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Deformation Quantizations With Separation of Variables on a Kähler Manifold

Alexander V. Karabegov

Joint Institute for Nuclear Research, Dubna 141980, Russia
E-mail: karabegov@main1.jinr.dubna.su

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Abstract: We give a simple geometric description of all formal differentiable deformation quantizations on a Kähler manifold M such that for each open subset $U \subset M$ $\star$-multiplication from the left by a holomorphic function and from the right by an antiholomorphic function on U coincides with the pointwise multiplication by these functions. We show that these quantizations are in 1–1 correspondence with the formal deformations of the original Kähler metrics on M.

1. Introduction

Formal deformation quantization on a symplectic manifold M (see [1]) is a structure of associative algebra on the space of formal series $C^\infty(M)[[\nu]]$ such that the multiplication in this algebra (denoted by $\star$ and named $\star$-multiplication) is a deformation of the point-wise product of functions on M and the commutator corresponding to the $\star$-multiplication is a deformation of the Poisson bracket $\{\cdot, \cdot\}$ on M.

Deformation quantization is called differentiable if the $\star$-product is given by a formal series of bidifferential operators. Differentiable $\star$-product can be restricted to any open subset $U \subset M$.

The formal product $\star$ can be thought of as an asymptotic expansion in a parameter $\hbar$ of some hypothetical family of noncommutative associative products $\{*_\hbar\}$ of $\hbar$-dependent operator symbols on M such as Weyl symbols, Wick and anti-Wick symbols or their generalizations for curved phase spaces (see [2]). These symbol products have to satisfy the following correspondence principle. For the functions φ, ψ on M $\varphi *_\hbar \psi \to \varphi\psi$ and $\hbar^{-1}(\varphi *_\hbar \psi - \psi *_\hbar \varphi) \to i\{\varphi, \psi\}$ as $\hbar \to 0$.

Only a very limited number of examples is known where deformation quantization appears in such a way from some concrete family of symbol products $\{*_\hbar\}$.

Berezin's quantization on Kähler manifolds (see [2]) provides important examples of differentiable deformation quantizations via the asymptotic expansion of the product of covariant symbols. Such deformation quantizations were obtained on the orbits of compact semisimple Lie groups in [7 and 3] and on bounded symmetric domains in [6 and 4].

Consider now the most simple example of Berezin's Kähler quantization via polynomial Wick symbols on $\mathbf{C}^m$ obtained by the normal ordering of creation–annihilation operators.

The monomial $(z^1)^{k_1}\cdots(z^m)^{k_m}(\bar{z}^1)^{l_1}\cdots(\bar{z}^m)^{l_m}$ is called the Wick symbol of the operator $a_1^{k_1}\cdots a_m^{k_m}(a_1^*)^{l_1}\cdots(a_m^*)^{l_m}$ written in the Wick normal form, where a_i and a_j^* are creation and annihilation operators respectively, subject to the standard commutation relations $[a_j^*, a_i] = \hbar\delta_{ij}$. The $*_\hbar$-product of two polynomial Wick symbols f, g induced by the operator product is given by the following well known formula:

$$f *_\hbar g = \sum_\alpha \frac{\hbar^{|\alpha|}}{\alpha!}\left(\frac{\partial}{\partial\bar{z}}\right)^\alpha f \left(\frac{\partial}{\partial z}\right)^\alpha g\,, \tag{1}$$

where the usual multi-index notation is used.

The asymptotic expansion of $f *_\hbar g$ as $\hbar \to 0$ is already given by formula (1) and thus defines a differentiable deformation quantization on $\mathbf{C}^m$ with a formal $\bigstar$-product $\bigstar$ defined by the same formula (1) with $\hbar$ replaced by the formal parameter ν.

Let us make the following trivial observation. If we multiply an operator with a polynomial Wick symbol from the left by the creation operator a_i or from the right by the annihilation operator a_j^* its Wick symbol gets point-wise multiplied by z^i or $\bar{z}^j$ respectively. It follows now that $*_\hbar$-multiplication of Wick symbols enjoys the following property of separation of variables into holomorphic and anti-holomorphic ones. The $*_\hbar$-multiplication from the left by holomorphic polynomial $f(z)$ or from the right by anti-holomorphic polynomial $g(\bar{z})$ coincides with the point-wise multiplication by these polynomials.

The $*_\hbar$-multiplication of Berezin's covariant symbols on bounded symmetric domains also has the property of separation of variables. Unfortunately, this property has no analog on compact Kähler manifolds since there are no nonconstant holomorphic or anti-holomorphic functions on such manifolds.

It turns out that the property of separation of variables for Kähler quantization holds "locally" and "asymptotically" as $\hbar \to 0$ at least in the known cases. Namely, the deformation quantizations obtained from Berezin's quantization on $\mathbf{C}^m$ and in [3, 4, 6, 7] have the following property of separation of variables.

For each open subset $U \subset M$, $\bigstar$-multiplication from the left by a holomorphic function and from the right by an antiholomorphic function on U coincides with the pointwise multiplication by these functions.

