Commun. Math. Phys. 167, 593–605 (1995)

Communications in
Mathematical
Physics

Separation of Variables in the Classical $SL(N)$ Magnetic Chain

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Received: 6 December 1993/in revised form: 11 April 1994

Abstract: We propose an elementary construction of separation of variables for the classical integrable $SL(N)$ magnetic chain

1. Introduction

In his paper [S1] E. Sklyanin constructed the separation of variables in the classical integrable $SL(3)$ magnetic chain and conjectured the existence of a similar construction in the $SL(N)$ case. This paper is devoted to the proof of the Sklyanin's conjecture. Before giving its precise formulation, we shall review some basic facts on the $SL(N)$ magnet, following [FT, S1].

The model can be described in terms of the monodromy matrix

$$T(u) = Z(u - \delta_M + L^{(M)}) \dots (u - \delta_1 + L^{(1)}) , \tag{1.1}$$

where Z is an invertible $N \times N$ constant matrix with distinct eigenvalues, δ_j $(j = 1, \dots, M)$ are some fixed numbers, and $L^{(j)}$ are traceless matrices constituted by variables $L^{(j)}_{\mu\nu}$ $(\mu, \nu = 1, \dots, N)$ with the Poisson brackets given by

$$\{L^{(j)}_{\mu_1\nu_1}, L^{(k)}_{\mu_2\nu_2}\} = (L^{(j)}_{\mu_1\nu_2}\delta_{\mu_2\nu_1} - L^{(j)}_{\mu_2\nu_1}\delta^{(j)}_{\mu_1\nu_2})\delta_{jk} . \tag{1.2}$$

In the generic case, the Poisson bracket (1.2) is non-degenerate on the $MN(N-1)$-dimensional manifold.

$$\det(u + L^{(j)}) = \prod_{k=1}^{N} (u + \lambda_k^{(j)}) = 0, \quad j = 1, \dots, M ,$$

★ Recipient of a Dov Biegun Postdoctoral Fellowship where $\lambda_k^{(j)}$ are distinct fixed numbers and $\sum_{k=1}^{N} \lambda_k^{(j)} = 0$. The Poisson brackets for entries of the monodromy matrix (1.1) can be written down in a compact form [FT],

$$\left\{ \overset{1}{T}(u), \overset{2}{T}(v) \right\} = \frac{1}{u-v} \left[\mathscr{P}, \overset{1}{T}(u)\, \overset{2}{T}(v) \right], \tag{1.3}$$

where $\overset{1}{T} = T \otimes \mathbf{1}$, $\overset{2}{T} = \mathbf{1} \otimes T$ and $\mathscr{P}$ is the permutation operator in $\mathbb{C}^N \otimes \mathbb{C}^N$. We shall denote entries of $T(u)$ by $t_j^k(u)$ (the lower index corresponds to the number of the row). Writing equality (1.3) element by element, one gets

$$\{t_{j_1}^{k_1}(u), t_{j_2}^{k_2}(v)\} = \frac{1}{u-v} (t_{j_2}^{k_1}(u) t_{j_1}^{k_2}(v) - t_{j_1}^{k_2}(u) t_{j_2}^{k_1}(v)) \,. \tag{1.4}$$

Consider

$$\det(T(u) + \lambda) = \lambda^N + \tau_1(u)\lambda^{N-1} + \cdots + \tau_N(u) \,.$$

Then $\tau_j(u)$ is a polynomial in u of degree jM. It can be shown (see e.g. [FT, RSe2, S1]) that non-leading coefficients of polynomials $\tau_j(u)$ $(j = 1, \ldots, N)$ form a commutative, with respect to the Poisson bracket (1.3), family of $MN(N-1)/2$ independent Hamiltonians, which entrains the complete integrability of the model. Now we are able to formulate Sklyanin's conjecture.

Conjecture. There exist functions $\mathscr{A}$ and $\mathscr{B}$ on $GL(N)$, such that (a) $\mathscr{A}(T)$ is an algebraic function and $\mathscr{B}(T)$ is a polynomial of degree $D = M \cdot \dfrac{N(N-1)}{2}$ of the matrix elements t_j^k, and (b) the variables x_i, P_i $\left(i = 1, \ldots, M \cdot \dfrac{N(N-1)}{2}\right)$, defined from the equations.

$$\mathscr{B}(T(x_i)) = 0, \quad P_i = \mathscr{A}(T(x_i)) \,,$$

have the Poisson brackets

$$\{x_i, x_j\} = \{P_i, P_j\} = 0, \quad \{P_j, x_k\} = P_j \delta_{jk}$$

and, besides, are bound to the Hamiltonians $\tau_v(u)$ by the relations

$$\det(P_j - T(x_j)) = 0 \,.$$

After establishing in Sect. 2 necessary notations and preliminary lemmas, we prove this conjecture in Sect. 3. The proof is quite elementary, though rather cumbersome. We give it in more or less full detail, having in mind the possible quantum counterpart which in the case $N = 2$ was studied in [S2, S3].

The similar conjecture have been formulated in [S1] for the Gaudin model which can be considered as a degenerate case of the $SL(N)$ magnetic chain [G, S1, S4]. This model has been treated in various contexts by many authors (see, e.g., [RSe1, RSe2, Sch, AHH1, AHH2]). We should mention especially the recent work [AHH2], where Darboux coordinates for the Gaudin model have been constructed and the Liouville–Arnold integration for corresponding isospectral flows on rational orbits in the loop algebra have been performed. In the concluding Sect. 4 we show how the construction of Sect. 2 may be used for finding separated Darboux coordinates for the Gaudin model and compare our approach with that of [AHH2].

