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Invariant Measures for the 2D-Defocusing Nonlinear Schrödinger Equation

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Abstract: Consider the $2D$ defocusing cubic NLS $iu_t + \Delta u - u|u|^2 = 0$ with Hamiltonian $\int(|\nabla\phi|^2 + \frac{1}{2}|\phi|^4)$. It is shown that the Gibbs measure constructed from the Wick ordered Hamiltonian, i.e. replacing $|\phi|^4$ by $:|\phi|^4:$, is an invariant measure for the appropriately modified equation $iu_t + \Delta u - [u|u|^2 - 2(\int |u|^2 dx)u] = 0$. There is a well defined flow on the support of the measure. In fact, it is shown that for almost all data ϕ the solution u, $u(0) = \phi$, satisfies $u(t) - e^{it\Delta}\phi \in C_{H^s}(\mathbb{R})$, for some $s > 0$. First a result local in time is established and next measure invariance considerations are used to extend the local result to a global one (cf. [B2]).

Introduction

Consider the Wick ordering $H_N = \int |\nabla u|^2 + \frac{1}{2}\int |u|^4 - 2a_N \int |u|^2 + a_N^2$ of the $2D$-Hamiltonian $\int |\nabla u|^2 + \frac{1}{2}\int |u|^4$ corresponding to the $2D$-defocusing cubic NLS.[1] It is shown that the solutions $u_N = u_N^\omega$ of the Cauchy problem

$$\begin{cases} (u_N)_t = i\frac{\partial H_N}{\partial \bar{u}} \equiv \Delta u_N - P_N(u_N|u_N|^2) + 2a_N u_N = 0 \\ u_N = P_N u_N\,,\ \ u_N(0) = \sum_{|n|\leqq N} \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle} \end{cases} \tag{i}$$

converge weakly for all time, for almost all ω.[2] Here $\{g_n(\omega)\,|\,n \in \mathbf{Z}\}$ are independent L^2-normalized complex Gaussians and P_N denotes the usual Dirichlet projection on the trigonometric system.

In fact, there is some $s > 0$, such that

$$u_N(t) - e^{2ic_N(\omega)t} \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i(\langle x,n\rangle + |n|^2 t)}\,, \tag{ii}$$

[1] u is a complex function.

[2] We ignore for notational simplicity the problem of the zero Fourier mode in (i). This problem may be avoided replacing $|n|$ by $(|n|^2 + \kappa)^{1/2}$, $\kappa > 0$ (redefining the Laplacian).

$$c_N(\omega) = \sum_{|n|<N} \frac{|g_n(\omega)|^2 - 1}{|n|^2} \tag{iii}$$

converges in $H^s(\mathbf{T}^2)$, for all time t.

The study of (i) mainly reduces to the truncation independent equation

$$u_t = \Delta u - \left[u|u|^2 - 2\left(\int_{\mathbf{T}^2} |u|^2\, dx \right) u \right], \tag{iv}$$

where the expression between brackets has to be considered as the usual cubic term $u|u|^2$ with suppression of certain square-terms (which are obviously divergent for the data considered in (i) when $N \to \infty$).

The main point is that (iv) is well posed for typical elements in the support of the Gibbs measure, or, equivalently, for data $\phi_\omega = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle}$, almost surely in ω. Once a local result is obtained, one proceeds as in [B2], using the invariance of the Gibbs measure $e^{-H_N(\phi)} \Pi\, d\phi$ for the flow of the truncated equations (i), to get the results on solutions for all time. The limit flow for $N \to \infty$ and the flow of (iv) have the normalized Gibbs measure $d\mu = \lim_{N\to\infty} e^{-H_N(\phi)} \Pi\, d\phi$ as an invariant measure.

This problem was considered in the paper [L-R-S]. The present work extends the one-dimensional result in [B2] to the $2D$-defocusing case.

1. Wick Ordered Hamiltonian for Cubic $2D$-NLS (Defocusing Case)

We first recall the process of Wick ordering the $|u|^4$-nonlinearity (we are in the complex case). This Wick ordered Hamiltonian will lead to the modification of the cubic nonlinearity appearing in (iv) above. For the general theory of Wick ordering, the reader may consult [G-J].

The Wick ordered "truncated" Hamiltonians are given by

$$H_N = \int |\nabla u_N|^2 + \frac{1}{2} \int |u_n|^4 - 2a_N \int |u_N|^2 + a_N^2,$$

where

$$a_N = \sum_{\substack{|n|<N \\ n\neq 0}} \frac{1}{|n|^2} = \left\| \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i\langle n,x\rangle} \right\|^2_{L^2(dx\, d\omega)}.$$

The corresponding Gibbs measure is

$$e^{-H_N} \Pi\, d\phi_N = \exp\left[-\frac{1}{2} \int |\phi_N|^4 + 2a_N \int |\phi_N|^2 - a_N^2 \right] \underbrace{\exp\left(-\int |\nabla\phi_N|^2\right) \Pi\, d\phi_N}_{\text{Wiener measure}}.$$

Denoting $\phi = \phi_N$, one has

$$-\frac{1}{2} \int |\phi|^4 + 2a_N \int |\phi|^2 - a_N^2 = -\frac{1}{2} \int (|\phi|^2 - 2a_N)^2 + a_N^2 < (\log N)^2. \tag{0}$$

Fourier expansion yields

$$-\frac{1}{2}\int|\phi|^4+2a_N\int|\phi|^2-a_N^2=-\frac{1}{2}\sum_{\substack{n_1-n_2+n_3-n_4=0\\ n_1\neq n_2,n_4}}\frac{g_{n_1}(\omega)}{|n_1|}\frac{\overline{g_{n_2}(\omega)}}{|n_2|}\frac{g_{n_3}(\omega)}{|n_3|}\frac{\overline{g_{n_4}(\omega)}}{|n_4|}\quad(1)$$

$$-\left(\sum_{|n|<N}\frac{|g_n(\omega)|^2}{|n|^2}\right)^2+2a_N\cdot\sum_{|n|<N}\frac{|g_n(\omega)|^2}{|n|^2}-a_N^2\quad(2)$$

$$+\frac{1}{2}\sum_{|n|<N}\frac{|g_n(\omega)|^4}{|n|^4}\quad(3)$$

$$\sum_{|n|<N}\frac{|g_n(\omega)|^2}{|n|^2}=\sum_{|n|<N}\frac{|g_n(\omega)|^2-1}{|n|^2}+a_N\,,$$

hence

$$(2)=-\left(\sum\frac{|g_n(\omega)|^2-1}{|n|^2}\right)^2.$$

Thus (1), (2), (3) are finite a.s. in ω.

Also for $N>N_0$, there is the following distributional inequality:

$$\mathbf{P}_\omega\left[\left|\left(-\frac{1}{2}\int|\phi_N|^4+2a_N\int|\phi_N|^2-a_N^2\right)\right.\right.$$
$$\left.\left.-\left(-\frac{1}{2}\int|\phi_{N_0}|^4+2a_{N_0}\int|\phi_{N_0}|^2-a_{N_0}^2\right)\right|>1\right]<Ce^{-N_0^\delta}\quad(4)$$

(for some $\delta>0$).

