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A Local-to-Global Singularity Theorem for Quantum Field Theory on Curved Space-Time

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(with an Appendix by Rainer Verch[2]**)**

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To a special friend, who saved my life when I was younger, without whom I could not have written this paper.

Abstract: We prove that if a reference two-point distribution of positive type on a time orientable curved space-time (CST) satisfies a certain condition on its wave front set (the "class $\mathscr{P}_{M,g}$ condition") and if any other two-point distribution (i) is of positive type, (ii) has the same antisymmetric part as the reference modulo smooth function and (iii) has the same local singularity structure, then it has the same global singularity structure. In the proof we use a smoothing, positivity-preserving pseudo-differential operator the support of whose symbol is restricted to a certain conic region which depends on the wave front set of the reference state. This local-to-global theorem, together with results published elsewhere, leads to a verification of a conjecture by Kay that for quasi-free states of the Klein-Gordon quantum field on a globally hyperbolic CST, the local Hadamard condition implies the global Hadamard condition. A counterexample to the local-to-global theorem on a strip in Minkowski space is given when the class $\mathscr{P}_{M,g}$ condition is not assumed.

1. Introduction

In the quantum field theory (QFT) of a Klein-Gordon scalar field on a globally hyperbolic curved space-time (CST) [2, 11], the Hadamard condition [18, 5] is believed to be a "physically necessary" condition on the two-point distribution of a quasi-free or more general state [13, 12, 28, 26]. Some reasons for this belief arose from investigations into the point-splitting renormalization technique used in defining observables quadratic in the field operators on such space-times. It was discovered that the Hadamard condition is sufficient for point-splitting renormalization to yield a stress-energy tensor $T_{\mu\nu}(x)$ that satisfies a set of properties encapsulating what is meant by "physically meaningful." These are called the *Wald axioms* [43, 44]. The *local Hadamard condition* (LH) specifies the asymptotic behavior of the two-point distribution $\omega_2(x_1, x_2)$ for x_1 close to x_2 to be

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$$\omega_2(x_1,x_2) \sim \lim_{\epsilon\to 0^+} \frac{1}{2\pi^2}\left(\frac{u(x_1,x_2)}{\sigma_\epsilon(x_1,x_2)} + v(x_1,x_2)\ln\sigma_\epsilon(x_1,x_2)\right) + w(x_1,x_2)\ ,$$

where u and v are certain smooth functions determined by the metric g, the function w is smooth and determined by the "physics" (the state-dependent part), and $\sigma_\epsilon(x_1,x_2) = \sigma(x_1,x_2) + 2(T(x_1) - T(x_2))\epsilon i + \epsilon^2$, where $\sigma(x_1,x_2)$ is minus the signed square of the geodesic distance from x_1 to x_2 and T is a global time coordinate function on M. The *global Hadamard condition* (GH) is the requirement that in addition the two-point distribution has singularities only at points x_1, x_2 which are connected by a null geodesic within a causal normal neighborhood of a Cauchy hypersurface. This condition is given rigorous meaning in [28]. Once (GH) has been defined, (LH) can be described as the requirement that for each space-time point x, there is a neighborhood U_x of x such that (GH) holds on $U_x \times U_x$.

In the local algebra approach [15, 17] to quantized fields on CST [7, 9, 28, 25] an issue first raised by Kay [22, 23] is whether all globally Hadamard states are locally quasiequivalent. Lüders and Roberts [34] made progress on this question on Robertson-Walker space-times and recently Verch [42] has proven local quasiequivalence on any globally hyperbolic CST. Another issue (see e.g., [14]) is whether the global Hadamard condition determines a folium of states that satisfies the "principle of local definiteness" [16, 10]. Again, progress on this question was made by Lüders and Roberts [34] and more recently Verch [42] has provided a proof that for ultrastatic space-times, quasi-free Hadamard states satisfy local definiteness. In the present work, however, these algebraic issues are sidestepped and instead only certain properties of general two-point distributions (such as their singularity structure) are investigated.

Fredenhagen and Haag [10] have investigated whether the laws of quantum gravity can be expressed *locally* in such a way that *global* laws can be recovered from them. Specifically, they considered a local algebraic framework on a manifold M with unspecified metric and used extensions of sheaf theoretic ideas to noncommutative algebras to demonstrate the reconstructibility of a globally defined state from its local germs, given suitable restrictions on the algebra and states of the theory. (The strongest restriction was that the state satisfy a "Reeh-Schlieder property.") A related question asked in [10] is whether specifying the (local) germs of a globally defined folium can be sufficient to permit reconstruction of the global folium.

Partly motivated by consideration of an analog to this latter *local-to-global* question in the context of quasi-free states satisfying the Klein-Gordon equation on a generic globally hyperbolic curved space-time, Kay [23] arrived (on the basis of a variety of evidence [27]) at the following

Conjecture 1.1 (Kay). *Suppose that ω is a quasi-free state satisfying the Klein-Gordon equation on a globally hyperbolic space-time. If the two-point distribution ω_2 has the usual commutator and positivity properties and is locally Hadamard, then ω_2 is globally Hadamard.*

This is a reworded version of Conjecture 2 and its Reformulation in [23] and is referred to simply as *Kay's conjecture* in this paper. Later, confidence in this conjecture was strengthened when it was shown by Gonnella and Kay [14] that several potential counterexamples to Conjecture 1.1 failed to be so. One such example, considered earlier and in a different context by Najmi and Ottewill [36], was a locally Hadamard, Klein-Gordon two-point distribution having space-like separated singularities and hence not globally Hadamard. In [14] it was shown that this two-point distribution, along with other examples having space-like separated singularities, violate the necessary positivity conditions required for them to be two-point distributions of states.

Another set of two-point distributions, which had also been considered earlier and in a different context by Allen [1], were bisolutions of the Klein-Gordon equation on de Sitter space having extra space-like separated singularities. It is pointed out in [14] that these examples manifestly satisfied the positivity requirements but were not locally Hadamard. (The coefficient of the leading term of the asymptotic expansion was strictly larger than that required by the Hadamard condition.)

An important consideration leading Kay [27] to regard Conjecture 1.1 as an analog of Fredenhagen and Haag's local-to-global question in terms of folia was the belief that sufficiently strong extra singularities for a space-like separated pair of points would force a locally Hadamard state out of the folium of globally Hadamard states on any bounded region containing this pair. See [24, 25] for related discussion and conjectures.

The main goal of this paper is to present a "local-to-global" theorem more general than Kay's conjecture. Results published elsewhere [37, 38] are employed in showing that Kay's conjecture is a special case of this local-to-global theorem. The language of micro-local analysis is used throughout and is found to apply very naturally to this problem and to yield a quite general answer. This is not the first paper on quantized fields on curved space-time in which micro-local methods are used: Dimock's work on the scattering operator for a scalar field on curved space-time [6] makes significant use of the distinguished parametrix theory of Duistermaat and Hörmander [8]. Also see [35] for even earlier examples of papers in which pseudo-differential operators were used in the context of quantum field theory on stationary and static space-times.

The "distributional approach" to quantized fields on curved space-time used throughout this paper is outlined in Sect. 2. Section 3 presents a brief introduction to micro-local analysis and lists some results that will be useful in later sections. In Sect. 4 the class $\mathscr{P}_{M,g}$ condition for a time orientable CST (M, g) and the main local-to-global singularity theorem are stated, followed by an outline of the proof of this theorem, the proof being contained in Sects. 5, 6, 7, and 8. (These sections are summarized in Sect. 4.) In Sect. 9, Kay's conjecture is verified using the existence of globally Hadamard quasi-free states on an arbitrary globally hyperbolic curved space-time and the equivalence (for quasi-free Klein-Gordon states) of the global Hadamard condition with a certain wave front set spectral condition (WFSSC) introduced in [37, 38] which is in turn stronger than the class $\mathscr{P}_{M,g}$ condition. In fact an even stronger statement than Conjecture 1.1 is proven: any two-point distribution of positive type which has the local Hadamard singularity structure and whose commutator is i times the difference of the advanced and retarded fundamental solutions of the Klein-Gordon operator (modulo a smooth function) must have the global Hadamard singularity structure. (Note that we do not assume the quasi-free property or that the Klein-Gordon equation is satisfied.) In Sect. 10, we demonstrate the necessity for the reference state to satisfy the class $\mathscr{P}_{M,g}$ condition, by displaying a counterexample to the local-to-global theorem on a strip in Minkowski space when this condition is not assumed. Section 11 discusses some implications of the local-to-global theorem, in particular the strengthening of the belief that the Hadamard condition is physically distinguished, and a corollary displaying some dependence among the axioms on curved space-time.

2. Distributional Approach to Quantized Fields on Curved Space-Time

A pair (M, g) is a *(curved) space-time* (CST) if M is a smooth n-dimensional pseudo-Riemannian manifold ($n \geq 2$) equipped with a smooth metric tensor field g of signature $(+-\cdots-)$. The metric g determines the notions of time-like, null, and space-like vectors $v \in T_x(M)$ at a point $x \in M$ by the conditions $g_x(v,v) > 0$, $g_x(v,v) = 0$, and $g_x(v,v) < 0$ respectively, where g_x is the value of the metric tensor field at x. Time-like, null, or space-like curves on (M, g) are smooth curves on M whose tangent vectors at every point on the curve are time-like, null, or space-like respectively. A *geodesic* is a (parametrized) curve whose tangent vector is parallel transported along itself. Points $x_1, x_2 \in M$ are *causally related* if x_1 and x_2 can be connected by a time-like or null curve in M. They are *space-like separated* if they are not causally related. They are *null related* if they may be connected by a null geodesic. The *closed light cone* V_x at x consists of all nonzero time-like and null vectors in $T_x(M)$. Clearly V_x decomposes into two components at each x. A *time orientable* CST is one in which a continuous global designation of "future" component of the closed light cone can be made. In this case the future/past (also called forward/backward) closed light cone at x is denoted by $V_x^{\pm}$. A CST (M, g) with a hypersurface S such that every inextendible causal curve in M intersects S precisely once is labelled *globally hyperbolic*. Every globally hyperbolic CST is necessarily time orientable. Some of these definitions are as in Hawking and Ellis [19] and Chapter 8 of Wald [45]. Also, a covector $k \in T_x^*(M)$ is called dual to $v \in T_x(M)$ if $k = g_x(\cdot, v)$.