2. Notations and Preliminary Lemmas

Let Ψ be a $(n-1)\times(n-1)$ complex matrix with entries ψ_j^k $(j,k=1,\ldots,n-1)$ and columns $\psi^{(k)}$ $(k=1,\ldots,n-1)$. We shall denote by $\Psi\begin{pmatrix} k_1 & \ldots & k_m \\ j_1 & \ldots & j_m \end{pmatrix}$ the determinant of the matrix $(\psi_{j_s}^{k_r})_{s,r=1}^m$ and by $\Psi\begin{pmatrix} \hat{p}_1 & \ldots & \hat{p}_l \\ \hat{q}_1 & \ldots & \hat{q}_l \end{pmatrix}$ the determinant of the matrix obtained after deleting $p_1^{\text{th}},\ldots,p_l^{\text{th}}$ columns and $q_1^{\text{th}},\ldots,q_l^{\text{th}}$ rows of Ψ. The following formula can be easily verified:

$$\sum_{j=2}^{n-1}(-1)^j\,\Psi\begin{pmatrix} 1 & 2 & \ldots & l \\ j & i_1 & \ldots & i_{l-1} \end{pmatrix}\Psi\begin{pmatrix} \widehat{n-2} & \widehat{n-1} \\ \hat{1} & \hat{j} \end{pmatrix}=0\,, \tag{2.1}$$

for any $l=1,\ldots,n-3$, $i_1,\ldots,i_{l-1}\in\{1,\ldots,n-1\}$. For a fixed vector $\xi\in\mathbb{C}^{n-1}$, let us consider $(n-1)\times(n-1)$ matrices

$$\begin{aligned} S&=S(\Psi,\xi)=[\xi,\Psi\xi,\ldots,\Psi^{n-2}\xi]\,,\\ S^{(j)}&=S^{(j)}(\Psi,\xi)=[\psi^{(j)},\xi,\Psi\xi,\ldots,\Psi^{n-3}\xi]\;(j=1,\ldots,n-1)\,. \end{aligned} \tag{2.2}$$

The following auxiliary lemmas will prove useful in the sequel.

Lemma 1. *Let* $1\leqq j\leqq n-1$. *Then*

$$\begin{aligned} &(-1)^i\det S^{(i)}(\Psi,\xi)\cdot S\begin{pmatrix} \widehat{n-1} \\ \hat{j} \end{pmatrix}+(-1)^{j-1}\det S^{(j)}(\Psi,\xi)\cdot S\begin{pmatrix} \widehat{n-1} \\ \hat{\imath} \end{pmatrix}\\ &\quad=\det S(\Psi,\xi)\cdot S\begin{pmatrix} \widehat{n-2} & \widehat{n-1} \\ \hat{\imath} & \hat{j} \end{pmatrix}. \end{aligned} \tag{2.3}$$

Proof. By the definition of $S(\Psi,\xi)$,

$$\sum_{k=1}^{n-2}(-1)^k(\Psi^{k-1}\xi)_l S\begin{pmatrix} \hat{k} & \widehat{n-1} \\ \hat{\imath} & \hat{j} \end{pmatrix}=\begin{cases} (-1)^i S\begin{pmatrix} \widehat{n-1} \\ \hat{j} \end{pmatrix}, & l=i \\ (-1)^{j-1} S\begin{pmatrix} \widehat{n-1} \\ \hat{\imath} \end{pmatrix}, & l=j \\ 0, & \text{otherwise.} \end{cases}$$

Then the left-hand side of (2.3) is equal to

$$\begin{aligned} &\sum_{k=1}^{n-2}(-1)^k(\det S^{(i)}\cdot(\Psi^{k-1}\xi)_i+\det S^{(j)}\cdot(\Psi^{k-1}\xi)_j)S\begin{pmatrix} \hat{k} & \widehat{n-1} \\ \hat{\imath} & \hat{j} \end{pmatrix}\\ &\quad=\sum_{k=1}^{n-2}(-1)^k\left(\sum_{l=1}^{n-2}\det S^{(l)}\cdot(\Psi^{k-1}\xi)_l S\begin{pmatrix} \hat{k} & \widehat{n-1} \\ \hat{\imath} & \hat{j} \end{pmatrix}\right). \end{aligned}$$

But the definition of $S^{(l)}$ implies that

$$\sum_{l=1}^{n-2}\det S^{(l)}(\Psi^{k-1}\xi)_l=\det[\Psi^k\xi,\xi,\Psi\xi,\ldots,\Psi^{n-3}\xi]=(-1)^{n-2}\delta_{k,n-2}\det S(\Psi,\xi)\,,$$

which proves (2.3).

Let us define the constant

$$c_n = \frac{(-1)^{\frac{(n-1)n}{2}}}{(n-1)!\dots 2!} \tag{2.4}$$

and the set $I_n(m)$ of multiindices (α):

$$I_n(m) = \{(\alpha) = (\alpha_j^k)_{j=1}^{n}\ {}_{k=1}^{j-1} \mid \alpha_j^k \in \{1, \dots, m\}\} . \tag{2.5}$$

