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# Irreducible Highest Weight Representations of Quantum Groups $U_q(gl(n, C))$

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**Abstract.** Explicit recurrence formulas of canonical realization (boson representation) for quantum enveloping algebras $U_q(gl(n, C))$ are given. Using them, irreducible highest weight representations of $U_q(gl(n, C))$ are obtained as restriction of representation on Fock space to invariant subspace generated by vacuum as a cyclic vector.

## 1. Introduction

The question of irreducible representations of quantum enveloping algebras was recently treated in a number of papers [1]. For irreducible highest weight representations (h.-w.irreps) it is known that their properties do not substantially differ (at least if $q$ is not a root of unity) from the usual Lie algebra case. Especially it is proved [2] that a h.-w.irrep of the quantum enveloping algebra is uniquely determined (up to isomorphism) by its highest weight. It is also known for which highest weights these representations are finite dimensional.

These results were obtained by generalization of methods from the theory of highest weight representations of semisimple Lie algebras by means of which the construction of the explicit form of the highest weight representation for quantum enveloping algebras is, in principle, possible too.

In this paper we perform, in fact, this construction for the quantum enveloping algebra $U_q(gl(n, C)) \supset U_q(sl(n, C))$ defined in [3]. We do not, however, describe the details of construction (for the case of simple Lie algebras see [4]) but present the final formulas for direct verification.

In these formulas the generators of the algebra $U_q(gl(n + 1, C))$ are expressed by means of $n$ canonical boson pairs, one complex parameter and auxiliary representation of the algebra $U_q(gl(n, C))$. This recurrence character of formulas (1) is, by our opinion, its first interesting feature. It makes it possible, e.g. to obtain for special weights a simpler form of representation in comparison with general cases (see Concluding Remarks). The second advantage is that the invariant subspace with vacuum as cyclic vector is an irreducible one.

Usually the h.-w.irreps are constructed using factorisation of certain standard representation by maximal invariant subspace [2]; we believe that our representations could be more convenient for practical use.

## 2. Two Theorems and Irreducibility Lemma

Through our paper we will use the following notation: $a_\alpha, a_\alpha^+$-creation-annihilation boson pairs; $[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}$; $\alpha, \beta = 1, 2, 3, \ldots, n$, $a^+a \equiv \sum_{\alpha=1}^{n} a_\alpha^+ a_\alpha$, $\mathscr{F}_n = \mathscr{F} \otimes \mathscr{F} \otimes \ldots \otimes \mathscr{F}$, *n-times*; $\mathscr{F}$-Fock space, $\bar{A}_j \equiv q^{-2a_j^+a_j-2} a_j$, $A_j \equiv f_q(a_j^+ a_j) a_j^+$, where $f_q(x) = q^2 \dfrac{1-q^{4x}}{x(1-q^4)}$ if $q^4 \neq 1$ and for $q^4 = 1$, $f_q(x) = 1$.

Let $U_q(gl(n, C))$ be the associative *C-algebra* generated by

$$(e_1, \ldots, e_{n-1}; f_1, \ldots, f_{n-1}; l_1^{\pm 1}, \ldots, l_n^{\pm 1})$$

with relations

$$\begin{gathered}
l_\alpha l_\beta = l_\beta l_\alpha\,, \qquad l_\alpha e_j = q^{-\delta_{\alpha j}+\delta_{\alpha j+1}} e_j l_\alpha\,, \qquad l_\alpha f_j = q^{\delta_{\alpha j}-\delta_{\alpha j+1}} f_j l_\alpha\,, \\
[e_i, f_j] = \delta_{ij} \frac{(l_{i+1} l_i^{-1})^2 - (l_{i+1} l_i^{-1})^{-2}}{q^2 - q^{-2}}\,, \\
e_i^2 e_j - (q^2 + q^{-2}) e_i e_j e_i + e_j e_i^2 = 0\,, \qquad |i-j| = 1\,, \\
f_i^2 f_j - (q^2 + q^{-2}) f_i f_j f_i + f_j f_i^2 = 0\,, \qquad |i-j| = 1\,, \\
[e_i, e_j] = [f_i, f_j] = 0\,, \qquad |i-j| > 1\,,
\end{gathered}$$

where $i, j = 1, 2, \ldots, n-1$; $\alpha, \beta = 1, 2, \ldots, n$.