For the test function space on a space-time (M, g), we use in this paper the space of smooth complex-valued functions of compact support $C_0^\infty(M)$. The dual space of $C_0^\infty(M)$ with respect to the metric volume form on (M, g) is the space of *distributions* on M and is denoted $\mathscr{D}'(M)$. See Sect. 6.3 of [21] for definitions and further discussion of distributions on a manifold.

Let $\mathscr{D}_m(M)$ denote $\bigotimes^m C_0^\infty(M)$ for $m \geq 1$ and define $\mathscr{D}_0(M) = \mathbb{C}$. For a collection of functions $\{f_m\}_{m\geq 0}$, where $f_m \in \mathscr{D}_m(M)$ and only a finite number of the f_m do not vanish, define $f = \bigoplus_{m=0}^\infty f_m$. With involution defined as $f^* = \bigoplus_{m=0}^\infty f_m^*$, where $f_m^*(x_1, \ldots, x_m) = \overline{f_m(x_m, \ldots, x_1)}$, and the product of f and $g = \bigoplus_{m=0}^\infty g_m$ defined as $f \times g = \bigoplus_{m=0}^\infty (f \times g)_m$, where $(f \times g)_m(x_1, \ldots, x_m) = \sum_{i=0}^m f_i(x_1, \ldots, x_i) g_{m-i}(x_{i+1}, \ldots, x_m)$, the set of all such f becomes an involutive algebra $\mathscr{B}(M)$, called the *Borchers algebra on M*. See [3, 10].

Let $\mathscr{D}_m'(M)$ denote the space $\bigotimes^m [\mathscr{D}'(M)]$, the dual of $\mathscr{D}_m(M)$. The direct sum topology is given to $\mathscr{B}(M) = \bigoplus_{m=0}^\infty \mathscr{D}_m(M)$. If μ is in $\mathscr{B}'(M)$, the dual of $\mathscr{B}(M)$ with respect to this topology, then for each $m \geq 0$ the *m-point distributions (or functions)* are $\mu_m = \mu|_{\mathscr{D}_m(M)} \in \mathscr{D}_m'(M)$. If $\omega \in \mathscr{B}'(M)$ satisfies $\omega_0 = 1$ and the positivity condition $\omega(f^* \times f) \geq 0$ then ω is a *state*. Suppose in addition that ω satisfies the local commutativity condition

$$\omega(\cdots \otimes f \otimes g \otimes \cdots) = \omega(\cdots \otimes g \otimes f \otimes \cdots) \tag{1}$$

for supp f and supp g space-like separated. (This is a statement of the independence of measurements (commensurability) of observables at space-like separation, a typical quantum mechanical restriction.) Then one may think of the m-point distributions $\omega_m(x_1, \ldots, x_m)$ as representing the expectation values of the product of m field operators $\Phi_\omega(x_1), \ldots, \Phi_\omega(x_m)$ with respect to some vector Ω_ω in a Hilbert space $\mathscr{H}_\omega$, an interpretation made available by an analog of the Wightman reconstruction theorem [3, 40], which is here given the generic label of "GNS construction" [4]. We call a triple (M, g, ω) whose ω satisfies these properties a *quantum field model on the CST* (M, g).

A state ω is *quasi-free* if the m-point distributions satisfy $\omega_{2m+1} = 0$ for $m \geq 0$ and

$$\omega_{2m}(f^1 \otimes \cdots \otimes f^{2m}) = \sum_{\pi \in \Pi_m} \omega_2(f^{\pi_1} \otimes f^{\pi_2}) \cdots \omega_2(f^{\pi_{2m-1}} \otimes f^{\pi_{2m}}) \qquad \text{(QF)}$$

for $m \geq 1$, where Π_m is the set of permutations $\pi: \{1, \ldots, 2m\} \to \{1, \ldots, 2m\}$ such that $\pi_1 < \pi_3 < \cdots < \pi_{2m-1}$ and $\pi_1 < \pi_2,\ \pi_3 < \pi_4, \ldots, \pi_{2m-1} < \pi_{2m}$. The main focus of research in quantum field theory on CST has been on states constructed from a linear wave equation via canonical quantization on CST [2]. These states turn out to satisfy (QF).

The fact that a quasi-free state ω is determined entirely by its two-point distribution leads one to direct particular attention to ω_2. Two general properties of ω_2, as implied by those for a (not necessarily quasi-free) state ω of a quantum field model on (M, g), are as follows:

Positive Type: For any $f \in C_0^\infty(M)$,

$$\omega_2(\bar{f} \otimes f) \geq 0\ . \qquad \text{(PT)}$$

This follows from the generic positivity condition on ω which in turn corresponds to the positive definiteness of the inner product on the Hilbert space $\mathscr{H}_\omega$ obtained by GNS construction from ω.

For any two-point distribution u, the *symmetric (anti-symmetric) part* is defined by

$$u_\pm(f \otimes g) = \frac{1}{2}\left(u(f \otimes g) \pm u(g \otimes f)\right)\ .$$

Equation (1) implies the following necessary condition on ω_2:

Local Commutativity: For any $f, g \in C_0^\infty(M)$ such that supp f and supp g are space-like separated,

$$(\omega_2)_-(f \otimes g) = 0\ . \qquad \text{(LC)}$$

The properties (PT) and (LC) make sense for any space-time (M, g), even possibly one that is not time orientable, and are two of the basic properties for ω_2 that are necessary for the state ω to yield a physically meaningful field Φ_ω by the GNS construction. We suggest in Sect. 11 that on a time orientable CST a certain "wave front set spectral condition" is a third such condition. There may be more, however.

A *Klein-Gordon quantum field model on* (M, g) is a quantum field model (M, g, ω) such that ω in addition satisfies:

Klein-Gordon: For any $f, g \in C_0^\infty(M)$,

$$\omega_2\left((\Box + m^2)f \otimes g\right) = \omega_2\left(f \otimes (\Box + m^2)g\right) = 0\ . \qquad \text{(KG)}$$

Here, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, where ∇_μ is the covariant derivative on the pseudo-Riemannian manifold (M, g). The term m^2 may be replaced by a more general potential $V(x)$ and a first derivative term $-ib^\mu(x)\nabla_\mu$ may be added.

In order to have a well-posed Cauchy problem for the Klein-Gordon equation, we assume in this paper (as is usually done) that the CST for a Klein-Gordon quantum field model is globally hyperbolic. For states satisfying (KG) on a globally hyperbolic

CST we also assume in this paper the property (QF) for ω as well as the following property:

Commutator: For any $f, g \in C_0^\infty(M)$,

$$(\omega_2)_-(f \otimes g) = \frac{\mathrm{i}}{2}\Delta(f \otimes g)\,, \tag{Com}$$

where $\Delta = \Delta_A - \Delta_R$ and Δ_A and Δ_R are the advanced and retarded fundamental solutions of the inhomogeneous Klein-Gordon equation. These distributions are uniquely determined by their support properties [31, 32, 33]. Condition (Com) is a direct consequence of canonically quantizing a scalar field satisfying the Klein-Gordon equation. Clearly it implies (LC).

3. Micro-Local Preliminaries

The definitions adopted for the distribution spaces $\mathscr{D}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$, and $\mathscr{E}'(\mathbb{R}^n)$ on $\mathbb{R}^n$ and $\mathscr{D}'(M)$ and $\mathscr{E}'(M)$ on a manifold M may be found in [21] or Appendix B of [37].

The convention for the Fourier transform of $f \in \mathscr{S}(\mathbb{R}^p)$, denoted by a hat $\hat{}$, is chosen to be

$$\hat{f}(k) = \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \mathrm{e}^{\mathrm{i}\langle x,k\rangle} f(x)\, dx$$

for $k \in \mathbb{R}^p$. Here dx is shorthand for the Lebesgue measure $dx^1 \cdots dx^p$ on $\mathbb{R}^p$. The inverse Fourier transform of f, denoted by a check $\check{}$, is then $\check{f} = \hat{f}^-$, where g^- is defined by $g^-(x) = g(-x)$. Recall that $\hat{}$ maps $\mathscr{S}(\mathbb{R}^p)$ isomorphically to itself.

Following [21], given an open set $X \subset \mathbb{R}^p$ one defines the space of *symbols* $S^m_{\rho,\delta}(X \times \mathbb{R}^q)$ on $X \times \mathbb{R}^q$ of order m and type ρ, δ, where $m \in \mathbb{R}$, $0 < \rho \leq 1$ and $0 \leq \delta < 1$, to be the space of smooth functions a on $X \times \mathbb{R}^q$ such that for any compact set $K \subset X$ and multi-indices α, β, there is a constant $C_{\alpha,\beta,K}$ such that

$$\sup_{x \in K} |D_x^\alpha D_k^\beta a(x,k)| \leq C_{\alpha,\beta,K}(1 + |k|)^{m - \rho|\beta| + \delta|\alpha|}$$

for all $k \in \mathbb{R}^p$, where $D_x^\alpha = D_{x^1}^{\alpha_1} \cdots D_{x^p}^{\alpha_p}$, $D_k^\beta = D_{k_1}^{\beta_1} \cdots D_{k_q}^{\beta_q}$, $D_{x^i} = -\mathrm{i}\partial_{x^i}$, and $D_{k_j} = -\mathrm{i}\partial_{k_j}$. Here, $|k|$ is the Euclidean norm of k, namely,

$$|k| = \left(\sum_{i=1}^p (k_i)^2\right)^{\frac{1}{2}}\,.$$

If $\rho + \delta = 1$ this space is denoted by $S^m_\rho(X \times \mathbb{R}^q)$ and if $\rho = 1$ and $\delta = 0$, it is called $S^m(X \times \mathbb{R}^q)$.

Given a symbol b in $S^m_{\rho,\delta}(\mathbb{R}^p \times \mathbb{R}^p)$, where the second copy of $\mathbb{R}^p$ is considered the dual of the first, the *pseudo-differential operator* B with symbol b is defined on $u \in \mathscr{S}(\mathbb{R}^p)$ by

$$Bu(x) = \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \mathrm{e}^{-\mathrm{i}\langle x,k\rangle} b(x,k)\hat{u}(k)dk$$

for $x \in \mathbb{R}^p$. The spaces of pseudo-differential operators with symbols in $S^m_{\rho,\delta}(\mathbb{R}^p \times \mathbb{R}^p)$, $S^m_\rho(\mathbb{R}^p \times \mathbb{R}^p)$ and $S^m(\mathbb{R}^p \times \mathbb{R}^p)$ are denoted $P^m_{\rho,\delta}(\mathbb{R}^p)$, $P^m_\rho(\mathbb{R}^p)$ and $P^m(\mathbb{R}^p)$

respectively. B is also defined on $v \in \mathscr{S}'(\mathbb{R}^p)$ by $(Bv)(u) = v(Bu)$ for all $u \in \mathscr{S}(\mathbb{R}^p)$.