Lemma 2. *Let $S(\Psi, \xi)$, $S^{(j)}(\Psi, \xi)$ be defined by (2.1), (2.2), and let Ψ' be the $(n-1)\times n$ matrix with columns $\psi^{(1)}, \dots, \psi^{(n-1)}, \xi$. Then*

$$\det S(\Psi, \xi) = c_{n-1} \sum_{(\alpha)\in I_{n-1}(n-1)} \Psi'\begin{pmatrix} \alpha_{n-1}^1 & \dots & \alpha_{n-1}^{n-2} & n \\ 1 & \dots & n-2 & n-1 \end{pmatrix}$$
$$\cdot \prod_{k=1}^{n-2} \Psi'\begin{pmatrix} \alpha_{n-k-1}^1 & \dots & \alpha_{n-k-1}^{n-k-2} & n \\ \alpha_{n-k}^1 & \dots & & \alpha_{n-k}^{n-k-1} \end{pmatrix}, \tag{2.6}$$

$$\det S^{(l)}(\Psi, \xi) = c_{n-2} \sum_{(\alpha)\in I_{n-2}(n-1)} \Psi'\begin{pmatrix} l & \alpha_{n-2}^1 & \dots & \alpha_{n-2}^{n-3} & n \\ 1 & & \dots & & n-1 \end{pmatrix}$$
$$\cdot \prod_{k=2}^{n-2} \Psi'\begin{pmatrix} \alpha_{n-k-1}^1 & \dots & \alpha_{n-k-1}^{n-k-2} & n \\ \alpha_{n-k}^1 & \dots & & \alpha_{n-k}^{n-k-1} \end{pmatrix}, \tag{2.7}$$

$$S\begin{pmatrix} \widehat{n-1} \\ \hat{j} \end{pmatrix} = c_{n-2} \sum_{(\alpha)\in I_{n-2}(n-2)} \Psi'\begin{pmatrix} \alpha_{n-2}^1 & & \dots & \alpha_{n-2}^{n-3} & n \\ 1 & \dots j-1\, j+1 \dots & & & n-1 \end{pmatrix}$$
$$\cdot \prod_{k=2}^{n-2} \Psi'\begin{pmatrix} \alpha_{n-k-1}^1 & \dots & \alpha_{n-k-1}^{n-k-2} & n \\ \alpha_{n-k}^1 & \dots & & \alpha_{n-k}^{n-k-1} \end{pmatrix}. \tag{2.8}$$

Proof. By definition,

$$\det S(\Psi, \xi) = \sum_{\sigma\in\mathfrak{S}_{n-1}} (-1)^{\operatorname{sgn}\sigma} \xi_{\sigma(1)} (\Psi\xi)_{\sigma(2)} \dots (\Psi^{n-2}\xi)_{\sigma(n-1)} .$$

where $\mathfrak{S}_{n-1}$ is the group of permutations of $n-1$ elements. Therefore,

$$\det S(\Psi, \xi) = \sum_{\sigma\in\mathfrak{S}_{n-1}} (-1)^{\operatorname{sgn}\sigma} \sum_{\alpha_{n-1}^1, \dots, \alpha_{n-1}^{n-2}=1}^{n-1} \xi_{\sigma(1)} \psi_{\sigma(2)}^{\alpha_{n-1}^1} \dots \psi_{\sigma(n-1)}^{\alpha_{n-1}^{n-2}}$$
$$\cdot \xi_{\alpha_{n-1}^1} (\Psi\xi)_{\alpha_{n-1}^2} \dots (\Psi^{n-3}\xi)_{\alpha_{n-1}^{n-2}}$$
$$= (-1)^{n-2} \Psi'\begin{pmatrix} \alpha_{n-1}^1 & \dots & \alpha_{n-1}^{n-2} & n \\ 1 & \dots & & n-1 \end{pmatrix} \xi_{\alpha_{n-1}^1} \dots (\Psi^{n-3}\xi)_{\alpha_{n-1}^{n-2}}$$
$$= (-1)^{n-2} \sum_{1 \le \alpha_{n-1}^1 < \dots < \alpha_{n-1}^{n-2} \le n-1} \Psi'\begin{pmatrix} \alpha_{n-1}^1 & \dots & \alpha_{n-1}^{n-2} & n \\ 1 & \dots & & n-1 \end{pmatrix}$$
$$\cdot S\begin{pmatrix} 1 & \dots & n-2 \\ \alpha_{n-1}^1 & \dots & \alpha_{n-1}^{n-2} \end{pmatrix}$$

$$= \frac{(-1)^{n-2}}{(n-2)!} \sum_{\alpha_{n-1}^1, \ldots, \alpha_{n-1}^{n-2}=1}^{n-1} \Psi' \begin{pmatrix} \alpha_{n-1}^1 & \ldots & \alpha_{n-1}^{n-2} & n \\ 1 & \ldots & & n-1 \end{pmatrix}$$

$$\cdot S \begin{pmatrix} 1 & \ldots & n-2 \\ \alpha_{n-1}^1 & \ldots & \alpha_{n-1}^{n-2} \end{pmatrix} .$$

Applying the same procedure to $S\begin{pmatrix} 1 & \ldots & n-2 \\ \alpha_{n-1}^1 & \ldots & \alpha_{n-1}^{n-2} \end{pmatrix}$ etc., we obtain formula (2.6). Formulas (2.7), (2.8) can be proven in the same way.

In the sequel we shall deal with polynomial functions on matrices that have a form

$$F(\Psi) = \text{const.} \cdot \prod_{k=1}^{m} \Psi \begin{pmatrix} i_k^1 & \ldots & i_k^{n_k} \\ j_k^1 & \ldots & j_k^{n_k} \end{pmatrix} . \tag{2.9}$$

Given a function of the form (2.9), we denote by $F_{j_k^s \to l}(\Psi)$ (resp. $F^{i_k^s \to l}(\Psi)$) the function (2.9) with the index j_k^s (resp. i_k^s) substituted by l.

3. Separation of Variables

For the monodromy matrix $T(u)$ given by (1.1) consider the principal submatrix $T_{N-1}(u) = (t_j^k(u))_{j,k=1}^{N-1}$ and the vector $t^{(N)}(u) = (t_j^N(u))_{j=1}^{N-1}$ formed by the first $(N-1)$ entries of the last column of $T(u)$.

