About the Regularizing Properties of the Non-cut-off Kac Equation

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Abstract: We prove in this work that under suitable assumptions, the solution of the spatially homogeneous non-cut-off Kac equation (or of the spatially homogeneous non cut-off 2D Boltzmann equation with Maxwellian molecules in the radial case) becomes very regular with respect to the velocity variable as soon as the time is strictly positive.

1. Introduction

In the upper atmosphere, a gas is described by the nonnegative density $f(t,x,v)$ of particles which at time t and point x, move with velocity v. Such a density satisfies the Boltzmann equation (cf. [Ce], [Ch, Co], [Tr, Mu]):

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f) , \tag{1.1}$$

where Q is a quadratic collision kernel acting only on the variable v and taking in account any collisions preserving momentum and kinetic energy:

$$Q(f)(v) = \int_{v_* \in \mathbb{R}^3} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \{ f(v')f(v'_*)$$

$$- f(v)f(v_*) \} B(|v - v_*|, \theta) \sin \theta \, d\phi d\theta dv_* , \tag{1.2}$$

with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma , \tag{1.3}$$

$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma , \tag{1.4}$$

$$\cos \theta = \sigma \cdot \frac{v - v_*}{|v - v_*|} , \tag{1.5}$$

and B is a nonnegative cross section. When the collisions in the gas come out of an inverse power law interaction in $\frac{1}{r^s}$ (with $s \geq 2$), the cross section writes

$$B(x, \theta) = x^{\frac{s-5}{s-1}} b(\theta) , \qquad (1.6)$$

where $b \in L^\infty_{\text{loc}}(]0, \pi])$ and

$$\sin \theta\, b(\theta) \sim K(s)\theta^{-\frac{s+1}{s-1}} \qquad (1.7)$$

for some $K(s) > 0$ when $\theta \to 0$.

Most of the mathematical work about the Boltzmann equation is made under the assumption of angular cut-off of Grad (cf. [Gr]), which means that b in Eq. (1.6) is supposed to satisfy $\sin \theta\, b(\theta) \in L^1([0, \pi])$. Note that for inverse power laws in $\frac{1}{r^s}$ with $s \geq 2$, this assumption never holds (because of the singularity appearing in Eq. (1.7)).

For example, the existence of a global renormalized solution to the full Boltzmann equation (1.1) is known under this assumption (cf. [DP, L]), but it is also the case with most of the works concerning the spatially homogeneous Boltzmann equation (cf. [A 1], [A 3], [Ee], [De 1]):

$$\frac{\partial f}{\partial t}(t, v) = Q(f)(t, v) , \qquad (1.8)$$

with the noticeable exception of [A 2], where existence is proved for the non-cut-off equation $(1.2)-(1.8)$ when $s > 3$.

We shall now concentrate on this spatially homogeneous equation (1.8). When the cut-off assumption is made, it is possible to write

$$Q(f) = Q^+(f) - f L f , \qquad (1.9)$$

where

$$Q^+(f)(v) = \int\limits_{v_* \in \mathbb{R}^3} \int\limits_{\theta=0}^{\pi} \int\limits_{\phi=0}^{2\pi} f(v')f(v'_*)B(|v - v_*|, \theta) \sin \theta\, d\phi d\theta dv_* , \qquad (1.10)$$

and

$$Lf = A * f , \qquad (1.11)$$

with

$$A(x) = 2\pi \int\limits_{\theta=0}^{\pi} B(|x|, \theta) \sin \theta\, d\theta . \qquad (1.12)$$

Then, the solution $f(t, v)$ of Eq. (1.8) can be written under the form

$$f(t, v) = f(0, v)e^{-\int\limits_0^t Lf(\tau, v)d\tau} + \int\limits_0^t Q^+(f)(s, v)\, e^{-\int\limits_s^t Lf(\tau, v)d\tau}\, ds . \qquad (1.13)$$

But the operator Q^+ is known to be regularizing with respect to the variable v (at least when $f \in L^2(\mathbb{R}^3_v)$, and when B satisfies some properties) (cf. [L 1]). Therefore, if $f(0, v)$ is not regular (for example if it belongs to $L^2(\mathbb{R}^3_v)$ but not to $H^1(\mathbb{R}^3_v)$), the solution $f(t, v)$ of Eq. (1.8) will at best keep the regularity of $f(0, v)$

when $t > 0$. In particular, no regularizing effect is expected for the solution of the cut-off homogeneous Boltzmann equation (1.8).

On the other hand, one can hope some regularizing properties for the solution of the non-cut-off homogeneous Boltzmann equation (1.8), (1.2) (when (1.6), (1.7) holds).

One of the reasons of assuming such a conjecture is that an asymptotics of the Boltzmann equation when the cross section is concentrating on the grazing collisions (these collisions are those that are neglected when the cut-off assumption is made) leads to the Fokker–Planck–Landau equation (cf. [De 2], [Dg, Lu]), which is known to induce regularizing effects (or at least compactness properties, even in the spatially inhomogeneous case (cf.[L 2])).

This article is devoted to the proof of such a conjecture in the simpler case of spatially homogeneous Kac equation. We recall that the original Kac model is used to describe a one-dimensional spatially homogeneous gas in which the collisions preserve the mass and the energy, but not the momentum (cf. [K], [MK]).

Note also that the theorems of Sects. 2, 3 and 4 hold for the spatially homogeneous non cut-off 2D radially symmetric solutions of the Boltzmann equation with Maxwellian molecules, as is shown in Appendix C.

In the Kac model, the nonnegative density $f(t, v)$ satisfies

$$\frac{\partial f}{\partial t}(t, v) = K(f)(t, v) \, , \tag{1.14}$$

where

$$K(f)(t, v) = \int\limits_{v_* \in \mathbb{R}} \int\limits_{\theta = -\pi}^{\pi} \{f(v'')f(v_*'') - f(v)f(v_*)\} \frac{d\theta}{2\pi} dv_* \, , \tag{1.15}$$

and

$$v'' = v \cos \theta - v_* \sin \theta \, , \tag{1.16}$$

$$v_*'' = v \sin \theta + v_* \cos \theta \, . \tag{1.17}$$

The analysis leading to Eq. (1.13) still holds in this case. Therefore, one can at best hope that the regularity of $f(0, v)$ is conserved for the solution $f(t, v)$ of Eq. (1.14) when $t > 0$. This affirmation is indeed easily proved when $f(0, v) \in L^1((1 + |v|^2)dv)$ (cf. Theorem A.1 of Appendix A), but also in the more difficult case when $f(0, v)$ lies in some Hölder spaces (cf. [G]) (note also the results in the same spirit for the Boltzmann equation of [We]).

We will therefore concentrate in this work on the equation

$$\frac{\partial f}{\partial t}(t, v) = K_\beta(f)(t, v) \, , \tag{1.18}$$

where

$$K_\beta(f)(t, v) = \int\limits_{v_* \in \mathbb{R}} \int\limits_{\theta = -\pi}^{\pi} \{f(v'')f(v_*'') - f(v)f(v_*)\} \beta(|\theta|) \, d\theta dv_* \, , \tag{1.19}$$

with

$$\beta(x) \sim x^{-\alpha} \tag{1.20}$$

When $x \to 0^+$ and $\alpha \in]1, 3]$.

This kernel is obtained by analogy with the non cut-off kernel (1.2), (1.6), (1.7) of Boltzmann equation.

However, the analysis in the case when $\alpha = 3$ (corresponding to the Coulombian interaction in the case of the Boltzmann equation (cf. [Dg, Lu])) is very different from the analysis when $\alpha \in]1,3[$. Therefore, we will only consider in the sequel the latter case.

We begin in Sect. 2 by proving that the existence of a solution holds for Eq. $(1.18)-(1.20)$. We prove then in Sect. 3 our main theorem. Namely, if $f(0,v) \in L^1(\mathbb{R}_v,(1+|v|^\gamma)dv)$ for all $\gamma > 0$, the solution $f(t,v)$ of Eq. $(1.18)-(1.20)$ lies in $C^\infty(\mathbb{R}_v)$ for all $t > 0$. Finally, in Sect. 4, we consider the case when only a finite number of moments are known to be initially bounded for f. The reader will also find for the sake of completeness some classical results used throughout this work in Appendix A and B at the end of the paper, Appendix C being devoted to the extension of the results to the 2D radially symmetric Boltzmann equation with Maxwellian molecules.

2. Existence for the Non cut-off Kac Equation

We prove in this section the following theorem:

Theorem 2.1. *Let $f_0 \geq 0$ be an initial datum such that*

$$\int_{v\in\mathbb{R}} f_0(v)\left(1+|v|^2+|\log f_0(v)|\right)dv < +\infty , \tag{2.1}$$

and let $\beta \geq 0$ be a cross section satisfying the following property:

$$\exists \beta_0, \beta_1 > 0, \alpha \in]1,3[, \quad \forall x \in]0,\pi], \quad \beta_0|x|^{-\alpha} \leq \beta(x) \leq \beta_1|x|^{-\alpha} . \tag{2.2}$$

Then, there exists a nonnegative solution $f(t,v) \in L^\infty([0,+\infty[_t; L^1(\mathbb{R}_v,(1+|v|^2)dv))$ to Eq. (1.18), (1.19), (2.2) with initial datum f_0 in the following sense: For all functions $\phi \in W^{2,\infty}(\mathbb{R}_v)$, we have

$$\frac{\partial}{\partial t} \int_{v\in\mathbb{R}} f(t,v)\phi(v)dv = \int_{v\in\mathbb{R}} \int_{v_*\in\mathbb{R}} K^\phi(v,v_*)f(t,v)f(t,v_*)dv_*dv , \tag{2.3}$$

where

$$K^\phi(v,v_*) = \int_{\theta=-\pi}^{\pi} \{\phi(v'')-\phi(v)\}\beta(|\theta|)d\theta . \tag{2.4}$$

The conservation of mass

$$\int_{v\in\mathbb{R}} f(t,v)\,dv = \int_{v\in\mathbb{R}} f_0(v)\,dv \tag{2.5}$$

holds for these solutions, but the energy may decrease.