The subalgebra generated by $(e_i, f_i, k_i^{\pm 1} \equiv (l_{i+1} l_i^{-1})^{\pm 1})$ is the algebra $U_q(sl(n, C))$. The Hopf structure algebra on $U_q(sl(n, C))$ is defined in [3], but we do not need it. The algebra $U_q(gl(n, C))$ is more convenient for our purpose than $U_q(sl(n, C))$.

**Theorem 1.** (a) *Let the operators* $(e_1, \ldots, e_{n-1}; f_1, \ldots, f_{n-1}; l_1, \ldots, l_n)$ *generate the representation of the algebra* $U_q(gl(n, C))$ *on the vector space* $V$. *Then the operators*

$$\begin{aligned}
F_i &= \bar{A}_{i+1} A_i \otimes l_{i+1}^{-2} l_i^2 - q^2 \otimes f_i\,, \\
E_i &= \bar{A}_i A_{i+1} \otimes 1 - q^{2(a_{i+1}^+ a_{i+1} - a_i^+ a_i - 1)} \otimes e_i\,, \\
L_\alpha &= q^{a_\alpha^+ a_\alpha} \otimes l_\alpha\,, \qquad L_{n+1} = q^{-a^+a + \Lambda_{n+1}} \otimes 1\,, \\
E_n &= q^{2a^+a+2n} \bar{A}_n \otimes 1\,, \qquad\qquad (1) \\
F_n &= \frac{1}{q^2 - q^{-2}} [q^{-4a^+a + 2\Lambda_{n+1} - 2n + 4} A_n \otimes l_n^{-2} - q^{-2\Lambda_{n+1} - 2n} A_n \otimes l_n^2] \\
&\quad + \sum_{k=1}^{n-1} \left( q^{-4a^+a + 2\Lambda_{n+1} - 2n + 2 + 2k} \prod_{l=0}^{k-1} q^{2a_{n-l}^+ a_{n-l}} A_{n-k} \right) \otimes X_{n-k}\,,
\end{aligned}$$

*where*

$$\begin{gathered}
1 \le i \le n-1\,, \qquad 1 \le \alpha \le n\,, \qquad \Lambda_{n+1} \in C \quad \text{and} \quad X_{n-1} = l_n^{-2} e_{n-1}\,, \\
X_{n-k} = -q^{-2} [q^{-2} X_{n-k+1} e_{n-k} - e_{n-k} X_{n-k+1}]\,, \qquad k = 2, \ldots, n-1\,,
\end{gathered}$$

*generate the representation of* $U_q(gl(n+1, C))$ *on the space* $\mathscr{F}_n \otimes V$.

(b) *If the representation of $U_q(gl(n,C))$ is the highest-weight one with the highest weight vector $v_0$, then the representation of $U_q(gl(n+1,C))$ is also a highest-weight representation with the highest weight vector $|0\rangle \otimes v_0$.*

*Proof.* (a) By direct verification; some useful relations are collected in Appendix.
(b) If the highest weight of the representation of $U_q(gl(n,C))$ is $(q^{\Lambda_1}, q^{\Lambda_2}, \ldots, q^{\Lambda_n})$ $[l_\alpha v_0 = q^{\Lambda_\alpha} v_0, e_i v_0 = 0]$, then from (1) we can immediately deduce

$$E_\alpha(|0\rangle \otimes v_0) = 0\,, \qquad L_\alpha(|0\rangle = q^{\Lambda_\alpha}(|0\rangle \otimes v_0)\,, \qquad L_{n+1}(|0\rangle = q^{\Lambda_{n+1}}(|0\rangle \otimes v_0)\,.$$

It means that the weight of the representation of $U_q(gl(n+1,C))$ is $(q^{\Lambda_1}, q^{\Lambda_2}, \ldots, q^{\Lambda_n}, q^{\Lambda_{n+1}})$. □