Using definitions (2.1), (2.2), one can construct matrices

$$S(u) = S(T_{N-1}(u), t^{(N)}(u)) ,$$
$$S^{(j)}(u) = S^{(j)}(T_{N-1}(u), t^{(N)}(u)) . \tag{3.1}$$

Define

$$B(u) = \mathscr{B}(T(u)) = \det S(u) ,$$
$$A(u) = \mathscr{A}(T(u)) = \frac{\det S^{(1)}(u)}{S\begin{pmatrix} \widehat{N-1} \\ \hat{1} \end{pmatrix}(u)} . \tag{3.2}$$

Clearly, $B(u)$ is the polynomial of degree $M \cdot \frac{N(N-1)}{2}$. Using, if necessary, a similarity transform $QT(u)Q^{-1}$, and taking advantage of the nondegeneracy of spectrum of Z, we may state that the leading coefficient of $B(u)$ is nonzero, and therefore $B(u)$ has $M \cdot \frac{N(N-1)}{2}$ zeroes.

Theorem 1. *The variables x_j, P_j $\left(j = 1, \ldots, M\frac{N(N-1)}{2} \right)$, defined from the equations*

$$\mathscr{B}(T(x_j)) = 0, \quad P_j = \mathscr{A}(T(x_j)) , \tag{3.3}$$

have the Poisson brackets

$$\{x_j, x_k\} = \{P_j, P_k\} = 0, \quad \{P_j, x_k\} = P_j \delta_{jk} . \tag{3.4}$$

Furthermore, P_j is an eigenvalue of the matrix $T(x_j)$, i.e.

$$\det(P_j - T(x_j)) = 0 .$$

The rest of the section will be devoted to the proof of this theorem.

First of all, let us calculate the Poisson bracket of two polynomial functions of T that have a form (2.9):

$$F(u) = F(T(u)) = \prod_{k=1}^{m} T\begin{pmatrix} i_k^1 & \cdots & i_k^{n_k} \\ j_k^1 & \cdots & j_k^{n_k} \end{pmatrix}(u) ,$$

$$G(v) = G(T(v)) = \prod_{s=1}^{\mu} T\begin{pmatrix} \alpha_s^1 & \cdots & \alpha_s^{\nu_s} \\ \beta_s^1 & \cdots & \beta_s^{\nu_s} \end{pmatrix}(v) .$$

Lemma 3.

$$(u-v)\{F(u), G(v)\} = \sum F_{j_k^l \to \beta_s^q}(u) G_{\beta_s^q \to j_k^l}(v) - \sum F^{i_k^l \to \alpha_s^q}(u) G^{\alpha_s^q \to i_k^l}(v) , \qquad (3.5)$$

where summation is over all possible pairs (j_k^l, β_s^q) and (i_k^l, α_s^q).

Proof. It suffices to calculate the Poisson bracket for determinants $T\begin{pmatrix} i^1 & \cdots & i^n \\ j^1 & \cdots & j^n \end{pmatrix}(u)$ and $T\begin{pmatrix} \alpha^1 & \cdots & \alpha^\nu \\ \beta^1 & \cdots & \beta^\nu \end{pmatrix}(v)$, then (3.5) will follow by the Leibnitz rule. It follows from (1.4) that

$$\left\{T\begin{pmatrix} i^1 & \cdots & i^n \\ j^1 & \cdots & j^n \end{pmatrix}(u), T\begin{pmatrix} \alpha^1 & \cdots & \alpha^\nu \\ \beta^1 & \cdots & \beta^\nu \end{pmatrix}(v)\right\}$$

$$= -\sum_{l=1}^{n}\sum_{q=1}^{\nu} T\begin{pmatrix} i^1 & \cdots i^{l-1}\alpha^q i^{l+1} \cdots & i \\ j^1 & \cdots & j^n \end{pmatrix}(u)$$

$$\cdot T\begin{pmatrix} \alpha^1 & \cdots \alpha^{q-1} i^l \alpha^{q-1} \cdots & \alpha^\nu \\ \beta^1 & \cdots & \beta^\nu \end{pmatrix}(v)$$

$$+ \sum_{l,p=1}^{n}\sum_{q,r=1}^{\nu} (-1)^{j^p + i^l} t_{\beta^r}^{i^l}(u) T\begin{pmatrix} i^1 \cdots i^{l-1} i^{l+1} \cdots i^n \\ j^1 \cdots j^{l-1} j^{l+1} \cdots j^n \end{pmatrix}(u)$$

$$\cdot(-1)^{\beta^r + \alpha^q} t_{j^p}^{\alpha^q}(v) T\begin{pmatrix} \alpha^1 \cdots \alpha^{q-1}\alpha^{q+1} \cdots \alpha^\nu \\ \beta^1 \cdots \beta^{q-1}\beta^{q+1} \cdots \beta^\nu \end{pmatrix}(v) .$$

The second term in the previous expression is equal to

$$\sum_{p=1}^{n}\sum_{r=1}^{v} T\begin{pmatrix} i^1 & \cdots & i^n \\ j^1 & \cdots j^{p-1}\beta^r j^{p+1} \cdots & j^n \end{pmatrix}(u)\, T\begin{pmatrix} \alpha^1 & \cdots & \alpha^\nu \\ \beta^1 & \cdots \beta^{r-1} j^p \beta^{r+1} \cdots & \beta^\nu \end{pmatrix}(v) ,$$

which shows that (3.5) holds true.