Moreover, if for some $p \in \mathbb{N}$, there exists $C_{2.1} > 0$ such that

$$\int_{v\in\mathbb{R}} f_0(v)\left(1+|v|^{2p}\right)dv \leq C_{2.1} , \tag{2.6}$$

one can find $C_{2.2} > 0$ such that for all $t \geq 0$,

$$\int_{v\in\mathbb{R}} f(t,v)\left(1+|v|^{2p}\right)dv \leq C_{2.2} . \tag{2.7}$$

Finally, if assumption (2.6) *holds for some* $p \geq 2$, *the conservation of energy*

$$\int\limits_{v \in \mathbb{R}} f(t,v)|v|^2 dv = \int\limits_{v \in \mathbb{R}} f_0(v)|v|^2 dv \qquad (2.8)$$

holds.

Remark. The analogue of this theorem is proved in [A 2] for the Boltzmann equation (with $s > 3$). The proof given is very similar to that of [A 2].

The sense to give to the right term of Definition (2.4) will become clear in the sequel. Note however that because of the singularity of β, this term is not defined if ϕ is not regular ($W^{2,\infty}$).

Proof of Theorem 2.1. We introduce for all $n \in \mathbb{N}^*$ the truncated sequence

$$\beta_n = \beta \wedge n \,. \qquad (2.9)$$

Note that because of assumption (2.2), there exists for all $\gamma > \alpha - 1$ a strictly positive $C_{2.3}(\gamma)$ such that for all $n \in \mathbb{N}^*$,

$$\int\limits_{\theta=-\pi}^{\pi} \left(|1 - \cos\theta|^{\gamma/2} + |\sin\theta|^\gamma\right) \beta_n(|\theta|)\,d\theta \leq C_{2.3}(\gamma)\,. \qquad (2.10)$$

It is also clear that

$$\int\limits_{\theta=-\pi}^{\pi} \left(|1 - \cos\theta|^{\gamma/2} + |\sin\theta|^\gamma\right) |\beta(|\theta|) - \beta_n(|\theta|)|d\theta \xrightarrow[n \to +\infty]{} 0\,. \qquad (2.11)$$

Then, we consider the (unique) nonnegative solution $f_n(t,v)$ of the classical Kac equation

$$\frac{\partial f_n}{\partial t}(t,v) = K_{\beta_n}(f_n)(t,v) \qquad (2.12)$$

with initial datum f_0 (for the existence and uniqueness of such a solution, cf. Theorem A.1 of Appendix A). This solution is known to satisfy the conservation of mass and energy, and the entropy inequality (cf. Theorem A.1 and A.2 of Appendix A):

$$\int\limits_{v \in \mathbb{R}} f_n(t,v)\,dv = \int\limits_{v \in \mathbb{R}} f_0(v)\,dv\,, \qquad (2.13)$$

$$\int\limits_{v \in \mathbb{R}} f_n(t,v)|v|^2 dv = \int\limits_{v \in \mathbb{R}} f_0(v)|v|^2 dv\,, \qquad (2.14)$$

$$\int\limits_{v \in \mathbb{R}} f_n(t,v) \log f_n(t,v) dv \leq \int\limits_{v \in \mathbb{R}} f_0(v) \log f_0(v)\,dv\,. \qquad (2.15)$$

It is now classical (cf. [De 3] for example) that Eq. (2.13) – (2.15) ensure the existence of a constant $C_{2.4}$ such that

$$\int\limits_{v \in \mathbb{R}} f_n(t,v) \left(1 + |v|^2 + |\log f_n(t,v)|\right) dv \leq C_{2.4}\,. \qquad (2.16)$$

Because of the Dunford–Pettis theorem (cf. [B]) and of estimate (2.16), one can extract from $(f_n)_{n \in \mathbb{N}}$ a subsequence still denoted by $(f_n)_{n \in \mathbb{N}}$ and converging to a function f in $L^\infty([0, +\infty[_t; L^1(\mathbb{R}_v))$ weak $*$.

Moreover, for all $q \in L^1([0, +\infty[)$ and all $\psi \in L^\infty_{\mathrm{loc}}([0, +\infty[_t \times \mathbb{R}_v)$ such that

$$\lim_{|v| \to +\infty} \sup_{t \in \mathbb{R}} \frac{|\psi(t,v)|}{|v|^2} = 0\,, \qquad (2.17)$$

we have

$$\int\limits_0^{+\infty} q(t) \int\limits_{v \in \mathbb{R}} f_n(t,v)\psi(t,v)\,dv dt \xrightarrow[n \to +\infty]{} \int\limits_0^{+\infty} q(t) \int\limits_{v \in \mathbb{R}} f(t,v)\psi(t,v)\,dv dt . \qquad (2.18)$$

Denoting for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$,

$$K_n^\phi(v,v_*) = \int\limits_{\theta=-\pi}^{\pi} \{\phi(v'') - \phi(v)\}\beta_n(|\theta|)\,d\theta , \qquad (2.19)$$

it is clear that (using the change of variables $(v,v_*,\theta) \to (v'',v_*'',-\theta)$),

$$\frac{\partial}{\partial t}\int\limits_{v \in \mathbb{R}} f_n(t,v)\phi(v)\,dv = \int\limits_{v \in \mathbb{R}}\int\limits_{v_* \in \mathbb{R}} K_n^\phi(v,v_*)f_n(t,v)f_n(t,v_*)\,dv_* dv . \qquad (2.20)$$

We shall now prove that when $\phi \in W^{2,\infty}(\mathbb{R}_v)$, it is possible to pass to the limit in Eq. (2.20) and to obtain Eq. (2.3). We begin by the

Lemma 1. *There exists a constant $C_{2.5} > 0$ (depending on α) and a sequence $C_{2.5}(n)$ converging to 0 such that the following estimates hold:*

1. *for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$,*

$$|K_n^\phi(v,v_*)| \leqq C_{2.5}\left(1 + |v|^{\frac{\alpha+5}{4}} + |v_*|^{\frac{\alpha+5}{4}}\right)\|\phi\|_{W^{2,\infty}(\mathbb{R}_v)} , \qquad (2.21)$$

2. *for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$,*

$$|K^\phi(v,v_*) - K_n^\phi(v,v_*)| \leqq C_{2.5}(n)\left(1 + |v|^{\frac{\alpha+5}{4}} + |v_*|^{\frac{\alpha+5}{4}}\right)\|\phi\|_{W^{2,\infty}(\mathbb{R}_v)} . \qquad (2.22)$$

Proof of Lemma 1. Note that

$$\phi(v'') - \phi(v) = \phi(v\cos\theta - v_*\sin\theta) - \phi(v)$$

$$= (v(\cos\theta - 1) - v_*\sin\theta)\,\phi'(v) + (v(\cos\theta - 1) - v_*\sin\theta)^2$$

$$\times \int\limits_{u=0}^1 (1-u)\phi''(v + u(v(\cos\theta - 1) - v_*\sin\theta))\,du . \qquad (2.23)$$

Therefore, for all $\delta \in]0,1[$,

$$|\phi(v'') - \phi(v) + v_*\sin\theta\phi'(v)| \leqq |\phi(v'') - \phi(v) + v_*\sin\theta\phi'(v)|^{1-\delta}$$

$$\times |v(\cos\theta - 1)\phi'(v) + (v(\cos\theta - 1) - v_*\sin\theta)^2$$

$$\times \int\limits_{u=0}^1 (1-u)\phi''(v + u(v(\cos\theta - 1) - v_*\sin\theta))\,du|^\delta$$

$$\leqq 8(1 + |v_*|^{1-\delta})(3\|\phi\|_{W^{1,\infty}(\mathbb{R}_v)})^{1-\delta}\left(|\cos\theta - 1|^\delta + |\sin\theta|^{2\delta}\right)$$

$$\times \left(1 + |v|^{2\delta} + |v_*|^{2\delta}\right)\|\phi\|_{W^{2,\infty}(\mathbb{R}_v)}^\delta$$

$$\leqq C_{2.6}\|\phi\|_{W^{2,\infty}(\mathbb{R}_v)}\left(|\cos\theta - 1|^\delta + |\sin\theta|^{2\delta}\right)\left(1 + |v|^{1+\delta} + |v_*|^{1+\delta}\right) \qquad (2.24)$$

for some strictly positive constant $C_{2.6}$. But $\theta \to \sin\theta$ is odd and therefore

$$|K_n^\phi(v, v_*)| = \left| \int_{\theta=-\pi}^{\pi} \{\phi(v'') - \phi(v)\}\beta_n(|\theta|)\,d\theta \right|$$

$$= \left| \int_{\theta=-\pi}^{\pi} \{\phi(v'') - \phi(v) + v_*\sin\theta\phi'(v)\}\beta_n(|\theta|)\,d\theta \right|$$

$$\leq C_{2.6} \|\phi\|_{W^{2,\infty}(\mathbb{R}_v)} \int_{\theta=-\pi}^{\pi} \left(|\cos\theta - 1|^\delta + |\sin\theta|^{2\delta} \right) \beta_n(|\theta|)\,d\theta$$

$$\times \left(1 + |v|^{1+\delta} + |v_*|^{1+\delta} \right). \tag{2.25}$$

We now use Eq. (2.10) with $\delta = \frac{1+\alpha}{4}$, and obtain

$$|K_n^\phi(v.v_*)| \leq 2\,C_{2.3} \left(\frac{1+\alpha}{2} \right) C_{2.6} \left(1 + |v|^{\frac{\alpha+5}{4}} + |v_*|^{\frac{\alpha+5}{4}} \right) \|\phi\|_{W^{2,\infty}(\mathbb{R}_v)}, \tag{2.26}$$

which clearly implies estimate (2.21). In order to get estimate (2.22), we use exactly the same proof, except that Eq. (2.10) is replaced by Eq. (2.11).

