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Yang–Mills Fields on Cylindrical Manifolds and Holomorphic Bundles II

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Abstract: We give complex holomorphic descriptions of Yang–Mills instantons on tubular four manifolds with nontrivial circle bundles over Riemann surfaces as section.

0. Introduction

Let Y be a nontrivial circle bundle. By the discussion in [8], we know that instantons on $Y \times R$ can be divided into three classes, namely those with flat limits without holonomy along the fibre circle of Y, those with flat limits with holonomy along the fibre circle and those with mixed limits. In [8], we give complex holomorphic descriptions of instantons on $Y \times R$ whose flat limits have trivial holonomy along the fibre circle. In this sequel, we give a complex holomorphic description of instantons whose flat limits have nontrivial holonomy along the fibre circle. The holomorphic data used to describe these instantons is basically different from that in [8], due to the holonomies of the flat limits along the fibre circle of Y. Nevertheless the method used to establish these results is similar to the one used in [8].

We assume the reader is familiar with [8] and shall make constant references to [8], and we shall continue to use the notation introduced in [8].

1. Some Definitions and Statements of the Main Results

Let Y be a circle bundle with non-trivial Chern class over some Riemann surface Σ. Let L and S be the associated line bundle and ruled surface, and also let Σ_0 and Σ_∞ be the two divisors in S as before. By Lemma 3.1 of [8], there is a metric g on Y and a holomorphic structure on $Y \times R$ such that the tube metric $g + dt \otimes dt$ on $Y \times R$ is a Hermitian metric and is conformal to a Kaehler metric. Moreover $Y \times R$ as a complex manifold can be compactified to a ruled surface.

For our main results, first we need to look at the behaviour of this Hermitian tube metric and the Kaehler metric under certain natural maps between tubes $Y \times R$ for different circle bundles Y.

Let $\tilde{Y}$ be the circle bundle over Σ with first Chern class $c_1(\tilde{Y}) = 1$ if $c_1(Y) > 0$ and -1 if $c_1(Y) < 0$, and $\tilde{L}$ be the holomorphic line bundle associated to $\tilde{Y}$ in the above construction of the Kaehler metric on $\tilde{Y} \times R$. Clearly we can take Y to be $\tilde{Y}^{\otimes\kappa}$, and take the associate holomorphic line bundles $L = \tilde{L}^{\otimes\kappa}$. There is an obvious κ-fold covering map from $\tilde{Y}$ to Y, or equivalently a branched one from $\tilde{L}$ to L, which induces a local biholomorphic map from $\tilde{Y} \times R \subset \tilde{L}$ to $Y \times R \subset L$. Let $\tilde{S}$ and S be the ruled surfaces associated with $\tilde{L}$ and L respectively, then the covering map extends to a covering map from $\tilde{S}$ to S, branched over Σ_0 and Σ_∞. Note that under this covering map, a point $(\tilde{y}, \tilde{t})$ in $\tilde{Y} \times R$ is taken to some point $(y, t = m\tilde{t})$ rather than $(y, t = \tilde{t})$ in $Y \times R$.

The following lemma says that we can adjust the Kaehler forms on the Riemann surface Σ by a constant to make the Kaehler metrics constructed on $\tilde{Y} \times R \subset \tilde{L}$ and $Y \times R \subset L$ conformal under the covering map.

Lemma 1.1. *Let $\tilde{\Omega} = \tilde{r}^{-2\tilde{\lambda}}(\tilde{r}^2\tilde{g} + d\tilde{r} \otimes d\tilde{r})$ and $\Omega = r^{-2\lambda}(r^2 g + dr \otimes dr)$ be the Kaehler metrics on $\tilde{Y} \times R$ and $Y \times R$ as in Lemma* 3.1. *If the Kaehler form Θ on Σ used in the construction of Ω is chosen to be $\kappa^2\tilde{\Theta}$ for the corresponding form $\tilde{\Theta}$ in $\tilde{\Omega}$, then $\Omega = \kappa^2\tilde{\Omega}$ under the covering map.*

Proof. This is not too difficult. First, $r = \tilde{r}^\kappa$ under the map. Secondly, the connection form θ on L is pulled back to $\kappa\tilde{\theta}$ on $\tilde{L}$ for the connection form $\tilde{\theta}$ on $\tilde{L}$, thus the metric $g = \Theta + (-i\theta) \otimes (-i\theta)$ on Y is pulled back to $\kappa^2\tilde{g}$ for the corresponding $\tilde{g}$ on $\tilde{Y}$, by the hypothesis that $\Theta = \kappa^2\tilde{\Theta}$. It is also simple to see that $\lambda = \frac{\tilde{\lambda}-1}{\kappa} + 1$, by the fact that $F_\theta = \kappa F_{\tilde{\theta}}$. It then follows by simple calculation that under the covering map

$$\Omega_0 = \kappa^2\tilde{\Omega}_0$$

and

$$\Omega = \kappa^2\tilde{\Omega}\ .$$

Note that the identity map of L and of any bundle on L is lifted to a cyclic transformation group of order κ of $\tilde{L}$, or of the pull-back bundle on $\tilde{L}$ in the bundle case, through the covering map. We shall denote the transformation group, of $\tilde{L}$ or of the pull-back bundle on it, indiscriminately by G_κ. The action of G_κ on $\tilde{L}$ extends holomorphically over Σ_0 and Σ_∞, on which it acts trivially.

Next we define two kind of complex holomorphic objects.