Lemma 4. *For any u and v*

$$\{B(u), B(v)\} = 0 . \qquad (3.6)$$

Proof. Due to (3.1), (3.2) and (2.6),

$$B(u) = c_{n-1} \sum_{(\alpha)\in I_{N-1}(N-1)} B((\alpha), u) ,$$

where

$$B((\alpha),u) = T\begin{pmatrix}\alpha^1_{N-1} & \dots & \alpha^{N-2}_{N-1} & N\\ 1 & \dots & & N-1\end{pmatrix}(u)$$
$$\cdot\prod_{k=1}^{N-2} T\begin{pmatrix}\alpha^1_{N-k-1} & \dots & \alpha^{N-k-2}_{N-k-1} & N\\ \alpha^1_{N-k} & \dots & & \alpha^{N-k-1}_{N-k}\end{pmatrix}(u)\,. \qquad (3.7)$$

(Here we have used the fact that the matrix Ψ' from Lemma 2 is now simply the matrix obtained from $T(u)$ by deleting the last row.) Using Lemma 3, let us calculate the Poisson bracket of $B((\alpha),u)$ and $B((\beta),v)$ for arbitrary fixed $(\alpha),(\beta)\in I_{N-1}(N-1)$. The summation in the first sum in (3.5) for $\{B((\alpha),u),B((\beta),v)\}$ is carried out over all possible pairs (α^q_p,β^s_r), (α^q_p,j), (i,β^s_r) *and* (i,j), where i (resp. j) is one of the lower indices in the first factor in the representation (3.7) for $B((\alpha),u)$ (resp. $B((\beta),v)$). Note, however, that the term corresponding to the pair (α^q_p,j) is equal to $B((\alpha),u)\cdot B((\beta),v)$ if $\alpha^q_p=j$ and zero otherwise. The same is true also for the pairs (i,β^s_r) and (i,j). Thus, the first sum in the expression (3.5) for $(u-v)\{B((\alpha),u),B((\beta),v)\}$ is equal to

$$((N-1)+2((N-2)+(N-3)+\dots+1))B((\alpha),u)\cdot B((\beta).v)$$
$$+\sum B_{\alpha^q_p\to\beta^s_r}((\alpha),u)\cdot B_{\beta^s_r\to\alpha^q_p}((\beta),v)$$
$$=(N-1)^2B((\alpha),u)\cdot B((\beta),v)+\sum B_{\alpha^q_p\to\beta^s_r}((\alpha),u)\cdot B_{\beta^s_r\to\alpha^q_p}((\beta),v)\,.$$

Similarly, one can see that the second sum in the expression (3.5) for functions $B((\alpha),u)$ and $B((\beta),v)$ is equal to

$$(N-1)^2B((\alpha),u)\cdot B((\beta),v)+\sum B^{\alpha^q_p\to\beta^s_r}((\alpha),u)\cdot B^{\beta^s_r\to\alpha^q_p}((\beta),v)\,.$$

Hence

$$(u-v)\{B(u),B(v)\}=c^2_{n-1}\sum_{(\alpha),(\beta)\in I_{N-1}(N-1)}\Bigg(\sum(B_{\alpha^q_p\to\beta^s_r}((\alpha),u)\cdot B_{\beta^s_r\to\alpha^q_p}((\beta),v)$$
$$-B^{\alpha^q_p\to\beta^s_r}((\alpha),u)\cdot B^{\beta^s_r\to\alpha^q_p}((\beta),v))\Bigg)\,. \qquad (3.8)$$

But for every pair (α), $(\beta)\in I_{N-1}(N-1)$ there exists also the pair $(\tilde\alpha),(\tilde\beta)\in I_{N-1}(N-1)$, such that $\tilde\alpha^\nu_\mu=\alpha^\nu_\mu$ $((\mu,\nu)\neq(p,q))$, $\tilde\alpha^q_p=\beta^s_r$ and $\tilde\beta^\nu_\mu=\beta^\nu_\mu$ $((\mu,\nu)\neq(r,s))$, $\tilde\beta^s_r=\alpha^q_p$. Then it is easily seen from (3.7) that

$$B_{\tilde\alpha^q_p\to\tilde\beta^s_r}((\tilde\alpha),u)\cdot B_{\tilde\beta^s_r\to\tilde\alpha^q_p}((\tilde\beta),v)=B^{\alpha^q_p\to\beta^s_r}((\alpha),u)\cdot B^{\beta^s_r\to\alpha^q_p}((\beta),v)\,.$$

Therefore every term appears in the sum (3.8) twice, but with different signs, which implies that $\{B(u),B(v)\}=0$.

Lemma 5. *There exist algebraic functions* $U(u,v)=U(T(u),T(v))$, $W(u,v)=W(T(u),T(v))$ *such that*

$$\{A(u),A(v)\}=\frac{1}{u-v}(B(u)\cdot U(u,v)+U(v,u)B(v))\,, \qquad (3.9)$$

$$\{B(u),A(v)\}=\frac{1}{u-v}(B(u)A(v)-W(u,v)B(v))\,, \qquad (3.10)$$

and

$$W(u,u) \equiv A(u)\,. \tag{3.11}$$

Proof. Denote

$$C(u) = \det S^{(1)}(u)\,, \quad D(u) = S\begin{pmatrix} \widehat{N-1} \\ \hat{1} \end{pmatrix}(u)\,. \tag{3.12}$$

Then $A(u) = \dfrac{C(u)}{D(u)}$. We have to calculate all the Poisson brackets $\{B(u), C(v)\}$, $\{B(u), D(v)\}$, $\{C(u), D(v)\}$, $\{C(u), C(v)\}$ and $\{D(u), D(v)\}$. Repeating essentially the same arguments as in the proof of the previous lemma, one can see that

$$(u-v)\{B(u), C(v)\}$$
$$= B(u)C(v) - c_{N-1}\sum_{j=1}^{N-1}\left(\sum_{l=2}^{N-1}(l-1)\sum_{(\alpha)\in I_{N-1}(N-1)} B_{\alpha_l^1 \to j}^{\alpha_l^1 \to 1}((\alpha), u)\right)\cdot \det S^{(j)}(v)$$

and

$$(u-v)\{B(u), D(v)\}$$
$$= c_{N-1}\sum_{j=1}^{N-1}(-1)^j\left(\sum_{l=2}^{N-1}(l-1)\sum_{(\alpha)\in I_{N-1}(N-1)} B_{\alpha_l^1 \to j}^{\alpha_l^1 \to 1}((\alpha), u)\right)\cdot S\begin{pmatrix} \widehat{N-1} \\ \hat{j} \end{pmatrix}(v)\,.$$