Lemma 2. *There exists a constant $C_{2.7} > 0$ (depending on α) such that when $\phi \in W^{2,\infty}(\mathbb{R}_v)$ satisfies*

$$|||\phi||| = \sup_{v\in\mathbb{R}} \frac{|\phi'(v)|}{1+|v|} < +\infty, \tag{2.27}$$

one has the following estimate:

$$|K_n^\phi(v, v_*)| \leq C_{2.7} \left(\|\phi''\|_{L^\infty(\mathbb{R}_v)} + |||\phi||| \right) \left(1 + |v|^2 + |v_*|^2 \right). \tag{2.28}$$

Proof of Lemma 2. According to Eq. (2.23),

$$\left| \phi(v'') - \phi(v) + v_*\sin\theta\phi'(v) \right| = |v(\cos\theta - 1)\phi'(v)$$

$$+ (v(\cos\theta - 1) - v_*\sin\theta)^2 \int_{u=0}^{1} (1-u)\phi''(v + u(v(\cos\theta - 1) - v_*\sin\theta))\,du|$$

$$\leq |\cos\theta - 1||v||\phi'(v)| + 4(|\cos\theta - 1| + |\sin\theta|^2)(|v|^2 + |v_*|^2)\,\|\phi''\|_{L^\infty(\mathbb{R}_v)}$$

$$\leq C_{2.8}(|\cos\theta - 1| + |\sin\theta|^2)(1 + |v|^2 + |v_*|^2)(\|\phi''\|_{L^\infty(\mathbb{R}_v)} + |||\phi|||) \tag{2.29}$$

for some constant $C_{2.8} > 0$.

Using now the oddity of $\theta \to \sin\theta$ as in Lemma 1, and estimate (2.10) for $\gamma = 2$, we get estimate (2.28).

We now come back to the proof of Theorem 2.1.

Suppose that $\phi \in W^{2,\infty}(\mathbb{R}_v)$, $q \in L^1([0,+\infty[)$, and $v \in \mathbb{R}$. Then, because of Lemma 1,

$$
\left| \int_{t=0}^{+\infty} \int_{v_* \in \mathbb{R}} K_n^{\phi}(v,v_*) f_n(t,v_*)\, dv_* q(t)\, dt - \int_{t=0}^{+\infty} \int_{v_* \in \mathbb{R}} K^{\phi}(v,v_*) f(t,v_*)\, dv_* q(t)\, dt \right|
$$

$$
\leqq \int_{t=0}^{+\infty} \int_{v_* \in \mathbb{R}} |K_n^{\phi}(v,v_*) - K^{\phi}(v,v_*)| f_n(t,v_*)\, dv_* q(t)\, dt
$$

$$
+ \left| \int_{t=0}^{+\infty} \int_{v_* \in \mathbb{R}} K^{\phi}(v,v_*)\{f_n(t,v_*) - f(t,v_*)\} dv_* q(t)\, dt \right|
$$

$$
\leqq C_{2.5}(n)\, \|q\|_{L^1([0,+\infty[_t)} \|\phi\|_{W^{2,\infty}(\mathbb{R}_v)} \int_{v_* \in \mathbb{R}} \left(1 + |v|^{\frac{\alpha+5}{4}} + |v_*|^{\frac{\alpha+5}{4}}\right) f_n(t,v_*)\, dv_*
$$

$$
+ \left| \int_{t=0}^{+\infty} \int_{v_* \in \mathbb{R}} K^{\phi}(v,v_*)\{f_n(t,v_*) - f(t,v_*)\} dv_* q(t)\, dt \right| . \tag{2.30}
$$

But the first term of Eq. (2.30) tends to 0 because of estimate (2.16). Moreover, because of Lemma 1, we have for all $v \in \mathbb{R}$,

$$
\lim_{|v_*| \to +\infty} \frac{|K^{\phi}(v,v_*)|}{|v_*|^2} = 0 . \tag{2.31}
$$

Therefore, estimate (2.18) ensures that the second term of Eq. (2.30) tends to 0. Finally, we obtain for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$ and $v \in \mathbb{R}$, the convergence in $L^{\infty}([0,+\infty[_t)$ weak * of

$$
L_n^{\phi}(t,v) = \int_{v_* \in \mathbb{R}} K_n^{\phi}(v,v_*) f_n(t,v_*)\, dv_* \tag{2.32}
$$

towards

$$
L^{\phi}(t,v) = \int_{v_* \in \mathbb{R}} K^{\phi}(v,v_*) f(t,v_*)\, dv_* . \tag{2.33}
$$

We now observe that for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$ and $v \in \mathbb{R}$, the sequence

$$
\Theta_n^{\phi}(v,v_*) = \frac{\partial^2 K_n^{\phi}}{\partial v_*^2}(v,v_*) \tag{2.34}
$$

is bounded in $L^{\infty}(\mathbb{R}_v)$. More precisely,

$$
\left| \frac{\partial^2 K_n^{\phi}}{\partial v_*^2}(v,v_*) \right| = \left| \int_{\theta=-\pi}^{\pi} \sin^2\theta\, \phi''(v'')\beta_n(|\theta|)\, d\theta \right|
$$

$$
\leqq C_{2.3}(2)\, \|\phi''\|_{L^{\infty}(\mathbb{R}_v)} . \tag{2.35}
$$

Moreover,

$$
\left| \frac{\partial K_n^{\phi}}{\partial v_*}(v,v_*) \right| = \left| \int_{\theta=-\pi}^{\pi} \sin\theta\, \phi'(v'')\beta_n(|\theta|)\, d\theta \right|
$$

$$
= \left| \int_{\theta=-\pi}^{\pi} -\sin\theta\{\phi'(v'') - \phi'(v)\}\beta_n(|\theta|)\, d\theta \right|
$$

$$\leqq \int\limits_{\theta=-\pi}^{\pi} |\sin\theta|\{|v||\cos\theta - 1| + |v_*||\sin\theta|\}$$

$$\times \int\limits_{u=0}^{1} |\phi''(v + u(v(\cos\theta - 1) - v_*\sin\theta))|\, du\, \beta_n(|\theta|)\, d\theta$$

$$\leqq 2\, C_{2.3}(2)\, \|\phi''\|_{L^\infty}\, (1 + |v| + |v_*|) \tag{2.36}$$

because of estimate (1.10).

Therefore, using Lemma 2, for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$ and $v \in \mathbb{R}$,

$$\left|\frac{\partial L_n^\phi}{\partial t}(t,v)\right| = \left|\frac{\partial}{\partial t}\int\limits_{v_*\in\mathbb{R}} K_n^\phi(v,v_*)\, f_n(t,v_*)\, dv_*\right|$$

$$= \left|\int\limits_{w\in\mathbb{R}}\int\limits_{v_*\in\mathbb{R}} K_n^{K_n^\phi(v,\,\cdot\,)}(w,v_*)\, f_n(t,w)\, f_n(t,v_*)\, dv_*dw\right|$$

$$\leqq \int\limits_{w\in\mathbb{R}}\int\limits_{v_*\in\mathbb{R}} C_{2.7}\left(\left\|\frac{\partial^2 K_n^\phi}{\partial v_*^2}(v,\,\cdot\,)\right\|_{L^\infty(\mathbb{R}_{v_*})} + |||K_n^\phi(v,\,\cdot\,)|||\right)$$

$$\times\left(1 + |w|^2 + |v_*|^2\right) f_n(t,w)\, f_n(t,v_*)\, dv_*dw$$

$$\leqq C_{2.4}^2 C_{2.7}\{C_{2.3}(2)\, \|\phi''\|_{L^\infty(\mathbb{R}_v)} + 2C_{2.3}(2)C_{2.5}(1 + |v|)\, \|\phi''\|_{L^\infty(\mathbb{R}_v)}\}\ . \tag{2.37}$$

It is also clear that

$$|L_n^\phi(t,v)| \leqq \left|\int\limits_{v_*\in\mathbb{R}} C_{2.5}\left(1 + |v|^{\frac{\alpha+5}{4}} + |v_*|^{\frac{\alpha+5}{4}}\right) f_n(t,v_*)\, dv_*\right| \|\phi\|_{W^{2,\infty}(\mathbb{R}_v)}$$

$$\leqq C_{2.4}C_{2.5}\left(1 + |v|^{\frac{\alpha+5}{4}}\right) \|\phi\|_{W^{2,\infty}(\mathbb{R}_v)}\ . \tag{2.38}$$

Therefore, for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$ and $v \in \mathbb{R}$, the sequence $L_n^\phi(\,\cdot\,,v)$ is bounded in $W^{1,\infty}([0,+\infty[_t)$. Using now the weak convergence (2.33) and Rellich theorem (cf. [B]), it is clear that for all $\phi \in W^{2,\infty}(\mathbb{R}_v)$, and a.e. $(t,v) \in [0,+\infty[_t \times \mathbb{R}_v$, the sequence L_n^ϕ tends to L^ϕ. Therefore, for all $q \in L^1([0,+\infty[)$ and all $T > 0$ such that Supp $q \subset [0,T]$,

$$\left|\int\limits_{t=0}^{+\infty}\left\{\int\limits_{v\in\mathbb{R}}\int\limits_{v_*\in\mathbb{R}} K_n^\phi(v,v_*)f_n(t,v)f_n(t,v_*)\, dvdv_*\right.\right.$$

$$\left.\left. - \int\limits_{v\in\mathbb{R}}\int\limits_{v_*\in\mathbb{R}} K^\phi(v,v_*)\, f(t,v)\, f(t,v_*)\, dvdv_*\right\}q(t)\, dt\right|$$

$$= \left|\int\limits_{t=0}^{+\infty}\left\{\int\limits_{v\in\mathbb{R}} L_n^\phi(t,v)\, f_n(t,v)\, dv - \int\limits_{v\in\mathbb{R}} L^\phi(t,v)\, f(t,v)\, dv\right\}q(t)\, dt\right|$$

$$\leqq \sup\limits_{t\in[0,T]}\left\{\int\limits_{v\in\mathbb{R}} \left|L_n^\phi(t,v) - L^\phi(t,v)\right| f_n(t,v)\, dv\right\} \|q\|_{L^1([0,+\infty[_t)}$$

$$+ \left|\int\limits_{t=0}^{+\infty}\left\{\int\limits_{v\in\mathbb{R}} L^\phi(t,v)\, (f_n(t,v) - f(t,v))\, dv\right\}q(t)\, dt\right|\ . \tag{2.39}$$

But according to estimate (2.38),

$$\lim_{|v_*| \to +\infty} \sup_{t \in [0,+\infty[} \sup_{n \in \mathbb{N}^*} \frac{|L_n^{\phi}(t, v_*)|}{|v_*|^2} = 0 . \tag{2.40}$$

Therefore, estimate (2.18) ensures that the second term of (2.39) tends to 0.