First consider the following class of holomorphic bundles. Let $\mathcal{E}_p$ be the set of rank two holomorphic bundles on S with trivial first Chern class such that their restrictions over Σ_0 and Σ_∞ are assigned with filtration structures, i.e., holomorphic line sub-bundles in our case of rank two. Such bundles are now called (quasi-) parabolic bundles ([2, 7, 14]) following the original work of Metha and Seshadri ([14]). Consider holomorphic bundle isomorphisms which respect the filtrations over Σ_0 and Σ_∞ between bundles in $\mathcal{E}_p$. Such bundle isomorphisms define an equivalence relation in $\mathcal{E}_p$ and preserves its subset $\mathcal{E}_p(i,j)$ consisting of elements of $\mathcal{E}_p$ such that the Chern class of the line bundle L_0 in the filtration over Σ_0 is $c_1(L_0) = -\frac{ic_1(L)}{|c_1(L)|}$ and that of the line bundle in the filtration over Σ_∞ is $c_1(L_\infty) = \frac{jc_1(L)}{|c_1(L)|}$ over Σ_∞. Define $\mathcal{M}_p$ and $\mathcal{M}_p(i,j)$ to be the resulting spaces of the equivalence classes by the action, corresponding respectively to $\mathcal{E}_p$ and $\mathcal{E}_p(i,j)$.

Next we consider another class of holomorphic bundles. Let $\tilde{\mathcal{E}}$ be a holomorphic bundle on $\tilde{S}$ satisfying the following conditions:

(1) The restrictions of $\tilde{\mathcal{E}}$ over Σ_0 and Σ_∞ are unitary flat with some given prefered unitary structures and split holomorphically and orthogonally into two compatible flat line sub-bundles, i.e.,

$$\tilde{\mathcal{E}}|_{\Sigma_0} = L_{\rho_0} \oplus L_{\rho_0}^{-1} ,$$

and

$$\tilde{\mathcal{E}}|_{\Sigma_\infty} = L_{\rho_\infty} \oplus L_{\rho_\infty}^{-1} ,$$

where L_{ρ_0} and L_{ρ_∞} are the holomorphic flat line bundles on Σ, under the natural identifications of Σ_0 and Σ_∞ with Σ, associated with some flat connections $\rho_0 \in \mathcal{F}_i$ and $\rho_\infty \in \mathcal{F}_j$ respectively as in Sect. 1 of [8].

(2) There is a holomorphic action of G_κ on $\tilde{\mathcal{E}}$ covering the action of G_κ on $\tilde{S}$.

(3) L_{ρ_0} and $L_{\rho_0}^{-1}$ are the eigenspaces of the generator $t_0 \in G_\kappa$ on $\tilde{\mathcal{E}}|_{\Sigma_0}$ with eigenvalues $e^{\frac{i}{\kappa}2\pi i}$ and $e^{-\frac{i}{\kappa}2\pi i}$ respectively, and L_{ρ_∞} and $L_{\rho_\infty}^{-1}$ are the eigenspaces of t_0 on $\tilde{\mathcal{E}}|_{\Sigma_\infty}$ with eigenvalues $e^{\frac{j}{\kappa}2\pi i}$ and $e^{-\frac{j}{\kappa}2\pi i}$.

Let $\tilde{\mathcal{E}}_c(i,j)$ be the set of such bundles. As in [8], we consider two kinds of equivalence relations in $\tilde{\mathcal{E}}_c(i,j)$. First we define an equivalence relation in $\tilde{\mathcal{E}}_c(i,j)$ by defining two such bundles to be equivalent if they are holomorphically isomorphic by a G_κ equivariant isomorphism which respects the splittings of the two bundles over the ends Σ_0 and Σ_∞. Such an isomorphism necessarily restricts to isomorphisms between the holomorphic flat line bundles with the same eigenvalues in the splittings of the two bundles over Σ_0 and Σ_∞. Define $\widetilde{\mathcal{M}}_c(i,j)$ to be the space of equivalence classes of elements in $\tilde{\mathcal{E}}_c(i,j)$.

Second we consider isomorphisms which are also unitary over one end. If $c_1(Y) > 0$, we define a new equivalence relation in $\tilde{\mathcal{E}}_c(i,j)$ by defining two bundles to be equivalent if they are holomorphically isomorphic by an equivariant isomorphism which respects the splittings over the two ends and the given preferred unitary structures over the end Σ_0. Define $\widetilde{\mathcal{M}}_{uc}(i,j)$ to be the resulting space of equivalence classes of $\tilde{\mathcal{E}}_c(i,j)$. Similarly if $c_1(Y) < 0$, we define the space $\widetilde{\mathcal{M}}_{cu}(i,j)$.

With these definitions, the main results of this article can be stated as follows.

Proposition 1.2. *There is a* 1–1 *correspondence between* $\mathcal{M}_p(i,j)$ *and* $\widetilde{\mathcal{M}}_c(i,j)$.

Theorem 1.3. *There is an injective map from* $\mathcal{M}(i,j)$ *to* $\widetilde{\mathcal{M}}_{uc}(i,j)$ *if* $c_1(Y) > 0$, *and to* $\widetilde{\mathcal{M}}_{cu}(i,j)$ *if* $c_1(Y) < 0$.

