V)])'$, $\varepsilon \mapsto \lambda_\varepsilon$, with $\langle \lambda_\varepsilon, a\rangle = (\varepsilon|a)$ for $a \in \Gamma(V)$, then one can easily show that this map is well-defined and bijective. By virtue of Theorem 6.4, we see that the *$\Gamma(V)$-characters also classifies the Virasoro elements*, if they exist.

6.2. Characterizing the Virasoro Element. All of the discussion in the last subsection would not have been worthwhile, if there were no new examples of Virasoro elements beyond the known ones in the case in which $\Gamma(V)$ is reductive. But where do we look for new examples? A partial answer is provided by Theorem 6.3. It tells us that we should look at those prevoa V satisfying the hypotheses, and look at elements $\omega \in V[2]$ for which

$$[\omega(n+1), a(m)] = -ma(n+m) \tag{6.15}$$

holds for all $a \in V[1]$ and integers m, n. In fact, as we will see, this condition actually characterizes the existence of a Virasoro element.

Definition 6.5. *Let $(g, (|))$ be a finite dimensional self-dual pair in $\mathscr{L}$, $\{u_i\}$ be dual bases $(u_i|u^j) = (u^j|u_i) = \delta^j_i$. If ϕ is any linear map of g, we let $\Omega^\phi_g = u_i\phi u^i$ (sum over i) be an element of the enveloping algebra of g. In a context in which only one pair $(g, (|))$ is being considered, we write Ω^ϕ instead of Ω^ϕ_g. When ϕ is the identity map, we write Ω instead of Ω^{id}. The operator representing Ω^ϕ in the adjoint representation is denoted $\mathrm{ad}_g\Omega^\phi_g$, or simply ad Ω^ϕ.*

Throughout this subsection, we assume that a prevoa V is of minimal type one with finite dimensional self-dual $\Gamma(V)$. Thus $\{u_i\}$ and $\{u^i\}$ are dual bases of $\Gamma(V)$.

Since V is generated by $V[1]$, the most general element of $V[2]$ must take the form

$$\omega = u_i(-1)\phi u^i + v(-2)\mathbf{1}\,, \tag{6.16}$$

where ϕ is some linear map on $V[1]$, and v is some element of $V[1]$. We claim that, without loss of generality, one can assume that ϕ is symmetric with respect to $(|)_{\Gamma(V)}$. For if we let $\phi^\dagger$ be the adjoint of ϕ, it is easy to show that

$$u_i(-1)(\phi - \phi^\dagger)u^i = [u_i, \phi u^i](-2)\mathbf{1}\,. \tag{6.17}$$

Thus Eq. (6.16) becomes

$$\omega = \frac{1}{2}u_i(-1)(\phi + \phi^\dagger)u^i + (\frac{1}{2}[u_i, \phi u^i] + v)(-2)\mathbf{1} . \qquad (6.18)$$

So from now on, ϕ in Eq. (6.16) *is assumed symmetric.*

It is obvious that every graded piece $V[n]$ of V is a $\Gamma(V)$-module in a natural way, i.e. every $a \in \Gamma(V) = V[1]$ acts by the operator $a(0)$. We denote the subspace of $\Gamma(V)$-invariants by $V[n]^{\Gamma(V)}$.

Theorem 6.6. *Let $\omega = u_i(-1)\phi u^i + v(-2)\mathbf{1}$ where ϕ is some (symmetric) linear map on $V[1]$, and $v \in V[1]$. Then the following are equivalent:*

(i) *For all $a \in V[1]$ and integers m, n,*

$$[\omega(n+1), a(m)] = -ma(n+m) .$$

(ii) *For all $a \in V[1]$,*

$$Y(\omega, z)a(w) \sim \frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{z-w} .$$

(iii) *$v = 0$ and for all $a \in V[1]$,*

$$2u_i(w)(\phi u^i|a) + [\phi u^i, [u_i, a]](w) = a(w) ,$$

$$\partial[\phi u^i, [u_i, a]](w) - 2 : u_i(w)[\phi u^i, a](w) : = 0 .$$

(iv) *$v = 0$ and*

$$2\phi + \mathrm{ad}_{\Gamma(V)}\Omega^{\phi}_{\Gamma(V)} = \mathrm{id} ,$$

$$\omega \in V[2]^{\Gamma(V)} .$$

(v) *ω is a Virasoro element with $v = 0$, and central charge $c = 2\mathrm{tr}\ \phi$.*

Proof. (i)$\Longleftrightarrow$(ii) : This is a direct application of Lemma 2.7 to the commutator given by (i) and the operator product expansion in (ii).
(ii)$\Longleftrightarrow$(iii) : By definition, $Y(\omega, z) =: u_i(z)(\phi u^i)(z) : + \partial v(z)$. By applying repeatedly

$$[a(z)^+, b(w)] = (a|b)\iota_{z,w}(z-w)^{-}2 + [a, b](w)\iota_{z,w}(z-w)^{-1} \qquad (6.19)$$

for $a, b \in V[1]$, we can easily get the operator product expansion:

$$\begin{aligned} Y(\omega, z)a(w) \sim{}& 2u_i(w)(\phi u^i|a)(z-w)^{-2} + 2\partial u_i(w)(\phi u^i|a)(z-w)^{-1} \\ &+ 2 : u_i(w)[\phi u^i, a](w) : (z-w)^{-1} + [\phi u^i, [u_i, a]](w)(z-w)^{-2} \\ &+ 2(v|a)(z-w)^{-3} - [v, a](w)(z-w)^{-2} . \qquad (6.20) \end{aligned}$$

Note that the invariant form $(|)$ is non-degenerate.Thus comparing Eq. (6.20) with part (ii), we see that (ii) and (iii) are equivalent.
(iii)$\Longleftrightarrow$(iv) : By Lemma 2.3(i), the first equation of part (iii) is equivalent to

$$2u_i(\phi u^i|a) + [\phi u^i[u_i, a]] = a . \qquad (6.21)$$

By the symmetry of ϕ, this becomes the first equation of part (iv).