In [5] a simple geometric construction of some formal differentiable deformation quantization with separation of variables on an arbitrary Kähler manifold was introduced. In particular cases it provides the quantizations considered in [3, 4, 6, 7], thus giving their autonomous intrinsic description in terms of the Kähler metrics alone.

In this paper we shall show that *all* quantizations with separation of variables on a Kähler manifold can be obtained by a slightly generalized construction from [5] and are naturally parametrized by geometric objects, the formal deformations of the original Kähler metrics.

2. Definition of Deformation Quantization with Separation of Variables

Define a formal differentiable deformation quantization on a symplectic manifold M (see [1]).

Let $\{C_r(\cdot,\cdot)\}$, $r=0,1,2,\ldots$ be a family of bidifferential operators on M, i.e., of differential operators which map $C^\infty(M)\otimes C^\infty(M)$ to $C^\infty(M)$. Define a binary operation $\star$ in the space of formal power series $\mathscr{F}=C^\infty(M)[[\nu]]$, posing for $f=\sum_{r=0}^\infty \nu^r f_r$ and $g=\sum_{r=0}^\infty \nu^r g_r$,

$$f\star g=\sum_{r=0}^{\infty}\nu^r\sum_{i+j+k=r}C_i(f_j,g_k)\,. \qquad (2)$$

The operation $\star$ defines a formal deformation quantization on the symplectic manifold M, if it is associative and for $f,g\in C^\infty(M)$ holds

$$C_0(f,g)=fg,\ C_1(f,g)-C_1(g,f)=i\{f,g\}\,, \qquad (3)$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on M, corresponding to the symplectic structure.

In such a case the operation $\star$ is called a $\star$-product.

All the deformation quantizations considered in this paper are formal differentiable, so in the sequel we will not mention it explicitly.

Since a $\star$-product is given by differential operators, it is local, that is, it can be restricted to any open subset $U\subset M$. The restriction of $\star$ defines a $\star$-product in the space $\mathscr{F}(U)=C^\infty(U)[[\nu]]$.

If there is given a deformation quantization on M then for each open subset $U\subset M$ in the space $\mathscr{F}(U)$ the algebras $\mathscr{L}(U)$ and $\mathscr{R}(U)$ of the left and right $\star$-multiplication operators act, respectively. For $f,g\in\mathscr{F}(U)$ define the operators $L_f\in\mathscr{L}(U)$ and $R_g\in\mathscr{R}(U)$ by the relations $L_f g=R_g f=f\star g$.

The operators from $\mathscr{L}(U)$ commute with the operators from $\mathscr{R}(U)$, $[L_f,R_g]=0$.

For $U=M$ denote $\mathscr{L}=\mathscr{L}(M)$, $\mathscr{R}=\mathscr{R}(M)$.

Let $\mathscr{D}(U)$ be the algebra of the formal series of differential operators of the form $\tilde{A}=\sum_{r=0}^\infty \nu^r A_r$, where A_r are differential operators on U with smooth coefficients. These series act as linear operators on the space $\mathscr{F}(U)$, for $\tilde{A}=\sum_{r=0}^\infty \nu^r A_r$ and $f=\sum_{r=0}^\infty \nu^r f_r$,

$$\tilde{A}f=\sum_{r=0}^{\infty}\nu^r\sum_{s=0}^{r}A_{r-s}f_s\,.$$

Since one can take a pointwise product of the elements of $\mathscr{F}(U)$, $\mathscr{F}(U)$ is included in $\mathscr{D}(U)$ as the algebra of pointwise multiplication operators. It follows from the definition of $\star$-product that $\mathscr{L}(U)$ and $\mathscr{R}(U)$ are subalgebras of $\mathscr{D}(U)$.

Further, we will refer sometimes to formal series of functions, operators etc., as to formal functions, operators, or even omit the word formal, which must not lead to a misunderstanding.

Let M be a Kähler manifold of complex dimension m with a Kähler form ω_0 of the type $(1,1)$.

Definition. *A deformation quantization on the Kähler manifold M is called a deformation quantization with separation of variables if, for any open subset $U\subset M$ and functions $a,b,f\in C^\infty(U)$, such that a is holomorphic and b antiholomorphic, $a\star f=a\cdot f$, $f\star b=f\cdot b$ holds.*

If on M there is defined a deformation quantization with separation of variables, then for a holomorphic function a and antiholomorphic function b on an arbitrary open subset $U\subset M$, the operators L_a and R_b are the operators of pointwise multiplication by the functions a and b respectively, $L_a=a$ and $R_b=b$. If, moreover, U is a coordinate chart with holomorphic coordinates $z^1,\dots,z^m$, then, since for $f \in \mathscr{F}(U)$ the operator L_f commutes with $R_{\bar z^l} = \bar z^l$, it contains only partial derivatives by z^k. Similarly, the operator R_f contains only partial derivatives by $\bar z^l$.