Our original goal is to describe instantons in $\mathcal{M}(i,j)$ in terms of bundles in $\mathcal{E}_p(i,j)$ with some extra unitary structures over the ends, using Proposition 1.2 as an intermediate step. However we have not been able to decide what kind of unitary structures on bundles in $\mathcal{M}_p(i,j)$ correspond to the given preferred unitary structures over Σ_0 and Σ_∞ on bundles in $\tilde{\mathcal{E}}_c(i,j)$ under the correspondence in Proposition 1.2, therefore we are unable to obtain at this stage a correspondence like the one in Proposition 1.2 for the space $\widetilde{\mathcal{M}}_{uc}(i,j)$ or $\widetilde{\mathcal{M}}_{cu}(i,j)$. Thus we only obtain the partial results here. The problem mentioned is essential to a proper general definition of the "admissable" Hermitian metrics for general parabolic bundles on higher dimensional manifolds ([2]).

2. The Correspondence Between $\widetilde{\mathcal{M}}_c(i,j)$ and $\mathcal{M}_p(i,j)$

In this section, we prove Proposition 1.2.

The construction in this section essentially exhibits the equivalence between what is now called an orbifold bundle or V-bundle and a parabolic bundle with rational weights ([7, 13]). What perhaps is more important is that it shows how on higher dimensions, in contrast to dimension 1 ([2, 7, 14]), the global topological information of the relevant divisor in the base manifold is reflected in the parabolic structures over the divisor when one goes from an orbifold bundle to a parabolic bundle and vice versa.

Let $\tilde{\mathcal{E}}$ be a holomorphic bundle on $\tilde{S}$ representing an element of $\widetilde{\mathcal{M}}_c(i,j)$. We shall construct from $\tilde{E}$ a holomorphic bundle on S which represents an element of $\mathcal{M}_p(i,j)$ in a canonical way. The bundle is a holomorphic extension over Σ_0 and Σ_∞ of the quotient bundle $\mathcal{E}|_{Y\times R}$ on $Y \times R$ of $\tilde{E}|_{\tilde{Y}\times R}$ by G_κ. We now illustrate how the extension is constructed. Again we do this for one end, say Σ_∞.

Let $\tilde{U}$ be a local coordinate neighbourhood of some point at Σ_∞. Adjust $\tilde{U}$ if necessary; we suppose $\tilde{U}$ is G_κ invariant and as before comes from a trivialization of the holomorphic line bundle $\tilde{L}$. Thus $\tilde{U}$ covers a similar local coordinate neighbourhood U of S at Σ_∞. Let $(\tilde{z},\tilde{w})$ be the complex coordinates on $\tilde{U}$ and (z,w) on U so that the covering map is given by $z=\tilde{z}^\kappa, w=\tilde{w}$. We first extend $\mathcal{E}|_{Y\times R}$ locally on each such U, then show that the local extensions patch together to give a global extension of $\mathcal{E}$.

To construct the local extension on U, choose a holomorphic basis $g'_{\tilde{U}}=\{e'_1,e'_2\}$ of $\tilde{\mathcal{E}}|_{\tilde{U}}$ such that $e_1|_{\tilde{U}\cap\Sigma_\infty}$ and $e_2|_{\tilde{U}\cap\Sigma_\infty}$ are local bases of L_{ρ_∞} and $L^{-1}_{\rho_\infty}$ respectively. Let

$$g_{\tilde{U}}=\sum_{0\leqq l\leqq\kappa-1} t_0^l g'_{\tilde{U}}\begin{pmatrix} e^{-\frac{lj2\pi i}{\kappa}} & 0\\ 0 & e^{\frac{lj2\pi i}{\kappa}}\end{pmatrix},$$

where as before t_0 denote the generator of G_κ covering the multiplication by $e^{\frac{2\pi i}{\kappa}}$ on the $\tilde{z}$ plane. Clearly $g_{\tilde{U}}(0,\tilde{w})=\kappa g'_{\tilde{U}}(0,\tilde{w})$, thus replacing by a smaller $\tilde{U}$ if necessary, $g_{\tilde{U}}$ is also a holomorphic basis of $\tilde{\mathcal{E}}$ on $\tilde{U}$ and has the same property as $g'_{\tilde{U}}$ with respect to the spliting of $\tilde{\mathcal{E}}$ over Σ_∞. Consider the action of G_κ on $g_{\tilde{U}}$, by construction

$$t_0 g_{\tilde{U}}=g_{\tilde{U}}\begin{pmatrix} e^{\frac{j2\pi i}{\kappa}} & 0\\ 0 & e^{-\frac{j2\pi i}{\kappa}}\end{pmatrix}.$$

Thus

$$h_{\tilde{U}}=g'_{\tilde{U}}\begin{pmatrix} \tilde{z}^j & 0\\ 0 & \tilde{z}^{-j}\end{pmatrix}$$

is a G_κ invariant holomorphic basis of $\tilde{\mathcal{E}}|_{\tilde{U}\setminus\Sigma_\infty}$. So it descends to a holomorphic basis h_U of $\mathcal{E}|_{U\setminus\Sigma_\infty}$. We extend $\mathcal{E}|_{U\setminus\Sigma_\infty}$ over the whole U by defining h_U to be a holomorphic basis of the extension bundle on U or, in another words, by patching

$\mathcal{E}|_{U\setminus\Sigma_\infty}$ with the trivial C^2 bundle on U by identifying h_U with the obvious trivial basis of the latter. We denote the extended bundle still by $\mathcal{E}$. In case confusion may arise, we shall precede it with the term "the extended."

The local extension thus constructed is independent of the particular choice of the local basis $g_{\tilde{U}}$ of $\tilde{\mathcal{E}}$ on $\tilde{U}$ we start with. Moreover, there is a distinguished holomorphic line sub-bundle of the restriction over $U \cap \Sigma_\infty$ of the extended bundle $\mathcal{E}$ on U, namely the one defined by the first section of the basis h_U, in the sense that it is also independent of the choice involved. These facts will follow from the following arguments for the more general fact that the local extensions of $\mathcal{E}|_{Y\times R}$ just constructed patch together to give a global extension of $\mathcal{E}$ over Σ_∞ and the local line sub-bundles on Σ_∞ defined in the local extensions patch together to give a global line sub-bundle of the restriction of the extended $\mathcal{E}$.