By Lemmas 2.3(i) and 2.5, the second equation of part (iii) is equivalent to

$$[\phi u^i, [u_i, a]](-2)\mathbf{1} - 2u_i(-1)[\phi u^i, a](-1)\mathbf{1} = 0\,. \tag{6.22}$$

Now computing $a(0)\omega$ (with $v = 0$), one sees that it is equal to the left-hand side of Eq. (6.22). Thus parts (iii) and (iv) are equivalent.
(iv)$\Longrightarrow$(v) : Since we have already shown the equivalence of parts (i)–(iv), we can use any one of them here as hypotheses. Applying part (i) to the case $a = u_i$, we get

$$[Y(\omega,z),\, u_i(w)^-] = u_i(z)\iota_{z,w}(z-w)^{-2}\,. \tag{6.23}$$

Applying part (ii) to the case $a = \phi u^i$, we get

$$\begin{aligned} Y(\omega,z)(\phi u^i)(w) &= (\phi u^i)(w)\iota_{z,w}(z-w)^{-2} \\ &\quad + \partial(\phi u^i)(w)\iota_{z,w}(z-w)^{-1} \\ &\quad + :Y(\omega,z)(\phi u^i)(w):\,. \end{aligned} \tag{6.24}$$

Part (iii) implies that $v = 0$. So we can write

$$Y(\omega,z) =: u_i(z)(\phi u^i)(z):\,. \tag{6.25}$$

We also know that for $a, b \in V[1]$, we have

$$a(z)b(w) = (a|b)\iota_{z,w}(z-w)^{-2} + [a,b](w)\iota_{z,w}(z-w)^{-1} + :a(z)b(w):\,. \tag{6.26}$$

We now apply Eqs (6.23)–(6.26) to compute the product:

$$\begin{aligned} &Y(\omega,z)Y(\omega,w) \\ &= Y(\omega,z)(u_i(w)^-(\phi u^i)(w) + (\phi u^i)(w)u_i(w)^+) \\ &= (u_i|\phi u^i)\iota_{z,w}(z-w)^{-4} + [u_i, \phi u^i](w)\iota_{z,w}(z-w)^{-3} \\ &\quad + :u_i(w)(\phi u^i)(w):\,\iota_{z,w}(z-w)^{-2} + :u_i(z)\phi u^i(w):\,\iota_{z,w}(z-w)^{-2} \\ &\quad + :u_i(w)\partial(\phi u^i)(w):\,\iota_{z,w}(z-w)^{-1} + :u_i(w)Y(\omega,z)(\phi u^i)(w):\,. \end{aligned} \tag{6.27}$$

The term with $(z-w)^{-3}$ vanishes by the symmetry of ϕ. Extracting the singular part of the right-hand side of Eq. (6.27), we get

$$Y(\omega,z)Y(\omega,w) \sim (tr\ \phi)(z-w)^{-4} + 2Y(\omega,w)(z-w)^{-2} + \partial Y(\omega,w)(z-w)^{-1}\,. \tag{6.28}$$

Now apply Lemma 2.7, we get

$$[\omega(n+1),\, \omega(m+1)] = (n-m)\omega(n+m+1) + \frac{2\mathrm{tr}\ \phi}{12}(n^3-n)\delta_{n+m,0}\,. \tag{6,29}$$

This verifies Definition 6.1(iii). Applying part (iii) again, we get

$$[\omega(0),\, a(z)] = \frac{d}{dz}a(z) \tag{6.30}$$

for all $a \in V[1]$. Since V is generated by $V[1]$, Corollary 2.6 tells us that for any $b \in V$, $Y(b,z)$ is a linear combination of

$$Y(a_1(-i_1-1)\cdots a_n(-i_n-1)\mathbf{1},z) = \frac{1}{i_1!}\cdots\frac{1}{i_n!} : \left(\frac{d}{dz}\right)^{i_1} a_1(z)\cdots\left(\frac{d}{dz}\right)^{i_n} a_n(z): . \tag{6.31}$$

Now Definition 6.1(i) can be easily checked by induction on n, using Eqs (6.30), Similarly, applying part (iii) again:

$$[\omega(1), a(m)] = -ma(m) . \tag{6.32}$$

one can check as well that Definition 6.1(ii) also holds.
(v)$\Longrightarrow$(i) : Suppose ω is a Virasoro element with $V = 0$. By Proposition 6.2, there is a unique $\Gamma(V)$-character λ such that

$$[\omega(n+1), a(m)] = -ma(m+n) + n(n+1)\langle\lambda, a\rangle\delta_{n+m,0} \tag{6.33}$$

for all $a \in V[1]$ and integers m, n. Using Lemma 2.7, we translate this into

$$Y(\omega,z)a(w) \sim a(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1} + 2\langle\lambda, a\rangle(z-w)^{-3} . \tag{6.34}$$

But Eq. (6.20) also holds with $v = 0$. Comparing it with Eq. (6.34), we see that $\langle\lambda, a\rangle = 0$ for all $a \in V[1]$. Thus Eq. (6.33) coincides with part (i). This completes our proof. □

From now on, given ϕ we denote $u_i(-1)\phi u^i \in V[2]$ as ω_ϕ. Combining Theorem 6.6 and Theorem 6.3, we have

Corollary 6.7. *V admits a Virasoro element* iff *there is a symmetric map ϕ satisfying*

$$2\phi + \mathrm{ad}_{\Gamma(V)}\Omega^{\phi}_{\Gamma(V)} = \mathrm{id} , \tag{6.35}$$

$$\omega_\phi \in V[2]^{\Gamma(V)} . \tag{6.36}$$

Thus we need to solve those two conditions above.

A few comments about the conditions are in order. Based on the known cases of prevoa V in which $\Gamma(V)$ is a simple Lie algebra, one is tempted to guess that in the general case, there might be a Virasoro element of the form (again sum over i)

$$\omega = ku_i(-1)u^i , \tag{6.37}$$

where k is some fixed scalar, i.e. $\phi = k$ id. Unfortunately, the first condition in Corollary 6.7 does not hold in this case unless ad Ω acts semi-simply on $\Gamma(V)$. This means that ω above does not give a Virasoro element *in general.* For example, if we let g be the self-dual double of a finite dimensional simple Lie algebra h and consider the prevoa $V = I(g, \mathbf{C})$, then Theorem 4.8 tell us that $\Gamma(V) = (g, (|))$ which is self-dual. In this case, one can easily show that $\mathrm{ad}_g\Omega_g$ is a *non-trivial square zero operator* on g. Thus Eq. (6.37) is a poor choice in the general case.

In the case when $\Gamma(V)$ is simple, we know that $(|)_{\Gamma(V)}$ is a scalar χ times the Killing form, say $\langle|\rangle$. One can easily verify that the two conditions in Corollary 6.7 have the solution $\phi = \frac{\chi}{2(\chi + h^v)}$, provided that $\chi + -h^v$, where h^v is the dual Coxeter number of $\Gamma(V)$. This gives us the Virasoro element

$$\omega = \frac{\chi}{2(\chi + h^v)}u_i(-1)u^i , \tag{6.38}$$

where the u_i and u^i are dual bases with respect to $(|)_{\Gamma(V)}$. This is of course the well-known formula of Suguwara–Sommerfield.

When $\Gamma(V)$ is abelian, it follows from Corollary 6.7 that $\phi - \frac{1}{2}$. This means that

$$\omega = \frac{1}{2}u_i(-1)u^i \ . \tag{6.39}$$

This gives the original Virasoro action.

6.3. Existence of the Virasoro Element. Our task in this subsection is to solve Eq. (6.35), (6.36). *To simplify notations, we denote $\Gamma(V)$ as g in this subsection.* We begin with an observation which generalizes the fact that the second Casimir of the self-dual Lie algebra g, lies in the center $Z(g)$ of $U(g)$.