Denoting

$$W_j(u) = c_{N-1}\sum_{l=2}^{N-1}(l-1)\sum_{(\alpha)\in I_{N-1}(N-1)} B_{\alpha_l^1 \to j}^{\alpha_l^1 \to 1}((\alpha), u)\,, \tag{3.13}$$

we obtain

$$\{B(u), A(v)\} = \frac{\{B(u), C(v)\}D(v) - \{B(u), D(v)\}C(v)}{D(v)^2}$$
$$= \frac{1}{u-v}\left(B(u)A(v) - \frac{1}{D(v)^2}\sum_{j=1}^{N-1} W_j(u)\left(\det S^{(j)}(v)D(v)\right.\right.$$
$$\left.\left. + (-1)^j S\begin{pmatrix} \widehat{N-1} \\ \hat{j} \end{pmatrix}(v)C(v)\right)\right)\,.$$

But due to Lemma 1 and definitions (3.2), (3.12),

$$\det S^{(j)}(v)D(v) + (-1)^j S\begin{pmatrix} \widehat{N-1} \\ \hat{j} \end{pmatrix}(v)C(v) = (-1)^{j+1} S\begin{pmatrix} \widehat{N-2} & \widehat{N-1} \\ \hat{1} & \hat{j} \end{pmatrix}\cdot B(v)\,.$$

Thus we show that

$$\{B(u), A(v)\} = \frac{1}{u-v}(B(u)A(v) - W(u,v)B(v))\,,$$

where

$$W(u,v) = \frac{1}{D^2(v)}\sum_{j=2}^{N-1}(-1)^{j+1}W_j(u)S\begin{pmatrix} \widehat{N-2} & \widehat{N-1} \\ \hat{1} & \hat{j} \end{pmatrix}(v)\,. \tag{3.14}$$

To prove the identity $W(u,u) \equiv A(u)$, let us note first that, similarly to the formulas of Lemma 2, one has

$$S\begin{pmatrix} 1 & 2 & \dots & l \\ j_1 & j_2 & \dots & j_l \end{pmatrix}(u)$$
$$= c_l \sum_{(\beta)\in I_l(N-1)} \begin{pmatrix} \beta_l^1 & \dots & \beta_l^{l-1} & n \\ j_1 & \dots & & j_l \end{pmatrix}(u) \prod_{k=0}^{l-2} T\begin{pmatrix} \beta_{l-k}^l & \dots & \beta_{l-k}^{l-k-1} & n \\ \beta_{l-k}^1 & \dots & & \beta_{l-k}^{l-k-1} \end{pmatrix}(u)$$

.

Then by the definition (3.7) of $B((\alpha),u)$,

$$\sum_{(\alpha)\in I_{N-1}(N-1)} B_{\alpha_l^1\to j}^{\alpha_l^1\to 1}((\alpha),u) = \sum_{\alpha_l^2,\dots,\alpha_l^{l-1}=1}^{N-1} \varphi(\alpha_l^2,\dots,\alpha_l^{l-1};u) S\begin{pmatrix} 1 & 2 & \dots & l-1 \\ j & \alpha_l^2 & \dots & \alpha_l^{l-1} \end{pmatrix}(u)$$

for some coefficients $\varphi(\alpha_l^2,\dots,\alpha_l^{l-1},u)$ that we do not need to specify, since by (2.1),

$$\sum_{j=2}^{N-1} S\begin{pmatrix} 1 & 2 & \dots & l-1 \\ j & \alpha_l^2 & \dots & \alpha_l^{l-1} \end{pmatrix}(u) S\begin{pmatrix} \widehat{N-2} & \widehat{N-1} \\ \hat{1} & \hat{j} \end{pmatrix}(u) = 0 \quad (l = 1,\dots,n-2)\,,$$

and therefore, due to (3.13), (3.14),

$$W(u,u) = \frac{c_{N-1}}{D^2(u)} \sum_{j=2}^{N-2} (-1)^{j+1}(N-2) \sum_{(\alpha)\in I_{N-1}(N-1)} B_{\alpha_{n-1}^1\to j}^{\alpha_{n-1}^1\to 1}((\alpha),u) S\begin{pmatrix} \widehat{N-1} \\ \hat{j} \end{pmatrix}(u)\,.$$

But (3.7) shows that

$$B_{\alpha_{n-1}^1\to j}^{\alpha_{n-1}^1\to 1}((\alpha),u) = 0$$

if one of the indices $\alpha_{n-1}^2,\dots,\alpha_{n-1}^{n-2}$ is equal to 1 or j. Hence

$$\sum_{(\alpha)\in I_{N-1}(N-1)} B_{\alpha_{n-1}^1\to j}^{\alpha_{n-1}^1\to 1}((\alpha),u)$$
$$= (-1)^j (N-3)!\, T\begin{pmatrix} \hat{j} \\ \hat{N} \end{pmatrix}(u)$$
$$\times \sum_{(\alpha)\in I_{N-1}(N-1)} T\begin{pmatrix} \alpha_{N-2}^1 & \dots & \alpha_{N-2}^{N-3} & N \\ 2 & \dots & & N-1 \end{pmatrix}(u)$$
$$\cdot \prod_{k=2}^{N-2} T\begin{pmatrix} \alpha_{N-k-1}^1 & \dots & \alpha_{N-k-2}^{N-k-2} & N \\ \alpha_{N-k}^1 & \dots & & \alpha_{N-k}^{N-k-1} \end{pmatrix}(u)$$
$$\overset{(2.8),(3.12)}{=} (-1)^j (N-3)!\, c_{N-2}^{-1} D(u)\, T\begin{pmatrix} \hat{j} \\ \hat{N} \end{pmatrix}(u)\,.$$