Again following [21], if $v \in \mathscr{E}'(\mathbb{R}^p)$, then $\Sigma(v)$ is defined to be the complement in $\mathbb{R}^p \backslash \{0\}$ of the set of all $k \in \mathbb{R}^p \backslash \{0\}$ for which there is an open conic neighborhood $\mathscr{C}_k$ of k such that $\hat{v}$ is of *rapid decrease* in $\mathscr{C}_k$, by which we shall mean that for any integer N, there exists a constant C_N such that for all $\xi \in \mathscr{C}_k$,

$$|\hat{v}(\xi)| \leq C_N(1 + |\xi|)^{-N} .$$

Also, for $u \in \mathscr{D}'(\mathbb{R}^p)$, the set $\Sigma_x(u)$ is defined for $x \in \mathbb{R}^p$ to be

$$\Sigma_x(u) = \bigcap_{\substack{\phi \in C_0^\infty \\ \phi(x) \neq 0}} \Sigma(\phi u) .$$

Remark. This definition is natural because of:

Lemma 3.1 (Lemma 8.1.1 of [21]). *$\Sigma(\phi v) \subset \Sigma(v)$ for any smooth ϕ of compact support, and any $v \in \mathscr{E}'(\mathbb{R}^p)$.* □

Hence "squeezing" the support of ϕ to x "squeezes" the set $\Sigma(\phi u)$ to $\Sigma_x(u)$.

Definition 3.2. *The* **wave front set** WF(u) *of $u \in \mathscr{D}'(\mathbb{R}^p)$ is the set*

$$\text{WF}(u) = \{(x, k) \in T^*(\mathbb{R}^p)\backslash \mathbf{0}: k \in \Sigma_x(u)\} ,$$

where $\mathbf{0}$ stands for the zero section $\mathbb{R}^p \times \{0\}$ of the cotangent bundle $T^*(\mathbb{R}^p) = \mathbb{R}^p \times \mathbb{R}^p$.

It follows from the definition of WF(u) for $u \in \mathscr{D}'(\mathbb{R}^p)$ that [21]

$$\pi_1(\text{WF}(u)) = \text{sing supp } u ,$$

where the set sing supp u, called the *singular support* of $u \in \mathscr{D}'(\mathbb{R}^p)$, is the complement in $\mathbb{R}^p$ of the largest open set on which u is smooth and π_1 is the projection onto the first variable. Roughly speaking, if (x, k) is a point in the wave front set of u, then x specifies the location of a singularity of u and k its "direction of propagation." This definition extends to distributions on manifolds ($u \in \mathscr{D}'(M)$), in which case WF(u) is an invariantly defined closed conic subset of $T^*(M)\backslash \mathbf{0}$ and $\mathbf{0}$ is the zero section of the cotangent bundle $T^*(M)$ [20].

The following result is reproduced from [21], to which the reader is referred for a proof.

Proposition 3.3 (Proposition 8.1.3 of [21]). *If $v \in \mathscr{E}'(\mathbb{R}^p)$ then*

$$\pi_2(\text{WF}(v)) = \Sigma(v) ,$$

where π_2 is the projection onto the second variable. □

4. Local-to-Global Singularity Theorem

Let (M, g) be a time orientable curved space-time, not necessarily globally hyperbolic, and choose a particular time orientation (continuous global designation of closed "future" light cone) for M. Let

$$\mathscr{R}_{M,g} = \{((x_1, k_1), (x_2, k_2)) \in T^*(M) \times T^*(M):\ k_1 \in (V_{x_1}^+)^d,\ k_2 \in (V_{x_2}^-)^d\}\ ,$$

where $(x_i, k_i) \in T^*(M),\ i = 1, 2$ means that $x_i \in M$ and $k_i \in T^*_{x_i}(M)$, and $(V_x^\pm)^d$ is the set of covectors dual to elements of $V_x^\pm$. Note that $\mathscr{R}_{M,g}$ is invariantly defined.

Definition 4.1. *Let* (M, g) *be a time orientable CST with time orientation chosen. A two-point distribution* $\mu_2 \in \mathscr{D}_2'(M)$ *is of* **class** $\mathscr{P}_{M,g}$ *if*

$$\mathrm{WF}(\mu_2) \subset \mathscr{R}_{M,g}\ .$$

In the special case $M = \mathbb{R}^n$, define $\mathscr{R}_n$ to be

$$\mathscr{R}_n = \{((x_1, k_1), (x_2, k_2)) \in T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^n): (k_1)_0 > 0,\ (k_2)_0 < 0\}\ ,$$

where the natural coordinate representation for covectors on $\mathbb{R}^n$ is chosen, i.e., $k_i = ((k_i)_0, (k_i)_1, \ldots, (k_i)_{n-1})$. (The slice of $\mathscr{Q} = \pi_2 \mathscr{R}_n = \{(k_1, k_2): (k_1)_0 > 0,\ (k_2)_0 < 0\}$ in the $(k_1)_0$-$(k_2)_0$ plane is shown in Fig. 1 in Sect. 5.)

Definition 4.2. *A two-point distribution* $\mu_2 \in \mathscr{D}_2'(\mathbb{R}^n)$ *is of* **class** $\mathscr{P}_n$ *if*

$$\mathrm{WF}(\mu_2) \subset \mathscr{R}_n\ .$$

Note that this is not an invariantly defined class on $\mathbb{R}^n$ since $\mathscr{R}_n$ is not invariantly defined with respect to global coordinate transformations of $\mathbb{R}^n$. For any Lorentzian metric g on $\mathbb{R}^n$ having the property that $(V_x^\pm)^d \subset \{k \in \mathbb{R}^n:\ k_0 \gtrless 0\}$, with respect to the natural coordinates of $\mathbb{R}^n$, one can see that $\mathscr{R}_{\mathbb{R}^n,g} \subset \mathscr{R}_n$, so that $\mathscr{P}_n$ is a larger class than $\mathscr{P}_{\mathbb{R}^n,g}$ in this case.

The main goal of this paper is to demonstrate that if one has a "reference" two-point distribution μ_2 of class $\mathscr{P}_{M,g}$ which is of positive type (PT) and satisfies local commutativity (LC), then if another two-point distribution ω_2 is of positive type (PT) and has the same antisymmetric part as μ_2 modulo C^∞, and in addition differs from μ_2 by a C^∞ function on sets of the form $U \times U$, then ω_2 differs from μ_2 by a C^∞ function *globally*. In fact it turns out that to prove this statement it is not crucial (except to maintain physicality) to require that μ_2 satisfies (LC). The proof that local smoothness of the difference $\omega_2 - \mu_2$ implies global smoothness of $\omega_2 - \mu_2$ depends only on the (PT) requirements for μ_2 and ω_2, the local smoothness of the difference $\omega_2 - \mu_2$, the global smoothness of $(\omega_2 - \mu_2)_-$, and the requirement that μ_2 is of class $\mathscr{P}_{M,g}$.

The method of proof is to first prove the result on flat space-time $\mathbb{R}^n$, with the class $\mathscr{P}_n$ condition replacing the class $\mathscr{P}_{M,g}$ condition, then to extend the result to a curved space-time (with the class $\mathscr{P}_{M,g}$ condition) by using the coordinate charts on the manifold to map back to $\mathbb{R}^n$. Specifically, in Sects. 5 to 8 the following statement is proven:

Theorem 4.3 (Local-to-global). *Let* (M, g) *be a time orientable space-time, not necessarily globally hyperbolic. Suppose that* $\mu_2 \in \mathscr{D}_2'(M)$ *is of class* $\mathscr{P}_{M,g}$ *and satisfies (PT), and that* $\omega_2 \in \mathscr{D}_2'(M)$ *satisfies*

1. ω_2 *is of positive type,*
2. $(\omega_2 - \mu_2)_- \in C^\infty$, *and*
3. for all $x \in M$, *there is a neighborhood* U_x *of* x *such that* $(\omega_2 - \mu_2)|_{U_x \times U_x}$ *is smooth.*

Then $\omega_2 - \mu_2$ *is globally smooth.*

Sections 5 to 7 treat the case $M = \mathbb{R}^n$, and where "class $\mathscr{P}_n$" replaces "class $\mathscr{P}_{M,g}$."

Following is an outline of the proof of Theorem 4.3 for the case $M = \mathbb{R}^n$. First, in Sect. 5, for each $c \in (0,1)$, a pseudo-differential operator A_c with symbol a_c is constructed which preserves local smoothness and positivity of tempered two-point distributions and such that a_c has support inside a certain conic region (namely the region cone supp a_c, a slice of which is pictured in Fig. 1). Given any distribution $\mu_2 \in \mathscr{D}_2'(\mathbb{R}^n)$ such that $\pi_2 \mathrm{WF}(\mu_2)$ has support in the region $\mathscr{Q} = \pi_2 \mathscr{R}_n$ (i.e., a μ_2 of class $\mathscr{P}_n$), the value of c can be chosen (to be c_0, say) so that $A := A_{c_0}$ acts as a smoothing operator on μ_2. If ω_2 is a two-point distribution of positive type with the same local singularity structure as μ_2 and if we take $\chi = \phi_0 \otimes \phi_0$ where ϕ_0 is an arbitrary smooth cutoff function (so that $\chi\omega_2$ is also of positive type), then $A\chi\omega_2$ is locally smooth and of positive type. In Sect. 6 the Cauchy-Schwartz inequality for $A\chi\omega_2$ is used to show that $A\chi\omega_2$ is smooth everywhere. In Sect. 7, the global singularity structure of ω_2 is recovered from that of $A\chi\omega_2$ as follows. It is shown that $\widehat{\chi\omega_2}$ is of rapid decrease in certain directions determined by the conic support of the symbol of A, and that the Cauchy-Schwartz inequality for $\chi\omega_2$ extends the directions of rapid decrease to a larger set. The rapid decrease of $\widehat{\chi\mu_2}$ in these directions and the symmetries of $u = \omega_2 - \mu_2$ imply that $\widehat{\chi u}$ decreases rapidly in *all* directions, so that χu and hence u are smooth globally. Hence ω_2 has the same global singularity structure as μ_2.