Lemma 6.8 *If ϕ intertwines the adjoint module g, then $\Omega^\phi \in Z(g)$.*

Proof. Let $a \in g$. Then for each i, we have

$$[a, u_i] = a^j_i u_j \tag{6.40}$$

for some constants a^j_i. Using the non-degenerate invariant form of g, we get

$$a^j_i = ([a, u_i]|u^j) = -(u_i|[a, u^j]) \ , \tag{6.41}$$

implying that

$$[a, u^j] = -a^j_i u^i \ . \tag{6.42}$$

Using Eq. (6.40), (6.42) and our hypothesis, we have

$$\begin{aligned} [a, u_i\phi u^i] &= [a, u_i]\phi u^i + u_i[a, \phi u^i] \\ &= a^j_i u_j \phi u^i - u_i \phi a^i_j u^j = 0 \ . \quad \square \end{aligned} \tag{6.43}$$

Proposition 6.9 *Suppose ϕ satisfies Eq.* (6.35). *Then the following are equivalent*:
(i) *ϕ intertwines the adjoint module*;
(ii) $\Omega^\phi \in Z(g)$;
(iii) $(2 + \mathrm{ad}\Omega) \circ \phi = \mathrm{id}$.

Proof. (i) $\Rightarrow$ (ii) is given by Lemma 6.8.
(ii) $\Rightarrow$ (iii) : Obviously if Ω^ϕ commutes with $\Gamma(V)$, then ad Ω^ϕ intertwines the adjoint module.

Thus by Eq. (6.35), so does ϕ . But if ϕ intertwines, then for every $a \in \Gamma(V)$ we have

$$\mathrm{ad}\Omega^\phi a = [\phi u^i, [u_i, a]] = [u^i, [u_i, \phi a]] = (\mathrm{ad}\Omega) \circ \phi a \ . \tag{6.44}$$

Thus Eq. (6.35) becomes

$$2\phi + (\mathrm{ad}\Omega) \circ \phi = \mathrm{id} \tag{6.45}$$

(iii) $\Longrightarrow$ (i): Since $\Omega \in Z(g)$, ad Ω intertwines the adjoint module. Now (iii) implied that ϕ also intertwines. $\square$

In preparation for the next theorem, consider the subspace $V^{(2)}$ of V spanned by $\mathbf{1}$ $a, a(-1)b$, where $a, b \in V[1]$. Let $O(V^{(2)})$ be the subspace of $V^{(2)}$ spanned by elements of the form $[a, b](-2)\mathbf{1} + [a, b] \in V^{(2)}$, where $a, b \in g$. It is important to note that the only homogeneous element in $O(V^{(2)})$ is zero. It is easy to show that $V^{(2)}$, $O(V^{(2)})$ are g-submodules of V.

Let $T^i g$ be the i^{th} tensor power of our Lie algebra g. Let $T^{(2)}g = \bigoplus_{0 \leqq i \leqq 2} T^i g$. Then $T^{(2)}g$ carries a g-action induced by the adjoint action of g. Let $O(T^{(2)}g)$ be the subspace of $T^{(2)}g$ spanned by $a \otimes b - b \otimes a - [a,b]$, where $a, b \in g$. Then it is obvious that the kernel of the natural map $Tg \to Ug$, restricted to tensors of at most rank 2, is given by $O(T^{(2)}g)$. Thus we write $U^{(2)}g = T^{(2)}g/O(T^{(2)}g)$ as a subspace of Ug.

We now define a linear map f: $T^{(2)}g \longrightarrow V^{(2)}$ which maps 1, $a, a \otimes b$ onto **1**, $a, b(-1)a$ respectively, for all $a, b \in g$. Then $O(T^{(2)}g)$ is a g-submodule of $T^{(2)}g$;f is a g-module map which maps $O(T^{(2)}g)$ into $O(V^{(2)})$. This induces the surjective map $f : U^{(2)}g \longrightarrow V^{(2)}/O(V^{(2)})$.

Proposition 6.10. *if* $\Omega^\phi \in Z(g)$, *then* $\omega_\phi \in V[2]^g$.

Proof. Let $a \in g$ and $[a, \Omega^\phi] = 0$. The by the symmetry ϕ, we have

$$[a, \phi u^i]u_i + \phi u^i[a, u_i] = 0 \,. \tag{6.46}$$

Under the map $f : U^{(2)}g \longrightarrow V^{(2)}/O(V^{(2)})$, this equation turns into

$$u_i(-1)[a, \phi u^i] + [a, \phi u_i](-1)\phi u^i \equiv 0 \mod O(V^{(2)}) \,. \tag{6.47}$$

The left-hand side of this equation is obviously $a(0)\omega_\phi$ which is homogeneous of weight 2. By definition, the only homogeneous element in $O(V^{(2)})$ is zero. This completes the proof. □

Theorem 6.11. *If* $2 + \text{ad}\Omega$ *is invertible, then* V *admits a Virasoro element given by* $\omega = u_i(-1)(2 + \text{ad}\Omega)^{-1}u^i$. *Moreover, its central charge is* $c = 2\text{tr}(2 + \text{ad}\Omega)^{-1}$

Proof. By hypothesis, $(2 + \text{ad}\Omega)$ is invertible. Call the inverse ϕ. Obviously ad Ω, and hence ϕ, is a symmetric intertwining map of the adjoint module g. By Lemma 6.8, $\Omega^\phi \in Z(g)$. Thus Eq. (6.36) follows from Proposition 6.10.

Since ϕ intertwines the adjoint module, we have ad $\Omega^\phi = \text{ad}\Omega \circ \phi$. This means that

$$\text{id} = (2 + \text{ad}\,\Omega) \circ \phi = 2\phi + \text{ad}\,\Omega^\phi, \tag{6.48}$$

which gives Eq (6.35).

Finally, the formula for the central charge of ω is given by Theorem 6.6 (v). □

Combining Theorems 6.4 and 6.11, we get

Corollary 6.12. *If* $2 + \text{ad}\Omega$ *is invertible, then the set of Virasoro elements of* V *is the affine subspace of* $V[2]$ *given by* $\{u_i(-1)(2 + \text{ad}\,\Omega)^{-1}u^i + L_{-1}\varepsilon | \varepsilon \in \text{cent}(g)\}$. *Moreover, the element corresponding to* ε *has central charge* $c = 2\text{tr}(2 + \text{ad}\Omega)^{-1} + 12(\varepsilon|\varepsilon)$.

We now consider the converse of Theorem 6.11. Here we need a further assumption on V. Recall that in Subsect. 4.4, we studied a natural prevoa map $\widehat{i_v} : I(\Gamma(V), \mathbf{C}) \longrightarrow V$. Let's denote by $\hat{i}_V[n]$ the restriction of the map to the subspace of homogeneous elements of weight n. By construction, this restriction map is injective for $n = 0, 1$.