To prove (4), one considers the different terms (1), (2), (3) and uses the standard moment inequalities for linear combinations of products of Gaussians (obtained from hypercontractivity estimates). The contribution of expression (1) above to the difference in (4) is given by

$$\sum_{\substack{n_1-n_2+n_3-n_4=0\\ n_1\neq n_2,n_4\\ \max|n_j|>N_0}}\frac{g_{n_1}(\omega)}{|n_1|}\frac{\overline{g_{n_2}(\omega)}}{|n_2|}\frac{g_{n_3}(\omega)}{|n_3|}\frac{\overline{g_{n_4}(\omega)}}{|n_4|}.$$

Since these are products of 4 Gaussians, there is equivalence of the $L^2(d\omega)$-norm and the Orlicz norm $L^\psi(d\omega)$, with $\psi(\lambda)=e^{\lambda^{1/2}}-1$. Since the $\{g_n(\omega)\}$ are independent complex Gaussians, one clearly gets for the $L^2(d\omega)$-norm

$$\left\{\int_{|x_1|>N_0}(1+|x_1|)^{-2}(1+|x_2|)^{-2}(1+|x_3|)^{-2}(1+|x_1-x_2+x_3|)^{-2}dx_1dx_2dx_3\right\}^{1/2}$$
$$\leqq N_0^{-1/2}.$$

The contribution of term (2) to the difference in (4) is

$$\left(\sum_{|n|\leqq N}\frac{|g_n(\omega)|^2-1}{|n|^2}\right)\left(\sum_{N_0<|n|\leqq N}\frac{|g_n(\omega)|^2-1}{|n|^2}\right).$$

Since $\int|g_n(\omega)|^2d\omega=1$, the norm is estimated by $(\sum_{|n|>N_0}\frac{1}{|n|^4})^{1/2}\lesssim N_0^{-1}$.

Similarly, term (3) contributes for $\sum_{|n|>N_0} \frac{1}{|n|^4} \sim N_0^{-2}$. Estimate (4) easily follows.

One deduces the following stability estimate ($\lambda > 2$)

$$\mathbf{P}_\omega \left[-\frac{1}{2} \int |\phi_N|^4 + 2a_N \int |\phi_N|^2 - a_N^2 > \lambda \right] \leqq e^{-e^{\delta\sqrt{\lambda}}} . \tag{5}$$

Proof. Choose N_0 with $(\log N_0)^2 < \frac{\lambda}{2}$. From (0) applied with $N = N_0$ and (4), (5) follows.

Hence, the renormalized Gibbs measure is a weighted Wiener measure with density in $\bigcap_{p<\infty} L^p$.

2. Truncated NLS

$$u_t^N = i\frac{\partial H_N}{\partial \overline{u^N}} \Rightarrow iu_t^N - \Delta u^N + P_N(u^N|u^N|^2) - 2a_N u^N = 0 .$$

Rewrite equation ($u = u^N$) as

$$iu_t - \Delta u + 2(\textstyle\int |u|^2 - a_N)u + P_N(u|u|^2 - 2u \int |u|^2) = 0 , \tag{6}$$

where $\int |u|^2 - a_N = \sum_{|n|\leqq N} \frac{|g_n(\omega)|^2 - 1}{|n|^2} = c_N(\omega)$ is time invariant, and converges to $c_\infty(\omega) < \infty$ a.s. in ω. Define $u_N = e^{2ic_N(\omega)t} \cdot v_N$, reducing Eq. (6) to

$$iv_t - \Delta v + P_N(v|v|^2 - 2v \textstyle\int |v|^2) = 0 . \tag{7}$$

The nonlinear term is given by

$$P_N\{ \sum_{n_2 \neq n_1, n_3} \widehat{v}(n_1)\overline{\widehat{v}(n_2)}\widehat{v}(n_3)\, e^{i\langle n_1 - n_2 + n_3, x\rangle} \} \tag{8}$$

$$- \sum_{|n|<N} \widehat{v}(n)|\widehat{v}(n)|^2\, e^{i\langle n,x\rangle} . \tag{9}$$

3. Cauchy Problem

$$\begin{cases} iu_t - \Delta u + P_N(u|u|^2 - 2u \int |u|^2) = 0 \\ u = P_N u,\ u(0) = \phi_N(x) = \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i\langle n,x\rangle} \end{cases} \tag{10}$$

on time interval $[0, \tau]$.

Proposition. *The Cauchy problem*

$$\begin{cases} iu_t - \Delta u + (u|u|^2 - 2u \int |u|^2) = 0 \\ u(0) = \phi_\omega(x) = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle} \end{cases} \tag{11}$$

is well posed on $[0, \tau]$ *except for* ω *in a set of measure* $\leqq e^{-\frac{1}{\tau^\delta}}$ ($\delta > 0$) *and the solution* u *is the (distributional) limit of the solutions* u_N *of* (10) *when* $N \to \infty$.

In fact

$$u - \sum \frac{g_n(\omega)}{|n|} e^{i(\langle x,n\rangle + |n|^2 t)}$$

is the limit in $L^\infty_{H^s(\mathbf{T}^2)}[0,\tau]$ *of*

$$u_N - \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i(\langle x,n\rangle + |n|^2 t)}$$

for some $s > 0$.

Corollary. *Solutions of*

$$\begin{cases} u_t^N = i\frac{\partial H_N}{\partial \bar{u}^N} \\ u^N(0) = \phi_N(\omega) \end{cases} \tag{12}$$

for $t \in [0,\tau]$ *and* $N \to \infty$ *converge for* ω *outside a set of measure* $\leqq e^{-\frac{1}{\tau^\delta}}$. *In fact*

$$u_N - e^{2ic_N(\omega)t} \cdot \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i(\langle x,n\rangle + t|n|^2)} \tag{13}$$

converges in $L^\infty_{H^s(\mathbf{T}^2)}[0,\tau]$ *when* $N \to \infty$ *for those* ω.

Using invariant Gibbs measures $e^{-H_N} \Pi\, d\phi_N$ (forming a convergent sequence to a measure $\mu \sim$ Wiener measure) and probabilistic considerations, one shows next that a.s. in the ω solution $u_N = u_{N,\omega}$ of (12) converges on $[0,\infty[$ and also (13) converges in H^s for all t. The limiting flow leaves μ invariant since $e^{-H_N}\Pi\, d\phi_N$ is invariant under the flow of the truncated equation (12). The reasoning followed here is completely analogous to the argument in [B2] for the 1-dimensional NLS.