In Sect. 8, coordinate charts on the manifold are used to map back to $\mathbb{R}^n$ and thereby show that the difference $(\omega_2 - \mu_2)(x_1', x_2')$ is smooth for x_1', x_2' in neighborhoods of $x_1, x_2 \in M$ respectively.

5. The Smoothing Operators A_c

We construct a class of pseudo-differential operators A_c corresponding to each c in the interval $(0,1)$ as follows. Fix $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ with the properties that supp $\psi_0 \subset \{k \in \mathbb{R}^n : |k| \leq 1\}$, $\psi_0(k) > 0$ for $|k| < 1$, and $\psi_0 = 1$ on the set $\{k \in \mathbb{R}^n : |k| \leq \frac{1}{2}\}$. The function ψ_0 is chosen to depend only on $|k|$. Fix $c \in (0,1)$ and for nonzero $\lambda \in \mathbb{R}^n$ let $\psi_{\lambda,c}(k) = \psi_0(\frac{k-\lambda}{c|\lambda|})$. Note that $\psi_{\lambda,c}(k) \neq 0$ for k in the open ball of radius $c|\lambda|$ with center at λ. If $\lambda \neq 0$, we have $c|\lambda| < |\lambda|$ and the support of $\psi_{\lambda,c}$ does not include $k = 0$.

Fix $\sigma > 0$ throughout this paper and let R_σ be the set $\{\lambda \in \mathbb{R}^n : \lambda_0 \geq 0, |\lambda| \geq \sigma\}$. For each $c \in (0,1)$ define the function a_c for $k_1, k_2 \in \mathbb{R}^n$ by

$$a_c(k_1, k_2) = \int_{R_\sigma} d\lambda\, \psi_{\lambda,c}(-k_1)\psi_{\lambda,c}(k_2). \tag{2}$$

This integral is well-defined since the support of $\psi_{\lambda,c}(k)$ with respect to λ for k fixed is compact. The slice of supp a_c in the $(k_1)_0$-$(k_2)_0$ plane is shown in Fig. 1.

Now denoting $(k_1, k_2) \in \mathbb{R}^{2n}$ by k, the operator A_c on $u \in \mathscr{S}'(\mathbb{R}^{2n})$ for $x = (x_1, x_2) \in \mathbb{R}^{2n}$ is defined by

$$A_c u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\mathrm{i}\langle x,k\rangle} a_c(k) \hat{u}(k)\, dk. \tag{3}$$

Here $\langle x, k\rangle = \langle x_1, k_1\rangle + \langle x_2, k_2\rangle$.

Following are some definitions needed for this and later sections.

Definition 5.1. *A set* $\mathscr{U} \subset \mathbb{R}^p$ *is* **conic** *iff* $sk \in \mathscr{U}$ *for all* $s \geq 0$ *whenever* $k \in \mathscr{U}$. *The* **conic extension** *of a set* $\mathscr{V} \subset \mathbb{R}^p$ *is the smallest closed conic set containing* $\mathscr{V}$ *and is denoted* cone $\mathscr{V}$. *If* b *is a function on* $\mathbb{R}^p$ *then* cone supp b *is called the* **conic support** *of* b.

See Fig. 1 for the slice of cone supp a_c in the $(k_1)_0$-$(k_2)_0$ plane.

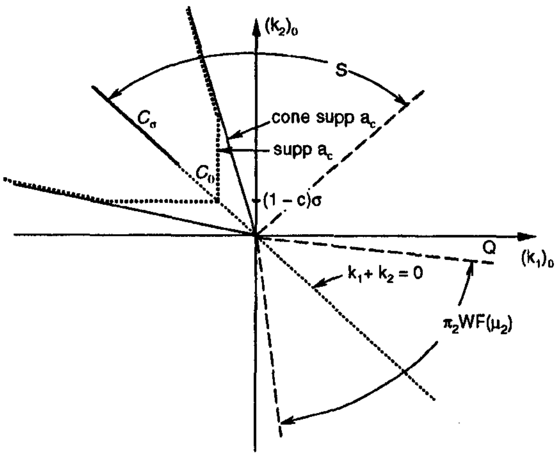

Fig. 1. Slices in the $(k_1)_0$-$(k_2)_0$ plane of supp a_c, cone supp a_c, $\{(k_1, k_2): k_1+k_2 = 0\}$, $\mathscr{C}_\sigma = \{(k_1, k_2): k_1+k_2 = 0,\ (k_2)_0 \geq 0,\ |k_2| \geq \sigma\}$, $\mathscr{C}_0 = \{(k_1, k_2): k_1+k_2 = 0,\ (k_2)_0 \geq 0\}$, $\mathscr{Q} = \{(k_1, k_2): (k_1)_0 > 0,\ (k_2)_0 < 0\}$, $\pi_2\mathrm{WF}(\mu_2)$ and S

Definition 5.2. *A distribution* $u \in \mathscr{D}_2'(M)$ *is* **locally smooth** *iff for each point* $x \in M$, *there is a neighborhood* U_x *of* x *such that* $u|_{U_x \times U_x}$ *is smooth.*

We summarize the desired properties of a_c and A_c in the following

Lemma 5.3 (Properties of a_c and A_c). *In the following* $c \in (0, 1)$ *is fixed except for property (e).*

(a) $a_c \in C^\infty(\mathbb{R}^{2n})$.
(b) $a_c \in S^n(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$, *hence* $A_c \in P^n(\mathbb{R}^{2n})$.
(c) A_c *maps* $\mathscr{S}'(\mathbb{R}^{2n})$ *into itself.*
(d) If $(k_1, k_2) \in \mathscr{C}_\sigma := \{(k_1, k_2): k_1 + k_2 = 0,\ (k_2)_0 \geq 0,\ |k_2| \geq \sigma\}$, *we have*

$$a_c(k_1, k_2) > 0\ .$$

(e) Given a μ_2 *satisfying the class* $\mathscr{P}_n$ *condition, the constant* $c \in (0, 1)$ *can be chosen small enough that*

$$\text{cone supp } a_c \cap \pi_2\mathrm{WF}(\mu_2) = \emptyset\ .$$

(f) If $u \in \mathscr{S}'(\mathbb{R}^{2n})$ *is locally smooth, then so is* $A_c u$.
(g) If $u \in \mathscr{S}'(\mathbb{R}^{2n})$ *is of positive type, then so is* $A_c u$.

Proof. Property (c) is obvious once (a) and (b) have been shown. Property (a) follows from the property that the integrand $\psi_{\lambda,c}(-k_1)\psi_{\lambda,c}(k_2)$ is smooth with respect to (k_1, k_2) and is smooth and compactly supported with respect to λ.

For $\mu > 0$, we have

$$\begin{aligned} a_c(\mu k_1, \mu k_2) &= \int_{R_\sigma} d\lambda\, \psi_{\lambda,c}(-\mu k_1)\psi_{\lambda,c}(\mu k_2) \\ &= \int_{R_\sigma} d\lambda\, \psi_{\lambda/\mu,c}(-k_1)\psi_{\lambda/\mu,c}(k_2) \\ &= \mu^n \int_{R_{\sigma/\mu}} d\lambda\, \psi_{\lambda,c}(-k_1)\psi_{\lambda,c}(k_2), \end{aligned}$$

where the following scaling property of $\psi_{\lambda,c}$ has been used:

$$\psi_{\lambda,c}(\mu k) = \psi_0\left(\frac{\mu k - \lambda}{c|\lambda|}\right) = \psi_0\left(\frac{k - \lambda/\mu}{c|\lambda/\mu|}\right) = \psi_{\lambda/\mu,c}(k)\,.$$

Note that when $|k_1|, |k_2|$ are large enough the condition $|\lambda| \geq \sigma$ in the integral in the definition of a_c in Eq. (2) is superfluous because the integrand already vanishes for $|\lambda| < \sigma$. Hence for large $|k_1|, |k_2|$ and $\mu \geq 1$, the last integral in the above calculation is $a_c(k_1, k_2)$ and it has been shown that $a_c(\mu k_1, \mu k_2) = \mu^n a_c(k_1, k_2)$. This proves (b). In particular, A_c is a pseudo-differential operator with symbol a_c homogeneous of degree n for large k.

Choose $(k_1, k_2) \in \mathscr{C}_\sigma$. Then $k_2 \in R_\sigma$ and

$$\psi_{\lambda,c}(-k_1)\psi_{\lambda,c}(k_2) = \left[\psi_{\lambda,c}(k_2)\right]^2$$

is strictly positive for $\lambda = k_2$. Since $\lambda = k_2$ is in the range of integration and the integrand in Eq. (2) is positive valued and smooth in λ, we have proven (d).

The proof for property (e) is rather long and is contained in Appendix A. Also, R. Verch has discovered a construction of a simpler pseudo-differential operator, which may be used in the proof of the main theorem. This construction, as well as the proof of the relevant properties, is found in Appendix B.

Property (f) follows easily from the pseudo-local property of pseudo-differential operators, namely sing supp $(A_c u) \subset$ sing supp u. See p. 39 of Taylor [41].