3. Deformation of Kähler Metrics Corresponding to Quantization With Separation of Variables

With each deformation quantization with separation of variables on a Kähler manifold M with a Kähler form ω_0, we canonically associate a formal deformation of the Kähler metrics ω_0, i.e., a formal series $\omega = \omega_0 + \nu\omega_1 + \nu^2\omega_2 + \cdots$ such that $\omega_1, \omega_2, \dots$ are closed but not necessarily nondegenerate forms of the type $(1,1)$ on M.

On a contractible coordinate chart U, there exists a Kähler potential $\Phi_0 \in C^\infty(U)$ such that $\omega_0 = i\partial\bar\partial\Phi_0 = i g_{kl} dz^k \wedge d\bar z^l$, where $g_{kl} = \partial^2\Phi_0/\partial z^k \partial \bar z^l$. Here as well as below we use the tensor rule of summation over repeated indices. The Kähler potential Φ_0 is defined up to a summand of the form $a + b$, where a is a holomorphic and b an antiholomorphic function on U.

Denote by (g^{lk}) the inverse matrix to (g_{kl}). The Poisson bracket of the functions $f, g \in C^\infty(U)$ can be expressed as follows:

$$\{f,g\} = i g^{lk} \left(\frac{\partial f}{\partial z^k} \frac{\partial g}{\partial \bar z^l} - \frac{\partial g}{\partial z^k} \frac{\partial f}{\partial \bar z^l} \right).$$

On M let a deformation quantization be defined. Introduce bidifferential operators $D_r(\cdot\,,\cdot)$, such that for $u, v \in C^\infty(M)$, $D_r(u,v) = C_r(u,v) - C_r(v,u)$ holds. From (3) it follows that $D_0 = 0$, and $D_1 = i\{\cdot\,,\cdot\}$. Thus for $f = \sum_{r=0}^\infty \nu^r f_r$ and $g = \sum_{r=0}^\infty \nu^r g_r$,

$$f \star g - g \star f = \sum_{r=1}^{\infty} \nu^r \sum_{i+j+k=r} D_i(f_j, g_k)\,. \tag{4}$$

Lemma 1. *Let U be a contractible coordinate chart on M. The system of equations for an unknown function u on U, $D_1(u, z^k) = f^k$, $k = 1,\dots,m$, where $f^k \in C^\infty(U)$, has a solution if and only if for all k, k' $D_1(f^k, z^{k'}) = D_1(f^{k'}, z^k)$ holds. Then the solution u is determined up to a holomorphic summand.*

Proof. By using the fact that $D_1 = i\{\cdot\,,\cdot\}$, the lemma can easily be reduced to the assertion that the solvability condition of the equation $\bar\partial u = g_{kl} f^k d\bar z^l$ is a $\bar\partial$-closedness of the form $g_{kl} f^k d\bar z^l$.

Proposition 1. *On a Kähler manifold M with a Kähler form ω_0 let a formal deformation quantization with separation of variables be defined. Then on each contractible coordinate chart $U \subset M$ there exist formal functions $u^1,\dots,u^m \in \mathscr{F}(U)$ such that $u^k \star z^{k'} - z^{k'} \star u^k = \nu\delta^{kk'}$, where δ is the Kronecker symbol.*

Proof. We will construct, say, the function $u = u^1$. Let $u = u_0 + \nu u_1 + \nu^2 u_2 + \dots$. The coefficients u_r have to satisfy the following system of equations,

$$\sum_{r=0}^{\infty} \nu^r \sum_{s=0}^{r-1} D_{r-s}(u_s, z^k) = \nu\delta^{1k},\ k = 1,\dots,m\,. \tag{5}$$

Equating the coefficients at the same powers of ν on the left-hand and right-hand sides of (5) we get, at $r = 1$, the equations $D_1(u_0, z^k) = \delta^{1k}$, $k = 1, \ldots, m$. Taking into account that $D_1 = i\{\cdot, \cdot\}$, it is easy to check that the function $u_0 = \partial\Phi_0/\partial z^1$ satisfies these equations. For $r > 1$, the obtained equations are as follows,

$$\sum_{s=0}^{r-1} D_{r-s}(u_s, z^k) = 0, \; k = 1, \ldots, m\,. \tag{6}$$

We construct the functions u_s step by step using Eqs. (6) and Lemma 1. Assume that for $s < n$ the functions u_s are constructed and satisfy Eqs. (6) for $r \leqq n$. We are going to show that the function u_n can be found from Eqs. (6) for $r = n + 1$, which we rewrite in the following form,

$$D_1(u_n, z^k) = -\sum_{s=0}^{n-1} D_{n-s+1}(u_s, z^k) = 0, \; k = 1, \ldots, m\,. \tag{7}$$

It follows from Lemma 1 that Eqs. (7) can be solved for unknown u_n if the sum

$$\sum_{s=0}^{n-1} D_1(D_{n-s+1}(u_s, z^k), z^{k'})$$

is symmetric with respect to the permutation of the indices k and k'. The Jacoby identity for the $\bigstar$-commutator (4) is reduced to the identities

$$\sum_{i=1}^{r-1} D_i(D_{r-i}(f, g), h) + \text{cyclic permutation of } f, g, h = 0 \tag{8}$$

for any smooth functions f, g, h. Setting in (8) $f = u_s$, $g = z^k$, $h = z^{k'}$, $r = n - s + 2$, and taking into account that since z^k pair-wise $\bigstar$-commute, $D_r(z^k, z^{k'}) = 0$ holds, we get

$$\sum_{i=1}^{n+1-s} (D_i(D_{n-i-s+2}(u_s, z^k), z^{k'}) - D_i(D_{n-i-s+2}(u_s, z^{k'}), z^k)) = 0\,. \tag{9}$$