4. Estimates on (11)

Consider the integral equation associated to (11)

$$u(t) = S(t)\phi + i\int_0^t S(t-\tau)[(u|u|^2 - 2u\int |u|^2)(\tau)]d\tau\,, \tag{14}$$

where $S(t) = e^{it\Delta}$. Consider the norm (space-time on $[0,\tau]$)

$$|||u|||_s = \left(\sum_n \int d\lambda (1+|n|^2)^s (1+|\lambda - |n|^2|)|\hat{u}(n,\lambda)|^2\right)^{1/2}. \tag{15}$$

Here $\hat{u}(n,\lambda)$ denotes the Fourier transform u, in the sense that

$$u(x,t) = \sum_n \int d\lambda\, e^{i(\langle n,x\rangle + \lambda t)}\hat{u}(n,\lambda) \quad \text{for } (x,t) \in \mathbf{T}^2 \times [0,\tau]$$

(strictly speaking, $\hat{u}$ is not uniquely defined and (15) should be understood as a restriction norm). The exposition below will be closely related to [B1], which the reader may wish to consult for more background and details.

We will show that for ω outside an exceptional set of size $< e^{-\frac{1}{\tau^\delta}}$, the transformation

$$u \mapsto S(t)\phi + i \int_0^t S(t-\tau)[(u|u|^2 - 2u \textstyle\int |u|^2)(\tau)]\, d\tau \tag{16}$$

defines an $|||\ \ |||_s$-contraction on the set $S(t)\phi + (|||\ \ |||_s$-ball).

Write, cf. (8), (9),

$$\begin{aligned} u|u|^2 - 2u \textstyle\int |u|^2 = &\sum_{n_2 \neq n_1, n_3} \widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\, e^{i\langle n_1 - n_2 + n_3, x\rangle} \\ &- \sum_n \widehat{u}(n)\, |\widehat{u}(n)|^2\, e^{i\langle n,x\rangle}\,. \end{aligned} \tag{17}$$

The contribution of the second term in (17) is immediate. We consider the first as a trilinear expression, replacing the $\widehat{u}, \overline{\widehat{u}}, \widehat{u}$ factors by $\widehat{u}_1, \overline{\widehat{u}_2}, \widehat{u}_3$ resp. We limit each Fourier transform to a dyadic region $|n_i| \sim N_i$ $(i = 1,2,3)$. Denote w the first term in (17). Since

$$\int_0^t S(t-\tau)w(\tau)d\tau = -i \sum_{n\in\mathbf{Z}^2} \int d\lambda \left\{ \widehat{w}(n,\lambda) e^{i\langle n,x\rangle} \frac{e^{i\lambda t} - e^{i|n|^2 t}}{\lambda - |n|^2} \right\},$$

there is an estimate of

$$\left|\left|\left| \int_0^t S(t-\tau)w(\tau)d\tau \right|\right|\right|_s$$

by (cf. [B1])

$$\left\{ \sum_n \int d\lambda \frac{|n|^{2s} |\widehat{w}(n,\lambda)|^2}{|\lambda - |n|^2|} \right\}^{1/2} + \left\{ \sum_n |n|^{2s} \left(\int \frac{|\widehat{w}(n,\lambda)|}{|\lambda - |n|^2|} d\lambda \right)^2 \right\}^{1/2}, \tag{18}$$

where the denominator $|\lambda - |n|^2|$ means $|\lambda - |n|^2| + 1$ (because estimates are local in time).

For each of the u_i $(i = 1,2,3)$, there are 2 possibilities,

$$u_i = \frac{1}{N_i} \sum_{|n|\sim N_i} g_n(\omega) e^{i(\langle x,n\rangle + t|n|^2)}, \tag{I}$$

$$|||u_i|||_s < 1\,, \tag{II}$$

decomposing as $S(t)\phi + (H^s$-ball).

Denote N^1, N^2, N^3 the decreasing ordering of $\{N_1, N_2, N_3\}$ and u^1, u^2, u^3 the corresponding u_i-factors. The estimates from [B] permit to bound (18) by

$$\tau^c \cdot \exp \frac{\log N^2}{\log\log N^2} \cdot |||u^1|||_s\, |||u^2|||_0\, |||u^3|||_0\,. \tag{19}$$

This estimate appears in [B1] in the discussion of the $2D$ cubic NLS. The main underlying (Strichartz-type) inequality is inequality (26) below. The factor $\exp \frac{\log N}{\log\log N}$ appears from bounding the number of lattice points on a circle of radius $\leqq N$.

The exponent s will be a sufficiently small positive number. It follows from (19) that the following cases are taking care of

$$u^1(\mathrm{II}),\ u^2(\mathrm{II}), \tag{20}$$

$$u^1(\mathrm{II}),\ u^3(\mathrm{II}) \text{ and } \log N^2 \sim \log N^3 . \tag{21}$$

If $|||v|||_s \leqq 1$, we may clearly write v as

$$\int d\lambda'(|\lambda'|+1)^{-1/2}\left(\sum_n n^{2s}(1+|\lambda'|)|\widehat{v}(n,n^2+\lambda')|^2\right)^{1/2}\left[e^{i\lambda' t}\sum_n a_{\lambda'}(n)e^{i(\langle n,x\rangle+t|n|^2)}\right], \tag{22}$$

where $a_{\lambda'}(n)=\frac{\hat{v}(n,n^2+\lambda')}{\left(\sum_n n^{2s}|\hat{v}(n,n^2+\lambda')|^2\right)^{1/2}}$, hence $\sum_n n^{2s}|a_{\lambda'}(n)|^2=1$. Also be Hölder's inequality,

$$\int_{|\lambda'|<K}(1+|\lambda'|)^{-1/2}\left(\sum_n n^{2s}(1+|\lambda'|)|\widehat{v}(n,n^2+\lambda')|^2\right)^{1/2}d\lambda' < (\log K)^{1/2}. \tag{23}$$

Next, we aim to bound the range of λ'. Observe that we may assume (restricting ω) that say

$$\left\|\frac{1}{N}\sum_{|n|\sim N} g_n(\omega)\, e^{i\langle x,n\rangle}\right\|_\infty \leqq c\log N \tag{24}$$

for all N. Hence,

$$\|u_i\|_\infty \leqq c\log N_i \quad \text{if } u_i \text{ is of type (I)}. \tag{25}$$

Recall also the main estimate used in the Cauchy problem for the 2D-cubic NLS (Strichart-type inequality)

$$\left\|\sum_{|n-n_0|<N} a_n\, e^{i(\langle x,n\rangle+t|n|^2)}\right\|_{L^4(\mathbf{T}^3)} < \exp\frac{\log N}{\log\log N}\cdot\left(\sum|a_n|^2\right)^{1/2}. \tag{26}$$

This L^4-inequality reduces to lattice point counting on circles and the exponential factor bounds the divisor function. For details, see again [B1].