We can use the formulas (1) to obtain sets of h.-w. representations. Substituting into (1) the trivial representation $U_q(gl(n,C))$, i.e. a one-dimensional representation in which $e_i = f_i = 0$, $l_i = w_i l_n$, $l_n = q^{\Lambda_n}$, $w_i^4 = 1$, we obtain a h.-w. representation of $U_q(gl(n+1,C))$ on $\mathcal{F}_n$ with highest weight $(w_1 q^{\Lambda_n}, w_2 q^{\Lambda_n}, \ldots, q^{\Lambda_n}, q^{\Lambda_{n+1}})$ . This representation depends on two continuous parameters $\Lambda_n$ and $\Lambda_{n+1}$ and $n$ canonical pairs. Let us denote the set of all such representations by $S_1^{(n)}$.

In the next step we substitute into (1) representations from $S_1^{(n-1)}$; we obtain representations of $U_q(gl(n+1,C))$ on the space $\mathcal{F}_n \otimes \mathcal{F}_{n-1} \approx \mathcal{F}_{2n-1}$ depending on three continuous parameters, and expressed by means of $2n-1$ canonical pairs with highest weight $(w_1 q^{\Lambda_{n-1}}, w_2 q^{\Lambda_{n-1}}, \ldots, q^{\Lambda_{n-1}}, q^{\Lambda_n}, q^{\Lambda_{n+1}})$; the set of these representations we denote by $S_2^{(n)}$. This procedure can be continued. In general, the representation in $S_k^{(n)}$, $k = 1, 2, \ldots, n$ depends on $k+1$ continuous parameters and $\frac{k}{2}(2n-k+1)$ canonical pairs. In the case $k = n$ we obtain the "full" number of independent parameters and the general highest weight $(q^{\Lambda_1}, q^{\Lambda_2}, \ldots, q^{\Lambda_n}, q^{\Lambda_{n+1}})$. In all cases the h.-w. vector is equal to the vacuum vector of the corresponding Fock space.

The representations just considered are reducible in general, but contain irreducible invariant subspaces.

**Lemma.** *Let a representation of $U_q(gl(n,C))$ have the following property (Property P): for any $x$ from representation space $V$ there exists such (common) $x_0 \in V$ so that $x_0 \in U_q(gl(n,C))x$. Then for representation $U_q(gl(n+1,C))$ given by Eqs.* (1),

$$|0\rangle \otimes x_0 \in U_q(gl(n+1,C))\tilde{x}\,,$$

*where $\tilde{x}$ is any vector from the representative space $\mathcal{F}_n \otimes V$.*

*Proof of Lemma.* We define

$$\bar{B}_i \equiv \left( \prod_{k=i}^{n} q^{-2a_k^+ a_k} a_i \right) \otimes 1\,,$$

and prove easily the relation

$$E_{i-1}\bar{B}_i - q^{-2}\bar{B}_i E_{i-1} = \bar{B}_{i-1}\,, \qquad i = 2, 3, \ldots, n\,. \tag{2}$$

As $\bar{B}_n \equiv q^{2\Lambda_{n+1}-2n+2} L_{n+1}^2 E_n \in U_q(gl(n+1,C))$, Eqs. (2) give $\bar{B}_i \in U_q(gl(n+1,C))$ for all $i = 1, 2, \ldots, n$.

Take any

$$\tilde{x} = \sum_{k_1,\ldots,k_n} |k_1, \ldots, k_n\rangle \otimes v_{k_1,\ldots,k_n} \in \mathcal{F}_n \otimes V\,.$$

Let $(\bar{k}_1, \ldots, \bar{k}_n)$ be a "highest degree" of this sum, understood in the following sense:

$$\bar{k}_1 = \max\{k_1 : v_{k_1,\ldots,k_n} \neq 0\}\,,$$
$$\bar{k}_2 = \max\{k_2 : v_{\bar{k}_1,k_2\ldots,k_n} \neq 0\}\,,$$
$$\bar{k}_n = \max\{k_n : v_{\bar{k}_1,\ldots,\bar{k}_{n-1},k_n} \neq 0\}\,.$$