Proposition 6.13. *if* $\widehat{i_v}$ [2] *is injective, then* $\omega_\phi \in V[2]^g$ *implies that* $\Omega^\phi \in Z(g)$.

Proof. First let's consider the $g = \Gamma(V)$-module map $f : U^{(2)}g \longrightarrow V^{(2)}/O(V(2))$ introduced above. This map is surjective and sends Ω^{ϕ} to $\omega_{\phi} + O(V^{(2)})$ by construction. We will show that under the hypothesis here, f is also injective. If we can do this, then it follows that f^{-1} pulls the coset $\omega_{\phi} + O(V^{(2)})$ back to Ω^{ϕ}.

Suppose $\alpha^{ij}u_iu_j + \beta \in U^{(2)}g$ (sum i, j) is in the kernel of f, for some $\beta \in g = \Gamma(V) = V[1]$. Without loss of generality, we may assume that (α^{ij}) is a symmetric scalar matrix, for the antisymmetric part can be absorbed by β. Then under the action of f, we get

$$\alpha^{ij}u_j(-1)u_i + \beta \equiv 0 \mod O(V^{(2)})\,. \tag{6.49}$$

This means that the left-hand side takes the form $\beta(-2)\mathbf{1} + \beta$. This gives

$$\beta(-2)\mathbf{1} = \alpha^{ij}u_j(-1)u_i\,. \tag{6.50}$$

By hypothesis, $\hat{i}_V$ [2] is injective. In particular, $V[2]$ has a basis of the form $\{u_i(-1)u_j + u_j(-1)u_i, u_i(-2)\mathbf{1}(\dim V\,[1])\geqq i\geqq j\geqq 0\}$. This means that both sides of Eq. (6.50) must vanish. It follows from Eq. (6.49) that $\beta \in O(V^{(2)})$. But β has weight 1, while zero is the only homogeneous element in $O(V^{(2)})$. Thus $\beta = 0$. This completes the proof. □

Theorem 6.14. *Suppose $\hat{i}_V$ [2] is injective. Then V admits a Virasoro element* iff *$2{+}\mathrm{ad}\Omega$ is invertible.*

Proof. Theorem 6.11 gives the "if" part. Suppose V admits a Virasoro element. Then by Corollary 6.7, V admits a Virasoro element ω_{ϕ} satisfying conditions (6.35), (6.36). Now by Proposition 6.13, we get $\Omega^{\phi} \in Z(g)$. By Proposition 6.9, finally, we see that $(2 + \mathrm{ad}\Omega)$ is invertible. □

We now recast the results in this section in more familiar notations.

Corollary 6.15. *Let $(g, (|))$ be any finite dimensional self-dual pair, and χ be any nonzero scalar. Then the prevoa $I(g, \mathbf{C}_{\chi})$ admits a Virasoro element* iff $2\chi + \mathrm{ad}_g\Omega_g$ *is invertible. If exists a Virasoro element is given by $\omega = u_i(-1)(2\chi + \mathrm{ad}_g\Omega_g)^{-1}\,u^i$, where $\{u_i\}$ and $\{u^i\}$ are dual bases of $(g, (|))$. Moreover, ω has central charge $c = 2\chi\mathrm{tr}(2\chi + \mathrm{ad}_g\,\Omega_g\,)^{-1}$.*

Proof. We take $V = I(g, \mathbf{C}_{\chi})$ and use the fact that (Theorem 4.8) $\Gamma(V) \cong (g, \chi(|))$, and hence $\Omega_{\Gamma(V)} = \chi^{-1}\Omega_g, \{\chi^{-\frac{1}{2}}u_i\}, \{\chi^{-\frac{1}{2}}u^i\}$ are dual bases of $\Gamma(V)$. Clearly $\widehat{i_V}$ is an isomorphism in this case. Thus the first statement of our claim follows from Theorem 6.14. Applying Theorem 6.11, we get the Virasoro element

$$\omega = \chi^{-1}u_i(-1)(2 + \chi^{-1}\mathrm{ad}_g\Omega_g)^{-1}u^i\,. \tag{6.51}$$

The central charge is computed the same way we did in Theorem 6.6. □

6.4. Some Special Cases. In Sect. 5.3, we study the notion of the double extension of a pair $(g, (|))$ in category, $\mathscr{L}$, by a Lie algebra h acting in g by anti-symmetric derivations. This construction offers a large pool of concrete examples of self-dual objects in the category $\mathscr{L}$. In turn, these examples lead to new examples of prevoas V of type one for which the radical $\mathrm{rad}V = 0$. Then in the last section, restricting ourselves to cases in which $\Gamma(V)$ is finite dimensional, we classify the Virasoro elements on V. Thus we obtain a large class of new *vertex operator algebras.*

In this subsection, we wish to consider the Virasoro structures on these new voas. We try to illuminate the new ones by comparing them to the ones that are well-known. First let's recover some well-known examples. We call a scalar μ a critical value of the self-dual pair $(g,(|))$ if $2\mu + \mathrm{ad}_g\Omega_g$ is singular.

Let $g = \bigoplus_i g_{(i)}$ be a finite dimensional semisimple Lie algebra with simple components $g_{(i)}$. Let $\langle|\rangle_i$ be the standard Killing form on $g_{(i)}$; $\{u_{j(i)}\}_j, \{u^j_{(i)}\}$ be some dual bases with respect to $\langle|\rangle_i$, and $h^v_{(i)}$ be the dual Coxeter number of $g_{(i)}$. Then we have

$$\Omega_g \sum_i u_{j(i)} u^j_{(i)} = \sum_i \Omega_{g(i)} ,$$
$$ad_g \Omega_g|_{g(i)} = 2h^v_{(i)} . \tag{6.52}$$

Let $\langle|\rangle$ denote the standard bilinear form of g. Thus given any nonzero scalar χ, the critical values of of the pair $(g, \chi\langle|\rangle)$ are $-\chi^{-1}h^v_{(i)}$. By Theorem 4.8, the prevoa $V = I(g, \mathbf{C}_\chi)$ has $\Gamma(V) \cong (g, \chi\langle|\rangle)$. Since this pair is self-dual, every quotient V/J of V by an ideal J has $\Gamma(V/J) = \Gamma(V)$, according to Proposition 3.8. Thus $\Gamma(V/J)$ has the same critical values $-\chi^{-1}h^v_{(i)}$. Hence if $\chi \neq -h^v_{(i)}$ for any i, then Theorem 6.11 tells us that V/J admit the Virasoro element

$$\omega = \sum_i \frac{1}{2(\chi + h^v_{(i)})} u_{j(i)}(-1)u^j_{(i)} , \tag{6.53}$$

which we call the Suguwara–Sommerfield formula. Since cent (g) is trivial, it follows from Theorem 6.4 that Eq. (6.53) also gives the *only* Virasoro element of $I(g, \mathbf{C}_\chi)$ and its quotients. Moreover the central charge of ω, by Theorem 6.11, is given by