Hence

$$\left\|\sum_{|n-n_0|<N}\int d\lambda\, a(n,\lambda)e^{i(\langle x,n\rangle+\lambda t)}\right\|_{L^4(\mathbf{T}^2\times[0,1])}$$
$$\ll N^\varepsilon\left(\sum_n\int d\lambda(1+|\lambda-|n|^2|)\,|a(n,\lambda)|^2\right)^{1/2}. \tag{27}$$

To prove (27), write $n = n_0 + n_1, |n_1| < N$ and $\lambda = |n|^2 + \lambda_1$. Estimate for $|\lambda - |n|^2| < K$

$$\left\| \sum_{|n-n_0|<N} \int_{|\lambda-|n|^2|<K} d\lambda \{a(n,\lambda) e^{i(\langle x,n\rangle+\lambda t)}\} \right\|_{L^4(\mathbf{T}^2\times[0,1])}$$

$$\leqq \int_{|\lambda_1|<K} d\lambda_1 \left\| \sum_{|n_1|<N} a(n_0+n_1, |n_0+n_1|^2+\lambda_1) e^{i(\langle x+2n_0t, n_1\rangle + |n_1|^2 t} \right\|_{L^4(\mathbf{T}^2\times[0,1])} \ll \text{by (26)}$$

$$N^\varepsilon \int_{|\lambda_1|<K} d\lambda_1 \left(\sum_{|n_1|<N} |a(n_0+n_1, |n_0+n_1|^2+\lambda_1)|^2 \right)^{1/2}$$

$$\leqq N^\varepsilon (\log K)^{1/2} \left(\sum_n \int d\lambda (1 + |\lambda - |n|^2|) |a(n,\lambda)|^2 \right)^{1/2} .$$

This bound is conclusive, except if $\log K \gg \log N$. Now the range $|\lambda - |n|^2| > N^{20}$ may be trivially estimated, writing from the triangle and Hausdorff–Young inequality w.r.t. the t-variable

$$\left\| \sum_{|n-n_0|<N} \int_{|\lambda-|n|^2|>N^{20}} d\lambda\, a(n,\lambda) e^{i(\langle x,n\rangle+\lambda t)} \right\|_{L^4(dx\,dt)}$$

$$\leqq \sum_{|n-n_0|<N} \left\{ \int_{|\lambda-|n|^2|>N^{20}} |a(n,\lambda)|^{4/3} d\lambda \right\}^{3/4} .$$

This expression is bounded by $N^2 N^{-5}$, from Hölder's inequality, which establishes (27).

By interpolation, for $2 \leqq p \leqq 4$,

$$\left\| \sum_{|n-n_0|<N} \int d\lambda\, a(n,\lambda) e^{i(\langle x,n\rangle+\lambda t)} \right\|_{L^p(\mathbf{T}^2\times[0,1])}$$

$$\ll N^\varepsilon \left(\sum_n \int d\lambda (1 + |\lambda - |n|^2|)^{2-\frac{4}{p}} |a(n,\lambda)|^2 \right)^{1/2} . \tag{28}$$

Consider first a triplet (u_1, u_2, u_3), where u^1 is (II) and hence u^2 is (I) (otherwise we are in case (20)). We estimate using duality (18) by

$$(N^1)^s \int u^1 \cdot u^2 \cdot u^3 \cdot v\,, \tag{29}$$

where

$$v = \sum_n \int d\lambda \frac{v(n,\lambda)}{|\lambda - |n|^2|^{1/2}} e^{i(\langle n,x\rangle+\lambda t)} \quad \text{or} \quad v = \sum_n \int d\lambda \frac{v(n)}{|\lambda - |n|^2|} e^{i(\langle n,x\rangle+\lambda t)}$$

with $\sum_n \int d\lambda |v(n,\lambda)|^2 \leqq 1$ and $\sum_n |v(n)|^2 \leqq 1$. Applying (28) with $p=3$ and (24), estimate (29) by

$$\begin{aligned}
(N^1)^s \sum_{J\in\mathscr{J}} \int |P_J u^1| \cdot |u^2| \cdot |u^3| \cdot |P_J v| &\leqq (N^1)^s \sum_{J\in\mathscr{J}} \|P_J u^1\|_3 \cdot \|u^2\|_\infty \|u^3\|_3 \|P_J v\|_3 \\
&\ll (N^1)^s (N^2)^\varepsilon \sum_{J\in\mathscr{J}} \left(\sum_{n\in J} \int d\lambda |\lambda - |n|^2|^{2/3} |\widehat{u^1}(n,\lambda)|^2\right)^{1/2} \\
&\quad \times \left(\sum_n \int d\lambda |\lambda - |n|^2|^{2/3} |\widehat{u^3}(n,\lambda)|^2\right)^{1/2} \left(\sum_{n\in J} \int d\lambda |\lambda - |n|^2|^{-1/3+} |\widehat{v}(n,\lambda)|^2\right)^{1/2} \\
&\leqq (N^2)^\varepsilon \left(\sum_{|n|\sim N^1} \int d\lambda |n|^{2s} |\lambda - |n|^2|^{2/3} |\widehat{u^1}(n,\lambda)|^2\right)^{1/2} \\
&\quad \cdot \left(\sum_{|n|\sim N_3} \int d\lambda |\lambda - |n|^2|^{2/3} |\widehat{u^3}(n,\lambda)|^2\right)^{1/2} \\
&\quad \times \left(\sum_n \int d\lambda |\lambda - |n|^2|^{-1/3+} |\widehat{v}(n,\lambda)|^2\right)^{1/2} .
\end{aligned} \tag{30}$$

Here $\mathscr{J}$ denotes a partition of the set $[|n| \sim N^1]$ in intervals J of size $\sim N^2$ and P_J is the corresponding Fourier restriction operator in the x-variable.

Thus the preceding estimate (30) is conclusive provided for some u_i of type (II) we consider the contribution of $\widehat{u_i}|_{[|\lambda_i - |n_i|^2| \gg (N^2)^\varepsilon]}$ or if the denominator $\lambda - |n|^2$ in (18) satisfies $|\lambda - |n|^2| \gg (N^2)^\varepsilon$. Hence we may in the estimate of (18) assume

$$|\lambda - |n|^2| \ll (N^2)^\varepsilon \quad \text{and} \quad |\lambda_i - |n_i|^2| \ll (N^2)^\varepsilon \quad \text{if } u_i \text{ of type (II)}. \tag{31}$$

It follows from (22), (23) that, up to introducing a factor $\log N_2$ in estimating (18), the u_i of type (II) may be taken of the form

$$u_i = e^{i\lambda_i' t} \sum_{|n|\sim N_i} a_i(n) e^{i(\langle n,x\rangle + t|n|^2)}, \tag{32}$$

where

$$|\lambda_i'| \ll N_2^\varepsilon \quad \text{and} \quad \sum n^{2s} |a_i(n)|^2 \leqq 1 . \tag{33}$$

Thus (18) with w the first term in (17) is bounded considering an expression of the form

$$\begin{aligned}
&(\log N^2)^2 \left\{ \sum_{|n|\leqq N^1} \left| \sum_{\substack{n=n_1-n_2+n_3, n_2\neq n_1,n_3 \\ |n|^2=|n_1|^2-|n_2|^2+|n_3|^2+\mu}} a_1(n_1) a_2(n_2) a_3(n_3) \right|^2 \right\}^{1/2} \\
&(|n_i| \sim N_i,\ i=1,2,3),
\end{aligned} \tag{34}$$

where $|\mu| \ll (N^2)^\varepsilon$, $\sum |a^1(n)|^2 \leqq 1$, $a^2(n) = \frac{g_n(\omega)}{N^2}$, $a^3(n) = \frac{g_n(\omega)}{N^3}$ or $\sum |a^3(n)|^2 \leqq (N^3)^{-2s}$.