For $h \in C^\infty$ let h^- be the function $x \mapsto h(-x)$. If $f \in \mathscr{S}(\mathbb{R}^n)$ it follows that

$$A_c u(\bar{f} \otimes f) = [a_c \hat{u}]^\vee (\bar{f} \otimes f) = a_c \hat{u}((\bar{f})^\vee \otimes \check{f}) = \hat{u}\left(a_c(\overline{\hat{f}} \otimes \hat{f}^-)\right)\,.$$

Here the check symbol $^\vee$ denotes the inverse Fourier transform. Now

$$\begin{aligned} \left[a_c(\overline{\hat{f}} \otimes \hat{f}^-)\right](k_1, k_2) &= a_c(k_1, k_2)\overline{\hat{f}(k_1)}\hat{f}(-k_2) \qquad (4) \\ &= \int_{R_\sigma} d\lambda\, \psi_{\lambda,c}(-k_1)\overline{\psi_{\lambda,c}(k_2)\hat{f}(k_1)}\hat{f}(-k_2) \\ &= \int_{R_\sigma} d\lambda\, \overline{\hat{g}_\lambda(k_1)}\hat{g}_\lambda(-k_2) \\ &= \left(\int_{R_\sigma} d\lambda\, \overline{\hat{g}_\lambda} \otimes \hat{g}_\lambda^-\right)(k_1, k_2), \end{aligned}$$

where $\hat{g}_\lambda(k) = \hat{f}(k)\psi_{\lambda,c}(-k) \in C_0^\infty(\mathbb{R}^n)$ and hence is in $\mathscr{S}(\mathbb{R}^n)$. We have used the property that $\psi_{\lambda,c}$ is real-valued. Note that Eq. (4), together with $\overline{\hat{f}} \otimes \hat{f}^- \in \mathscr{S}(\mathbb{R}^{2n})$ and $a_c \in S^n(\mathbb{R}^{2n})$, implies that $\int_{R_\sigma} d\lambda\, \overline{\hat{g}_\lambda} \otimes \hat{g}_\lambda^- \in \mathscr{S}(\mathbb{R}^{2n})$. Hence,

$$A_c u(\bar{f} \otimes f) \;=\; \hat{u}\left(\int_{R_\sigma} d\lambda\, \overline{\hat{g}_\lambda} \otimes \hat{g}_\lambda^-\right). \tag{5}$$

Since the integral in Eq. (5) can be approximated by a Riemann sum, and since

$$\hat{u}(\overline{\hat{g}_{\lambda_j}} \otimes \hat{g}_{\lambda_j}^-) = u(\overline{g_{\lambda_j}} \otimes g_{\lambda_j}) \geq 0$$

for each term $\overline{\hat{g}_{\lambda_j}} \otimes \hat{g}_{\lambda_j}^-$ in the Riemann sum, continuity of $\hat{u}$ on $\mathscr{S}(\mathbb{R}^{2n})$ implies $A_c u(\bar{f} \otimes f) \geq 0$ and property (g) is proven. □

6. Local-to-Global Smoothness

Now we demonstrate that the positivity of a two-point distribution v leads to the conclusion, via the Cauchy-Schwartz inequality for v, that local smoothness of v implies global smoothness of v.

Proposition 6.1. *If $v \in \mathscr{D}'(\mathbb{R}^{2n})$ is locally smooth and of positive type, then v is globally smooth.*

Proof. The positivity of v implies the Cauchy-Schwartz inequality:

$$|v(f \otimes g)|^2 \leq v(f \otimes f)v(g \otimes g)$$

for real-valued $f, g \in C_0^\infty(\mathbb{R}^n)$. Now let f^p be a sequence of functions such that $f^p \in C_0^\infty(\mathbb{R}^n)$, $f^p \geq 0$, $\int_{\mathbb{R}^n} f^p = 1$, and supp $f^p \to \{0\}$. In other words $f^p \to \delta$ in the topology of $\mathscr{D}'(\mathbb{R}^n)$. Let $f_x^p(y) = f^p(y - x)$. Then

$$|v((f_x^p - f_x^q) \otimes g)|^2 \leq v((f_x^p - f_x^q) \otimes (f_x^p - f_x^q))v(g \otimes g)\,.$$

Choose an $x_0 \in \mathbb{R}^n$. Choose a positive integer N_0 and V, W, U (open) neighborhoods of x_0 with $\bar{V}, \bar{W}$ compact, $\bar{V} \subset W$, and $\bar{W} \subset U$ such that (i) v is smooth on $U \times U$ (so that v is uniformly continuous on $\bar{W}$), and (ii) for $p, q \geq N_0$ and $x \in \bar{V}$, we have supp $(f_x^p - f_x^q) \subset \bar{W}$. Given $\epsilon > 0$, choose δ so that $x_1, x_2, y_1, y_2 \in \bar{W}$ and $0 < |x_1 - y_1| < \delta$, $0 < |x_2 - y_2| < \delta$ imply that $|v(x_1, x_2) - v(y_1, y_2)| < \epsilon$. Also choose $N \geq N_0$ so that for all $p, q \geq N$, whenever $x \in \bar{V}$ and $x_1 \in$ supp $(f_x^p - f_x^q)$ we have $|x_1 - x| < \delta$. Then, since $\int dx_1 dx_2\, v(x, x)[f^p(x_1 - x) - f^q(x_1 - x)][f^p(x_2 - x) - f^q(x_2 - x)] = 0$ (we can perform the integration with respect to x_1 explicitly) and $v(f \otimes f)$ is positive, we have, for $x \in \bar{V}$ and $p, q \geq N$,

$$\begin{aligned}
&v\left((f_x^p - f_x^q) \otimes (f_x^p - f_x^q)\right) \\
&= \Big|\int dx_1 dx_2\, [v(x_1, x_2) - v(x, x)][f^p(x_1 - x) - f^q(x_1 - x)] \times \\
&\qquad [f^p(x_2 - x) - f^q(x_2 - x)]\Big| \\
&\leq \int dx_1 dx_2\, \epsilon |f^p(x_1 - x) - f^q(x_1 - x)||f^p(x_2 - x) - f^q(x_2 - x)| \\
&\leq 4\epsilon.
\end{aligned}$$

Hence $w_p(x,g) \equiv v(f_x^p, g)$ is uniformly Cauchy on the compact set $\bar{V}$ and, by completeness of $\mathbb{R}$, the sequence $\{w_p(x,g)\}$ converges uniformly on $\bar{V}$ to some number, call it $w(x,g)$.

Doing the same for the second argument results in a function $w(x,y)$ with $x, y \in \mathbb{R}^n$ such that $w(x,y)$ is the uniform limit on compact sets of $v(f_x^p \otimes f_y^q)$ as $p, q \to \infty$. To show that $w \in C^\infty$ we replace the test functions in the Cauchy-Schwartz inequality by $f = \partial^\alpha f_x^k - \partial^\alpha f_x^l$, $g = \partial^\beta g_y^p - \partial^\beta g_y^q$ and use an analogous argument as above to conclude that $w_{p,q}^{\alpha,\beta}(x,y) := v(\partial^\alpha f_x^p \otimes \partial^\beta f_y^q)$ is uniformly Cauchy on compact sets as $p, q \to \infty$, and hence has a uniform limit on compact sets, call it $w^{\alpha,\beta}(x,y)$.

Now clearly $w_{p,q}$ and $w_{p,q}^{\alpha,\beta}$ are smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore, $w_{p,q}^{\alpha,\beta}(x,y) = \partial_x^\alpha \partial_y^\beta w_{p,q}(x,y)$. Hence $\partial_x^\alpha \partial_y^\beta w_{p,q} \to w^{\alpha,\beta}$ uniformly on compact sets as $p, q \to \infty$.

It is now straightforward to argue that the derivatives $\partial_x^\alpha \partial_y^\beta w$ exist and are equal to $w^{\alpha,\beta}$ (see, for example, the proof of Theorem V.9 of Reed and Simon [39]). Finally, it is clear that v is representable by w, which completes the proof of the global smoothness of v. □

7. Proof of Main Theorem on $\mathbb{R}^n$

Proof of Theorem 4.3 for the case $M = \mathbb{R}^n$ (with class $\mathscr{P}_n$ replacing class $\mathscr{P}_{M,g}$). Let μ_2 and ω_2 be as in Theorem 4.3. Let $u = \omega_2 - \mu_2$. Since μ_2 is of class $\mathscr{P}_n$, by property (e) of Lemma 5.3 there is a c_0 for which cone supp a_{c_0} does not intersect $\pi_2 \mathrm{WF}(\mu_2)$. Hence supp a_{c_0} and $\pi_2 \mathrm{WF}(\mu_2)$ are also disjoint. Denote a_{c_0} by a and A_{c_0} by A.

Fix some real-valued function $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ and let $\chi = \phi_0 \otimes \phi_0 \in C_0^\infty(\mathbb{R}^{2n})$. Since $u \in \mathscr{D}'(\mathbb{R}^{2n})$ is locally smooth, so is χu and since $\chi u \in \mathscr{E}'(\mathbb{R}^{2n})$, property (f) of Lemma 5.3 implies that $A\chi u$ is also locally smooth. Furthermore $\chi\omega_2, \chi\mu_2 \in \mathscr{E}'(\mathbb{R}^{2n})$ are of positive type, and property (g) of Lemma 5.3 implies that $A\chi\omega_2, A\chi\mu_2$ are also.

Since μ_2 is of class $\mathscr{P}_n$ and $\mathrm{WF}(\chi\mu_2) \subset \mathrm{WF}(\mu_2)$, we obtain that $\pi_2 \mathrm{WF}(\chi\mu_2)$ and cone supp a are disjoint. By Proposition 3.3, we have that $\widehat{\chi\mu_2}(k_1, k_2)$ is of rapid decrease (see Sect. 3) in cone supp a. Thus, by property (b) of Lemma 5.3, the function $a\widehat{\chi\mu_2}$ is of rapid decrease in all of $\mathbb{R}^{2n} \backslash \{0\}$, and so $A\chi\mu_2 \in C^\infty$.

The preceding shows that $A\chi\omega_2 = A\chi u + A\chi\mu_2 \in \mathscr{S}'(\mathbb{R}^{2n})$ is locally smooth and of positive type. By Proposition 6.1 we have $A\chi\omega_2 \in C^\infty$.

Now let $\mathscr{C}_0$ be the conic set defined by

$$\mathscr{C}_0 = \{(k_1, k_2) \neq 0 \colon k_1 + k_2 = 0, (k_2)_0 \geq 0\} \, .$$

Note that $\mathscr{C}_0$ is the conic extension of $\mathscr{C}_\sigma$ for any $\sigma > 0$. Properties (b) and (d) of Lemma 5.3 and the global smoothness of $A\chi\omega_2$ then imply that $\widehat{\chi\omega_2}$ is of rapid decrease in $\mathscr{C}_0$. See Fig. 1 for the slices of $\mathscr{C}_\sigma$ and $\mathscr{C}_0$ in the $(k_1)_0$-$(k_2)_0$ plane.