Let $\tilde{U}_1$ and $\tilde{U}_2$ be two local coordinate neighbourhoods as above. Suppose $\tilde{U}_1 \cap \tilde{U}_2$ is non-empty, otherwise it is trivial. Let $\tilde{z}_2 = \tilde{z}_1\tilde{g}_{12}$. By our choice of the coordinates, $\tilde{g}_{12}$'s are the transition functions of the holomorphic line bundle $\tilde{L}^{-1}$, the minus sign corresponding to the fact that we are working around Σ_∞ rather than Σ_0.

Let $g_{\tilde{U}_1}$ and $g_{\tilde{U}_2}$ be the local bases of $\tilde{\mathcal{E}}$ on $\tilde{U}_1$ and on $\tilde{U}_2$ with which we start the local extensions of $\mathcal{E}|_{Y\times R}$ on U_1 and U_2 respectively. Let h_{U_1} and h_{U_2} the corresponding local bases for $\mathcal{E}$ on $U_1 \setminus \Sigma_\infty$ and on $U_2 \setminus \Sigma_\infty$ as above. Consider the transition between h_{U_1} and h_{U_2}. First we have

$$g_{\tilde{U}_2} = g_{\tilde{U}_1}\tilde{T}'_{12}$$

for some holomorphic $Gl(2,C)$ valued matrix function $\tilde{T}'_{12}$ on $\tilde{U}_1 \cap \tilde{U}_2$ which is upper triangular on $\tilde{U}_1 \cap \tilde{U}_2 \cap \Sigma_\infty$. Note that the first diagonal entry of $\tilde{T}'_{12}$ is the transition of the line bundle L_{ρ_∞} in the splitting of $\tilde{\mathcal{E}}$ on Σ_∞. It follows that

$$h_{\tilde{U}_2} = g_{\tilde{U}_2}\begin{pmatrix} \tilde{z}_2^j & 0 \\ 0 & \tilde{z}_2^{-j} \end{pmatrix} = h_{\tilde{U}_1}\begin{pmatrix} \tilde{z}_1^{-j} & 0 \\ 0 & \tilde{z}_1^j \end{pmatrix}\tilde{T}'_{12}\begin{pmatrix} \tilde{z}_2^j & 0 \\ 0 & \tilde{z}_2^{-j} \end{pmatrix}.$$

So the transition function between $h_{\tilde{U}_1}$ and $h_{\tilde{U}_2}$ is

$$\tilde{T}_{12} = \begin{pmatrix} \tilde{z}_1^{-j} & 0 \\ 0 & \tilde{z}_1^j \end{pmatrix}\tilde{T}'_{12}\begin{pmatrix} \tilde{z}_2^j & 0 \\ 0 & \tilde{z}_2^{-j} \end{pmatrix}.$$

It is defined on $\tilde{U}_1 \cap \tilde{U}_2 \setminus \Sigma_\infty$, holomorphic and G_κ invariant, $h_{\tilde{U}_1}$ and $h_{\tilde{U}_2}$ being holomorphic G_κ invariant bases of $\tilde{\mathcal{E}}$ on $\tilde{U}_1 \cap \tilde{U}_2 \setminus \Sigma_\infty$. Thus it descends to a holomorphic $GL(2,C)$ valued function T_{12} on $U_1 \cap U_2 \setminus \Sigma_\infty$. Clearly T_{12} is the transition function between h_{U_1} and h_{U_2} on $U_1 \cap U_2 \setminus \Sigma_\infty$.

It suffices to show that T_{12} extends holomorphically over $U_1 \cap U_2 \cap \Sigma_\infty$ and is upper triangular there, for it then follows that our local extensions indeed patch together to give a global extension of $\mathcal{E}|_{Y\times R}$, and that the local line sub-bundles also patch together to give a global holomorphic line sub-bundle of $\mathcal{E}|_{\Sigma_\infty}$. We then define a parabolic structure with weights $-\frac{1}{2} < -\frac{j}{\kappa} < \frac{j}{\kappa} < \frac{1}{2}$ of $\mathcal{E}$ over Σ_∞ by defining the parabolic structure to be the one determined by the line bundle constructed above and by assigning the weights $\frac{j}{\kappa}$ to the line bundle and $-\frac{j}{\kappa}$ to $\mathcal{E}|_{\Sigma_\infty}$.

From the definition of T_{12}, it is easy to see that its lower triangular entry tends to 0 as z_1, or equivalently z_2, goes to 0. Also the first of its diagonal entries, when lifted to $\tilde{U}_1 \cap \tilde{U}_2 \setminus \Sigma_\infty$, is the same as the corresponding entry in the matrix $\tilde{T}_{12}$ multiplied by $\tilde{g}_{12}^j$, and the second diagonal entry, when lifted to $\tilde{U}$, is the same as the corresponding entry of $\tilde{T}_{12}$ multiplied by $\tilde{g}_{12}^{-j}$. In particular they are bounded. Consequently these entries all extend holomorphically over Σ_∞, on which the lower triangular entry vanishes, the first diagonal entry is equal to the restriction of the corresponding entry of $\tilde{T}'_{12}$ over Σ_∞ multiplied by the function $\tilde{g}_{12}^j$ and the second one is equal to the restriction of the second diagonal entry of $\tilde{T}'_{12}$ multiplied by $\tilde{g}_{12}^{-j}$. Finally it is also clear from the construction that the upper triangular entry of T_{12}, which is holomorphic on $U_1 \cap U_2 \setminus \Sigma_\infty$, is bounded by $C \cdot |z_1^{-\frac{2j}{\kappa}}|$. Since $2j < \kappa$, $|z_1|^{-\frac{2j}{\kappa}} < |z_1|^{-1}$, it is actually bounded, being holomorphic. So it also extends holomorphically over $U_1 \cap U_2 \cap \Sigma_\infty$.