Next, assume u^1 of type (I). Estimate by (24), (28),

$$(29) \leqq (N^1)^s \|u^1\|_\infty \|u^2\|_3 \|u^3\|_3 \|v\|_3$$

$$\leqq (N^1)^s (\log N^1) \cdot (N^2)^\varepsilon \left(\sum_{|n|\sim N^2} \int d\lambda |\lambda - |n|^2|^{2/3} |\widehat{u^2}(n,\lambda)|^2 \right)^{1/2}$$

$$\times \left(\sum_{|n|\sim N^3} \int d\lambda |\lambda - |n|^2|^{2/3} |\widehat{u^3}(n,\lambda)|^2 \right)^{1/2}$$

$$\times \left(\sum_n \int d\lambda |\lambda - |n|^2|^{-1/3+} |\widehat{v}(n,\lambda)|^2 \right)^{1/2} . \tag{35}$$

Thus the estimate (35) is conclusive provided for some u_i of type (II), we consider the contribution of $\widehat{u}_i|_{[|\lambda_i - |n_i|^2| > (N^1)^{7s}]}$ or if the denominator $\lambda - |n|^2$ in (18) satisfies $|\lambda - |n|^2| > (N^1)^{7s}$. Thus in this case, (18) may be estimated assuming

$$|\lambda - |n|^2| < (N^1)^{7s} \quad \text{and} \quad |\lambda_i - |n_i|^2| < (N^1)^{7s} \quad \text{if } u_i \text{ is of type (II)},$$

and hence is bounded by

$$(\log N^1)^2 (N^1)^s \left\{ \sum_n \left| \sum_{\substack{n = n_1 - n_2 + n_3,\, n_2 \neq n_1, n_3 \\ |n|^2 = |n_1|^2 - |n_2|^2 + |n_3|^2 + \mu}} a_1(n_1) a_2(n_2) a_3(n_3) \right|^2 \right\}^{1/2} (|n_i| \sim N_i), \tag{36}$$

where $|\mu| < (N^1)^{7s}$, $a^1(n) = \frac{g_n(\omega)}{N^1}$ and $a^i(n) = \frac{g_n(\omega)}{N^i}$ or $\sum |a^i(n)|^2 \leqq (N^i)^{-2s}$ for $i = 2, 3$.

Observe that for $n = n_1 - n_2 + n_3$,

$$|n|^2 - (|n_1|^2 - |n_2|^2 + |n_3|^2) = 2\langle n_2 - n_1, n_2 - n_3 \rangle , \tag{37}$$

hence the second condition in the summation in (34), (36) may be written

$$\langle n_2 - n_1, n_2 - n_3 \rangle = \frac{\mu}{2} . \tag{38}$$

If $|n_2| > 10(|n_1| + |n_3|)$, $|\langle n_2 - n_1, n_2 - n_3 \rangle| \sim |n_2|^2$ and it follows from $|\mu| \ll (N^2)^\varepsilon$ or $|\mu| < (N^1)^{7s}$ that thus $|n_1| \sim N^1$ or $|n_3| \sim N^1$. Hence, we may assume $n_1 = N^1$, since the role of u_1, u_3 is identical. We assume here s small enough ($s < \frac{2}{7}$).

Our next aim is for given n and μ to estimate

$$\# \left\{ \underset{n_2 \neq n_1, n_3}{(n_1, n_2, n_3)} \,\middle|\, |n_i| \sim N_i \text{ and } n = n_1 - n_2 + n_3, \langle n_2 - n_1, n_2 - n_3 \rangle = \frac{\mu}{2} \right\} . \tag{39}$$

In the proof of Lemma 1 below, we will use some elementary facts about lattice points on circles in the plane. First, on a circle of radius R, there are at most $\exp \frac{\log R}{\log \log R} \ll R^\varepsilon$ lattice points. As already mentioned above, this bound is an estimate on the divisor function (considering factorization in the ring of Gaussian integers $a + bi$, $a, b \in \mathbf{Z}$). Secondly, if Γ is an arc on a circle of radius R and $|\Gamma| < cR^{1/3}$, then Γ may only contain two lattice points. Indeed, if there were 3 distinct elements P_1, P_2, P_3 in $\Gamma \cap \mathbf{Z}^2$, then

$$c\frac{|\Gamma|^3}{R} > 2 \text{ area triangle } (P_1, P_2, P_3) = \det \begin{bmatrix} 1 & P_1 \\ 1 & P_2 \\ 1 & P_3 \end{bmatrix} \in \mathbf{Z}\backslash\{0\}$$

leading to a contradiction. This last argument is the essence of Jarnick's theorem on the distribution of lattice points on strictly convex arcs (see [BP]).

Lemma 1.

$$(39) \ll \min\{N_2^3(N_1 \wedge N_3)^\varepsilon,\ N^2N^3(N^3)^\varepsilon\}\,. \tag{40}$$

(*Recall that N^1, N^2, N^3 is the decreasing ordering of N_1, N_2, N_3.*)

Proof.

(i) Fix n_2 and write $\langle n_1 - n_2, n_1 - n \rangle = -\frac{\mu}{2}$ as

$$\left| n_1 - \frac{n+n_2}{2}\right|^2 = -\frac{\mu}{2} + \left|\frac{n-n_2}{2}\right|^2 . \tag{41}$$

Thus (41) corresponds to the lattice points n_1 on a given circle with $|n_1| \sim N_1$. Their number is bounded by $\exp\frac{\log N_1}{\log\log N_1}$ (distinguish the cases $\log N_1 \geqq \log$ radius and $\log N_1 \ll \log$ radius; in the second case, the number is at most 2, by the triangle argument). This gives the first bound in (40).

(ii) Write the equation as $\langle n - n_1, n - n_3\rangle = \frac{\mu}{2}$ and assume $|n_3| \leqq |n_1|$. Write $n - n_3 = r(a,b)$, with a, b relatively prime, $v \neq 0$. It follows that

$$\langle n_1, (a,b)\rangle = -\frac{\mu}{2r} + \langle n, (a,b)\rangle\,. \tag{42}$$

If $a, b \neq 0$, the number of solutions of (42) in n_1 is at most $1 + \frac{N_1}{|a| \vee |b|}$.