Let S be the region

$$S = \{(k_1, k_2) \neq 0 \colon \ |k_2| \geq |k_1|, (k_2)_0 \geq 0\} \, .$$

Clearly S is a closed conic subset of $\mathbb{R}^{2n} \backslash \{0\}$ and $S \cap \mathscr{Q} = \emptyset$. See Fig. 1.

Although it is not apparent from Fig. 1, the sets supp a_c and cone supp a_c protrude slightly into the region $\mathscr{Q}$ in conic neighborhoods of points of the form $k_i = (0, (k_i)_1, (k_i)_2, (k_i)_3) \neq 0, i = 1, 2$. However, the choice of $a = a_{c_0}$ has been made according to property (e) of Lemma 5.3 so that cone supp a does not intersect $\pi_2 \mathrm{WF}(\mu_2)$ anywhere in $\mathbb{R}^{2n}\backslash\{0\}$.

Now we consider the Cauchy-Schwartz inequality for $\chi\omega_2$ expressed in terms of the Fourier transform $\widehat{\chi\omega_2}$:

$$|\widehat{\chi\omega_2}(\hat{f} \otimes \hat{g})|^2 \leq \widehat{\chi\omega_2}(\hat{f} \otimes \hat{f}^-)\widehat{\chi\omega_2}(\hat{g}^- \otimes \hat{g}) ,$$

where $\hat{f}, \hat{g}$ are arbitrary real-valued Schwartz test functions. Since χ is of compact support, $\widehat{\chi\omega_2}(k)$ is a smooth function of $k \in \mathbb{R}^{2n}$, so one may insert in the inequality sequences of positive-valued test functions $\{\hat{f}^i\}$, $\{\hat{g}^i\}$ which tend to the delta functions δ_{k_1} and δ_{k_2} in the topology of $\mathscr{D}'(\mathbb{R}^n)$. In this limit the inequality becomes

$$|\widehat{\chi\omega_2}(k_1, k_2)|^2 \leq \widehat{\chi\omega_2}(k_1, -k_1)\widehat{\chi\omega_2}(-k_2, k_2) . \tag{6}$$

The property $S \cap \mathscr{Q} = \emptyset$ implies by Proposition 3.3 that $\widehat{\chi\mu_2}$ is of rapid decrease in S. Furthermore, from $\chi\omega_2 \in \mathscr{E}'(\mathbb{R}^{2n})$ it follows (see e.g., [21]) that there exists an integer M and a constant C such that for all $k \in \mathbb{R}^{2n}\backslash\{0\}$, we have $|\widehat{\chi\omega_2}(k)| \leq C(1 + |k|)^M$. Hence for $(k_1, k_2) \in S$, we have $(-k_2, k_2) \in \mathscr{C}_0$, and Eq. (6) implies that for all N, there exists C_N such that

$$|\widehat{\chi\omega_2}(k_1, k_2)|^2 \leq \widehat{\chi\omega_2}(k_1, -k_1)\widehat{\chi\omega_2}(-k_2, k_2)$$

$$\leq C_N^2(1 + \sqrt{2}|k_1|)^M(1 + \sqrt{2}|k_2|)^{-M-2N} .$$

But $\sqrt{2}|k_1| \leq \sqrt{|k_1|^2 + |k_2|^2} = |k|$ and $\sqrt{2}|k_2| \geq |k|$ since $|k_2| \geq |k_1|$. So in S, we have the inequality $|\widehat{\chi\omega_2}(k)|^2 \leq C_N^2(1 + |k|)^M(1 + |k|)^{-M-2N} = C_N^2(1 + |k|)^{-2N}$. Hence $\widehat{\chi\omega_2}$ is of rapid decrease in S. Furthermore, the rapid decrease of $\widehat{\chi\mu_2}$ and $\widehat{\chi\omega_2}$ in S imply that of $\widehat{\chi u}$ in S.

Since ϕ_0 has been chosen in the definition of $\chi = \phi_0 \otimes \phi_0$ to be real-valued, χu has the following two symmetries. Firstly, $\widehat{\chi u}(k_1, k_2) = \widehat{\chi u}(k_2, k_1)$ modulo a term of rapid decrease, since $\chi = \phi_0 \otimes \phi_0$ is symmetric with respect to interchange of k_1 and k_2 and u has smooth antisymmetric part, by hypothesis. Secondly, it follows from the positivity of μ_2 and ω_2 that the $(\mu_2)_+, (\omega_2)_+$ are real and $(\mu_2)_-, (\omega_2)_-$ are imaginary. The hypothesis of the smoothness of u_- then implies that χu must be real-valued up to C^∞. This implies that $\overline{\widehat{\chi u}(k_1, k_2)} = \widehat{\chi u}(-k_1, -k_2)$ modulo a term of rapid decrease. The second symmetry extends the rapid decrease of $\widehat{\chi u}(k_1, k_2)$ from directions $(k_1, k_2) \neq 0$ for which $|k_2| \geq |k_1|$ and $(k_2)_0 \geq 0$ (i.e., points in S) to those directions for which $|k_2| \geq |k_1|$ and $(k_2)_0 \leq 0$. Hence all directions $(k_1, k_2) \neq 0$ with $|k_2| \geq |k_1|$ are directions of rapid decrease. The first symmetry extends the directions of rapid decrease from all $(k_1, k_2) \neq 0$ for which $|k_2| \geq |k_1|$ to those for which $|k_1| \geq |k_2|$. Hence all directions $(k_1, k_2) \neq 0$ are directions of rapid decrease. Therefore $\chi u \in C_0^\infty(\mathbb{R}^{2n})$. Since the support of ϕ_0 was arbitrary, $u \in C^\infty(\mathbb{R}^{2n})$. □

8. Extension to Curved Space-Time

Proof of Theorem 4.3. Let (M, g), μ_2 and ω_2 satisfy the hypotheses of Theorem 4.3. Set $u = \omega_2 - \mu_2$ as before. Choose any points $x_1, x_2 \in M$ such that $x_1 \neq x_2$, together with small enough contractible open neighborhoods U_1, U_2 containing x_1, x_2 such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. We wish to show that u is smooth when restricted to the neighborhood

$U_1 \times U_2$ of (x_1, x_2) in $M \times M$. Choose chart mappings ϕ_1, ϕ_2 sending U_1, U_2 to open subsets V_1, V_2 of $\mathbb{R}^n$ respectively, such that $\overline{V_1} \cap \overline{V_2} = \emptyset$ and

$$((\phi_1^{-1})^* k_1)_0 > 0 \text{ and } ((\phi_2^{-1})^* k_2)_0 > 0 \tag{7}$$

for any cotangent vectors $k_1 \in (V_{x_1'}^+)^d$, $k_2 \in (V_{x_2'}^-)^d$. Here x_1' and x_2' range in the sets U_1 and U_2 respectively. Such a choice of ϕ_1, ϕ_2 is always possible for small enough U_1, U_2 on the time orientable CST (M, g).

Now let $\psi_1 \in C_0^\infty(U_1), \psi_2 \in C_0^\infty(U_2)$ be positive-valued cutoff functions such that $\psi_1(x_1) \neq 0$ and $\psi_2(x_2) \neq 0$. Also let $\tilde{\psi}_1 = (\phi_1^{-1})^* \psi_1$ and $\tilde{\psi}_2 = (\phi_2^{-1})^* \psi_2$. The mapping ϕ^* on $C_0^\infty(V_1 \cup V_2)$ is defined to be

$$\phi^* f = \phi_1^*(f|_{V_1}) + \phi_2^*(f|_{V_2}) .$$

Also let $\tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2$, where $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are extended to functions in $C_0^\infty(V_1 \cup V_2)$ by defining them to be 0 outside of V_1 and V_2 respectively. Let $\tilde{\mu}_2$ be the following distribution on $\mathbb{R}^n \times \mathbb{R}^n$: if $\tilde{f}, \tilde{g} \in C_0^\infty(\mathbb{R}^n)$ then

$$\tilde{\mu}_2(\tilde{f} \otimes \tilde{g}) = \mu_2[\phi^*(\tilde{\psi}\tilde{f}) \otimes \phi^*(\tilde{\psi}\tilde{g})] .$$

Similarly, define $\tilde{\omega}_2(\tilde{f} \otimes \tilde{g}) = \omega_2[\phi^*(\tilde{\psi}\tilde{f}) \otimes \phi^*(\tilde{\psi}\tilde{g})]$ and $\tilde{u} = \tilde{\omega}_2 - \tilde{\mu}_2$.

Clearly, $\tilde{\mu}_2$ and $\tilde{\omega}_2$ are in $\mathscr{D}'(\mathbb{R}^{2n})$ and are of positive type. Also $(\tilde{\omega}_2 - \tilde{\mu}_2)_-$ is smooth and $\tilde{\omega}_2 - \tilde{\mu}_2$ is locally smooth, as follow directly from the corresponding properties of $\omega_2 - \mu_2$. Now since the wave front set transforms under diffeomorphisms of the manifold M as a subset of the cotangent bundle $T^*(M)$ (cf. Theorem 8.2.4 of [21]), we have

$$\mathrm{WF}(\tilde{\mu}_2) \subset (\phi^{-1} \otimes \phi^{-1})^* \mathrm{WF}((\psi \otimes \psi)\mu_2) .$$

Here $\psi = \psi_1 + \psi_2$, where ψ_1, ψ_2 are extended to $C_0^\infty(M)$ by defining them to be 0 outside U_1, U_2 respectively. Since μ_2 is of class $\mathscr{P}_{M,g}$ (Definition 4.1) and $\mathrm{WF}((\psi \otimes \psi)\mu_2) \subset \mathrm{WF}(\mu_2)$, the distribution $(\psi \otimes \psi)\mu_2$ is also of class $\mathscr{P}_{M,g}$ and by Eq. (7), $\tilde{\mu}_2$ must be of class $\mathscr{P}_n$ (Definition 4.2). Hence the results of Sects. 5, 6 and 7 apply and we conclude that $\tilde{\omega}_2 - \tilde{\mu}_2 \in C^\infty(\mathbb{R}^{2n})$. This means that $(\omega_2 - \mu_2)|_{(U_1 \cup U_2) \times (U_1 \cup U_2)} \in C^\infty$. Without loss of generality, U_1 and U_2 can be chosen so that $\omega_2 - \mu_2$ is smooth on $U_1 \times U_1$ and $U_2 \times U_2$. Then, since U_1 and U_2 are disjoint, it follows that $(\omega_2 - \mu_2)|_{U_1 \times U_2} \in C^\infty$, which is the desired result. Since the choice of the original points $x_1, x_2 \in M$ was arbitrary, it follows that $\omega_2 - \mu_2 \in C^\infty(M \times M)$. The proof of Theorem 4.3 is complete. □