We finally use Egorov's theorem, estimate (2.38), the equiintegrability of the sequence f_n (obtained by estimate (2.16) and the convergence a.e. of L_n^{ϕ} to L^{ϕ}, in order to obtain the convergence of the first term of (2.39) to 0.

As announced before, we can now pass to the limit in Eq. (2.20) and obtain the first part of Theorem 2.1.

In order to prove the second part of Theorem 2.1, we observe that if assumption (2.6) holds, then Theorem A.2 (cf. Appendix A) ensures the existence of $C_{A.3} > 0$ such that

$$\int_{v \in \mathbb{R}} f_n(t, v) \left(1 + |v|^{2p} \right) dv \leq C_{A.3} \tag{2.41}$$

(note that $C_{A.3}$ does not depend on n).

But estimates (2.18), (2.20) and (2.21) imply for a.e. $t \geq 0$ the convergence of $\int_{v \in \mathbb{R}} f_n(t, v) \chi(v) dv$ to $\int_{v \in \mathbb{R}} f(t, v) \chi(v) dv$ when $\chi \in C_c^2(\mathbb{R}_v)$.

Therefore, for all $R > 0, t > 0$,

$$\int_{|v| \leq R} f(t, v) \left(1 + |v|^{2p} \right) dv \leq C_{A.3}. \tag{2.42}$$

Then, estimate (2.7) holds because of Fatou's lemma.

Finally, we prove the conservation of mass (2.5). We observe that for some function $\chi_R \in C^2(\mathbb{R}_v)$ such that Supp $(\chi_R) \subset [-R-1, R+1]$,

$$\left| \int_{v \in \mathbb{R}} f(t, v) dv - \int_{v \in \mathbb{R}} f_0(v) dv \right| \leq \frac{1}{R^2} \int_{|v| \geq R} \{ f_n(t, v) + f(t, v) \} |v|^2 dv$$

$$+ \left| \int_{v \in \mathbb{R}} \chi_R(v) \{ f_n(t, v) - f(t, v) \} dv \right| \tag{2.43}$$

for any $R > 0$. But according to the properties used in the proof of estimate (2.42), estimate (2.43) ensures that the conservation of mass (2.5) holds.

In the same way, we can see that under assumption (2.6) with $p \geq 2$, the conservation of energy (2.8) holds.

3. Regularization Properties When All Polynomial Moments are Initially Bounded

This section is devoted to the proof of the following theorem:

Theorem 3.1. Let $f_0 \geq 0$ be an initial datum such that for all $p \in \mathbb{N}$, there exists $C_{3.1}(p) > 0$ satisfying

$$\int_{v \in \mathbb{R}} f_0(v) \left(1 + |v|^p + |log f_0(v)| \right) dv \leq C_{3.1}(p) , \tag{3.1}$$

and let $\beta \geq 0$ be a cross section satisfying estimate (2.2).

Then, if $f(t,v)$ is a nonnegative solution of Eq. (1.18), (1.19), (2.2) in the sense of Eq. (2.3) with initial datum f_0, we have for all $\bar{t} > 0$ and all $q \in \mathbb{N}$:

$$f(t,v) \in L^\infty([\bar{t}, +\infty[; C^q(\mathbb{R}_v)), \qquad (3.2)$$

or in abridged form,

$$f(t,v) \in L^\infty(]0, +\infty[; C^\infty(\mathbb{R}_v)). \qquad (3.3)$$

Proof of Theorem 3.1. According to Theorem 2.1, we know that for all $p \in \mathbb{N}$, there exists $C_{3.2}(p) > 0$ satisfying

$$\forall t \in [0, +\infty[, \quad \int_{v\in\mathbb{R}} f(t,v)\left(1 + |v|^p\right) dv \leqq C_{3.2}(p). \qquad (3.4)$$

Therefore, the Fourier transform

$$\hat{f}(t,\xi) = \int_{v\in\mathbb{R}} e^{-iv\xi} f(t,v)dv \qquad (3.5)$$

of f is such that for all $p \in \mathbb{N}$,

$$\left| \frac{\partial^p \hat{f}}{\partial \xi^p}(t,\xi) \right| \leqq C_{3.2}(p). \qquad (3.6)$$

But $v \to e^{-iv\xi}$ lies in $W^{2,\infty}(\mathbb{R}_v)$, and therefore it is possible to use Eq. (2.3).

Then, a simple calculation leads to the following equation for the Fourier transform of f:

$$\frac{\partial \hat{f}}{\partial t}(t,\xi) = \int_{\theta=-\pi}^{\pi} \left\{ \hat{f}(t,\xi\cos\theta)\hat{f}(t,\xi\sin\theta) - \hat{f}(t,0)\hat{f}(t,\xi) \right\} \beta(|\theta|)d\theta. \qquad (3.7)$$

Note that this equation is used in [G], and that it also appeared in [De 1], though for the Laplace transform of f. We rewrite it under the form

$$\frac{\partial \hat{f}}{\partial t}(t,\xi) = \frac{1}{2}\int_{\theta=-\pi}^{\pi} \left\{ \hat{f}(t,\xi\sin\theta) + \hat{f}(t,-\xi\sin\theta) - 2\hat{f}(t,0) \right\} \beta(|\theta|)d\theta\,\hat{f}(t,\xi)$$

$$+ \int_{\theta=-\pi}^{\pi} \left\{ \hat{f}(t,\xi\cos\theta) - \hat{f}(t,\xi) \right\} \hat{f}(t,\xi\sin\theta)\beta(|\theta|)d\theta. \qquad (3.8)$$

We now use the notations

$$a(t,\xi) = -\frac{1}{2}\int_{\theta=-\pi}^{\pi} \left\{ \hat{f}(t,\xi\sin\theta) + \hat{f}(t,-\xi\sin\theta) - 2\hat{f}(t,0) \right\} \beta(|\theta|)d\theta, \qquad (3.9)$$

and

$$b(t,\xi) = \int_{\theta=-\pi}^{\pi} \left\{ \hat{f}(t,\xi\cos\theta) - \hat{f}(t,\xi) \right\} \hat{f}(t,\xi\sin\theta)\beta(|\theta|)d\theta. \qquad (3.10)$$

Therefore,

$$\frac{\partial \hat{f}}{\partial t}(t,\xi) = -a(t,\xi)\hat{f}(t,\xi) + b(t,\xi), \qquad (3.11)$$

and

$$\hat{f}(t,\xi) = \hat{f}(0,\xi)e^{-\int_0^t a(\tau,\xi)d\tau} + \int_0^t b(s,\xi)e^{-\int_s^t a(\tau,\xi)d\tau} ds . \tag{3.12}$$

But

$$\hat{f}(t,\xi \sin \theta) + \hat{f}(t,-\xi \sin \theta) - 2\hat{f}(t,0) \leqq 0 , \tag{3.13}$$

because $f \geqq 0$. Therefore, $a(t,\xi)$ is real and

$$a(t,\xi) \geqq \frac{1}{2} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \{2\hat{f}(t,0) - \hat{f}(t,\xi \sin \theta) - \hat{f}(t,-\xi \sin \theta)\} \beta(|\theta|)d\theta . \tag{3.14}$$

Then, we make the change of variables

$$u = |\xi| \sin \theta . \tag{3.15}$$

We get

$$a(t,\xi) \geqq \frac{1}{2} \int_{u=-|\xi|}^{|\xi|} \{2\hat{f}(t,0) - \hat{f}(t,u) - \hat{f}(t,-u)\} \beta \left(\arcsin \left|\frac{u}{\xi}\right| \right) \frac{du}{|\xi|\sqrt{1 - |\frac{u}{\xi}|^2}} . \tag{3.16}$$

But for any $x \in [0,\pi]$,

$$\beta(x) \geqq \beta_0 |x|^{-\alpha} , \tag{3.17}$$

and therefore

$$a(t,\xi) \geqq \frac{\beta_0}{4} \int_{u=-|\xi|}^{|\xi|} \{2\hat{f}(t,0) - \hat{f}(t,u) - \hat{f}(t,-u)\} \left|\frac{u}{\xi}\right|^{-\alpha} \frac{du}{|\xi|}$$

$$\geqq \frac{\beta_0}{4}|\xi|^{\alpha-1} \int_{u=-|\xi|}^{|\xi|} \{2\hat{f}(t,0) - \hat{f}(t,u) - \hat{f}(t,-u)\} |u|^{-\alpha}du . \tag{3.18}$$

And since

$$2\hat{f}(t,0) - \hat{f}(t,u) - \hat{f}(t,-u) = -|u|^2 \int_{r=-1}^{1} (1 - |r|)\frac{\partial^2 \hat{f}}{\partial \xi^2}(t,ru)dr$$

$$= -|u|^2 \int_{r=-1}^{1} (1 - |r|)\mathscr{R}e \left(\frac{\partial^2 \hat{f}}{\partial \xi^2}(t,ru) \right) dr , \tag{3.19}$$

we get

$$a(t,\xi) \geqq \frac{\beta_0}{4}|\xi|^{\alpha-1} \int_{u=-|\xi|}^{|\xi|} |u|^{2-\alpha} \int_{r=-1}^{1} (1 - |r|)\mathscr{R}e \left(-\frac{\partial^2 \hat{f}}{\partial \xi^2}(t,ru) \right) drdu . \tag{3.20}$$

But

$$-\frac{\partial^2 \hat{f}}{\partial \xi^2}(t,0) = \int_{v\in\mathbb{R}} f(t,v)|v|^2dv , \tag{3.21}$$

and estimate (2.8) ensures that

$$-\frac{\partial^2 \hat{f}}{\partial \xi^2}(t,0) = \int\limits_{v\in\mathbb{R}} f_0(v)|v|^2 dv ,\tag{3.22}$$