Consider the case $a, b \neq 0$, $|n_3| > |a| \vee |b|$. Fix A, B, $|a| \sim A$, $|b| \sim B$. The number of n_3's satisfying $n - n_3 = r(a,b)$, $|n_3| > |a| \vee |b|$ is at most $\frac{N_3}{A \vee B}$. The corresponding number of n_1's is $\frac{N_1}{A\vee B}$. This gives the bound $\sum_{A,B} A \cdot B \cdot \frac{N_3}{A\vee B} \cdot \frac{N_1}{A \vee B} = N_1 N_3 \log N_3$.

Assume now n_3 satisfies $|n_3| < |a| \vee |b|$. Fix n_3, thus N_3^2 choices and estimate the number of n_1's by $1 + \frac{N_1}{|a|\vee|b|} < \frac{N_1}{N_3}$. Thus this contribution is bounded by N_1N_3. If $a = 0$ $(b \neq 0)$, n_3 is restricted to N_3 choices (n, n_3 with same first coordinate). The first coordinate of n_1 is arbitrary and the second defined by $\langle n - n_1, n - n_3\rangle = \frac{\mu}{2}$. This gives again a bound by N_1N_3.

Hence there is also the estimate by $N_1 N_3 \log(N_1 \wedge N_3)$.

(iii) Write the equation as $\langle n - n_3, n_3 - n_2\rangle = \frac{\mu}{2}$. Write $n - n_3 = r(a,b)$ with $r \neq 0, a, b$ relative prime. As in (ii), the contribution of $a, b \neq 0$, $|n_2|, |n_3| > |a| \vee |b|$, is estimated by $N_2N_3 \log(N_2 \wedge N_3)$. The contribution of $|n_3| < |a| \vee |b| < |n_2|$ is bounded by $N_3^2 \frac{N_2}{N_3} = N_2 N_3$ and the contribution of $|n_2| < |a| \vee |b|$ at most N_3^2. For $a = 0$ or $b = 0$, the number of possibilities is N_2N_3.

This yields the estimate $N_2N_3 \log(N_2 \wedge N_3) + N_3^2$.

From (i), (ii), (iii), it follows that

$$(39) \ll \min\{N_2^2(N_1\wedge N_3)^\varepsilon, N_1N_3(N_1\wedge N_3)^\varepsilon, N_2N_3(N_2\wedge N_3)^\varepsilon + N_3^2,$$

$$N_2N_1(N_2\wedge N_1)^\varepsilon + N_1^2\}\,. \tag{43}$$

In case $\{N^2,N^3\}=\{N_2,N_3\}$ and $N_3>N_2$, write $N_2^2(N_1\wedge N_3)^\varepsilon \leqq N_2^2N_3^\varepsilon < N_2N_3N_2^\varepsilon$ and similarly if $\{N^2,N^3\}=\{N_2,N_1\}$.

This proves the lemma.

Lemma 2. *Consider the set*

$$S=\{(n_1,n_2,n_3)\,|\,n_2\neq n_1,n_3 \text{ and } \langle n_2-n_1,n_2-n_3\rangle = \mu\}\,.$$

(i) *For fixed* n_1, $\#S(n_1)\ll N_2N_3(N_2\wedge N_3)^\varepsilon$ *and* $\#S(n_1)\ll N_3^2N_2^\varepsilon$.
(ii) *For fixed* n_2, $\#S(n_2)\ll N_1N_3(N_1\wedge N_3)^\varepsilon$.
(iii) *For fixed* n_1,n_2, $\#S(n_1,n_2)<N_3$.
(iv) *For fixed* n_1,n_3, $\#S(n_1,n_3)\ll N_2^\varepsilon$.

Proof.

(i) Fix n_1 and consider estimate (i) in Lemma 1, with $n\leftrightarrow n_1$, $n_1\leftrightarrow n_2$, $n_2\leftrightarrow n_3$. This gives the bound $N_3^2N_2^\varepsilon$. Apply next estimate (iii) of Lemma 1 with $n\leftrightarrow n_1$, $n_3\leftrightarrow n_2$, $n_2\leftrightarrow n_3$, giving the bound $N_2N_3(N_2\wedge N_3)^\varepsilon+N_2^2$. In case $N_2>N_3$, use the $N_3^2N_2^\varepsilon$ bound.
(ii) Follows from (ii) of Lemma 1.
(iii) Immediate.
(iv) Follows from lattice point estimate on circles.

We list the different (u_1,u_2,u_3)-cases to be considered in bounding (18). As mentioned earlier, we may assume $n_1=N^1$. Cases (20), (21) are already considered:

Case (a) : $n_1=N^1(\mathrm{II})$, $n_2=N^2(\mathrm{I})$, $n_3=N^3(\mathrm{II})$.
Case (b) : $n_1=N^1(\mathrm{II})$, $n_2=N^3(\mathrm{II})$, $n_3=N^2(\mathrm{I})$.
Case (c) : $n_1=N^1(\mathrm{I})$, $n_2=N^2(\mathrm{II})$, $n_3=N^3(\mathrm{II})$.
Case (d) : $n_1=N^1(\mathrm{I})$, $n_2=N^3(\mathrm{II})$, $n_3=N^2(\mathrm{II})$.
Case (e) : $n_1=N^1(\mathrm{II})$, $n_2=N^2(\mathrm{I})$, $n_3=N^3(\mathrm{I})$.
Case (f) : $n_1=N^1(\mathrm{II})$, $n_2=N^3(\mathrm{I})$, $n_3=N^2(\mathrm{I})$.
Case (g) : $n_1=N^1(\mathrm{I})$, $n_2=N^2(\mathrm{I})$, $n_3=N^3(\mathrm{II})$.
Case (h) : $n_1=N^1(\mathrm{I})$, $n_2=N^3(\mathrm{I})$, $n_3=N^2(\mathrm{II})$.
Case (i) : $n_1=N^1(\mathrm{I})$, $n_2=N^2(\mathrm{II})$, $n_3=N^3(\mathrm{I})$.
Case (j) : $n_1=N^1(\mathrm{I})$, $n_2=N^3(\mathrm{II})$, $n_3=N^2(\mathrm{I})$.
Case (k) : $n_1=N^1(\mathrm{I})$, $n_2=N^2(\mathrm{I})$, $n_3=N^3(\mathrm{I})$.
Case (l) : $n_1=N^1(\mathrm{I})$, $n_2=N^3(\mathrm{I})$, $n_3=N^2(\mathrm{I})$.

Consider first cases (k), (l) depending only on the data $\phi=\sum\frac{g_n(\omega)}{|n|}e^{i\langle n,x\rangle}$.