$$c = \sum_i \frac{\chi \dim g_{(i)}}{\chi + h^v_{(i)}} . \tag{6.54}$$

Now let's turn to the other extreme: let $(g,(|))$ be a finite dimensional self-dual pair with an abelian g. In this case, $\mathrm{ad}_g\Omega_g$ is of course zero. Let $\{u^i\}$, $\{u_i\}$ be some dual bases of $(g,(|))$, and χ be any nonzero scalar. Consider the prevoa $V = I(g, \mathbf{C}_\chi)$ and its quotients. Note that as before, any quotient V/J of V has $\Gamma(V/J) = (g, \chi(|))$. By Theorems 6.4 and 6.11 (the case $\phi = \frac{1}{2}$), the most general Virasoro elements of V/J are given by

$$\omega = \frac{1}{2\chi} u_i(-1)u^i + \varepsilon(-2)\mathbf{1} \tag{6.55}$$

with $\varepsilon \in \mathrm{cent}(g) = g$. Equation (6.55) is the Chodos–Thorn–Feigin–Funchs formula. The central charge of ω is given by Corollary 6.12:

$$c = \dim g + 12\chi(\varepsilon|\varepsilon)_g . \tag{6.56}$$

Thus we see that Theorems 6.4 and 6.11 unifies the above two extreme cases in a natural way. In particular, Eq. (6.53) and (6.55) are actually two special cases of a general formula. The same is true for Eq. (6.53) and (6.56).

We now consider a third special case: $(g,(|)_g)$ is the self-dual double of an arbitrary finite dimensional Lie algebra h, i.e. $g = h \ltimes h'$, where h acts on h' by the coadjoint action. Thus $(|)_g$ is the canonical pairing between h and h'. Let $\{u_i\}$ and

$\{u^{i'}\}$ be bases – dual to each other – of h and h' respectively. Then it is easy to check that

$$\Omega_g = u_i u^{i'} + u^{i'} u_i \,, \tag{6.57}$$

$$(\mathrm{ad}_g \Omega_g) a = -u^{i'} \circ \mathrm{ad}_h a \circ \mathrm{ad}_h u_i + u^{i'} \circ \mathrm{ad}_h [u_i, a] \in h' \,, \tag{6.58}$$

$$(\mathrm{ad}_g \Omega_g)|_{h'} = 0 \,. \tag{6.59}$$

In particular, we have

$$(\mathrm{ad}_g \Omega_g)^2 = 0 \,. \tag{6.60}$$

This implies that, for any nonzero scalar χ,

$$(2\chi + \mathrm{ad}_g \Omega_g)^{-1} = \frac{1}{4\chi^2}(2\chi - \mathrm{ad}_g \Omega_g) \,, \tag{6.61}$$

$$\mathrm{tr}\,(2\chi + \mathrm{ad}_g \Omega_g)^{-1} = \frac{1}{2\chi}\mathrm{dim} g = \chi^{-1}\mathrm{dim} h \,. \tag{6.62}$$

Equation (6.60) also implies that the only critical value of the pair $(g, \chi(|))$ is zero. Thus if we consider the prevoa $V = I(g, \mathbf{C}_\chi)$ and its quotients, we find that, by Theorem 6.11 (the case $\phi = (2 + \chi^{-1}\mathrm{ad}_g \Omega_g)^{-1}, I(g, \mathbf{C}_\chi)$ and its quotients always admit Virasoro elements. By Eqs. (6.59), (6.61) and Theorems 6.4, 6.11, every Virasoro element takes the form

$$\omega = \frac{1}{2\chi}(u_i(-1)u^{i'} + u^{i'}(-1)u_i) - u^{i'}(-1)(\mathrm{ad}_g \Omega_g) u_i + \varepsilon(-2)\mathbf{1} \,, \tag{6.63}$$

where $\varepsilon \in \mathrm{cent}(g)$. By Corollary 6.12 and Eq. (6.62), the central charge of ω is

$$c = 2\mathrm{dim} h + 12\chi(\varepsilon|\varepsilon)_g \,. \tag{6.64}$$

Even in the case above, there are yet two extreme subcases: (i) h is semisimple; (ii) h is the abelian. In subcase (i), we have cent $(g) = 0$. Thus there is a unique Virasoro element given by Eq. (6.63) with $\varepsilon = 0$. In this case, the central charge is independent of χ. In subcase (ii), we have cent $(g) = h \oplus h'$, and hence we have the "maximal" number of Virasoro elements.

We now go to the fourth special case which, in fact, subsumes the case of self-dual double (see Sect. 5.3). Consider the finite dimensional self-dual Lie algebra $\Delta = \Delta(g, h; \theta)$, obtained from the *double extension* of a self-dual pair $(g, (|)_g)$ by a Lie algebra h acting on $(g, (|)_g)$ via skew symmetric derivations, $\theta : h \longrightarrow \mathrm{Der}(g, (|)_g)$.

Obviously, this setting is much more general than the case of self-dual double. Thus it is more difficult to obtain detailed information about the Virasoro elements in this setting. For in general, $\mathrm{ad}_\Delta \Omega_\Delta$ no longer acts nilpotently. It is therefore difficult to describe the Virasoro elements of $V = I(\Delta, \mathbf{C}_\chi)$ in terms of the constituents g, h, θ etc. We will however give a few general properties of the operator $\mathrm{ad}_\Delta \Omega_\Delta$. These properties may simplify and facilitate the computation of the Virasoro elements in concrete situations.

Let $\{u^i\}$ and $\{u_i\}$ be dual bases of $(g, (|)_g)$. Let $\{v^{j'}\}$ and $\{v_j\}$ be bases – dual to each other – of h' and h respectively. To simplify notation, we will identify h, g, h' as the appropriate subspaces of Δ. For example, we regard a as the same thing as an element of the form $(a, 0, 0)$ in $\Delta = g \oplus g \oplus h'$. Thus $\{v_j, u^i, v^{i'}\}$ and $\{v^{j'}, u_i, v_j\}$ are regarded as two ordered bases of Δ which are dual to one another. Then we have

$$\Omega_\Delta = v_j v^{j'} + u_i u^i + v^{j'} v_j \ . \tag{6.65}$$

Note that we may also regard the self-dual double $\delta = \delta(h) = (h \ltimes h', (|)_\delta)$ of h, and $(g, (|)_g)$ canonically as subalgebras of $(\Delta, (|)_\Delta)$ in the category $\mathscr{L}$.