Then

$$W(u,u) = \frac{-c_{N-1}(N-2)!\,c_{N-2}^{-1}}{D(u)} \sum_{j=2}^{N-1} T\begin{pmatrix} \hat{j} \\ \hat{N} \end{pmatrix}(u) S\begin{pmatrix} \widehat{N-2} & \widehat{N-1} \\ \hat{1} & \hat{j} \end{pmatrix}$$
$$\overset{(2.4),(2.7)}{=} \frac{C(u)}{D(u)} = A(u)\,.$$

Thus we have proved equalities (3.10), (3.11). Absolutely analogously, one can show the validity of (3.9) with $U(u,v)$ equal to

$$U(u,v) = \frac{c_{N-2}}{D^2(u)D^2(v)} \sum_{j=2}^{N-1} (-1)^{j+1} S\begin{pmatrix} \widehat{N-2} & \widehat{N-1} \\ \hat{1} & \hat{j} \end{pmatrix}(u)$$
$$\cdot \sum_{(\alpha)\in I_{N-2}(N-1)} \left(\sum_{l=2}^{N-2} (l-1) D^{\alpha_l^1 \to 1}_{\alpha_l^1 \to j}((\alpha), v) C(v) - \sum_{l=2}^{N-3} (l-1) C^{\alpha_l^1 \to 1}_{\alpha_l^1 \to j}((\alpha), v) D(v) \right),$$

where the definition of $D((\alpha), v)$, $C((\alpha), v)$ is similar to (3.7). Now we are ready to present.

Proof of Theorem 1. Let x_j $\left(j = 1, \ldots, M\,\frac{N(N-1)}{2}\right)$ be roots of $B(u)$: $B(x_j) = \mathscr{B}(T(x_j)) = 0$. By Lemma 4,

$$\{x_i, x_j\} = 0 \quad \left(i,j = 1, \ldots, M\,\frac{N(N-1)}{2}\right).$$

Furthermore, since for any functions $f(u)$ and g,

$$\{f(x_i), g\} = \{f(u), g\}_{u=x_i} + f'(x_i)\{x_i, g\},$$

we have

$$\{x_i, P_j\} = \{x_i, A(x_j)\} = \{x_i, A(v)\}_{v=x_j} + A'(x_j)\{x_i, x_j\}$$
$$= \frac{1}{B'(x_i)} (\{B(x_i), A(v)\} - \{B(u), A(v)\}_{u=x_i})_{v=x_j}$$
$$= -\frac{1}{B'(x_i)} \{B(u), A(v)\}_{u=x_i, v=x_j}.$$

Due to (3.10)

$$\{B(u), A(v)\}_{u=x_i, v=x_j} = \frac{1}{x_i - x_j} (B(x_i) A(x_j) - W(x_i, x_j) B(x_j))$$
$$= A(x_j) \frac{B(x_i) - B(x_j)}{x_i - x_j} + B(x_j) \frac{A(x_j) - W(x_i, x_j)}{x_i - x_j}.$$

Using the definition of x_i and identity (3.11), we get

$$\{x_i, P_j\} = -A(x_j)\delta_{ij} = -P_j\delta_{ij}.$$

Similarly, by (3.9) and (3.10)

$$\{P_i, P_j\} = \{A(x_i), A(x_j)\}$$
$$= A'(x_i)\{x_i, A(x_j)\} - \frac{A'(x_j)}{B'(x_j)} \{A(u), B(v)\}_{u=x_i, v=x_j}$$
$$+ \{A(u), A(v)\}_{u=x_i, v=x_j} = 0.$$

Thus we have proved that the Poisson brackets for P's and x's are given by (3.4). It remains to show that $\det(T(x_i) - P_i) = 0$. Since $B(x_i) = \det S(T_{N-1}(x_i), t^{(N)}(x_i)) = 0$, there exists a vector-row $\xi \in \mathbb{C}^{n-1}$ such that $\xi T^k_{N-1}(x_i) t^{(N)}(x_i) = 0$ $(k = 0, \ldots, N-2)$. Then for any vector-row η from the subspace $L = \operatorname{span}\{\xi, \xi T_{N-1}(x_i), \ldots, \xi T^{N-2}_{N-1}(x_i)\}$, $\eta T_{n-1}(x_i) \in L$ and $\eta t^{(n)}(x_i) = 0$. This means that if $\eta = (\eta_1, \ldots, \eta_{N-1}) \in L$ is the left-eigenvector of $T_{N-1}(x_i)$, then $\eta' = (\eta_1, \ldots, \eta_{N-1}, 0)$ is the left-eigenvector of $T(x_i)$, and the corresponding eigenvalue is $\lambda = \dfrac{\eta_1 t^1_1(x_i) + \cdots + \eta_{N-1} t^1_{N-1}}{\eta_1}$. But η is also a solution of the equation $\eta S(T_{N-1}(x_i), t^{(N)}(x_i)) = \eta S(x_i) = 0$. Expressing coordinates of η through minors of the matrix $S(x_i)$, one can easily see that

$$\lambda = \frac{S^{(1)}(x_i)}{S\begin{pmatrix} \widehat{N-1} \\ \hat{1} \end{pmatrix}(x_i)} = A(x_i) = P_i\,.$$

This completes the proof of the theorem.