Summing up Eqs. (9) for $s = 0, 1, \ldots, n - 1$ and changing the order of summation, we get

$$\sum_{s=0}^{n-1} (D_1(D_{n-s+1}(u_s, z^k), z^{k'}) - D_1(D_{n-s+1}(u_s, z^{k'}), z^k))$$

$$= -\sum_{i=2}^{n+1} \sum_{s=0}^{n-i+1} (D_i(D_{n-i-s+2}(u_s, z^k), z^{k'}) - D_i(D_{n-i-s+2}(u_s, z^{k'}), z^k))\,. \tag{10}$$

It follows from the fact that $D_1(u_0, z^k) = \delta^{1k}$ $\bigstar$-commutes with $z^{k'}$ that $D_i(D_1(u_0, z^k), z^{k'}) = 0$, therefore the inner sum on the right-hand side of (10) at $i = n + 1$ is equal to zero. It follows from (6) that the inner sum on the right-hand side of (10) is equal to zero also for $1 < i < n + 1$, thus the right-hand side of (10) equals zero which proves the solvability of the system (7) for unknown s_n. The proposition is proved.

In a completely analogous way, one can find the formal functions $v^1,\dots,v^m \in \mathscr{F}(U)$ such that $v^l \bigstar \bar{z}^{l'} - \bar{z}^{l'} \bigstar v^l = -\nu\delta^{ll'}$.

Since $L_{z^k} = z^k$, it follows from Proposition 1 that $[L_{u^k}, z^{k'}] = \nu\delta^{kk'}$. Using the fact that the operators from $\mathscr{L}(U)$ contain only partial derivatives by z^k, the operators L_{u^k} and similarly, operators R_{v^l}, can be calculated explicitly.

Lemma 2. $L_{u^k} = u^k + \nu\partial/\partial z^k$, $R_{v^l} = v^l + \nu\partial/\partial\bar{z}^l$.

Introduce the formal differential forms $\alpha = -\sum_k u^k dz^k$ and $\beta = \sum_l v^l d\bar{z}^l$. Since the operators L_{u^k} and R_{v^l} commute, one gets $\partial u^k/\partial\bar{z}^l = \partial v^l/\partial z^k$, therefore $\bar{\partial}\alpha = \partial\beta$. Define the closed formal differential form $\omega = i\bar{\partial}\alpha = i\partial\beta$ of the type $(1,1)$. As follows from the proof of Proposition 1, the first term of the formal series ω coincides with ω_0, therefore ω is a deformation of the Kähler form ω_0.

Assume $\tilde{u}^1,\dots,\tilde{u}^m$ is another set of solutions of (5), and set $\tilde{\alpha} = -\sum_k \tilde{u}^k dz^k$. It follows from Lemma 2 and from the fact that the operators $L_{\tilde{u}^k}$ and R_{v^l} commute, that the form $i\bar{\partial}\tilde{\alpha}$ coincides with ω, that is, ω does not depend on the concrete choice of the solution of system (5). It is easy to show also that ω does not depend on the choice of coordinates on U.

It follows from the Poincaré $\bar{\partial}$-lemma that on a contractible coordinate chart $U \subset M$ there exists a formal series $\Phi = \Phi_0 + \nu\Phi_1 + \cdots \in \mathscr{F}$, which is a potential of the formal Kähler metrics $\omega = \omega_0 + \nu\omega_1 + \nu^2\omega_2 + \dots$. That means that for all $r \geqq 0$, $\omega_r = i\partial\bar{\partial}\Phi_r = i(\partial\Phi_r^2/\partial z^k\partial\bar{z}^l)dz^k \wedge d\bar{z}^l$.

Since $\omega = i\bar{\partial}\alpha = i\bar{\partial}(-\partial\Phi)$, then $\alpha + \partial\Phi$ is a $\bar{\partial}$-closed form of the type $(1,0)$. Therefore the coefficients of $\alpha + \partial\Phi$, which are equal to $\partial\Phi/\partial z^k - u^k$, are holomorphic. Now it is straightforward that $L_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \nu\partial/\partial z^k$ and, similarly, $R_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$.

Thus, starting from a given deformation quantization with separation of variables, we construct on each contractible chart $U \subset M$ a formal deformation ω of the Kähler form ω_0. It follows from the construction of the form ω that on the intersections of charts the local forms agree with each other and define a global form ω on M.

Theorem 1. *Each deformation quantization with separation of variables on a Kähler manifold M canonically corresponds to a formal Kähler metrics ω, which is a deformation of the Kähler metrics ω_0 on M. If Φ is a potential of the formal metrics ω on a coordinate chart $U \subset M$, then $L_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \nu\partial/\partial z^k$ and $R_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$.*

4. A Construction of the Quantization with Separation of Variables from Deformation of Kähler Metrics

Our goal is to generalize the construction of the deformation quantization announced in [5].