It is straightforward to compute the action of Ω_Δ on each of the subspaces: h, g, h'. For $a \in h,\ b \in g,\ a' \in h'$, we get

$$\begin{aligned}
(\mathrm{ad}_\Delta \Omega_\Delta)(a,0,0) &= (0, \left[\theta(a)u_i, u^i\right], \beta(\theta(a)u_i, u^i) + (\mathrm{ad}_\delta \Omega_\delta)a)\ ,\\
(\mathrm{ad}_\Delta \Omega_\Delta)(0,b,0) &= (0, (\mathrm{ad}_g \Omega_g)b, \beta(u_i, \left[u^i, b\right]))\ ,\\
(\mathrm{ad}_\Delta \Omega_\Delta)(0,0,a') &= 0\ .
\end{aligned} \tag{6.66}$$

In particular, it follows that every eigenvector corresponding to a nonzero eigenvalue λ must have the form

$$(0, b, \lambda^{-1}\beta(u_i, \left[u^i, b\right]))\ , \tag{6.67}$$

where b is an eigenvector of $\mathrm{ad}_g \Omega_g$:

$$(\mathrm{ad}_g \Omega_g) b = \lambda b\ . \tag{6.68}$$

This means that $\mathrm{ad}_\Delta \Omega_\Delta$ *is nilpotent* iff $\mathrm{ad}_g \Omega_g$ *is nilpotent.* The eigenvectors of $\mathrm{ad}_\Delta \Omega_\Delta$ with zero eigenvalue are those (a, b, a') satisfying

$$\begin{aligned}
(\mathrm{ad}_\Delta \Omega_\Delta)(a,b,a') =& (0, \left[\theta(a)u_i, u^i\right] + (\mathrm{ad}_g \Omega_g)b,\\
& \beta(\theta(a)u_i, u^i) + (\mathrm{ad}_\delta \Omega_\delta)a + \beta(u_i, \left[u^i, b\right]))\\
=& 0\ .
\end{aligned} \tag{6.69}$$

Let χ be a nonzero scalar such that both $2\chi + \mathrm{ad}_\Delta \Omega_\Delta$ and $2\chi + \mathrm{ad}_g \Omega_g$ are invertible. Let's consider the prevoa $I(\Delta, \mathbf{C}_\chi)$ and its quotients V/J of V by any ideal J. In this case, we have $\Gamma(V/J) = (\Delta, \chi(|)_\Delta)$. Then by Theorem 6.11 and Corollary 6.12, the set of Virasoro elements in V/J consists of

$$\begin{aligned}
\omega =& v_j(-1)(2\chi + \mathrm{ad}_\Delta \Omega_\Delta)^{-1} v^{j'} + u_i(-1)(2\chi + \mathrm{ad}_\Delta \Omega_\Delta)^{-1} u^i\\
& + v^{j'}(-1)(2\chi + \mathrm{ad}_\Delta \Omega_\Delta)^{-1} v_j + \varepsilon(-2)\mathbf{1}\ ,
\end{aligned} \tag{6.70}$$

where ε ranges over the center of Δ. Equation (6.70) can be expressed in terms of the constituents g, h, θ, of Δ using Eqs. (6.66). The central charge of ω is given by

$$\begin{aligned}
c &= 2\chi\, \mathrm{tr}_\Delta (2\chi + \mathrm{ad}_\Delta \Omega_\Delta)^{-1} + 12\chi(\varepsilon|\varepsilon)_\Delta\\
&= 2\dim h + 2\chi \mathrm{tr}_g (2\chi + \mathrm{ad}_g \Omega_g)^{-1} + 12\chi(\varepsilon|\varepsilon)_\Delta\ .
\end{aligned} \tag{6.71}$$

Note that $\dim g = \dim\Delta - 2\dim h$. Thus if g itself is a double extension of some lower dimensional Lie algebras in $\mathscr{L}$, then Eq. (6.71) may be viewed as a recursion formula in which $\mathrm{tr}_g(2\chi + \mathrm{as}_g \Omega_g)^{-1}$ can in turn be computed in terms of the constituents of G itself.

6.5. Representation Theory of $\hat{g}$. From the construction of the affine Lie algebra

$$\hat{g} = \mathbf{C}d \oplus (g \otimes \mathbf{C}\left[t, t^{-1}\right]) \oplus \mathbf{C}\zeta\ , \tag{6.72}$$

it is clear that the Lie algebra of derivations $\mathrm{Der}\mathbf{C}\left[t,t^{-1}\right]$ acts canonically on the subalgebra $\tilde{g} = g \otimes \mathbf{C}\left[t,t^{-1}\right] \oplus \mathbf{C}\zeta$. Namely, we have

$$t^n\frac{d}{dt}(a\otimes t^m,\zeta) = m(a\otimes t^{n+m-1},0)\,. \tag{6.73}$$

But the Virasoro algebra Vir projects onto $\mathrm{Der}\,\mathbf{C}\left[t,t^{-1}\right]$. The kernel of this projection is the one-dimensional center of Vir. Via this projection, Vir acts canonically on $\tilde{g}$ with zero central charge. Using this action, we define the semi-direct product algebra $\mathrm{Vir}\ltimes\tilde{g}$. Note that the affine Lie algebra $\hat{g}$ may be identified with a subalgebra of $\mathrm{Vir}\ltimes\tilde{g}$, with d identified with $-L_0$.

In this subsection, we will use the above action and the theory of the Virasoro elements developed in the last section to study the following two questions:

(i) Given a graded $\hat{g}$-module M, when can we extended the $\tilde{g}$-action to a $\mathrm{Vir}\ltimes\tilde{g}$-action?

(ii) When is the $\hat{g}$-module $I(g,\mathbf{C}_\chi)$ irreducible for generic values of χ?

We assume that $(g,(|))$ is a finite dimensional self-dual pair in the category $\mathscr{L}$. Thus the notations used in the last section remain valid here. As before, we restrict ourselves to the ($\mathbf{Z}$-graded) $\tilde{g}$-modules M in which $M[n]=0$ for all $n \ll 0$, and ζ acts by a nonzero scalar χ.

Proposition 6.16. *If $2\chi + \mathrm{ad}_g\Omega_g$ is invertible, then every $\tilde{g}$-module M of the type above extends to a $\mathrm{Vir}\ltimes\tilde{g}$-module.*

Proof. By Proposition 4.12, M is a module over the prevoa $I(g,\mathbf{C}_\chi)$. By hypothesis and Corollary 6.15, $I(g,\mathbf{C}_\chi)$ admits a Virasoro element of the form $\omega = u_i(-1)(2\chi + \mathrm{ad}_g\Omega_g)^{-1}u^i$. By Theorem 6.6, the coefficients of the vertex series $Y(\omega,z), Y(a,z)$ satisfy

$$[\omega(n+1),a(m)] = -ma(n+m)\,, \tag{6.74}$$

where $a\in g$. As operators on the $I(g,\mathbf{C}_\chi)$-module M, the $\omega(n+1)$ and the $a(m)$ must satisfy the same relations. But these relations coincide with the Lie bracket of the L_n and the $a\otimes t^m$ in the algebra $\mathrm{Vir}\ltimes\tilde{g}$. It follows that M carries the $\mathrm{Vir}\ltimes\tilde{g}$-action which extends the $\tilde{g}$-action. □

Note that the proof above also gives an explicit formula for the operators by which Vir acts. Namely

$$Y(\omega,z) =: u_i(z)\left[(2\chi+\mathrm{ad}_g\Omega_g)^{-1}u^i\right](z):\ . \tag{6.75}$$

We now move on to question (ii) above. We begin by defining a generalization of *the Kac–Casimir operator* [17]. As noted above, $\hat{g}$ can be identified as a subalgebra of $\mathrm{Vir}\ltimes\tilde{g}$ by identifying d with $-L_0$. But a priori, the extension given by Proposition 6.16 need not respect this identification. That is, the action of $-L_0$ on M given by the extension need not coincide with the original action of d on M. This is because one can shift the action of d by a scalar constant – hence change the action of $\hat{g}$-while preserving the $\tilde{g}$-module structure. However, once the action of d is fixed on M, the operator $d+L_0$ turns out to have a nice property.