9. Verification of Kay's Conjecture

Work published elsewhere [12, 8, 37, 38] shows the existence of globally Hadamard two-point distributions μ_2 satisfying (KG), (Com) and (PT) up to C^∞ on any globally hyperbolic space-time (M, g). According to Condition 3 of Theorem 5.1 of [38] (see also the Note Added in Proof), the global Hadamard condition for a μ_2 satisfying (KG) and (Com) is equivalent to the following condition:

Definition 9.1. *Let $\mu_2 \in \mathscr{D}_2'(M)$ where (M, g) is a time orientable curved space-time (not necessarily globally hyperbolic). Then μ_2 satisfies the* **wave front set spectral condition (WFSSC)** *if*

$$\begin{aligned} \mathrm{WF}(\mu_2) &= \{((x_1, k_1), (x_2, k_2)) \in (T^*(M) \times T^*(M))\backslash \mathbf{0}: \\ &\qquad (x_1, k_1) \sim (x_2, -k_2), k_1 \in (V_{x_1}^+)^d\}, \end{aligned} \tag{8}$$

where the equivalence relation $(x_1, k_1) \sim (x_2, k_2)$ means that x_1 and x_2 are null related by a null geodesic γ and the duals of k_1 and k_2 are the tangents to γ at x_1 and x_2 respectively.

Observe that the WFSSC requires $\mu_2(x_1, x_2)$ to have singularities at all points x_1, x_2 connected by a null geodesic, and if x_1 and x_2 are connected by more than one null geodesic, then there will be several directions for (k_1, k_2) (all null) in $\mathrm{WF}(\mu_2)$ at (x_1, x_2). This property is sufficient for a model satisfying a linear wave equation. For more general models one does not expect that these directions will be only null; they may also be time-like (see Köhler [29] and the upcoming discussion on his WFSSC).

Note that if $((x_1, k_1), (x_2, k_2)) \in \mathrm{WF}(\mu_2)$ then $k_1 \in (V_{x_1}^+)^d$, $k_2 \in (V_{x_2}^-)^d$, and so the set on the right side of Eq. (8) is a subset of $\mathscr{R}_{M,g}$. See Definition 4.1. Hence a μ_2 satisfying the WFSSC is of class $\mathscr{S}_{M,g}$. Theorem 4.3 then implies that any two-point distribution ω_2 satisfying (PT), (Com) and the local Hadamard condition is *globally* Hadamard. (This is all in 4 dimensions, but we foresee no obstacles to extending this to any dimension $n \geq 2$, provided that the appropriate replacements are made in the asymptotic expressions in the global Hadamard condition.) This verifies Kay's conjecture.

Indeed, what has been proven is stronger than Kay's conjecture:

Theorem 9.2. *Let (M, g) be a 4-dimensional globally hyperbolic (hence time orientable) curved space-time. Let $\omega_2 \in \mathscr{D}_2'(M)$ be the two-point distribution of a state (so that ω_2 is of positive type). Suppose also that ω_2 satisfies (Com) and the local Hadamard condition. Then ω_2 is globally Hadamard.* □

Note that ω_2 is not required to satisfy (KG) even up to C^∞. Furthermore, μ_2 and ω_2 need not be the two-point distributions of quasi-free states; they are allowed to be those of any two states μ and ω on (M, g). Finally, observe that the only reason that the space-time must be globally hyperbolic is in order that ω_2 can satisfy the global Hadamard condition (the statement of which requires global hyperbolicity). Otherwise the manifold need only be time orientable.

In [37] a condition more general than WFSSC was proposed (Property 4.9), which allowed k_1 to be any covector in $(V_{x_1}^+)^d$, and required $-k_2$ to be the parallel transport of k_1 along some causal geodesic from x_1 to x_2. Recently Köhler [29] has proposed a modification to the WFSSC designed to take into account (for non-linear theories) the possibility that more than one causal geodesic may connect x_1 and x_2 and that at such (x_1, x_2) the covectors (k_1, k_2) may not be null. (This was not considered in [37].) His condition is reproduced as follows:

Definition 9.3 ([29]; Definition 7). *The two-point distribution $\omega_2 \in \mathscr{D}_2'(M)$ satisfies the* **[modified] wave front set spectrum condition** *iff its wave front set* $\mathrm{WF}(\omega_2)$ *consists only of points $(x_1, k_1), (x_2, k_2) \in T^*(M)\backslash \mathbf{0}$ such that x_1 and x_2 are causally related and k_1 is in the dual of the closed forward light cone. Furthermore there are causal geodesics γ_i joining x_1 and x_2 and vectors l_i in the dual of the closed forward light cone, such that $\sum_i l_i = k_1$ and the parallel transported vectors l_i along γ_i sum up to $-k_2$.*

In the case that several null geodesics connect x_1 and x_2, this definition allows the covector k_1 to be split up into several parts, such that each part is in $(V_{x_1}^+)^d$ and such that after each part is parallel transported along a different null geodesic to x_2 and summed there, the covector $-k_2$ is obtained. Köhler presents new examples of Wightman fields (other than linear) on a manifold satisfying this more general condition and violating Definition 9.1. (For fields satisfying a linear wave equation, the WFSSC in Definition 9.1 is sufficient.) We observe that a local-to-global singularity theorem is true for examples satisfying this modified WFSSC since the class $\mathscr{P}_{M,g}$ condition can still be shown to follow from it. See also the dissertation of Köhler [30] where it is shown that the "supercurrent" (in a globally Hadamard product state) of an analog of the free Wess-Zumino model on Ricci flat (globally hyperbolic) space-times is a Wightman field satisfying the modified WFSSC in a nontrivial way.

10. A Counterexample on a Strip in $\mathbb{R}^4$

Consider the vacuum two-point distribution $\omega_2^0 \in \mathscr{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$, defined as the inverse Fourier transform of

$$\frac{1}{2\pi}\delta(k_1 + k_2)\theta((k_1)_0)\delta(k_1^2 - m^2)\,.$$

Let G be the symmetric part of ω_2^0:

$$G = (\omega_2^0)_+\,.$$

Recall that $\frac{\mathrm{i}}{2}\Delta$ is the antisymmetric part of ω_2^0, so that $\omega_2^0 = G + (\mathrm{i}/2)\Delta$. Take $\mu_2 = 2G + (\mathrm{i}/2)\Delta = \omega_2^0 + G$ and $\omega_2 = \omega_2^0 + (G + G^T)$. Here we define G^T as $G^T(f \otimes g) = G(f \otimes g^T)$, where $g^T(x_2^0, \mathbf{x}_2) = g(-x_2^0, \mathbf{x}_2)$ and where $(x_2^0, \mathbf{x}_2) = x_2 = (x_2^0, x_2^1, x_2^2, x_2^3)$. The globally hyperbolic space-time is chosen to be

$$M = \{x \in \mathbb{R}^4 : a < x^0 < b\}$$

for some fixed a, b such that $0 < a < b$. On M, the difference $\omega_2 - \mu_2$ is locally smooth, the antisymmetric part $(\omega_2 - \mu_2)_-$ vanishes, and μ_2, ω_2 are of positive type. The positivity is easy to see for μ_2, since it is the sum of distributions of positive type, namely ω_2^0 and G. To verify positivity for ω_2, choose $f \in C^\infty(\mathbb{R}^4)$ and decompose f into its even and odd parts f_+, f_- with respect to x^0, so that $f = f_+ + f_-$, $f_+^T = f_+$ and $f_-^T = -f_-$. Because of $G(f \otimes g) = G(f^T \otimes g^T)$ and the symmetry and time-translation invariance of G, we have $G(\bar{f}_+ \otimes f_-) = G(\bar{f}_- \otimes f_+) = 0$, and the same holds for G^T. Thus

$$\begin{aligned} \omega_2(\bar{f} \otimes f) &= \omega_2^0(\bar{f} \otimes f) + [G(\bar{f}_+ \otimes f_+) + G(\bar{f}_- \otimes f_-) \\ &\quad + G^T(\bar{f}_+ \otimes f_+) + G^T(\bar{f}_- \otimes f_-)] \\ &= \omega_2^0(\bar{f} \otimes f) + [G(\bar{f}_+ \otimes f_+) + G(\bar{f}_- \otimes f_-) \\ &\quad + G(\bar{f}_+ \otimes f_+) - G(\bar{f}_- \otimes f_-)] \\ &= \omega_2^0(\bar{f} \otimes f) + 2G(\bar{f}_+ \otimes f_+) \geq 0. \end{aligned}$$

(Note also that μ_2, ω_2 satisfy (KG).)

However, ω_2 has singularities at space-like separated points, namely the singularities arising from G^T (cf. the first example considered in Gonnella and Kay [14]).

Finally, one can show that

$$\pi_2 \mathrm{WF}(\mu_2) = \{(k_1, k_2) \neq 0 : k_1 + k_2 = 0\} .$$

Thus $\mathrm{WF}(\mu_2)$ is not a subset of $\mathscr{R}_n$, nor of $\mathscr{R}_{M,g}$ for our choice of (M, g). Hence if μ_2 does not satisfy the class $\mathscr{P}_n$ condition (and hence does not satisfy the class $\mathscr{P}_{M,g}$ condition) the local-to-global statement need not hold on this space-time. We expect similar counter-examples to exist for arbitrary time orientable space-times.

11. Discussion

There is already much evidence in the literature [5, 43, 44, 13, 12, 28, 42] that the global Hadamard condition (GH) should be regarded as a necessary condition for a quasi-free state on a Klein-Gordon quantum field model to be "physical." (GH) strongly appears to be distinguished among properties characterizing the asymptotics of quasi-free Klein-Gordon states for x_1 near to x_2. The resolution of Kay's conjecture provides further confidence in (GH) since with (GH) we have a local to global theorem (Theorem 9.2), whereas without (GH) no such theorem is expected to hold (Sect. 10). Moreover, Theorem 4.3 remains true with the class $\mathscr{P}_{M,g}$ condition replaced by Köhler's modified WFSSC. In the opinion of this author, this latter condition is a strong candidate for a generalization of the Hadamard condition for two-point distributions satisfying (PT) and (LC) on any quantum field model (free or self-interacting) on a time orientable space-time. Furthermore, an explicit formulation of a WFSSC for m-point distributions with $m \geq 3$ has been suggested by Köhler [30]. (A tentative attempt made in Chapter 4 of [37] has been shown by Köhler to be inadequate.) In any case we propose that for a quantum field model (M, g, ω) on a time orientable space-time (M, g) there is one more physically necessary condition on ω_2 besides (PT) and (LC), namely Köhler's modified WFSSC.