Remarks. 1. When $N = 2, 3$, variables (3.3) coincide with those introduced in [S1]. Note, however, that for $N \geqq 4$, the family $A(u)$ is not involutive.
2. It follows immediately from definitions (3.1), (3.2) that the polynomial $\mathscr{B}(T)$ and, therefore, coordinates x_i, are invariant under the action of $SL(N-1)$ by the similarity transform $QT_{N-1}Q^{-1}$ of the principal submatrix T_{N-1}. Applying Lemma 1 to the matrix $S(x_i)$, one can see that coordinate P_i are $SL(N-1)$-invariant too. In the case $N = 3$ $SL(2)$-invariance of separated variables was noticed by N. Reshetikhin (see [S1]).

4. Gaudin Model

Let us now compare the results of the previous section with the construction of the Darboux coordinates for the Gaudin model, which was proposed in the recent work [AHH2].

The model can be described in terms of the rational matrix function

$$\mathscr{T}(u) = \mathscr{Z} + \sum_{m=1}^{M} \frac{\mathscr{L}^{(m)}}{u - \delta_m}, \tag{4.1}$$

where $\mathscr{Z}$ is a constant matrix with distinct eigenvalues. Another case treated in [AHH2] – $\mathscr{Z} = 0$ – will not be considered here. Note also, that we assume matrices $\mathscr{L}^{(m)}$ to have distinct eigenvalues, while in [AHH2] genericity conditions are more relaxed: $\mathscr{L}^{(m)}$ are allowed to have multiple zero eigenvalue.

The Poisson brackets for entries of $\mathscr{T}(u)$ are given by

$$\left\{\overset{1}{\mathscr{T}}(u), \overset{2}{\mathscr{T}}(v)\right\} = \frac{1}{u - v}\left[\mathscr{P}, \overset{1}{\mathscr{T}}(u) + \overset{2}{\mathscr{T}}(v)\right]. \tag{4.2}$$

The Gaudin model can be considered as a degenerate case of the $SL(N)$ magnetic chain [G, S4]. Namely, consider polynomial (1.1) with

$$Z = 1 + \varepsilon\mathscr{Z} \quad \text{and} \quad L^{(m)} = \varepsilon\mathscr{L}^{(m)} \tag{4.3}$$

and let $p(u) = (u - \delta_1) \ldots (u - \delta_m)$. Then $p(u)^{-1} T(u) = 1 + \varepsilon \mathscr{T}(u) + O(\varepsilon^2)$ and the Poisson brackets (4.2) are obtained as a linearization of the quadratic Poisson brackets (1.3).

Let us define functions $\tilde{B}(u)$, $\tilde{A}(u)$ as follows:

$$\tilde{B}(u) = \mathscr{B}(\mathscr{T}(u)), \quad \tilde{A}(u) = \mathscr{A}(\mathscr{T}(u)), \tag{4.4}$$

by (3.1), (3.2), (2.2) with $T(u)$ replaced by $\mathscr{T}(u)$. Then for polynomial (1.1), (4.3) we have

$$\begin{aligned} B(u) &= \mathscr{B}(T(u)) = p(u)^d \varepsilon^d \tilde{B}(u) + O(\varepsilon^{d+1}), \\ A(u) &= \mathscr{A}(T(u)) = p(u)(1 + \varepsilon \tilde{A}(u) + O(\varepsilon^2), \end{aligned} \tag{4.5}$$

where $d = \dfrac{N(N-1)}{2}$. After substitution of (4.5) into (3.6), (3.8), (3.9) and considering the leading order in ε we obtain

$$\begin{aligned} \{\tilde{B}(u), \tilde{B}(v)\} &= 0, \\ \{\tilde{A}(u), \tilde{A}(v)\} &= \frac{1}{u-v}(\tilde{B}(u) \cdot \tilde{U}(u,v) + \tilde{U}(v,u) B(v)), \\ \{\tilde{B}(u), \tilde{A}(v)\} &= \frac{1}{u-v}(\tilde{B}(u) - \tilde{W}(u,v)\tilde{B}(v)), \end{aligned}$$

where $\tilde{W}(u,u) \equiv 1$. Exactly as it has been done in the proof of Theorem 1, one can show that the variables $\tilde{x}_j$, $\tilde{p}_j$ $\left(j = 1, \ldots, M\,\dfrac{N(N-1)}{2}\right)$ defined by

$$\tilde{B}(\tilde{x}_j) = 0, \quad \tilde{p}_j = \tilde{A}(\tilde{x}_j) \tag{4.6}$$

has the canonical Poisson brackets:

$$\{\tilde{x}_j, \tilde{x}_k\} = \{\tilde{p}_j, \tilde{p}_k\} = 0, \quad \{\tilde{p}_j, \tilde{x}_k\} = \delta_{jk}.$$

Thus we obtain the following analogue of Theorem 1.

Theorem 2. *The variables* $\tilde{x}_j, \tilde{p}_j$ $\left(j = 1, \ldots, M\,\dfrac{N(N-1)}{2}\right)$ *defined by* (4.4), (4.6) *are Darboux coordinates for the Gaudin model. Moreover, they satisfy the relation*

$$\det(\tilde{p}_j - \mathscr{T}(\tilde{x}_j)) = 0,$$

The Darboux coordinates for the Gaudin model were given in [AHH2] by the solutions (u_k, λ_k) of the set of polynomial equations

$$(\mathscr{T}(u) - \lambda)' V_0 = 0,$$

where V_0 is a fixed vector and the superscript $'$ denotes the transition to the matrix of cofactors. The construction of the previous section has allowed us to find a more explicit way to define the Darboux coordinates. In particular, as well as in the case of the $SL(N)$ magnetic chain the canonical coordinates are zeroes of the single polynomial in u. This important fact makes our construction more applicable for the investigation of the connection between the Bethe Ansatz and the separation of variables in the quantum case (cf. [S2, S3]). We are going to discuss this topic elsewhere.

Acknowledgements. The author is grateful to the referee who drew his attention to the works [AHH1, AHH2].

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Communicated by M. Jimbo