Assume that there is given a formal deformation ω of the Kähler metrics ω_0 on M.

Lemma 3. *Assume that on a contractible coordinate chart $U \subset M$, there is chosen a potential $\Phi = \Phi_0 + \nu\Phi_1 + \cdots \in \mathscr{F}$ of the formal metrics ω. Then the set of formal series of differential operators from $\mathscr{D}(U)$, which commute with the operators $\bar{z}^l$ and $\partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$, depends only on the metrics ω, rather than on the concrete choice of the potential.*

Proof. If $\Phi' \in \mathscr{F}$ is another potential of the metrics ω, then $\Phi' = \Phi + a + b$, where a and b are formal series of holomorphic and antiholomorphic functions, respectively. An operator which commutes with $\bar{z}^l$ and $\partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$, commutes also with multiplication by antiholomorphic functions. Therefore, it commutes with $\partial\Phi'/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l = (\partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l) + \partial b/\partial\bar{z}^l$, which implies the assertion of the lemma.

Denote the set of formal operators mentioned in Lemma 3 by $\mathscr{L}_\omega(U)$. Notice that $\mathscr{L}_\omega(U)$ is an operator algebra.

Let U be a coordinate chart on M with a potential Φ_0 of the Kähler metrics ω_0 defined on it. Denote by $S(U)$ the set of differential operators with smooth coefficients on U, which commute with multiplication by the antiholomorphic coordinates $\bar{z}^l$, i.e. which contain only partial derivatives by z^k.

Define the differential operators D^l on U, $D^l = g^{lk}\partial/\partial z^k = i\{\bar{z}^l, \cdot\}$.

Lemma 4. *For all $k, l, l' = 1, \ldots, m$ the following relations hold*:

(i) $[D^l, D^{l'}] = 0$;
(ii) $[D^l, \partial\Phi_0/\partial\bar{z}^{l'}] = \delta^l_{l'}$;
(iii) $\partial/\partial z^k = g_{kl}D^l$.

The assertion of the lemma can be checked by direct calculations.

It follows from Lemma 4 that any operator from $S(U)$ can be canonically represented as a sum of monomials of the form $a_{l_1\ldots l_s}D^{l_1}\cdots D^{l_s}$, where $a_{l_1\ldots l_s} \in C^\infty(U)$ is symmetric with respect to l_i.

Definition. *The twisted symbol of an operator $A \in S(U)$, which is represented in the canonical form $A = \sum a_{l_1\ldots l_s}D^{l_1}\cdots D^{l_s}$, is a polynomial in $\xi^1, \ldots, \xi^m$, $a(\xi) = \sum a_{l_1\ldots l_s}\xi^{l_1}\cdots\xi^{l_s}$ with coefficients in $C^\infty(U)$.*

From Lemma 4 easily follows

Lemma 5. *Let $a(\xi)$ be the twisted symbol of an operator $A \in S(U)$. Then the twisted symbol of the operator $[A, \partial\Phi_0/\partial\bar{z}^l]$ is equal to $\partial a/\partial\xi^l$.*

Consider a system of equations for an unknown operator $A \in S(U)$,

$$[A, \partial\Phi_0/\partial\bar{z}^l] = B_l, \quad l = 1, \ldots, m\,, \tag{11}$$

where $B_l \in S(U)$.

Lemma 6. *System* (11) *has solutions if and only if for all l, l' $[B_l, \partial\Phi_0/\partial\bar{z}^{l'}] = [B_{l'}, \partial\Phi_0/\partial\bar{z}^l]$. If A_0 is a partial solution of the system, then the general solution is of the form $A_0 + A_1$, where A_1 is an arbitrary multiplication operator.*

Proof. Pass to the twisted symbols a, b_l of the operators A, B_l, respectively. System (11) transforms to the equation $da = \sum_l b_l d\xi^l$, where $da = \sum_l(\partial a/\partial\xi^l)d\xi^l$. The assertion of the lemma is now reduced to a standard fact concerning differential forms with polynomial coefficients, which follows from Euler's identity.

Proposition 2. *Let ω_0 be a Kähler metrics on M and $U \subset M$ be a contractible coordinate chart. For each formal function $f = \sum \nu^r f_r \in \mathscr{F}(U)$, there exists a unique formal series of differential operators $\tilde{A}_f = \sum \nu^r A_r$ from $\mathscr{L}_{\omega_0}(U)$, such that $\tilde{A}_f 1 = f$. In particular, A_0 is a multiplication operator by the function f_0.*

Proof. Since $\tilde{A}_f$ commutes with antiholomorphic functions, all the operators A_r are in $S(U)$. Let Φ_0 be a potential of the metrics ω_0. The commutation condition of $\tilde{A}_f$ with $\partial\Phi_0/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$ is equivalent to the system of equations $[A_0, \partial\Phi_0/\partial\bar{z}^l] = 0$ and

$$[A_r, \partial\Phi_0/\partial\bar{z}^l] = [\partial/\partial\bar{z}^l, A_{r-1}] . \tag{12}$$