Moreover, if we denote

$$E = \int\limits_{v\in\mathbb{R}} f_0(v)|v|^2 dv ,\tag{3.23}$$

we get (thanks to estimate (3.6)), that for any η such that

$$|\eta| \leq \frac{E}{C_{3.2}(3)} ,\tag{3.24}$$

the estimate

$$\mathscr{R}e\left(-\frac{\partial^2 \hat{f}}{\partial \xi^2}(t,\eta)\right) \geq \frac{E}{2}\tag{3.25}$$

holds. But estimates (3.20) and (3.24) ensure that

$$a(t,\xi) \geq \frac{\beta_0}{16}E|\xi|^{\alpha-1}\int\limits_{u=-\inf(|\xi|,\frac{E}{C_{3.2}(3)})}^{\inf(|\xi|,\frac{E}{C_{3.2}(3)})} |u|^{2-\alpha}du$$

$$\geq \frac{\beta_0}{16(3-\alpha)}E|\xi|^{\alpha-1}2^{3-\alpha}\left(\inf\left(|\xi|,\frac{E}{C_{3.2}(3)}\right)\right)^{3-\alpha} .\tag{3.26}$$

Therefore, there exists $C_{3.3} > 0$, $C_{3.4} > 0$, such that when $|\xi| \geq C_{3.3}, t \geq 0$,

$$a(t,\xi) \geq C_{3.4}|\xi|^{\alpha-1} .\tag{3.27}$$

We will now use Eq. (3.12) and estimate (3.27) to prove Theorem 3.1 by induction.

Lemma 3. *We make the assumptions of Theorem 3.1. We suppose moreover that there exists $\delta \geq 0$ such that for all $t_1 > 0, \varepsilon_1 > 0$, we can find $C_{3.5}(\varepsilon_1,t_1) > 0$ satisfying*

$$\forall \xi \in \mathbb{R}, \quad \sup_{s\geq t_1} |\hat{f}(s,\xi)| \leq \frac{C_{3.5}(\varepsilon_1,t_1)}{1+|\xi|^{\delta-\varepsilon_1}} .\tag{3.28}$$

Then, for all $t_1 > 0$, $\varepsilon_1 > 0$, we can find $C_{3.6}(\varepsilon_1,t_1) > 0$ satisfying

$$\forall \xi \in \mathbb{R}, \quad \sup_{s\geq t_1} |\hat{f}(s,\xi)| \leq \frac{C_{3.6}(\varepsilon_1,t_1)}{1+|\xi|^{\delta+\frac{\alpha-1}{2}-\varepsilon_1}} .\tag{3.29}$$

Proof of Lemma 3. We fix $t_1 > 0, \varepsilon_1 > 0$. According to Eq. (3.12), for any $t \geq t_1$,

$$\hat{f}(t,\xi) = \hat{f}(0,\xi)e^{-\int\limits_0^t a(\tau,\xi)d\tau}$$

$$+ \int\limits_0^{\frac{t_1}{2}} b(s,\xi)e^{-\int\limits_s^t a(\tau,\xi)d\tau}\,ds + \int\limits_{\frac{t_1}{2}}^t b(s,\xi)\,e^{-\int\limits_s^t a(\tau,\xi)d\tau}\,ds .\tag{3.30}$$

Therefore, estimate (3.27) ensures that for any $t \geq t_1, |\xi| \geq C_{3.3}$,

$$|\hat{f}(t,\xi)| \leq |\hat{f}(0,\xi)|e^{-C_{3.4}t|\xi|^{\alpha-1}} + \sup_{s \in [0,\frac{t_1}{2}]} \frac{t_1}{2}|b(s,\xi)|e^{-C_{3.4}\frac{t_1}{2}|\xi|^{\alpha-1}}$$

$$+ \sup_{s \geq \frac{t_1}{2}} |b(s,\xi)| \int_{\frac{t_1}{2}}^{t} e^{-(t-s)C_{3.4}|\xi|^{\alpha-1}} ds . \tag{3.31}$$

But for all $s \in [0, +\infty[, \xi \in \mathbb{R}$,

$$|b(s,\xi)| = \left| \int_{\theta=-\pi}^{\pi} \{\hat{f}(s, \xi \cos \theta) - \hat{f}(s,\xi)\}\hat{f}(s, \xi \sin \theta) \beta(|\theta|) d\theta \right|$$

$$= \left| \int_{\theta=-\pi}^{\pi} \xi(\cos \theta - 1) \int_{u=0}^{1} \frac{\partial \hat{f}}{\partial \xi}(s, \xi + u\xi(\cos \theta - 1)) du \, \hat{f}(s, \xi \sin \theta) \beta(|\theta|) d\theta \right|$$

$$\leq C_{2.3}(2) C_{3.2}(0) C_{3.2}(1) |\xi| . \tag{3.32}$$

Therefore, estimate (3.31) implies that for any $t \geq t_1, |\xi| \geq C_{3.3}$,

$$|\hat{f}(t,\xi)| \leq C_{3.2}(0)e^{C_{3.4}t|\xi|^{\alpha-1}} + C_{2.3}(2)C_{3.2}(0)C_{3.2}(1)\frac{t_1}{2}|\xi|e^{-C_{3.4}\frac{t_1}{2}|\xi|^{\alpha-1}}$$

$$+ \sup_{s \geq \frac{t_1}{2}} \frac{|b(s,\xi)|}{C_{3.4}|\xi|^{\alpha-1}} . \tag{3.33}$$

According to assumption (3.28), we have for all $\varepsilon > 0$,

$$\forall \xi \in \mathbb{R}, \quad \sup_{s \geq \frac{t_1}{2}} |\hat{f}(s,\xi)| \leq \frac{C_{3.5}(\varepsilon, t_{\frac{1}{2}})}{1 + |\xi|^{\delta-\varepsilon}} . \tag{3.34}$$

Therefore, using Corollary B.3 of Appendix B and assumption (3.1), there exists for all $\varepsilon > 0$ a strictly positive constant $C_{3.7}(\varepsilon, t_1)$ such that

$$\forall \xi \in \mathbb{R}, \quad \sup_{s \geq \frac{t_1}{2}} |\hat{f}(s,\xi)| \leq \frac{C_{3.7}(\varepsilon, t_1)}{1 + |\xi|^{\delta-\varepsilon}} . \tag{3.35}$$

We now compute (for all $|\xi| \geq C_{3.3}, \varepsilon > 0$),

$$\sup_{s \geq \frac{t_1}{2}} |b(s,\xi)| = \sup_{s \geq \frac{t_1}{2}} \left| \int_{\theta=-\pi}^{\pi} \{\hat{f}(s, \xi \cos \theta) - \hat{f}(s,\xi)\}\hat{f}(s, \xi \sin \theta) \beta(|\theta|) d\theta \right|$$

$$\leq \sup_{s \geq \frac{t_1}{2}} \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} |\hat{f}(s, \xi \cos \theta) - \hat{f}(s,\xi)|^{1-(\frac{\alpha-1}{2}+\varepsilon)} |\xi|^{\frac{\alpha-1}{2}+\varepsilon} |\cos \theta - 1|^{\frac{\alpha-1}{2}+\varepsilon}$$

$$\times \left| \int_{u=0}^{1} \frac{\partial \hat{f}}{\partial \xi}(s, \xi + u\xi(\cos \theta - 1)) du \right|^{\frac{\alpha-1}{2}+\varepsilon} |\hat{f}(s, \xi \sin \theta)| \beta(|\theta|) d\theta$$

$$+ \sup_{s \geq \frac{t_1}{2}} \int_{|\theta \pm \frac{\pi}{2}| \leq \frac{\pi}{4}} |\hat{f}(s, \xi \cos \theta) - \hat{f}(s, \xi)| \, |\hat{f}(s, \xi \sin \theta)| \beta(|\theta|) \, d\theta$$

$$+ \sup_{s \geq \frac{t_1}{2}} \int_{\frac{3\pi}{4} \leq |\theta| \leq \pi} |\hat{f}(s, \xi \cos \theta) - \hat{f}(s, \xi)| \, |\hat{f}(s, \xi \sin \theta)| \beta(|\theta|) \, d\theta \,. \quad (3.36)$$

We now use estimates (3.34), (3.35) and resume the computation (for all $|\xi| \geq C_{3.3}, \varepsilon > 0$),

$$\sup_{s \geq \frac{t_1}{2}} |b(s, \xi)| \leq 2C_{2.3}(\alpha - 1 + 2\varepsilon)|\xi|^{\frac{\alpha-1}{2}+\varepsilon} C_{3.2}(0)$$

$$\times \left(\frac{C_{3.5}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}} \right)^{1 - \left(\frac{\alpha-1}{2}+\varepsilon \right)} \left(\frac{C_{3.7}(\varepsilon, t_1)}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}} \right)^{\frac{\alpha-1}{2}+\varepsilon}$$

$$+ 2\beta_1 \left(\frac{\pi}{4} \right)^{-\alpha} C_{3.2}(0) \frac{C_{3.5}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}} + 2\beta_1 \left(\frac{3\pi}{4} \right)^{-\alpha} C_{3.2}(0) \frac{C_{3.5}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}}$$

$$\leq 2C_{2.3}(2)|\xi|^{\frac{\alpha-1}{2}-\delta+2\varepsilon} C_{3.2}(0) 2^{\frac{\delta-\varepsilon}{2}} \left(C_{3.5} \left(\varepsilon, \frac{t_1}{2} \right) \right)^{1 - \frac{\alpha-1}{2}+\varepsilon} (C_{3.7}(\varepsilon, t_1))^{\frac{\alpha-1}{2}+\varepsilon}$$

$$+ 4\beta_1 \left(\frac{\pi}{4} \right)^{-\alpha} C_{3.2}(0)|\xi|^{-\delta+\varepsilon} 2^{\frac{\delta-\varepsilon}{2}} C_{3.5} \left(\varepsilon, \frac{t_1}{2} \right)$$

$$\leq C_{3.8}(\varepsilon, t_1)|\xi|^{\frac{\alpha-1}{2}-\delta+2\varepsilon} \quad (3.37)$$

for some strictly positive constant $C_{3.8}(\varepsilon, t_1)$.