Thus we have to estimate (36), where $a_i(n_i)=\frac{g_{n_i}(\omega)}{|n_i|}$, $|n_i|\sim N_i$. Assume n_1,n_2,n_3 distinct. We may assume ω satisfying

$$\Bigg|\sum_{\substack{n=n_1-n_2+n_3\\ |n|^2=|n_1|^2-|n_2|^2+|n_3|^2+\mu}}\frac{g_{n_1}(\omega)}{|n_1|}\frac{g_{n_2}(\omega)}{|n_2|}\frac{g_{n_3}(\omega)}{|n_3|}\Bigg|^2 \ll (N^1)^\varepsilon\cdot\sum_{\substack{n=n_1-n_2+n_3\\ |n|^2=|n_1|^2-|n_2|^2+|n_3|^2+\mu}} N_1^{-2}N_2^{-2}N_3^{-2}\,, \tag{43}$$

the ω-exceptional set being of size $e^{-c(N^1)^\delta}$. Summation of (43) yields for (36) the bound

$$(N^1)^\varepsilon (N^1)^s \Bigg\{ \sum_{\substack{n_1 \neq n_2 \neq n_3,\, |n_i| \sim N_i \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \frac{\mu}{2}}} N_1^{-2} N_2^{-2} N_3^{-2} \Bigg\}^{1/2} \ll (N^1)^{s-1/2+\varepsilon} \tag{44}$$

applying Lemma 2, (iii) with $n_1 \leftrightarrow n_3$. We take $s < \frac{1}{2}$.

Next assume $n_1 = n_3 \neq n_2$. The conditions $n = 2n_3 - n_2, |n_2 - n_3|^2 = \frac{\mu}{2}$ yield a number of terms at most $(N_2 \wedge N_3)^\varepsilon$. Hence the (43)-bound is still valid. Thus (44) gives a bound on (18) in cases (k), (l).

We analyze the cases (a)–(j). Some of them will require additional arguments.

Case (a). Use the estimate (34). The number of terms in the second summation is at most $N_2 N_3^{1+\varepsilon}$, by Lemma 1. Thus, by Hölder's inequality, Lemma 2 (iv),

$$(34) \ll (N^2)^\varepsilon \Bigg\{ N_2 N_3^{1+\varepsilon} \sum_{\substack{n_2 \neq n_1, n_3 \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \frac{\mu}{2}}} |a_1(n_1)|^2 N_2^{-2+\varepsilon} |a_3(n_3)|^2 \Bigg\}^{1/2} \tag{45}$$

$$\ll (N_2)^\varepsilon \left(\frac{N_3}{N_2} \right)^{1/2} N_3^{-s} \ll N_2^{\varepsilon - s} . \tag{46}$$

Observe that for $N_1 \gg N_2$, the w-expressions corresponding to different dyadic values of N_1 are orthogonal and hence the $|||\ |||_s$-norms of the corresponding contributions to the nonlinear term add up in l^2. This leads to a bound of the form $M^{\varepsilon - s} |||u_1|||_s\, |||u_3|||_s$, if we restrict $N_2 > M$. On the other hand, exploiting the small time interval $[0, \tau]$ and the $|\lambda - |n|^2|$-factor in the definition of the $|||\ |||_s$-norm, one also has an estimate of the form $M^C \tau^\delta |||u_1|||_s |||u_3|||_s$, using for instance a straightforward $L^4 \times L^4 \times L^4 \times L^4$ estimate (after projection) on (29) and $\|u_i\|_4 < M^{1/4} \tau^{1/4-} |||u_i|||$ for $i = 1$ or 3. Consequently an estimate $\tau^\delta |||u_1|||_s\, |||u_3|||_s$ in (18) is obtained, for some $\delta > 0$. We don't repeat those considerations again later on.

Case (b). Use estimate (34). Applying the Hölder's inequality in the inner summation w.r.t. the n_2-summation, $\sum_{n_2} |a_2(n_2)|^2 < N_2^{-2s}$. This gives

$$N_3^\varepsilon N_2^{-s} \Bigg\{ \sum_{n, n_2} \Bigg| \sum_{\substack{n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3 \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \frac{\mu}{2}}} a_1(n_1) \frac{g_{n_3}(\omega)}{|n_3|} \Bigg|^2 \Bigg\}^{1/2}$$

$$\ll N_3^\varepsilon N_2^{-s} \Bigg\{ \sum_{\substack{n_2 \neq n_1, n_3 \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \frac{\mu}{2}}} |a_1(n_1)|^2 N_3^{-2+\varepsilon} \Bigg\}^{1/2}$$

$$\ll N_3^\varepsilon N_2^{-s} (N_2 N_3 N_2^\varepsilon N_3^{-2})^{1/2} \ll N_3^{\varepsilon - s} . \tag{47}$$

Applying the estimate from Lemma 1 (i) (replacing n_1 by n_3) and Lemma 2, (i).

Case (*c*). Use estimate (36). Proceeding as in case (b), we estimate by

$$N_1^{\varepsilon+s}N_2^{-s}\Bigg\{\sum_{n,n_2}\Bigg|\sum_{\substack{n=n_1-n_2+n_3,n_2\neq n_1,n_3\\ \langle n_2-n_1,n_2-n_3\rangle=\frac{\mu}{2}}}\frac{g_{n_1}(\omega)}{|n_1|}a_3(n_3)\Bigg|^2\Bigg\}^{1/2}$$

$$\ll N_1^{\varepsilon+s}N_2^{-s}\Bigg\{\sum_{\langle n_2-n_1,n_2-n_3\rangle=\frac{\mu}{2}}N_1^{-2}|a_3(n_3)|^2\Bigg\}^{1/2}$$

$$\ll N_1^{\varepsilon+s}N_2^{-s}(N_1^{-2}N_3^{-2s}N_2N_1N_2^{\varepsilon})^{1/2}\ll N_1^{\varepsilon}\left(\frac{N_1}{N_2}\right)^{s-\frac{1}{2}}N_3^{-s}\,, \tag{48}$$

applying Lemma 1 (i) and Lemma 2 (i) with n_1 replaced by n_3.

Next, we make another estimate using the Gaussians $\{g_{n_1}(\omega)\,|\,|n_1|\sim N_1\}$. Applying Hölder's inequality with respect to n_3 in the inner summation, estimate by

$$N_1^{s+\varepsilon}N_3^{-s}\Bigg\{\sum_{n,n_3}\Bigg|\sum_{\substack{n=n_1-n_2+n_3,n_2\neq n_1,n_3\\ \langle n_2-n_1,n_2-n_3\rangle=\frac{\mu}{2}}}\frac{g_{n_1}(\omega)}{|n_1|}a_2(n_2)\Bigg|^2\Bigg\}^{1/2}. \tag{49}$$

Fix n_3, $|n_3|\sim N_3$. Define the matrix $\mathscr{G}=\mathscr{G}_\omega=(\sigma_{n,n_2})_{\substack{|n|<N_1,n\neq n_3\\ |n_2|<N_2}}$ by

$$\sigma_{n,n_2}=\begin{cases}N_1^{-1}g_{n+n_2-n_3}(\omega) & \text{if } \langle n_3-n,n_2-n_3\rangle=\frac{\mu}{2}, n_2\neq n_3\\ 0 & \text{otherwise}.\end{cases} \tag{50}$$