Find all the terms of the series $\tilde{A}_f$ step by step. It follows from Lemma 5 that A_0 is a multiplication operator, so $A_0 1 = f_0$ implies that $A_0 = f_0$. Assume that we have found all the operators A_r for $r < s$ which satisfy (12) and such that $A_r 1 = f_r$. Let us now show that A_s can be found from (12) for $r = s$, i.e., that the conditions of Lemma 6 on the right-hand side of (12) are satisfied,

$$\left[\left[\frac{\partial}{\partial\bar{z}^l}, A_{s-1}\right], \frac{\partial\Phi_0}{\partial\bar{z}^{l'}}\right] = \left[\left[\frac{\partial}{\partial\bar{z}^{l'}}, A_{s-1}\right], \frac{\partial\Phi_0}{\partial\bar{z}^l}\right].$$

It follows from the Jacoby identity for commutators that

$$\left[\left[\frac{\partial}{\partial\bar{z}^l}, A_{s-1}\right], \frac{\partial\Phi_0}{\partial\bar{z}^{l'}}\right] = \left[\left[\frac{\partial}{\partial\bar{z}^l}, \frac{\partial\Phi_0}{\partial\bar{z}^{l'}}\right], A_{s-1}\right] + \left[\frac{\partial}{\partial\bar{z}^l}, \left[A_{s-1}, \frac{\partial\Phi_0}{\partial\bar{z}^{l'}}\right]\right]$$

$$= \left[\frac{\partial^2\Phi_0}{\partial z^l\partial\bar{z}^{l'}}, A_{s-1}\right] + \left[\frac{\partial}{\partial\bar{z}^l}, \left[\frac{\partial}{\partial\bar{z}^{l'}}, A_{s-2}\right]\right].$$

It is easy to check that the last expression is symmetric with respect to the permutation of l and l'. Thus system (12) is solvable for $r = s$. Among its solutions there is the only one solution A_s such that $A_s 1 = f_s$. The assertion is proved.

Lemma 7. *For a given formal function $f = f_0 + \nu f_1 + \cdots \in \mathscr{F}(U)$ there exists a function $g = g_0 + \nu g_1 + \cdots \in \mathscr{F}(U)$ such that $\tilde{A}_f g = 1$ if and only if f_0 does not vanish on U. Then g is defined uniquely and $g_0 = 1/f_0$.*

Proof. Let $\tilde{A}_f = \sum \nu^r A_r$. The condition $\tilde{A}_f g = 1$ is equivalent to the system of equations $A_0 g_0 = 1$ and $A_0 g_r = -\sum_{s=0}^{r-1} A_{r-s} g_s$. According to Proposition 2, $A_0 = f_0$, therefore if f_0 does not vanish, all the functions g_r can be calculated step by step. That completes the proof.

Lemma 8. *Let the formal functions $f, g \in \mathscr{F}(U)$ be such that $\tilde{A}_f g = 1$. Then the operator $\tilde{A}_g$ is inverse to $\tilde{A}_f$ and, in particular, $\tilde{A}_g f = 1$.*

Proof. The operator $\tilde{A}_f\tilde{A}_g$ belongs to $\mathscr{L}_{\omega_0}(U)$. Since $\tilde{A}_f\tilde{A}_g 1 = \tilde{A}_f g = 1$, then $\tilde{A}_f\tilde{A}_g = \tilde{A}_1 = 1$. It follows from Lemma 7 that the coefficient at the zero power of ν of the formal series g does not vanish. Therefore, there exists a function $h \in \mathscr{F}(U)$ such that $\tilde{A}_g h = 1$, so $\tilde{A}_g\tilde{A}_h = 1$. Thus the operator $\tilde{A}_g$ has both left and right inverse operators which immediately implies the assertion of the lemma.

We will use some elementary facts about formal series. Let R be a vector space and $\tilde{R} = R[[\nu]]$ be the space of formal series with coefficients in R. There is a decreasing filtration in $\tilde{R}$, $\tilde{R} = \tilde{R}_0 \supset \tilde{R}_1 \supset \tilde{R}_2 \ldots$, where $\tilde{R}_n$ consists of the series of the form $\tilde{A} = \sum_{r=n}^{\infty} \nu^r A_r$, $A_r \in R$. An element $\tilde{A} \in \tilde{R}$ is of the order n, $\mathrm{ord}(\tilde{A}) = n$, if $\tilde{A} \in \tilde{R}_n \backslash \tilde{R}_{n+1}$. A series $\sum \tilde{A}_n$ with the elements $\tilde{A}_n \in \tilde{R}$, such that the order $\mathrm{ord}(\tilde{A}_n) \to \infty$ as $n \to \infty$, converges to an element of $\tilde{R}$ with respect to the topology defined by the filtration. If $\tilde{A} - \tilde{B}$ is of the order n, we write $\tilde{A} \equiv \tilde{B} \pmod{\nu^n}$.