We end our discussion of the local-to-global theorem by restating Theorem 9.2 as a converse of Theorem 6.6.2 of [8], demonstrating some dependence (mod C^∞) between the global Hadamard condition and the positivity condition:

Corollary 11.1. *If (M, g) is a 4-dimensional globally hyperbolic CST, and ω_2 is a locally Hadamard two-point distribution satisfying (KG) and (Com) mod C^∞, then the following statements are equivalent:*

1. *ω_2 is globally Hadamard.*
2. *ω_2 is of positive type mod C^∞.* □

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A. Proof of Lemma 5.3(e)

Following we present the proof of Lemma 5.3(e).

Define the sets S_1 and $S_{2,c}$ for $c \in (0,1)$ to be the intersections of the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ with the conic sets $\pi_2 \mathrm{WF}(\mu_2)$ and cone supp a_c respectively. Also let $R = S^{n-1} \cap \text{cone } R_\sigma = S^{n-1} \cap \{\lambda \in \mathbb{R}^n\colon\ \lambda_0 \geq 0\}$ and let $S_3 = S^{2n-1} \cap$ cone $\{(k_1, k_2)\colon\ k_1 + k_2 = 0,\ k_2 \in R\} = S^{2n-1} \cap \mathscr{C}_0$, where $\mathscr{C}_0$ is defined in Sect. 7. Clearly S_1, $S_{2,c}$, and S_3 are compact subsets in S^{2n-1} and R is compact in S^{n-1}, and $S_1 \cap S_3 = \emptyset$ since μ_2 is of class $\mathscr{P}_n$ by assumption. For $\lambda \in \mathbb{R}^n \backslash \{0\}$ and $c \in (0,1)$, let $b_{\lambda,c}(k_1,k_2) = \psi_{\lambda,c}(-k_1)\psi_{\lambda,c}(k_2)$. Also for $y \in R$ and $c \in (0,1)$ let $S_{4,y,c}$ be the intersection of S^{2n-1} with cone supp $b_{y,c}$. Note that cone supp $b_{y,c} =$ cone supp $b^s_{y,c}$ for $s > 0$, where $b^s_{y,c}(k_1,k_2) = b_{y,c}(sk_1, sk_2) = b_{y/s,c/s}(k_1,k_2)$. Hence cone supp $a_c \subset \cup_{\lambda \in \text{cone} R_\sigma}$cone supp $b_{\lambda,c} \subset \cup_{y \in R}$cone supp $b_{y,c}$, hence $S_{2,c} \subset \cup_{y\in R} S_{4,y,c}$.

Note that if $\lambda \neq 0$, then as c tends to 0 the conic support of $b_{\lambda,c}$ tends to the ray in the $(-\lambda, \lambda)$ direction. Furthermore, (as will be shown) for any $y \in R$, $S_{4,y,c}$ may be contained within a set of the form $B^S_{k_c}(w_y)$, where k_c is a constant depending only on c (not on y) which can be chosen so that $k_c \to 0$ as $c \to 0$ and where $w_y = (-y,y)/\sqrt{2}$. Here, $B^S_r(x)$ is the closed ball in S^{2n-1} with radius r and center $x \in S^{2n-1}$, with respect to the spherical metric $d(\cdot,\cdot)$, i.e., $B^S_r(x) = \{z \in S^{2n-1}\colon\ d(x,z) \leq r\}$. The distance $\mathrm{dist}(S_1, S_3) = \inf_{m_1 \in S_1, m_2 \in S_3} d(m_1, m_2)$ between the disjoint compact sets S_1, S_3 is clearly nonzero. If k is chosen to be *half* that distance, then for all $y \in R$, the set $B^S_k(w_y)$ will be disjoint from S_1. Then, for some c chosen so that for all $y \in R$, $S_{4,y,c} \subset B^S_k(w_y)$, we must have

$$S_{2,c} \subset \cup_{y\in R} S_{4,y,c} \subset \cup_{y \in R} B^S_k(w_y) \subset S_1^c\ .$$

This would then prove (e). What remains is to show that $\forall c$, $\exists k_c$ such that $\forall y \in R$, $S_{4,y,c} \subset B^S_{k_c}(w_y)$.

Note that if (k_1,k_2) is a point in the set $S_{4,y,c}$ (for $y \in R$) then the following relations are satisfied:

1. $|k_1|^2 + |k_2|^2 \leq 1$,
2. for some $s > 0$ we have $|k_1 + sy| \leq c|sy|$, $|k_2 - sy| \leq c|sy|$.

Clearly the set defined by the second pair of relations lies within the ball in $\mathbb{R}^{2n}$ given by

$$|(k_1,k_2) - (-sy, sy)| \leq c|(-sy,sy)|\ .$$

Hence $S_{4,y,c} \subset S^{2n-1} \cap \{(k_1,k_2)\colon\ \exists s > 0, (k_1,k_2) \in B_{c|(-sy,sy)|}(-sy,sy)\}$. Here $B_t(w)$ is the closed ball with radius t and center $w \in \mathbb{R}^{2n}$ with respect to the Euclidean metric $|\cdot|$ on $\mathbb{R}^{2n}$. The set on the RHS of the above inclusion is obtained by taking the union of the balls $B_{c|(-sy,sy)|}(-sy,sy)$ over all values of $s > 0$ and then intersecting with S^{2n-1}. It is clear (from drawing a sketch) that this union of balls is the solid closed cone (minus the vertex) whose angle from the axis direction $(-y,y)$ is $\arcsin c$. Hence when $k_c = \arcsin c$ then for any $y \in R$, $S_{4,y,c}$ lies within the set $B^S_{k_c}(w_y)$ in S^{2n-1}. This proves (e) of Lemma 5.3. □

B. Construction of Simple Smoothing Operator (by *Rainer Verch*)

For a given $\mu_2 \in \mathscr{E}'(\mathbb{R}^{2n})$ with $\mathrm{WF}(\mu_2) \subset \mathscr{R}_n$, a simple smoothing operator A with properties sufficient for the proof of the main theorem on $\mathbb{R}^n$ will be constructed in this Appendix.

Let $\mu_2 \in \mathscr{E}'(\mathbb{R}^{2n})$ with $\mathrm{WF}(\mu_2) \subset \mathscr{R}_n$ be given. Then there is some positive number α such that $\Sigma(\mu_2) = \pi_2\mathrm{WF}(\mu_2)$ is contained in a set of the form $K \times (-K)$, where K is some open conic set in $\mathbb{R}^n$ with the property that $k_0 > \alpha|\mathbf{k}|$ for all $k = (k_0, \mathbf{k}) \in K$.

Relative to this set K, we shall define A as the pseudo-differential operator of a symbol a, whose construction will be given as follows: Consider the by a unit in the negative k_0-direction shifted copy $K_1 := K - (1, \mathbf{0})$ of K. The two sets $\mathscr{N}_\ell := \mathbb{R}^n \backslash \overline{K}$, $\mathscr{N}_u := K_1$, form an open covering of $\mathbb{R}^n$; denote by ψ_ℓ, ψ_u a smooth partition of unity in $\mathbb{R}^n$ subordinate to this covering. Let j be a smooth monotone function on $\mathbb{R}$ taking non-negative values and with the property that $j(t) = 0$ for $t > 2$ and $j(t) = 1$ for $t < 1$. Then define the function

$$a(k_1, k_2) := j((k_1)_0)\psi_\ell(k_1)j(-(k_2)_0)\psi_\ell(-k_2), \quad k_1, k_2 \in \mathbb{R}^n,$$

on $\mathbb{R}^{2n}$. It is not difficult to check that a and all its derivatives are bounded, so we have $a \in S^0_{0,0}(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$, and we define A as the corresponding pseudo-differential operator acting on $\mathscr{S}'(\mathbb{R}^{2n})$.

Lemma B.1. *Properties of a and A are:*

1. $0 \leq a(k_1, k_2) \leq 1$ *for all* $k_1, k_2 \in \mathbb{R}^n$.
2. u *of positive type implies* Au *of positive type for all* $u \in \mathscr{S}'(\mathbb{R}^{2n})$.
3. (cone supp a) $\cap\, \pi_2\mathrm{WF}(\mu_2) = \emptyset$.
4. *If* $k_1 = -k_2 \in \mathbb{R}^n \backslash \{0\}$, *with* $(k_1)_0 \leq 0$ *and* $|k_1|^2 > \alpha^{-2} + 1$, *then* $a(k_1, k_2) = 1$.

Proof. 1) is clear from the definition. 2) follows as in the proof of Lemma 5.3(g), it is a straightforward consequence of the special form of a. 3) We have $\pi_2\mathrm{WF}(\mu_2) \subset K \times (-K)$, and a has support in $(\mathbb{R}^n \backslash K) \times (\mathbb{R}^n \backslash (-K))$ which is a conic set in $\mathbb{R}^{2n}$. 4) Observe that $(k_1)_0 \leq 0$ and $|k_1|^2 > \alpha^{-2} + 1$ entails $j((k_1)_0) = 1$ and $k_1 \in \mathbb{R}^n \backslash \overline{K_1}$, and we have $\psi_\ell \equiv 1$ on $\mathbb{R}^n \backslash \overline{K_1}$. So if in addition $k_1 = -k_2$, then $a(k_1, k_2) = a(k_1, -k_1) = j((k_1)_0)^2\psi_\ell(k_1)^2 = 1$. □

Remark. For the proof of the main theorem it is sufficient to have such a smoothing operator for each $\mu_2 \in \mathscr{E}'(\mathbb{R}^{2n})$. In the proof of the main theorem, the initially given $\mu_2 \in \mathscr{D}'(\mathbb{R}^{2n})$ is multiplied by a smooth spatial cut-off function χ of compact support, and the smoothing operator is only applied on $\chi\mu_2$.

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