The above discussion also shows that the line sub-bundle of $\mathcal{E}|_{\Sigma_\infty}$ constructed is precisely $\tilde{L}^{\otimes j} \otimes L_{\rho_\infty}$.

Similarly $\mathcal{E}$ can be extended over Σ_0 and the extended $\mathcal{E}$ has a parabolic structure of weights $-\frac{1}{2} < -\frac{i}{\kappa} < \frac{i}{\kappa} < \frac{1}{2}$ and the line subbundle defining the parabolic structure is $\tilde{L}^{-\otimes i} \otimes L_{\rho_0}$.

Up to now, we have constructed for any given element $\tilde{\mathcal{E}}$ of $\tilde{\mathcal{E}}_c(i,j)$ an element $\mathcal{E}$ of $\mathcal{E}_p(i,j)$. This defines a map from $\tilde{\mathcal{E}}_c(i,j)$ to $\mathcal{E}_p(i,j)$. It is also rather clear that the map descends to a map from $\widetilde{\mathcal{M}}_c(i,j)$ to $\mathcal{M}_p(i,j)$. Thus we have shown the correspondence in one direction.

To get the correspondence in the opposite direction, one simply reverse the above construction. The correspondence thus obtained is then trivially the inverse of the earlier one. This completes the proof of Proposition 1.2.

3. Equivariant Holomorphic Description of $\mathcal{M}(i,j)$

In this section, we first prove an analogue of Theorem 2.1 of [8] and then Theorem 1.3. The proof of the former is largely an application of Theorem 2.1 of [8] and the construction of the map of Theorem 1.3 is a refinement of the one for the corresponding map of Theorem 2.2 in [8].

First we prove the following analogue of Theorem 2.1 of [8].

Theorem 3.1. *Let $\tilde{\mathcal{E}}$ be a holomorphic bundle on $\tilde{S}$ representing some element of $\widetilde{\mathcal{M}}_{cu}(i,j)$ and let $\mathcal{E}$ be the quotient bundle on $Y \times R$ of $\tilde{\mathcal{E}}|_{\tilde{Y} \times R}$ by G_κ. There is a unique Hermitian metric H on $\mathcal{E}$ for which the following holds.*

(a) *The curvature F_H of the Chern connection A_H on $Y \times R$ satisfies*

$$i\Lambda F_H = 0 \ .$$

(b) *Let $\tilde{H}$ be the lift of H to $\tilde{\mathcal{E}}$. Then $\tilde{H}$ is bounded in the sense that for any smooth Hermitian metric $\tilde{K}$, say, on $\tilde{\mathcal{E}}$, $d(\tilde{H},\tilde{K}) < \infty$ where d denotes the distance in the space of Hermitian metrics as in* [8]. *Moreover, if $c_1(Y) > 0$, $\tilde{H}$ is an extension of the given unitary structures of $\tilde{\mathcal{E}}$ over Σ_0 and if $c_1(Y) < 0$, $\tilde{H}$ is an extension of the unitary structures of $\tilde{\mathcal{E}}$ over Σ_∞.*

(c) *If* $c_1(Y) > 0$, *the connection* A_H *on* $\mathcal{E}|_{Y\times R}$ *limits to the flat connection* ρ_0 *as* t *goes to* $-\infty$ *and if* $c_1(Y) < 0$, *it limits to the flat connection* ρ_∞ *as* t *goes to* ∞.

Proof. By Theorem 2.1 of [8], there is a unique Hermitian metric $\tilde{H}$ on $\tilde{\mathcal{E}}|_{\tilde{Y}\times R}$ satisfying (1), (2) and (3) of the theorem. By the uniqueness, $\tilde{H}$ is also G_κ invariant. To see this, one simply checks that the Hermitian metric $\tilde{H}' = t_0^l\tilde{H}$ also satisfies the properties of $\tilde{H}$. Here $t_0\tilde{H}$ is the pull-back of $\tilde{H}$, i.e.,

$$t_0\tilde{H}(\eta,\xi)(\tilde{z},\tilde{w}) = \tilde{H}(t_0\eta, t_0\xi)(e^{\frac{2\pi i}{\kappa}}\tilde{z},\tilde{w}) .$$

It follows that $\tilde{H}$ descends to a Hermitian metric H on $\mathcal{E}|_{Y\times R}$. We show that H satisfies the conditions of the theorem.

(a) follows from (1) for $\tilde{H}$.

(b) is just (2).

(c) is slightly more subtle . Flat connections on Y are lifted to flat connections on $\tilde{Y}$ by the covering map. The images of the lifting are the same on two kinds of flat connections on Y, namely those in $\cup_j\mathcal{F}_j$ and the reducible ones in $\mathcal{F}_+$. The fact that the asymptotic flat limits of A_H lie in $\cup_j\mathcal{F}_j$ rather than in $\mathcal{F}_+$ is reflected in the fact that the G_κ action on $\tilde{\mathcal{E}}|_{\Sigma_0\cup\Sigma_\infty}$ has eigenvalues $e^{\pm\frac{i}{\kappa}2\pi i}$ and $e^{\pm\frac{l}{\kappa}2\pi i}$. We show (c) in the case $c_1(Y) < 0$, i.e. A_H limits to ρ_∞ as $t \to \infty$. The other case is identical.