Definition 6.17. *Given a $\hat{g}$-module M, we call the operator $d+L_0$ on M (where L_0 is given by Proposition 6.16) the generalized Kac–Casimir operator on M. Note that by convention, the action of d on $I(g,\mathbf{C}_\chi)$ is fixed by $d\mathbf{1}=0$.*

Proposition 6.18. *If* $2\chi + \mathrm{ad}_g\Omega_g$ *is invertible, then the generalized Kac–Casimir operator on* M *commutes with the action of the prevoa* $I(g, \mathbf{C}_\chi)$.

Proof. Since $I(g, \mathbf{C}_\chi)$ is generated by $I(g, \mathbf{C}_\chi)$ [1] , it is enough to show that $d + L_0$ commutes with the $a(m)$, where $a \in I(g, \mathbf{C}_\chi)$ [1]. But this just follows from

$$[d + L_0, a(m)] = ma(m) - ma(m) = 0 \tag{6.76}$$

for all $a \in g$. □

Theorem 6.19. *The* $\hat{g}$*-module* $I(g, \mathbf{C}_\chi)$ *is irreducible for generic values of* χ.

Proof. Suppose otherwise. We will obtain a contradiction. Let's denote $I(g, \mathbf{C}_\chi)$ by V_χ, the vacuum vector of V_χ by $\mathbf{1}_\chi$. Now every homogeneous element v of weight n in V_χ can be written as $u\mathbf{1}_\chi$, for a unique $u \in U\hat{g}_+[n]$. Note also that whenever V_χ is reducible, it has a nonzero homogeneous singular vector $v \in V_\chi[n]$ for some $n > 0$, i.e. $a(m)v = 0$ for all $a \in g$ and $m > 0$.

If our supposition were true, then there would be a nonzero element $u \in U\hat{g}_+[n]$ for some $n > 0$, such that $u\mathbf{1}_\chi$ is singular *for all* χ. This can be easily shown using the determinant of the Shapovalov form). Suppose $2\chi + \mathrm{ad}_g\Omega_g$ is invertible. By Proposition 6.18, we have the operator $d + L_0$ which commutes with the action of $\tilde{g}$. Applying this operator on the singular vector $u\mathbf{1}_\chi$, we get

$$(d + L_0)u\mathbf{1}_\chi = u_i(0)(\phi_\chi u^i)(0)u\mathbf{1}_\chi - nu\mathbf{1}_\chi\,, \tag{6.77}$$

where $(2\chi + \mathrm{ad}_g\Omega)^{-1} = \phi_\chi$. But since $d + L_0$ commutes with u, the left-hand side of Eq. (6.77) is zero. By rearranging the terms slightly, we get

$$\left[u_i(0), \left[(\phi_\chi u^i)(0), u\right]\right]\mathbf{1}_\chi = nu\mathbf{1}_\chi\,. \tag{6.78}$$

Using the canonical isomorphism $I(g, \mathbf{C}_\chi) \cong U\hat{g}_+$, Eq, (6.78) gives a relation in $U\hat{g}_+$:

$$\left[u_i(0), \left[(\phi_\chi u^i)(0), u\right]\right] = nu\,, \tag{6.79}$$

which holds whenever $2\chi + \mathrm{ad}_g\Omega_g$ is invertible. Now $\phi_\chi = (2\chi + \mathrm{ad}_g\Omega_g)^{-1}$, where $\mathrm{ad}_g\Omega_g$ is a fixed linear operator on g. Thus the left-hand side of Eq. (6.79) clearly depends on χ, while the right-hand side does not. Hence our supposition must be false. □

6.6 The Automorphisms of $I(g, \mathbf{C})$. We assume here that for the finite dimensional self-dual pair $(g, (|)), 2 + \mathrm{ad}_g\Omega_g$ is invertible. We write $V = I(g, \mathbf{C})$.

Let $\mathrm{Aut}(V)$ denote the prevoa automorphism group of V, and $\mathrm{Aut}(\Gamma(V))$ be the automorphism group of the object $\Gamma(V)$ in category $\mathscr{L}$. It is clear from Theorem 3.7 that, every automorphism f of V induces an automorphism $\Gamma(f)$ of $\Gamma(V)$. By the functoriality of Γ, it preserves the identity, inverses as well as compositions. This means that we have a group homomorphism

$$\mathrm{Aut}(V) \longrightarrow \mathrm{Aut}(\Gamma(V))\,. \tag{6.80}$$

By an abuse of notation, we denote this homomorphism as Γ.

For every automorphism ψ of $\Gamma(V) = (g, (|))$, the functor $I(-, \mathbf{C})$ induces (Proposition 4.9 an automorphism $\hat{\psi}$ of the prevoa $V = I(g, \mathbf{C}_\chi)$. By construction (Sect. 4.4), it is clear that $\Gamma(\hat{\psi})$ is ψ itself. This means that the map (6.80) is surjective.

We claim that it is injective, i.e. every automorphism of V is determined by its restriction to $V[1]$. Let $f \in \mathrm{Aut}(V)$ with $\Gamma(f) = \mathrm{id}_{\Gamma(V)}$. By definition, we have $f(Y(a,z)b) = Y(fa,z)f\,b$, i.e. $f(a(n)b) = (fa)(n)f\,b$ for all $a, b \in V$ and integer n. Being generated by $V[1]$, V is spanned by $a_1(n_1)\cdots a_k(n_k)\mathbf{1}$ with the $a_i \in V[1]$. Since $\Gamma(f)$, which is the identity map, is also the restriction of f to $V[1]$, we have

$$f(a_1(n_1)\cdots a_k(n_k)\mathbf{1}) = (fa_1)(n_1)\cdots(fa_k)(n_k)f\mathbf{1} = a_1(n_1)\cdots a_k(n_k)\mathbf{1}\,. \tag{6.81}$$

Hence f is the identity map. To summarize we have

Proposition 6.20. *The map* $\mathrm{Aut}(V) \longrightarrow \mathrm{Aut}(\Gamma(V))$ *is a group isomorphism. In particular, every prevoa automorphism of* V *is determined by its value on* $V[1]$.

Proposition 6.21. *The Virasoro element* $\omega = u_i(-1)(2+\mathrm{ad}\Omega)^{-1}u^i$ *is fixed by* $\mathrm{Aut}(V)$.