For an arbitrary formal function $S = S_0 + \nu S_1 + \cdots \in \mathscr{F}(U)$ define its exponent, $e^S = e^{S_0} \sum_{n=0}^{\infty} (1/n!)(S - S_0)^n \in \mathscr{F}(U)$. The series in the definition of the exponent converges since $\mathrm{ord}((S - S_0)^n) \geqq n$.

Lemma 9. *For $S \in \mathscr{F}(U)$, $\partial e^S/\partial z^k = (\partial S/\partial z^k)e^S$ and $\partial e^S/\partial \bar{z}^l = (\partial S/\partial \bar{z}^l)e^S$. Moreover, for $S, T \in \mathscr{F}(U)$ the equality $e^S \cdot e^T = e^{S+T}$ holds.*

The proof is standard.

Proposition 3. *Let ω be a formal deformation of the Kähler metrics ω_0 on M and $U \subset M$ be a contractible coordinate chart. For each formal function $g = \sum \nu^r g_r \in \mathscr{F}(U)$ there exists a unique formal series of differential operators $\tilde{B}_g = \sum \nu^r B_r$ from $\mathscr{L}_\omega(U)$ such that $\tilde{B}_g 1 = g$.*

Proof. Let $\Phi = \Phi_0 + \nu\Phi_1 + \nu^2\Phi_2 + \cdots$ be a potential of ω. Set $S = \Phi_1 + \nu\Phi_2 + \ldots$. It follows from Lemma 9 that $e^{-S}(\partial\Phi_0/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l)e^S = \partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$. Since, moreover, $e^{-S}\bar{z}^l e^S = \bar{z}^l$ we get that $e^{-S}\mathscr{L}_{\omega_0}(U)e^S = \mathscr{L}_\omega(U)$. The operator $\tilde{B}_g$ exists if and only if there is a function $f \in \mathscr{F}(U)$ such that $e^{-S}(\tilde{A}_f)e^S = \tilde{B}_g$. It is enough for f to satisfy the relation $e^{-S}\tilde{A}_f(e^S) = g$ or $\tilde{A}_f(e^S) = e^S g$, which is equivalent to the equality $\tilde{A}_f\tilde{A}_{e^S} = \tilde{A}_{e^S g}$. From Lemmas 7 and 8 it follows that there is a function $h \in \mathscr{F}(U)$ such that the operator $\tilde{A}_h$ is inverse to $\tilde{A}_{e^S}$. Therefore $\tilde{A}_f = \tilde{A}_{e^S b}\tilde{A}_h$, and so $f = \tilde{A}_{e^S g}h$, which completes the proof.

According to Proposition 3, the mapping $f \mapsto \tilde{B}_f$ is a bijection of $\mathscr{F}(U)$ onto $\mathscr{L}_\omega(U)$. Since $\mathscr{L}_\omega(U)$ is an operator algebra, one can define in $\mathscr{F}(U)$ an associative product $\star$, carrying over to $\mathscr{F}(U)$ the operator product from $\mathscr{L}_\omega(U)$. For $f, g \in \mathscr{F}(U)$ by definition $\tilde{B}_{f\star g} = \tilde{B}_f\tilde{B}_g$. Applying both sides of the obtained equality to the constant 1, one gets $f \star g = \tilde{B}_f g$. That means that $\tilde{B}_f$ is a left multiplication operator in the algebra $\mathscr{F}(U)$ with the operation $\star$. Denote $L_f = \tilde{B}_f$.

Calculate the first two terms of the formal series of operators $L_{\bar{z}^l}$.

Lemma 10. $L_{\bar{z}^l} \equiv \bar{z}^l + \nu D^l \pmod{\nu^2}$.

Proof. Let $L_{\bar{z}^l} \equiv A + \nu B \pmod{\nu^2}$, then

$$\left[L_{\bar{z}^l}, \frac{\partial\Phi}{\partial\bar{z}^{l'}} + \nu\frac{\partial}{\partial\bar{z}^{l'}}\right] \equiv \left[A + \nu B, \frac{\partial\Phi_0}{\partial\bar{z}^{l'}} + \nu\left(\frac{\partial\Phi_1}{\partial\bar{z}^{l'}} + \frac{\partial}{\partial\bar{z}^{l'}}\right)\right] \pmod{\nu^2}. \tag{13}$$

The operators $L_{\bar{z}^l}$ and $\partial\Phi/\partial\bar{z}^{l'} + \nu\partial/\partial\bar{z}^{l'}$ commute, therefore the coefficients at the zero and first powers of ν on the right-hand side of (13) are equal to zero. First, $[A, \partial\Phi_0/\partial\bar{z}^{l'}] = 0$, therefore, according to Lemma 5, A is a multiplication operator. Since $L_{\bar{z}^l}1 = A1 = \bar{z}^l$ then $A = \bar{z}^l$. Taking into account that $A = \bar{z}^l$, we get the equation $[B, \partial\Phi_0/\partial\bar{z}^{l'}] = \delta^l_{l'}$. Since $B1 = 0$, from Lemma 5 follows that $B = D^l$. The lemma is proved.