We now use estimate (3.37) to precise estimate (3.33). We get for all $t \geq t_1, |\xi| \geq C_{3.3}, \varepsilon > 0$,

$$|\hat{f}(t, \xi)| \leq C_{3.2}(0)e^{-C_{3.4}t_1|\xi|^{\alpha-1}} + C_{2.3}(2) C_{3.2}(0) C_{3.2}(1) \frac{t_1}{2}|\xi|e^{-C_{3.4}\frac{t_1}{2}|\xi|^{\alpha-1}}$$

$$+ \frac{C_{3.8}(\varepsilon, t_1)}{C_{3.4}}|\xi|^{-\frac{\alpha-1}{2}\delta+2\varepsilon} \,. \quad (3.38)$$

Taking $\varepsilon = \frac{\varepsilon_1}{2}$, we get some strictly positive constant $C_{3.9}(\varepsilon_1, t_1)$ such that when $t \geq t_1, |\xi|$ is large enough,

$$|\hat{f}(t, \xi)| \leq C_{3.9}(\varepsilon_1, t_1)|\xi|^{-\frac{\alpha-1}{2}-\delta+\varepsilon_1} \,. \quad (3.39)$$

Finally, using estimate (3.6) for $p = 0$, we obtain estimate (3.29).

We now come back to the proof of Theorem 3.1. We already know (because of estimate (3.6) when $p = 0$) that assumption (3.28) holds when $\delta = 0$. Lemma 3 clearly implies by induction that for any $\bar{t} > 0, q \geq 0$, there exists a strictly positive constant $C_{3.10}(\bar{t}, q)$ such that

$$\forall \xi \in \mathbb{R}, \quad \sup_{t \geq \bar{t}} |\hat{f}(t, \xi)| \leq \frac{C_{3.10}(\bar{t}, q)}{1 + |\xi|^q} \,. \quad (3.40)$$

Using now the Sobolev inequalities (or more simply the fact that $H^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$), we get Theorem 3.1.

4. Regularization Properties when Some Polynomial Moments are Initially Bounded

We extend in this section the results of Sect. 3 when assumption (3.1) does not hold any more.

Theorem 4.1. *Let $f_0 \geq 0$ be an initial datum such that*

$$\exists r \in \mathbb{N}, r \geq 2, C_{4.1} > 0, \int_{v \in \mathbb{R}} f_0(v) \left(1 + |v|^{2r} + |\log f_0(v)|\right) dv \leq C_{4.1} , \qquad (4.1)$$

and let $\beta \geq 0$ satisfy (2.2). Then, if $f(t,v)$ is a nonnegative solution of Eq. (1.18), (1.19), (2.2) in the sense of Eq. (2.3) with initial datum f_0, we have for all $\bar{t} > 0$ and all $\varepsilon > 0$:

$$f(t,v) \in L^{\infty}([\bar{t}, +\infty[_t; H^{2r-\frac{1}{2}-\varepsilon}(\mathbb{R}_v)) . \qquad (4.2)$$

Corollary 4.2. *In particular, under the assumptions of Theorem 4.1, we have for all $\bar{t} > 0$ and all $\varepsilon > 0$:*

$$f(t,v) \in L^{\infty}([\bar{t}, +\infty[_t; C^{2r-2,1-\varepsilon}(\mathbb{R}_v)) . \qquad (4.3)$$

Proof of Theorem 4.1. Corollary 4.2 is a straightforward consequence of Theorem 4.1 and of classical Sobolev inequalities (cf. [B]).

We now prove Theorem 4.1. We use the same strategy as in Theorem 3.1. Estimates (3.4) and (3.6) still hold, but only for $p \leq 2r$. Moreover, Eq. (3.9), (3.10), (3.12) also hold, and lead to estimate (3.27) as in Theorem 3.1.

However, Lemma 3 is changed in the following way:

Lemma 4. *We make the assumptions of Theorem 4.1. We suppose moreover that there exists $\delta \geq 0$ such that for all $t_1 > 0, \varepsilon_1 > 0$, we can find $C_{4.2}(\varepsilon_1, t_1)$ satisfying*

$$\forall \xi \in \mathbb{R}, \quad \sup_{s \geq t_1} |\hat{f}(s, \xi)| \leq \frac{C_{4.2}(\varepsilon_1, t_1)}{1 + |\xi|^{\delta - \varepsilon_1}} . \qquad (4.4)$$

Then, for all $t_1 > 0, \varepsilon_1 > 0$, we can find $C_{4.3}(\varepsilon_1, t_1)$ satisfying

$$\forall \xi \in \mathbb{R}, \quad \sup_{s \geq t_1} |\hat{f}(s, \xi)| \leq \frac{C_{4.3}(\varepsilon_1, t_1)}{1 + |\xi|^{\delta\{1 - \frac{1}{2r}\frac{\alpha-1}{2}\} + \frac{\alpha-1}{2} - \varepsilon_1}} . \qquad (4.5)$$

Proof of Lemma 4. We fix $t_1 > 0, \varepsilon_1 > 0$. It is clear that estimate (3.33) still holds. However, using Theorem B.2 of Appendix B, we only get for all $\varepsilon > 0$ a strictly positive constant $C_{4.4}(\varepsilon, t_1)$ such that

$$\forall \xi \in \mathbb{R}, \quad \sup_{s \geq \frac{t_1}{2}} \left|\frac{\partial \hat{f}}{\partial \xi}(s, \xi)\right| \leq \frac{C_{4.4}(\varepsilon, t_1)}{1 + |\xi|^{\delta\{1 - \frac{1}{2r}\frac{\alpha-1}{2}\} + \frac{\alpha-1}{2} - \varepsilon_1}} . \qquad (4.6)$$

Then, we note that estimate (3.36) still holds, but estimate (3.37) becomes (for all $|\xi| \geq C_{3.3}, \varepsilon > 0$),

$$\sup_{s \geq \frac{t_1}{2}} |b(s, \xi)| \leq 2C_{2.3}(\alpha - 1 + 2\varepsilon)|\xi|^{\frac{\alpha-1}{2}+\varepsilon}C_{3.2}(0)$$

$$\times \left(\frac{C_{4.2}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}} \right)^{1-(\frac{\alpha-1}{2}+\varepsilon)} \left(\frac{C_{4.4}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta(1-\frac{1}{2r})-\varepsilon}} \right)^{\frac{\alpha-1}{2}+\varepsilon}$$

$$+ 2\beta_1 \left(\frac{\pi}{4} \right)^{-\alpha} C_{3.2}(0) \frac{C_{4.2}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}} + 2\beta_1 \left(\frac{3\pi}{4} \right)^{-\alpha} C_{3.2}(0) \frac{C_{4.2}(\varepsilon, \frac{t_1}{2})}{1 + |\frac{\sqrt{2}}{2}\xi|^{\delta-\varepsilon}}$$

$$\leq C_{4.5}(\varepsilon, t_1)|\xi|^{\frac{\alpha_1}{2}+\delta\{-1+\frac{\alpha-1}{2}\frac{1}{2r}\}+(2+\frac{\delta}{2r})\varepsilon} \tag{4.7}$$

for some strictly positive constant $C_{4.5}(\varepsilon, t_1)$.

Then, estimate (3.28) becomes for all $|\xi| \geq C_{3.3}, \varepsilon > 0$,

$$|\hat{f}(t, \xi)| \leq C_{3.2}(0)e^{-C_{3.4}t_1|\xi|^{\alpha-1}} + C_{2.3}(2)C_{3.2}(0)C_{3.2}(1)\frac{t_1}{2}|\xi|e^{-C_{3.4}\frac{t_1}{2}|\xi|^{\alpha-1}}$$

$$+ \frac{C_{4.5}(\varepsilon, t_1)}{C_{3.4}}|\xi|^{-\frac{\alpha-1}{2}-\delta\{1-\frac{\alpha-1}{2}\frac{1}{2r}\}+(2+\frac{\delta}{2r})\varepsilon} . \tag{4.8}$$

Taking $\varepsilon = \varepsilon_1 \left(2 + \frac{\delta}{2r} \right)^{-1}$, we get some strictly positive constant $C_{4.6}(\varepsilon, t_1)$ such that when $t \geq t_1, |\xi|$ is large enough,

$$|\hat{f}(t, \xi)| \leq C_{4.6}(\varepsilon, t_1)|\xi|^{-\frac{\alpha-1}{2}-\{1-\frac{\alpha-1}{2}\frac{1}{2r}\}\delta+\varepsilon_1} . \tag{4.9}$$

Then, Lemma 4 is obtained exactly as Lemma 3.

We now come back to the proof of Theorem 4.1. We already know that assumption (4.4) of Lemma 4 holds when $\delta = 0$. Moreover, using Lemma 4 by induction, we can see that for all $t_1 > 0, \varepsilon_1 > 0, n \in \mathbb{N}$, there exists a strictly positive constant $C_{4.7}(\varepsilon_1, t_1, n)$ such that

$$\forall \xi \in \mathbb{R}, \quad \sup_{t \geq t_1} |\hat{f}(t, \xi)| \leq \frac{C_{4.7}(\varepsilon_1, t_1, n)}{1 + |\xi|^{\delta_n - \varepsilon_1}} , \tag{4.10}$$

where $(\delta_n)_{n \in \mathbb{N}}$ is the sequence defined by

$$\delta_0 = 0 , \tag{4.11}$$

$$\delta_{n+1} = \delta_n \left\{ 1 - \frac{\alpha - 1}{2} \frac{1}{2r} \right\} + \frac{\alpha - 1}{2} . \tag{4.12}$$

But this sequence is strictly increasing and converges to $2r$. Therefore for all $\varepsilon > 0, \bar{t} > 0$, there exists $C_{4.8}(\varepsilon, \bar{t})$ such that

$$\forall \xi \in \mathbb{R}, \quad \sup_{s \geq \bar{t}} |\hat{f}(s, \xi)| \leq \frac{C_{4.8}(\varepsilon, \bar{t})}{1 + |\xi|^{2r - \frac{\varepsilon}{2}}} . \tag{4.13}$$

Finally, estimate (4.13) ensures that

$$f \in L^\infty([\bar{t}, +\infty[_t; H^{2r-\frac{1}{2}-\varepsilon}(\mathbb{R}_v)) , \tag{4.14}$$

and Theorem 4.1 is proved.