Then $(\bar{B}_n)^{\bar{k}_n}(\bar{B}_{n-1})^{\bar{k}_{n-1}}\ldots(\bar{B}_1)^{\bar{k}_1}\tilde{x} = \text{const}\,|0\rangle \otimes v_{\bar{k}_1,\ldots,\bar{k}_n}$ where const $\neq 0$. Let further $p \equiv p(e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}, l_1^{\pm 1}, \ldots, l_n^{\pm 1}) \in U_q(gl(n, C))$ is such that $pv_{\bar{k}_1,\ldots,\bar{k}_n} = v_0$.

We take $p(E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}, L_1^{\pm 1}, \ldots, L_n^{\pm 1}) \in U_q(gl(n+1, C))$ and due to relation $E_i(|0\rangle \otimes v) = |0\rangle \otimes e_i v F_i(|0\rangle \otimes v) = |0\rangle \otimes f_i v L_i^{\pm}(|0\rangle \otimes v) = |0\rangle \otimes l_i^{\pm} v$,

$$p(E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}, L_1^{\pm 1}, \ldots, L_n^{\pm 1})\,|0\rangle \otimes v_{\bar{k}_1,\ldots,\bar{k}_n} = |0\rangle \otimes v_0\,. \quad \square$$

Because the trivial representation of $U_q(gl(n, C))$ has the property $P$, the lemma can be applied to any representation from the set $\bigcup_{k=1}^{n} S_k^{(n)}$, and we obtain

**Theorem 2.** *Invariant subspaces of any representation from the set* $\bigcup_{k=1}^{n} S_k^{(n)}$ *with vacuum as the cyclic vector is an irreducible one.*

## 3. Concluding Remarks

(a) Maximal weight of the subalgebra $U_q(sl(n+1, C))$ in a representation from the set $S_n^{(n)}$ is $(q^{\Lambda_2-\Lambda_1}, q^{\Lambda_3-\Lambda_2}, \ldots, q^{\Lambda_{n+1}-\Lambda_n})$, i.e. we have constructed an irreducible h.-w. representation of $U_q(sl(n+1, C))$ for any highest weight. As such a representation is, up to isomorphism and $q$ not being a root of unity, unique [2], we have constructed the set of all highest weight irreducible representations for $U_q(sl(n+1, C))$.

(b) If highest weight for $U_q(gl(n+1, C))$ has the special form

$$(w_1 q^{\Lambda_k}, \ldots, w_{k-1} q^{\Lambda_k}, q^{\Lambda_k}, q^{\Lambda_{k+1}}, \ldots, q^{\Lambda_{n+1}})\,, \qquad w_i^4 = 1\,, \qquad i = 1, 2, \ldots, k-1\,,$$

we can choose at least two different (but equivalent, of course) forms of corresponding highest weight representation. It is either a representation from the set $S_n^{(n)}$, where the generators are expressed by means of $\frac{n}{2}\,(n+1)$-canonical pairs or from the set $S_{n+1-k}^{(n)}$ with generators depending on $\frac{1}{2}\,(n+1-k)\,(n+k)$-canonical pairs only. The advantage of the second type of representation is that its representative space is "smaller" in comparison with the first type ($\mathscr{F}_k \subset \mathscr{F}_n, k < n$) and, consequently, the characteristics of the irreducible subspace $U_q(gl(n, C))\,|0\rangle$ might be simpler.

(c) If $q$ is not a root of unity and $\Lambda_{n+1} - \Lambda_n, \Lambda_n - \Lambda_{n-1}, \ldots, \Lambda_{k+1} - \Lambda_k \in \{0, 1, 2, \ldots\}$, the corresponding representation from $S_k^{(n)}$ is finite-dimensional [2], and in this way we obtain the set of all such representations of $U_q(gl(n+1, C))$. The explicit matrix form of these representations, using Gelfand-Zetlin patterns, is described in [5]. In comparison with that construction we do not give an explicit definition of basis; we give only alternative forms of generators. Finding some basis in the representation space $U_q(gl(n+1, C))\,|0\rangle$ needs further effort. For $q = 1$ and

$n = 2$ we found such a basis in the isomorphic image of the Fock space onto Bargman space of analytic functions $\left[\text{i.e. after substituting } (a_i^+, a_i) \text{ with } \left(x_i, \frac{\partial}{\partial x_i}\right)\right]$ [7].