Let $\tilde{U}$ be a local neighbourhood of Σ_∞ in $\tilde{S}$ as before, and U be the corresponding neighbourhood of Σ_∞ in S. Similarly let $(\tilde{z},\tilde{w})$ and $(z = \tilde{z}^\kappa, w = \tilde{w})$ be the corresponding coordinates on $\tilde{U}$ and U. It suffices to show that the holonomy of A_H along the circle $|z| = r$ tends to $\begin{pmatrix} e^{\frac{i}{\kappa}2\pi i} & 0 \\ 0 & e^{-\frac{i}{\kappa}2\pi i}\end{pmatrix}$ as r goes to 0.

As in the last section, let $g_{\tilde{U}}$ be a local holomorphic basis of $\tilde{E}|_{\tilde{U}}$ such that

$$t_0 g_{\tilde{U}} = g_{\tilde{U}}\begin{pmatrix} e^{\frac{i}{\kappa}2\pi i} & 0 \\ 0 & e^{-\frac{i}{\kappa}2\pi i}\end{pmatrix} .$$

Let $e_{\tilde{U}} = g_{\tilde{U}}\begin{pmatrix} e^{j\tilde{\theta}i} & 0 \\ 0 & e^{-j\tilde{\theta}i}\end{pmatrix}$. It is a G_κ invariant basis of $\tilde{\mathcal{E}}|_{\tilde{U}\setminus\Sigma_\infty}$. Let e_U be the corresponding basis of $\mathcal{E}|_{U\setminus\Sigma_\infty}$. Let $\tilde{A}_g$ be the connection matrix form of $A_{\tilde{H}}$ under the basis $g_{\tilde{U}}$. As $\tilde{H}$ is only continuous at Σ_∞, we see that $\tilde{A}_g$ is only defined on $\tilde{U}\setminus\Sigma_\infty$. The point is the restriction of $\tilde{A}_g$ over the three manifolds $|\tilde{z}| = r$ limit to the zero form as r goes to 0 in the tube picture $\tilde{Y}\times R$.

To see this, let $\tilde{H}'$ be a Hermitian metric on $\tilde{\mathcal{E}}$ as in Lemma 5.2 [8]. If necessary we assume $\tilde{H}'$ is G_κ invariant, as we obviously can. Let $\tilde{A}'_g$ be the connection matrix form of $A_{\tilde{H}'}$ under the basis $g_{\tilde{U}}$. By the remark that follows Lemma 5.2 in [8], we see that

$$|\tilde{A}'_g|_{\Omega_0} = O(e^{-t}), \quad t \to \infty .$$

On the other hand, by Lemma 5.5 of [8],

$$|\tilde{A}_g - \tilde{A}_g'|_{\Omega_0} = O(e^{-\delta t}), \quad t \to \infty$$

for some positive number δ. It follows that

$$|\tilde{A}_g|_{\Omega_0} = O(e^{-\delta t}), \quad t \to \infty \, .$$

Thus if we normalize the three manifold $|\tilde{z}| = \tilde{r}$ to a standard neighbourhood in $\tilde{Y}$ and consider the restrictions of $\tilde{A}_g$ as forms on this standard neighbourhood in $\tilde{Y}$, then these forms indeed limit to zero matrix form as $\tilde{r}$ goes to 0.

Now let $\tilde{A}_e$ be the connection matrix form of $A_{\tilde{H}}$ under the basis $e_{\tilde{U}}$, then

$$\tilde{A}_e = \begin{pmatrix} j & o \\ 0 & -j \end{pmatrix} id\tilde{\theta} + \begin{pmatrix} e^{-j\tilde{\theta}i} & 0 \\ 0 & e^{j\tilde{\theta}i} \end{pmatrix} \tilde{A}_g \begin{pmatrix} e^{j\tilde{\theta}i} & 0 \\ 0 & e^{-j\tilde{\theta}i} \end{pmatrix} .$$

Since both the connection $A_{\tilde{H}}$ and the basis $e_{\tilde{U}}$ are G_κ invariant, the connection form $\tilde{A}_e$ is also G_κ invariant. Now the term $\begin{pmatrix} j & o \\ 0 & -j \end{pmatrix} id\tilde{\theta}$ is manifestly G_κ invariant, it follows that the term $\begin{pmatrix} e^{-j\tilde{\theta}i} & 0 \\ 0 & e^{j\tilde{\theta}i} \end{pmatrix} \tilde{A}_g \begin{pmatrix} e^{j\tilde{\theta}i} & 0 \\ 0 & e^{-j\tilde{\theta}i} \end{pmatrix}$ is also G_κ invariant. These three G_κ invariant terms descend on $U \setminus \Sigma_\infty$ respectively to A_e, $\begin{pmatrix} \frac{j}{\kappa} & o \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\theta$ and a term we denote by a, so

$$A_e = \begin{pmatrix} \frac{j}{\kappa} & o \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\theta + a \, .$$

Again normalize the three manifold $|z| = r$ to a standard neighbourhood in Y, then the proceeding discussion implies that the restrictions of a to $|z| = r$, considered as forms on this neighbourhood in Y tend to zero form as r goes to 0. Thus the holonomy of the restriction of A_H along any circle $|z| = r$ limits to

$$e^{\int_0^{2\pi} \begin{pmatrix} \frac{j}{\kappa} & o \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\theta} = \begin{pmatrix} e^{\frac{j}{\kappa}2\pi i} & 0 \\ 0 & e^{-\frac{j}{\kappa}2\pi i} \end{pmatrix}$$

as r goes to 0. This shows (c).