Appendix A: Standard Properties of the Classical Kac Equation

We prove in this appendix some classical facts about the spatially homogeneous Kac equation, and present some others that can easily be deduced from the theory of the Boltzmann equation (cf. [A 1]).

Theorem A.1. *Let $f_0 \geqq 0$ be an initial datum such that*

$$\int_{v \in \mathbb{R}} f_0(v) \left(1 + |v|^2\right) dv < +\infty . \tag{A.1}$$

Then, there exists a unique nonnegative solution $f(t,v)$ of Eq. (1.18), (1.19) in $L^\infty([0, +\infty[_t, L^1(\mathbb{R}_v, (1 + |v|^2)dv))$ with initial datum f_0 as soon as the cross section β in (1.19) belongs to $L^\infty([0, \pi])$.

This solution satisfies the conservation of mass and energy for all $t \geqq 0$:

$$\int_{v \in \mathbb{R}} f(t,v) \, dv = \int_{v \in \mathbb{R}} f_0(v) \, dv , \tag{A.2}$$

$$\int_{v \in \mathbb{R}} f(t,v)|v|^2 dv = \int_{v \in \mathbb{R}} f_0(v)|v|^2 dv . \tag{A.3}$$

Proof of Theorem A.1. We introduce the sequence $(f_n(t,v))_{n \in \mathbb{N}}$, defined by

$$f_0(t,v) = f_0(v) , \tag{A.4}$$

$$f_{n+1}(t,v) = f_0(v) + \int_{s=0}^{t} \int_{v_* \in \mathbb{R}} \int_{\theta=-\pi}^{\pi} \{f_n(s,v'')f_n(s,v''_*) $$
$$- f_{n+1}(s,v)f_{n+1}(s,v_*)\}\beta(|\theta|) \, d\theta dv_* ds , \tag{A.5}$$

and present a proof of existence in the Cauchy–Lipschitz style.

Note that it is easy to obtain (by induction) the conservation of mass and energy for f_n:

$$\int_{v \in \mathbb{R}} f_n(t,v) \, dv = \int_{v \in \mathbb{R}} f_0(v) \, dv , \tag{A.6}$$

$$\int_{v \in \mathbb{R}} f_n(t,v)|v|^2 dv = \int_{v \in \mathbb{R}} f_0(v)|v|^2 dv . \tag{A.7}$$

Therefore, it is possible to write explicitly f_{n+1} as a function of f_n, and the sequence (A.4), (A.5) is well defined. It is also clear that $f_n \geqq 0$.

Then, we define for all $n \in \mathbb{N}^*$,

$$u_n(t) = \int_{v \in \mathbb{R}} |f_n(t,v) - f_{n-1}(t,v)|(1 + |v|^2) \, dv . \tag{A.8}$$

We get

$$u_{n+1}(t) \leqq \int_{s=0}^{t} \int_{v \in \mathbb{R}} \int_{v_* \in \mathbb{R}} f_n(s,v_*)|f_n(s,v) - f_{n-1}(s,v)|$$

$$\times \left\{ \int_{\theta=-\pi}^{\pi} \left(1 + |v|^2 \cos^2 \theta + |v_*|^2 \sin^2 \theta\right) \beta(|\theta|)d\theta \right\} dvdv_* ds$$

$$+ \int_{s=0}^{t} \int_{v \in \mathbb{R}} \int_{v_* \in \mathbb{R}} f_{n-1}(s,v)|f_n(s,v_*) - f_{n-1}(s,v_*)|$$

$$\times \left\{ \int\limits_{\theta=-\pi}^{\pi} \left(1 + |v|^2 \cos^2 \theta^2 + |v_*|^2 \sin^2 \theta \right) \beta(|\theta|) d\theta \right\} dv dv_* ds$$

$$+ \int\limits_{s=0}^{t} \int\limits_{v\in\mathbb{R}} \int\limits_{v_*\in\mathbb{R}} f_{n+1}(s,v_*) |f_{n+1}(s,v) - f_n(s,v)|$$

$$\times (1 + |v|^2) \left\{ \int\limits_{\theta=-\pi}^{\pi} \beta(|\theta|) d\theta \right\} dv dv_* ds$$

$$+ \int\limits_{s=0}^{t} \int\limits_{v\in\mathbb{R}} \int\limits_{v_*\in\mathbb{R}} f_n(s,v) |f_{n+1}(s,v_*) - f_n(s,v_*)|$$

$$\times (1 + |v|^2) \int\limits_{\theta=-\pi}^{\pi} \{\beta(|\theta|) d\theta\} dv dv_* ds$$

$$\leqq C_{A.1} \int\limits_{s=0}^{t} \{u_n(s) + u_{n+1}(s)\} ds , \tag{A.9}$$

for some strictly positive constant $C_{A.1}$. Moreover, we can prove in the same way that for all $t \geqq 0$,

$$u_1(t) \leqq C_{A.2} t, \tag{A.10}$$

where $C_{A.2} > 0$.

But estimate (A.9) ensures that (when $t \in [0, T], n \geqq 1$),

$$u_{n+1}(t) \leqq \left(C_{A.1} + C_{A.1}^2 T e^{C_{A.1} T} \right) \int\limits_{s=0}^{t} u_n(s) ds . \tag{A.11}$$

Therefore, for all $T > 0, s \in [0, T], n \geqq 1$,

$$u_n(s) \leqq \left(C_{A.1} + C_{A.1}^2 T e^{C_{A.1} T} \right)^{n-1} \frac{s^n}{n!} C_{A.2} . \tag{A.12}$$

This estimate ensures that the sequence $(f_n)_{n\in\mathbb{N}}$ satisfies the Cauchy property in $L_{\text{loc}}^\infty([0, +\infty[, L^1(\mathbb{R}_v, (1 + |v|^2) dv))$. Its limit f clearly satisfies Eq. (1.18), (1.19). Moreover, $f \geqq 0$ and the conservation of mass and energy $f \in L^\infty([0, +\infty[, L^1(\mathbb{R}_v, (1 + |v|^2) dv))$.

The uniqueness of such a solution is then directly obtained by a Cauchy–Lipschitz type argument.

We now consider the polynomial moments of the solution of the spatially homogeneous Kac equation.

Theorem A.2. *Let $f_0 \geqq 0$ be an initial datum such that*

$$\int\limits_{v\in\mathbb{R}} f_0(v) \left(1 + |v|^2 + |\log f_0(v)| \right) dv < +\infty . \tag{A.13}$$

Then, for all $t \geqq s \geqq 0$, the unique nonnegative solution $f(t, v)$ of Eq. (1.18), (1.19) with initial datum f_0 (when the cross section β in (1.19) belongs to $L^\infty([0, \pi])$) satisfies

$$\int\limits_{v\in\mathbb{R}} f(t,v) \log f(t,v) dv \leqq \int\limits_{v\in\mathbb{R}} f(s,v) \log f(s,v) dv$$

$$\leqq \int\limits_{v\in\mathbb{R}} f_0(v) \log f_0(v) dv < +\infty . \tag{A.14}$$

Moreover, if

$$\exists r \in \mathbb{N}, \quad \int\limits_{v \in \mathbb{R}} f_0(v) \left(1 + |v|^{2r}\right) dv < +\infty \,, \tag{A.15}$$

there exists $C_{A.3} > 0$ *(independent of* β*) such that for all* $t \geq 0$:

$$\int\limits_{v \in \mathbb{R}} f(t,v) \left(1 + |v|^{2r}\right) dv \leq C_{A.3} \,. \tag{A.16}$$

Proof of Theorem A.2. For estimate (A.14), we refer to [A 1], where it is proved for the Boltzmann equation (for example for Maxwellian molecules with an angular cut-off).
We now prove estimate (A.16) in the case where $r = 2$. We can write

$$\frac{\partial}{\partial t} \int\limits_{v \in \mathbb{R}} f(t,v) |v|^4 dv = \int\limits_{v \in \mathbb{R}} \int\limits_{v_* \in \mathbb{R}} \int\limits_{\theta=-\pi}^{\pi} (|v \cos \theta - v_* \sin \theta|^4 - |v|^4)$$
$$\times f(t,v) f(t,v_*) \beta(|\theta|) \, d\theta dv_* dv$$
$$= \int\limits_{\theta=-\pi}^{\pi} \{\cos^4 + \sin^4 \theta - 1\} \beta(|\theta|) \, d\theta \int\limits_{v \in \mathbb{R}} f(t,v) |v|^4 dv \int\limits_{v \in \mathbb{R}} f(t,v) \, dv$$
$$+ 6 \int\limits_{\theta=-\pi}^{\pi} \cos^2 \theta \sin^2 \theta \beta(|\theta|) \, d\theta \left(\int\limits_{v \in \mathbb{R}} f(t,v) |v|^2 dv \right)$$
$$= \left\{ \int\limits_{\theta=-\pi}^{\pi} \cos^2 \theta \sin^2 \theta \beta(|\theta|) \, d\theta \right\} \left\{ -2 \int\limits_{v \in \mathbb{R}} f(t,v) |v|^4 dv \int\limits_{v \in \mathbb{R}} f(t,v) \, dv \right.$$
$$\left. + 6 \left(\int\limits_{v \in \mathbb{R}} f(t,v) |v|^2 dv \right)^2 \right\} \,. \tag{A.17}$$

Therefore, a simple application of the maximum principle yields

$$\int\limits_{v \in \mathbb{R}} f(t,v) |v|^4 dv \leq \sup \left(\int\limits_{v \in \mathbb{R}} f_0(v) |v|^4 dv, 3 \frac{(\int_{v \in \mathbb{R}} f(t,v) |v|^2 dv)^2}{\int_{v \in \mathbb{R}} (t,v) \, dv} \right) \,. \tag{A.18}$$

Finally, when $r > 2$, the same kind of computation can be done. Note that a rigorous proof is given in the case of the Boltzmann equation with Maxwellian molecules in [Tr].