Now we obtain the formula expressing the operator L_f, $f \in \mathscr{F}(U)$, via $L_{\bar{z}^l}$.

Proposition 4.

$$L_f = \sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{\partial\bar{z}}\right)^\alpha f \; (L_{\bar{z}} - \bar{z})^\alpha, \tag{14}$$

where α is a multi-index.

Proof. It follows from Lemma 10 that $\mathrm{ord}(L_{\bar{z}^l} - \bar{z}^l) = 1$, therefore $\mathrm{ord}((L_{\bar{z}} - \bar{z})^\alpha) = |\alpha|$ so the series in (14) converges. Denote temporarily the right-hand side of (14) by $\tilde{A}$. Since $(L_{\bar{z}^l} - \bar{z}^l)1 = 0$, then $\tilde{A}1 = f$, so to prove the proposition it is enough to show that $\tilde{A} \in \mathscr{L}_\omega(U)$. Let $\alpha = (i_1, \dots, i_m)$ be a multi-index. Introduce the following notation, $\alpha \pm l = (i_1, \dots, i_l \pm 1, \dots, i_m)$. Taking into account that $L_{\bar{z}^l} \in \mathscr{L}_\omega(U)$, one gets

$$\left[\left(\frac{\partial}{\partial \bar{z}}\right)^\alpha f, \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}\right] = \nu \left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha + l} f$$

and

$$\left[\frac{1}{\alpha!}(L_{\bar{z}} - \bar{z})^\alpha, \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}\right] = -\nu \frac{1}{(\alpha - l)!}(L_{\bar{z}} - \bar{z})^{\alpha - l},$$

which implies that

$$\left[\tilde{A}, \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}\right] = \nu \left(\sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha + l} f \; (L_{\bar{z}} - \bar{z})^\alpha \right.$$

$$\left. - \sum_\alpha \frac{1}{(\alpha - l)!}\left(\frac{\partial}{\partial \bar{z}}\right)^\alpha f \; (L_{\bar{z}} - \bar{z})^{\alpha - l}\right) = 0 .$$

The proposition is proved.

Example. Let us show now how the deformation quantization obtained from the Wick symbol product (see Introduction) can be derived from our formalism.

We take $M = \mathbf{C}^m$, $\omega = \omega_0 = i \sum_j dz^j \wedge d\bar{z}^j$ and $\Phi = \sum_j |z^j|^2$. Then $\bar{z}^j = \partial \Phi / \partial z^j$, thus $L_{\bar{z}^j} = \bar{z}^j + \nu \partial / \partial z^j$. Now formula (14) reduces to (1) with $\hbar$ replaced by the formal parameter ν.

It immediately follows from Proposition 4 and bilinearity of the product $\bigstar$ that the product $\bigstar$ is given by formula (2) for some bidifferential operators C_r. Let $u, v \in C^\infty(U)$. Calculate the operators C_0 and C_1 considering the first two terms of the series $u \bigstar v$ and taking into account Lemma 10,

$$u \bigstar v = L_u v \equiv uv + \nu \sum_l \frac{\partial u}{\partial \bar{z}^l} D^l v \,(\mathrm{mod}\, \nu^2) .$$

It follows that $C_0(u, v) = uv$ and $C_1(u, v) = \sum_l \partial u / \partial \bar{z}^l D^l v$, therefore

$$C_1(u, v) - C_1(v, u) = \sum_l \left(\frac{\partial u}{\partial \bar{z}^l} D^l v - \frac{\partial v}{\partial \bar{z}^l} D^l u\right)$$

$$= g^{lk}\left(\frac{\partial v}{\partial z^k}\frac{\partial u}{\partial \bar{z}^l} - \frac{\partial u}{\partial z^k}\frac{\partial v}{\partial \bar{z}^l}\right) = i\{u, v\} .$$

That means that the product $\bigstar$ is a $\bigstar$-product on the chart U with the Kähler metrics ω_0. It is clear from the construction of the product $\bigstar$ from the deformation of Kähler metrics ω that on the intersections of charts the products $\bigstar$ agree with each other and define a global deformation quantization with separation of variables on the Kähler manifold M. From Theorem 1 it follows that the deformation of Kähler metrics corresponding to the $\bigstar$-product $\bigstar$, coincides with ω. Thus we have stated the following

Theorem 2. *Deformation quantizations with separation of variables on a Kähler manifold M are in* 1–1 *correspondence with the formal deformations of the Kähler metrics* ω_0 *on M. If on M there is given a quantization with separation of variables corresponding to a formal deformation* ω *of the metrics* ω_0, *U is a contractible coordinate chart on M, and* Φ *is a potential of* ω *on U, then the operators of left* $\star$*-multiplication* $\mathscr{L}(U)$ *are characterized by the property that they commute with multiplication by antiholomorphic functions and with the operators* $R_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$. *Similarly, the operators of right* $\star$*-multiplication* $\mathscr{R}(U)$ *are characterized by the property that they commute with multiplication by holomorphic functions and with the operators* $L_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \nu\partial/\partial z^k$.

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