Finally the uniqueness follows from the corresponding uniqueness for $\tilde{H}$. This completes the proof of Theorem 3.1

By similar arguments to the ones in Sect. 5 of [8], there is an ASD connection A_{H_0} on E over $Y \times R$ which is compatible with the Hermitian metric H_0 on E and is gauge equivalent to A_H by a positive definite self-adjoint gauge transformation of $E|_{Y\times R}$ determined in a canonical way. It follows that there is a map from $\tilde{\mathcal{E}}_{cu}(i,j)$ to the set of gauge equivalence classes of ASD connections on (E, H_0).

Next we construct the map from $\mathcal{M}(i,j)$ to $\widetilde{\mathcal{M}_{cu}}(i,j)$ and prove Theorem 1.3. The construction is similar to the construction in Sect. 6 of [8], and we shall use some of the results there.

An instanton on $Y \times R$ is lifted to an instanton on $\tilde{Y} \times R$, and thus determines a complex holomorphic structure on the vector bundle where it sits. As in Sect. 6 of [8], the bundle is then extended over the ends in a natural way to give a holomorphic bundle on $\tilde{S}$ with unitary flat structures over the ends.

Theorem 3.2. *Let A be an instanton on (E,H_0) over $Y \times R$ with asymptotic flat limits $[A_{-\infty}] \in \mathcal{F}_i$ and $[A_\infty] \in \mathcal{F}_j$. Let $(\tilde{E},\tilde{H}_0)$ be the lift of (E,H_0) on $\tilde{Y} \times R$ and $\tilde{A}$ the lift of A by the covering map. Then the bundle $\tilde{E}$ with the holomorphic structure determined by $\tilde{A}$ extends holomorphically over Σ_0 and Σ_∞. The extended bundle, which we denote by $\tilde{\mathcal{E}}$ satisfies the following conditions.*

(a) *The bundle $\mathcal{E}$ splits holomorphically over Σ_0 and Σ_∞ as follows.*

$$\tilde{\mathcal{E}}|_{\Sigma_0} = L_{[A_{-\infty}]} \oplus L^{-1}_{[A_{-\infty}]} ,$$

and

$$\tilde{\mathcal{E}}|_{\Sigma_\infty} = L_{[A_\infty]} \oplus L^{-1}_{[A_\infty]} .$$

In particular $\tilde{\mathcal{E}}$ are unitarily flat over the two ends.

(b) *The Hermitian metric $\tilde{H}_0$ on $\tilde{\mathcal{E}}$ extends continuously over Σ_0 and Σ_∞ on which it is defined by the flat unitary structures of $\tilde{\mathcal{E}}$ in (1) over the ends.*

(c) *The G_κ action on $\tilde{E}$ extends over the two ends to an G_κ action on $\tilde{\mathcal{E}}$ which respects the splittings of $\tilde{\mathcal{E}}$ in (1). Moreover the action has eigenvalues $e^{\frac{i}{\kappa}2\pi i}$ on $L_{[A_{-\infty}]}$ and $e^{-\frac{i}{\kappa}2\pi i}$ on $L^{-1}_{[A_{-\infty}]}$, and has eigenvalues $e^{\frac{j}{\kappa}2\pi i}$ on $L_{[A_\infty]}$ and $e^{-\frac{j}{\kappa}2\pi i}$ on $L^{-1}_{[A_\infty]}$.*

Again we shall only prove the theorem for the end Σ_∞.

First we need an analogue of Lemma 6.2 of [8].

Lemma 3.3. *Let U be a local coordinate around Σ_∞ as before and let θ be the angular coordinate of the holmorphic coordinate z. Let A be an instanton on $Y \times R$. If $[A_\infty] \in \mathcal{F}_j$ for some j, then there exist a number $\delta > 0$ and a gauge g_U on U such that the connection form of A under the gauge is given by*

$$A = \begin{pmatrix} \frac{j}{\kappa} & 0 \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\theta + a ,$$

where a satisfies the following condition: If

$$\nabla^k a = a_{i_0 \cdots i_k} e_{i_0} \otimes \cdots e_{i_k} ,$$

then

$$|a_{i_0 \cdots i_k}| = O(\frac{1}{|z|^{\sigma(i_0,\dots,i_k)-\delta}}) ,$$

for any nonnegative interger k.

Proof of Lemma 3.3. As in Lemma 6.2 [8], choose a gauge on E over the end $Y \times [T,\infty)$, for some $T \gg 1$, such that

$$(*) \qquad |\nabla^k_{A_\infty}(A - A_\infty)|_{\Omega_0} = O(e^{-\delta t}) ,$$

for some number $\delta > 0$.

Since $[A_\infty] \in \mathcal{F}_j$, adjust U if neccessary, we can choose a local gauge g_U over U such that

$$A_\infty = \begin{pmatrix} \frac{j}{\kappa} & 0 \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\theta .$$

Here θ is the angular coordinate for z.

Let $a = A - A_\infty$, thus $A = \begin{pmatrix} \frac{j}{\kappa} & 0 \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\theta + a$. By the arguments in Lemma 6.2 of [8], it suffices to show that

$$|\nabla^k a|_{\Omega_0} = O(|z|^\delta), \quad k = 0, 1, \ldots .$$

Since $id\theta = \frac{1}{2}(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}})$, it is easy to see that this can be proved in the same way as we prove Lemma 6.2 of [8].