(d) Formulas (1) can be rewritten by means of operators

$$\bar{a}_q^{(i)} \equiv \bar{A}_i\,, \qquad a_q^{(i)} \equiv A_i\,, \qquad q^{2N_q^{(i)}} \equiv q^{2a_i^+ a_i}\,.$$

These operators fulfill commutation relations of the $q$-deformed Heisenberg algebra [1]:

$$\begin{gathered} \bar{a}_q^{(i)} a_q^{(j)} - q^{-2} a_q^{(j)} \bar{a}_q^{(i)} = \delta_{ij} q^{2N_q^{(i)}}\,, \\ \bar{a}_q^{(j)} q^{N_q^{(i)}} = q^{N_q^{(i)} + \delta_{ij}} \bar{a}_q^{(j)}\,, \qquad a_q^{(j)} q^{N_q^{(i)}} = q^{N_q^{(i)} - \delta_{ij}} a_q^{(j)}\,. \end{gathered} \tag{3}$$

Our operators, however, fulfill additional relations

$$\bar{a}_q^{(i)} a_q^{(i)} = \frac{q^{2(N_q^{(i)}+1)} - q^{-2(N_q^{(i)}+1)}}{q^2 - q^{-2}}\,. \tag{4}$$

so that they form the representation of the $q$-analogue of Weyl algebra $A^-(n)$ defined in [6]. It is possible to prove that relations (3)–(4) lead to formulas (1) independently of the representation used.

In ref. [6] the $q$-analogue of Weyl algebra was used for construction of "simple" quantum enveloping algebras. In the case of $U_q(sl(n+1, C))$ the generators are quadratic expressions in generators of $A^-(n+1)$ (no free parameter). Taking some standard representation of $A^-(n+1)$ irreducible representations of $U_q(sl(n+1, C))$ are obtained as a restriction to some explicitly defined subspaces in representative space; in this way only part of the set of all finite dimensional representations is obtained. Our representations of $U_q(gl(n+1, C))$ are expressed by means of generators of $A^-(n)$ and one free parameter or $A^-(n-1)$ and two free parameters, etc., and we then obtain the full set of finite dimensional representations of $U_q(gl(n+1, C))$.

(e) If $q$ is the root of unity then all our assertions remain true. However, as the simplest case of $U_q(gl(2, C))$ shows, we do not obtain the cyclic representations (see [10] for periodic and partially periodic representations of $SU(N)_q$).

(f) In the limit $q \to 1$ the only slightly complicated expression is the first part of the generator $F_n$ which must be rewritten in the form

$$\begin{aligned} &\frac{1}{q^2 - q^{-2}} [q^{-4a^+a + 2\Lambda_{n+1} - 2n + 4} A_n \otimes l_n^{-2} - q^{-2\Lambda_{n+1} - 2n} A_n \otimes l_n^2 \\ &\quad = q^{-2} \left( -a^+a + \frac{\Lambda_{n+1} - n + 2}{2} \right) f_q \left( -a^+a + \frac{\Lambda_{n+1} - n + 2}{2} \right) A_n \otimes l_n^{-2} \\ &\quad\quad + q^{-2} \frac{\Lambda_{n+1} + n}{2} f_q \left( -\frac{\Lambda_{n+1} + n}{2} \right) A_n \otimes l_n^2 + A_n \otimes \frac{l_n^{-2} - l_n^2}{q^2 - q^{-2}}\,. \end{aligned}$$