Appendix B: Interpolation Between Derivatives

We give here for the sake of completeness the proof of some classical results used in Sect. 2, 3 and 4.

Theorem B.1. *Let* f *lie in* $C^2(\mathbb{R})$ *and satisfy*
   1. *There exists* $C_{B.1} > 0, \alpha > 0$, *such that*

$$\forall x \in \mathbb{R}, \quad |f(x)| \leq \frac{C_{B.1}}{1 + |x|^\alpha} \,. \tag{B.1}$$

2. *There exists* $C_{B.2} > 0$, *such that*

$$\forall x \in \mathbb{R}, \quad |f''(x)| \leq C_{B.2} .$$ (B.2)

Then,

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq \sqrt{\frac{8\, C_{B.1} C_{B.2}}{1 + |x|^\alpha}} .$$ (B.3)

Proof of Theorem B.1. Suppose that

$$|f'(x_0)| > \sqrt{\frac{8\, C_{B.1} C_{B.2}}{1 + |x_0|^\alpha}} .$$ (B.4)

Then, because of estimate (B.2), for all $t \in [0, 1]$,

$$\left| f'\left(x_0 + t\, \text{sgn}\, (x_0) \sqrt{\frac{2\, C_{B.1}}{C_{B.2}(1 + |x_0|^\alpha)}}\right) - f'(x_0) \right| \leq \sqrt{\frac{2\, C_{B.1} C_{B.2}}{1 + |x_0|^\alpha}}.$$ (B.5)

But estimate (B.5) ensures that

$$\left| f'\left(x_0 + t\, \text{sgn}\, (x_0) \sqrt{\frac{2\, C_{B.1}}{C_{B.2}(1 + |x_0|^\alpha)}}\right) \right| \geq \sqrt{\frac{2 C_{B.1} C_{B.2}}{1 + |x_0|^\alpha}} .$$ (B.6)

Therefore,

$$I = \left| f'\left(x_0 + \text{sgn}\, (x_0) \sqrt{\frac{2\, C_{B.1}}{C_{B.2}(1 + |x_0|^\alpha)}}\right) - f'(x_0) \right|$$

$$\geq \frac{2\, C_{B.1}}{1 + |x_0|^\alpha} .$$ (B.7)

But

$$I \leq \left| f\left(x_0 + \text{sgn}\, (x_0) \sqrt{\frac{2\, C_{B.1}}{C_{B.2}(1 + |x_0|^\alpha)}}\right) \right| + |f(x_0)|$$

$$< \frac{2\, C_{B.1}}{1 + |x_0|^\alpha} .$$ (B.8)

Thus, we get a contradiction and conclude that Theorem B.1 holds.

Theorem B.2. *Let* $p \in \mathbb{N}, p \geq 2$, *and* f *lie in* $C^p(\mathbb{R})$. *If* f *satisfies the following properties*:

1. *There exists* $C_{B.3} > 0, \alpha > 0$, *such that*

$$\forall x \in \mathbb{R}, \quad |f(x)| \leq \frac{C_{B.3}}{1 + |x|^\alpha} .$$ (B.9)

2. *There exists* $C_{B.4} > 0$, *such that*

$$\forall q \in [1, p], \forall x \in \mathbb{R}, \quad |f^{(q)}(x)| \leq C_{B.4} .$$ (B.10)

Then, for all $\varepsilon > 0$, there exists $C_{B.5}(\varepsilon) > 0$, such that

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leqq \frac{C_{B.5}(\varepsilon)}{1 + |x|^{\alpha(1-\frac{1}{p})+\varepsilon}} \, . \tag{B.11}$$

Proof of Theorem B.2. We use Theorem B.1 and give a proof by induction. We know that if there exists $C_{B.6} > 0$ and a finite sequence $(u_q)_{q \in [0,p]}$ such that for all $q \in [0, p]$,

$$\forall x \in \mathbb{R}, \quad |f^{(q)}(x)| \leqq \frac{C_{B.6}}{1 + |x|^{u_q}} \, , \tag{B.12}$$

then there exists $C_{B.7} > 0$ such that for all $q \in [0, p]$,

$$\forall x \in \mathbb{R}, \quad |f^{(q)}(x)| \leqq \frac{C_{B.7}}{1 + |x|^{v_q}} \, , \tag{B.13}$$

where

$$v_0 = \alpha, \quad \forall i \in [1, p-1], v_i = \frac{1}{2}(u_{i-1} + u_{i+1}), \quad v_p = 0 \, . \tag{B.14}$$

Therefore, we define by induction the sequence

$$r_0(0) = \alpha, \quad \forall i \in [1, p], r_i(0) = 0 \, , \tag{B.15}$$

and

$$r_0(n+1) = \alpha, \quad \forall i \in [1, p-1], r_i(n+1) = \frac{1}{2}(r_{i-1}(n) + r_{i+1}(n)), \quad r_p(n+1) = 0 \, . \tag{B.16}$$

It is clear that for all $n \in \mathbb{N}$, there exists $C_{B.8}(n) > 0$, such that for all $q \in [0, p]$,

$$\forall x \in \mathbb{R}, \quad |f^{(q)}(x)| \leqq \frac{C_{B.8}(n)}{1 + |x|^{r_q(n)}} \, . \tag{B.17}$$

But for all $i \in [0, p]$, the sequence $(r_i(n))_{n \in \mathbb{N}}$ tends towards r_i, where

$$r_0 = \alpha, \quad \forall i \in [1, p-1], r_i = \frac{1}{2}(r_{i-1} + r_{i+1}), \quad r_p = 0 \, . \tag{B.18}$$

Therefore,

$$r_1 = \left(1 - \frac{1}{p}\right)\alpha, \tag{B.19}$$

which yields Theorem B.2.

Finally, using Theorem B.2 by induction, we get the

Corollary B.3. *Let f lie in $C^\infty(\mathbb{R})$ and satisfy:*
1. *There exists $C_{B.9} > 0, \alpha > 0$, such that*

$$\forall x \in \mathbb{R}, \quad |f(x)| \leqq \frac{C_{B.9}}{1 + |x|^\alpha} \, . \tag{B.20}$$

2. *For all $q \in \mathbb{N}$ there exists $C_{B.10}(q) > 0$, such that*

$$\forall x \in \mathbb{R}, \quad |f^{(q)}(x)| \leqq C_{B.10}(q) \, . \tag{B.21}$$

Then, for all $\varepsilon > 0$, there exists $C_{B.11}(\varepsilon) > 0$, such that

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leqq \frac{C_{B.5}(\varepsilon)}{1 + |x|^{\alpha - \varepsilon}}. \tag{B.22}$$

Appendix C: The Case of the Radially Symmetric 2D Boltzmann Equation with Maxwellian Molecules

We consider now the radially symmetric solutions of the 2D spatially homogeneous Boltzmann equation with Maxwellian molecules (Note that one can prove the existence of such solutions exactly as in Sect. 2, provided of course that the initial datum is radially symmetric). The corresponding Boltzmann kernel can be written

$$Q(f)(v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} \left\{ f(v')f(v'_*) - f(v)f(v_*) \right\} b(|\theta|) \sin\theta \, d\theta dv_* , \tag{C.1}$$

with

$$v' = \frac{v + v_*}{2} + R_\theta \left(\frac{v - v_*}{2} \right) , \tag{C.2}$$

$$v'_* = \frac{v + v_*}{2} - R_\theta \left(\frac{v - v_*}{2} \right) , \tag{C.3}$$

We suppose moreover that b satisfies

$$\sin\theta \, b(|\theta|) \sim K|\theta|^{-\gamma} \tag{C.4}$$

for some $K > 0$ when $\theta \to 0$ (this is the non-cut–off case) and that b is regular outside 0.

Using the fact that f is radially symmetric, one can recast the kernel Q under the form:

$$Q(f)(v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} \left\{ f\left(\frac{v}{2} + R_\theta \left(\frac{v}{2} \right) + \frac{v_*}{2} + R_{\theta - \pi} \left(\frac{v_*}{2} \right) \right) \right.$$

$$f\left(\frac{v}{2} + R_{\theta - \pi} \left(\frac{v}{2} \right) + \frac{v_*}{2} + R_\theta \left(\frac{v_*}{2} \right) \right) - f(v)f(v_*) \Big\} b(|\theta|) \sin\theta \, d\theta dv_*$$

$$= \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} \left\{ f\left(R_{\frac{\theta}{2}}(v) \cos\left(\frac{\theta}{2} \right) + R_{\frac{\theta}{2} - \frac{\pi}{2}}(v_*) \sin\left(\frac{\theta}{2} \right) \right) \right.$$

$$f\left(R_{\frac{\theta}{2} - \frac{\pi}{2}}(v) \sin\left(\frac{\theta}{2} \right) + R_{\frac{\theta}{2}}(v_*) \cos\left(\frac{\theta}{2} \right) \right) - f(v)f(v_*) \Big\} b(|\theta|) \sin\theta \, d\theta dv_*$$

$$= \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} \left\{ f\left(v \, \cos\left(\frac{\theta}{2} \right) - v_* \, \sin\left(\frac{\theta}{2} \right) \right) f\left(v \sin\left(\frac{\theta}{2} \right) + v_* \cos\left(\frac{\theta}{2} \right) \right) \right.$$

$$- f(v)f(v_*) \Big\} b(|\theta|) \sin\theta \, d\theta dv_* . \tag{C.5}$$

Thus, we can see that the equation is very similar to the Kac equation. The main difference is simply that now v is in $\mathbb{R}^2$ instead of $\mathbb{R}$. It is then possible to prove all the theorems of the previous sections with exactly the same proof.

Note however that for the 3D radially symmetric solutions of the spatially homogeneous Boltzmann equation with Maxwellian molecules, the analogy with the Kac equation is not so clear. This case shall be discussed in a future work.

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