Consider now the lift of $\tilde{A}$ on $\tilde{E}$. Let $\tilde{U}$ be the local coordinate neighbourhood of Σ_∞ on $\tilde{S}$ covering U. Let $\tilde{z}$ and $\tilde{\theta}$ be the corresponding coordinates, covering z and θ respectively. Under the invariant gauge $g_{\tilde{U}}$ of $\tilde{E}$ covering g_U, the connection form

$$\tilde{A} = \kappa \begin{pmatrix} \frac{j}{\kappa} & 0 \\ 0 & -\frac{j}{\kappa} \end{pmatrix} id\tilde{\theta} + \tilde{a} ,$$

where $\tilde{a}$ is the pull back of a. From Lemma 6.2 of [8], it follows that if

$$\nabla^k \tilde{a} = \tilde{a}_{i_0 \cdots i_k} \tilde{e}_{i_0} \otimes \cdots \tilde{e}_{i_k} ,$$

then

$$|\tilde{a}_{i_0 \cdots i_k}| = O(\frac{1}{|\tilde{z}|^{\sigma(i_0, \ldots, i_k) - \kappa\delta}}) .$$

Let $g_1 = \begin{pmatrix} e^{-j\tilde{\theta}i} & 0 \\ 0 & e^{j\tilde{\theta}i} \end{pmatrix}$. It is a well-defined gauge transform for the pull-back bundle on $\tilde{U} \setminus \{\tilde{z} = 0\}$. Under the new gauge $g_{\tilde{U}} g_1$, the connection form of $\tilde{A}$ is $g_1^{-1} \tilde{a} g_1$, which we denote by $\tilde{b}$. Since $id\tilde{\theta} = \frac{1}{2}(\frac{d\tilde{z}}{\tilde{z}} - \frac{d\bar{\tilde{z}}}{\bar{\tilde{z}}})$, as in Lemma 6.2 of [8], it is easy to see that $|\nabla^k g_1| = O(\frac{1}{|\tilde{z}|^k})$, hence b also satisfies the condition that if

$$\nabla^k \tilde{b} = \tilde{b}_{i_0 \cdots i_k} \tilde{e}_{i_0} \otimes \cdots \tilde{e}_{i_k} ,$$

then

$$|\tilde{b}_{i_0 \cdots i_k}| = O(\frac{1}{|\tilde{z}|^{\sigma(i_0, \ldots, i_k) - \kappa\delta}}) ,$$

Thus the connection $\tilde{A}$ satisfies Lemma 6.2 of [8]. Apply the extension construction in Sect. 6 of [8], we see that the bundle $\tilde{E}$ with the holomorphic structure determined by $\tilde{A}$ extends over Σ_∞ and similarly over Σ_0. It follows plainly from the construction that the extended bundle $\tilde{E}$ satisfies (a), (b). We now show (c).

As in Sect. 6 of [8], the extension of $\tilde{E}$ is first constructed locally on each $\tilde{U}$. In particular, on $\tilde{U}$ there is a local holomorphic gauge $g''_{\tilde{U}}$ of $\mathcal{E}$ such that the restriction of the first section of $g''_{\tilde{U}}$ on $\tilde{U} \cap \Sigma_\infty$ gives a local trivialization of the holomorphic line bundle $L_{[A_\infty]}$ and the second one gives a local trivialization of $L^{-1}_{[A_\infty]}$, and

$$g''_{\tilde{U}} = (g_{\tilde{U}} g_1) T_{\tilde{U}}(\tilde{z}, \tilde{w})$$

for some holomorphic $GL(2,C)$ valued function $T_{\tilde{U}}$ originally defined on $\tilde{U} \setminus \Sigma_\infty$ but extended holomorphically over $\tilde{U} \cap \Sigma_\infty$ on which it equals κI. Consider the action of G_κ under the basis $g''_{\tilde{U}}$. We have

$$(t_0 g''_{\tilde{U}})(t_0\tilde{z}, \tilde{w}) = g''_{\tilde{U}}(t_0\tilde{z}, \tilde{w}) T^{-1}_{\tilde{U}}(t_0\tilde{z}, \tilde{w}) \begin{pmatrix} e^{\frac{i}{\kappa}2\pi i} & 0 \\ 0 & e^{-\frac{i}{\kappa}2\pi i} \end{pmatrix} T_{\tilde{U}}(\tilde{z}, \tilde{w}) .$$

The function $T^{-1}_{\tilde{U}}(t_0\tilde{z}, \tilde{w}) \begin{pmatrix} e^{\frac{i}{\kappa}2\pi i} & 0 \\ 0 & e^{-\frac{i}{\kappa}2\pi i} \end{pmatrix} T_{\tilde{U}}(\tilde{z}, \tilde{w})$, defined and holomorphic on $\tilde{U} \setminus \Sigma_\infty$, clearly extends holomorphically over $\tilde{U} \cap \Sigma_\infty$. Consequently the action of G_κ extends to a holomorphic action on $\tilde{\mathcal{E}}|_{\tilde{U}}$ for each $\tilde{U}$ and the extended action satisfies (3) locally. By continuity, these local extensions give a global extension of G_κ action which satisfies (c). This completes the proof of Theorem 3.2 .

As in Sect. 6 of [8], we see that Theorem 3.2 gives a map from $\mathcal{M}(i,j)$ to $\widetilde{\mathcal{M}}_{cu}(i,j)$ and the map is injective. Thus we have the conclusion of Theorem 1.3.

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