Assuming now the existence of limits,

$$\lim_{q\to 1} \frac{l_i^2 - l_i^{-2}}{q^2 - q^{-2}} \equiv \tilde{e}_{ii}\,, \qquad \lim_{q\to 1} l_i \equiv 1\,,$$

and also the normal commutation relations of $gl(n, C)$ for

$$\lim_{q\to 1} e_i \equiv \tilde{e}_{i+1i}\,, \qquad \lim_{q\to 1} f_i \equiv \tilde{e}_{ii+1}\,,$$

we obtain from Eqs. (1),

$$
\begin{aligned}
E_{i+1i} &\equiv \lim_{q\to 1} E_i = a_i a_{i+1}^+ \otimes 1 - 1 \otimes \tilde{e}_{i+1i}\,, \\
E_{ii+1} &\equiv \lim_{q\to 1} F_i = a_{i+1} a_i^+ \otimes 1 - 1 \otimes \tilde{e}_{ii+1}\,, \\
E_{\alpha\alpha} &\equiv \lim_{q\to 1} \frac{L_\alpha^2 - L_\alpha^{-2}}{q^2 - q^{-2}} = a_\alpha^+ a_\alpha + \tilde{e}_{\alpha\alpha}\,, \\
E_{n+1n} &\equiv \lim_{q\to 1} E_n = a_n \otimes 1\,, \\
E_{nn+1} &\equiv \lim_{q\to 1} F_n = (-a^+a + \Lambda_{n+1} + 1)a_n^+ \otimes 1 + \sum_{\alpha=1}^{n} (-1)^{n-\alpha+1} a_\alpha^+ \otimes \tilde{e}_{n\alpha}\,, \\
E_{n+1n+1} &\equiv \lim_{q\to 1} \frac{L_{n+1}^2 - L_{n+1}^{-2}}{q^2 - q^{-2}} = -a^+a + \Lambda_{n+1}\,,
\end{aligned}
\tag{5}
$$

where $\tilde{e}_{nk} \equiv [\tilde{e}_{nk+1}, \tilde{e}_{k+1k}]$, $k = n-2,\, n-1,\, \dots,\, 1$. It is proved in [8] that these operators generate the representation of the Lie algebra $gl(n+1, C)$. So formulas (1) can be considered as $q$-deformation of formulas (5). Formulas of similar form were derived for all classical simple Lie algebras and even for most of their real noncompact forms (where moreover the operators are skew-symmetric) [9]. We believe that in all of these cases the $q$-deformation exists. Therefore formulas similar to (1) could be obtained for all classical simple quantum enveloping algebras.

## Appendix

Some useful relations for the proof of Theorem 1:

$$
q^{2\delta_{ki}-2\delta_{ki+1}} X_k e_i - e_i X_k = -q^2 \delta_{ki+1} X_i\,, \qquad l_\alpha X_j = q^{2\delta_{\alpha n}-\delta_{\alpha j}} X_j l_\alpha \quad,
$$

$$
X_j f_i - q^{2\delta_{n-1,i}} f_i X_j = \begin{cases} -\delta_{ij} q^{-2} X_{i+1} (l_{i+1}^{-1} l_i)^2 & \text{for } i \neq n-1 \\ Y_j & \text{for } i = n-1\,, \end{cases}
$$

where

$$
Y_{n-1} \equiv \frac{1}{q^2 - q^{-2}} (l_{n-1}^{-2} - l_n^{-4} l_{n-1}^2)\,, \qquad Y_{n-2} \equiv q^{-2} l_{n-1}^{-2} e_{n-2}\,,
$$

$$
Y_k \equiv q^{-2}(q^{-2} Y_{k+1} e_k - e_k Y_{k+1})\,, \quad k \leq n-3\,.
$$

$$
\begin{aligned}
&q^2 X_k Y_l - q^{-2} Y_l X_k + X_l Y_k - Y_k X_l \\
&\quad = -\delta_{ln-1} q^2 (q^2 + q^{-2}) l_n^{-4} l_{n-1}^2 X_k\,, \quad k < l \leq n-1\,, \\
&X_k Y_k - q^{-2} Y_k X_k = -\delta_{kn-1} q^2 (q^2 + q^{-2}) l_{n-1}^2 l_n^{-4} X_{n-1}\,.
\end{aligned}
$$

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