Proof. Let $f \in \mathrm{Aut}(V)$ and $a \in V[1]$. Then we have

$$\begin{aligned} f(\mathrm{ad}\Omega a) &= \Gamma(f)\left[u_i, \left[u^i, a\right]\right] \\ &= \left[fu_i, \left[fu^i, fa\right]\right] \\ &= \left[u_i, \left[u^i, fa\right]\right] \quad \text{using } (fu_i|fu^i) = \delta_i^j \\ &= \mathrm{ad}\Omega fa\,. \end{aligned} \tag{6.82}$$

This means that f commutes with $(2+\mathrm{ad}\Omega)^{-1}$ on $V\,[1]$. Thus we have

$$\begin{aligned} f\omega &= (fu_i)(-1)(2+\mathrm{ad}\Omega)^{-1}fu^i \\ &= u_i(-1)(2+\mathrm{ad}\Omega)^{-1}u^i \quad \text{using } (fu_i|fu^j) = \delta_i^j \\ &= \omega \quad \Box \end{aligned} \tag{6.83}$$

Theorem 6.22. *The Virasoro element* $\omega = u_i(-1)(2+\mathrm{ad}\Omega)^{-1}u^i + \varepsilon(-2)\mathbf{1}$, *corresponding to* $\varepsilon \in \mathrm{cent}(\Gamma(V))$, *is fixed by* $\mathrm{Aut}\,(V)$ iff ε *is fixed by* $\mathrm{Aut}(\Gamma(V))$. *The orbits of* $\mathrm{Aut}(\Gamma(V))$ *in* $\mathrm{cent}(\Gamma(V))$ *are in one-to-one correspondence with the equivalence classes of vertex operator algebra structures on* $V = I(g, \mathbf{C})$.

Proof. By Proposition 6.21, ω is fixed by $\mathrm{Aut}(V)$ iff $\varepsilon(-2)\mathbf{1}$ is. Suppose that $f \in \mathrm{Aut}(V)$ fixes $\varepsilon(-2)\mathbf{1}$.
Then we have

$$(f\varepsilon)(-2)\mathbf{1} = \varepsilon(-2)\mathbf{1}\,, \tag{6.84}$$

implying that

$$\frac{d}{dz}Y(f\varepsilon - \varepsilon, z) = 0\,. \tag{6.85}$$

Since $f\varepsilon - \varepsilon \in V\,[1]$, Eq. (6.85) means that $\Gamma(f)\varepsilon = f\varepsilon = \varepsilon$. Thus if $\varepsilon(-2)\mathbf{1}$ is fixed by all of $\mathrm{Aut}(V)$, then ε is fixed by all of $\mathrm{Aut}(\Gamma(V))$ by Proposition 6.20. The converse is similar.

Given the prevoa V, a voa structure on V is specified by a Virasoro element ω. Two voa structures, ω, ω' on V are equivalent iff V admits an automorphism sending ω to ω'. By Theorem 6.4, the set of Virasoro elements in V consists of $\omega + \varepsilon(-2)\mathbf{1}$ where ε ranges over cent$(\Gamma(V))$, and $\omega = u_i(-1)(2 + \mathrm{ad}\Omega)^{-1}u^i$. Now the second part of our claim follows from Proposition 6.21. □

6.7 Concluding Remarks. In the present work, the contributions that are made toward the understanding of vertex operator theory may be summarized as follows:

(i) establishing a precise correspondence between the category $\mathscr{L}$ of Lie algebras with invariant forms and the category $\mathscr{V}$ of prevoas, using a new functor Γ;
(ii) Classifying the simple minimal type ones in $\mathscr{V}$;
(iii) Classifying the Virasoro structures on each minimal type one V, with finite dimensional self-dual $\Gamma(V)$;
(iv) revealing new examples of vertex operator algebras;
(v) extending the $\hat{g}$-action to a Vir$\ltimes\hat{g}$-action.

Recall that the critical values of a finite dimensional self-dual pair $(g, (|))$ in $\mathscr{L}$, are the eigenvalues of $-\frac{1}{2}\mathrm{ad}_g\Omega_g$. Strictly speaking, we have done (iii) only for the case in which 1 is not a critical value of $\Gamma(V)$. The case of a general V for which $\Gamma(V)$ admits the critical value, 1, is as yet unsolved. The real difficultly, amounts be the fact that when the map $\widehat{i_V}$ [2] (see Theorem 6.14) fails to be injective, it is not clear how to solve the two linear conditions (6.35), (6.36) in Corollary 6.7. However, we believe that Theorem 6.14 still holds even when $\widehat{i_V}$ [2] is not injective. In other words, when $\Gamma(V)$ has 1 as a critical value, those two conditions (6.35), (6.36) should have no solution. We have verified this in the case when $\Gamma(V)$ is semisimple. It is worth noting that the conditions (6.35), (6.36) are invariant under the automorphism group Aut$(\Gamma(V))$, i.e. if ϕ is a solution, so are its Aut$(\Gamma(V))$-conjugates. This should be a useful fact for analysing the solutions to those conditions. It can be shown that when V is simple prevoa, i.e. $V = L(g, \mathbf{C})$, condition (6.35) implies (6.36). Thus the problem reduces to proving the following (purely Lie theoretic) conjecture:

$$2\phi + \mathrm{ad}\Omega^{\phi} = \mathrm{id}_g \textit{ has no solution } \phi \textit{ unless } 2 + ad\ \Omega \textit{ is invertible} \qquad (6.86)$$

Again, we have verified this in the case where g is simple.

Based on our knowledge about the semisimple case, we expect that interesting phenomena [5] should occur when 1 is a critical value of $\Gamma(V)$. For example when g is simple, $(|)$ is the standard bilinear form of g, and $V = I(g, \mathbf{C})$, we know that 1 is a critical value of $\Gamma(V)$ iff χ is equal to the negative of the dual Coxeter number of g. At this critical value, V admits a large algebra of intertwining operators. It is an interesting problem to realize something similar in the non-reductive case.

As pointed out in our introduction, one of the motivations behind this work is the attempt to understand some new examples which come up in conformal filed theory, quantum groups, and Chern–Simons–Witten theory. While having solved the problems we stated in the introduction, we have not attempted to resolve all of the issues raised there. This is not the purpose of the present work. The purpose here is to find the most general context in which those issues may be discussed. For instance, in view of the known connections between WZW voas and quantum groups via KZ equations in the reductive case, we may now at least *ask for* similar connections in the non-reductive case. A good point to begin may be to study WZW-type models based on certain Manin triple $(g \oplus g', g, g')$, where g is one of Drifel'd's Lie bialgebras [3].

Historically in representation theory and related subjects, non-reductive groups have received much less attention than their reductive cousins. So what we have done in the present work runs counter to the conventional wisdom. But the reward we get is a glimpse of a whole new world beyond reductive groups.

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