Estimate (49) by

$$N_1^{s+\varepsilon}N_3^{-s}N_2^{-s}N_3\|\mathscr{G}_\omega^*\mathscr{G}_\omega\|^{1/2} \tag{51}$$

and

$$\|\mathscr{G}\mathscr{G}^*\|\leqq\max_n\Bigg(\sum_{n_2}|\sigma_{nn_2}|^2\Bigg)+\Bigg(\sum_{n\neq n'}\Bigg|\sum_{n_2}\sigma_{nn_2}\bar{\sigma}_{n'n_2}\Bigg|^2\Bigg)^{1/2}. \tag{52}$$

The first term in (52) is bounded by $N_2N_1^{-2+\varepsilon}$. Write

$$\sum_{n\neq n'}\Bigg|\sum_{n_2}\sigma_{nn_2}\bar{\sigma}_{n'n_2}\Bigg|^2=N_1^{-4}\sum_{n\neq n'}\Bigg|\sum_{\substack{\langle n_3-n,n_2-n_3\rangle=\frac{\mu}{2}\\ \langle n_3-n',n_2-n_3\rangle=\frac{\mu}{2}\\ n_2\neq n_3}}g_{n+n_2-n_3}(\omega)\bar{g}_{n'+n_2-n_3}(\omega)\Bigg|^2, \tag{53}$$

which expression depends on the initial data ϕ_ω. Observe that the Gaussian 2-products in the inner sum are at most repeated twice. Hence (53) may be estimated by

$$N_1^{-4}\#\Big\{(n,n',n_2)\,|\,n\neq n',n\neq n_3,n'\neq n_3,n_2\neq n_3,\langle n_3-n,n_2-n_3\rangle=\frac{\mu}{2},$$
$$\langle n_3-n',n_2-n_3\rangle=\frac{\mu}{2}\Big\}. \tag{54}$$

The condition $\langle n_3-n,n_2-n_3\rangle=\frac{\mu}{2}$ allows $N_1N_2^{1+\varepsilon}$ pairs (n,n_2), by Lemma 2 (ii).

Next, since $n_2 \neq n_3$, there are at most N_1 possible choices for n'. The resulting estimate on (54) is $N_1^{-2} N_2^{1+\varepsilon}$. Consequently, from (51) and the preceding

$$(49) \ll N_1^{s+\varepsilon} N_3^{1-s} N_2^{-s} (N_1^{-2} N_2^{1+\varepsilon})^{1/4} = N_1^{\varepsilon - \frac{1}{4}} \left(\frac{N_1}{N_2}\right)^{s-\frac{1}{4}} N_3^{1-s} . \tag{55}$$

Combining (55) with the previous bound (48), one easily gets the bound $N_1^{-\frac{\varepsilon}{4}+\varepsilon}$ in Case (c).

Case (*d*). Same estimate applies as in Case (c).

Case (*e*). We have to estimate (34) with $n_1 \in J$, where J is a subinterval of length $\sim N_2$ in $[|n_1| \sim N_1]$. Thus

$$N_2^{\varepsilon} \left(\sum_{n \in \tilde{J}} \left| \sum_{\substack{n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3 \\ \langle n - n_1, n - n_3 \rangle = \mu}} a_1(n_1) \frac{\overline{g_{n_2}(\omega)}}{|n_2|} \frac{g_{n_3}(\omega)}{|n_3|} \right|^2 \right)^{1/2} , \tag{56}$$

where $(a_1(n_1))_{n_1 \in J}$ satisfies $\sum_{n_1 \in J} |a_1(n_1)|^2 \leqq 1$ and $\tilde{J}$ is a doubling of J. Define the matrix $\mathscr{G} = \mathscr{G}_\omega = (\sigma_{n,n_1})_{n \in \tilde{J}, n_1 \in J}$ by

$$\sigma_{n,n_1} = N_2^{-1} N_3^{-1} \sum_{\substack{n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3 \\ \langle n - n_1, n - n_3 \rangle = \mu}} g_{n_2}(\omega) g_{n_3}(\omega) , \tag{57}$$

where the summation extends over indices n_2, n_3.

Estimate (56) by $N_2^{\varepsilon} \|\mathscr{G}\mathscr{G}^*\|^{1/2}$ and

$$\|\mathscr{G}\mathscr{G}^*\| \leqq \max_n \sum_{n_1 \in J} |\sigma_{n n_1}|^2 + \left(\sum_{n \neq n'} \left| \sum_{n_1 \in J} \sigma_{n n_1} \overline{\sigma}_{n', n_1} \right|^2 \right)^{1/2} . \tag{58}$$

Since $n_2 \neq n_3$ in the summation (57), we get

$$\sum_{n_1 \in J} |\sigma_{n n_1}|^2 \ll (N_2 N_3)^{-2} \cdot$$
$$\#\{(n_1, n_2, n_3) \mid n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3, \langle n - n_1, n - n_3 \rangle = \mu\} \cdot N_2^{\varepsilon}$$
$$\ll (N_2 N_3)^{-2} N_3^2 N_2 N_2^{\varepsilon} \ll N_2^{\varepsilon - 1} , \tag{59}$$

assuming

$$\left|{\sum}' g_{n_2}(\omega) g_{n_3}(\omega)\right|^2 \ll N_2^{\varepsilon} {\sum}' 1 , \tag{60}$$

where $\sum'$ denotes the (57)-summation.

Write explicitly

$$\sum_{n \neq n'} \left| \sum_{n_1 \in J} \sigma_{n n_1} \overline{\sigma}_{n', n_1} \right|^2 = (N_2 N_3)^{-4} \sum_{n \neq n'} \left| \sum_{(*)} \overline{g_{n_2}(\omega)} g_{n_3}(\omega) g_{n_2'}(\omega) \overline{g_{n_3'}(\omega)} \right|^2 , \tag{61}$$

where $(*)$ refers to the set of $(n_1, n_2, n_3, n_2', n_3')$ such that

$$\begin{cases} n_1 \in J \\ n = n_1 - n_2 + n_3 \,,\; n_2 \neq n_1, n_3 \,,\; \langle n - n_1, n - n_3\rangle = \mu \\ n' = n_1 - n_2' + n_3' \,,\; n_2' \neq n_1, n_3' \,,\; \langle n' - n_1, n' - n_3'\rangle = \mu \,. \end{cases} \tag{62}$$

Consider the following cases:

case (i). The indices n_2, n_3, n_2', n_3' are distinct.
case (ii). $n_2 = n_2'$ $(n_3 \neq n_3')$.
case (iii). $n_3 = n_3'$ $(n_2 \neq n_2')$.
case (iv). $n_2 = n_3'$, $n_3 \neq n_2'$.
case (v). $n_2 \neq n_3'$, $n_3 = n_2'$.
case (vi). $n_2 = n_3'$, $n_3 = n_2'$.

Case (i). Denote $\sum^1_{(*)}$ the corresponding subsummation of $\sum_{(*)}$. Clearly each of the order 4 Gaussian products in $\sum^1_{(*)}$ can only appear a bounded number of times. Hence we may assume

$$\left| \sum_{(*)}^1 \overline{g_{n_2}}\, g_{n_3} g_{n_2'} \overline{g_{n_3'}} \right|^2 \ll N_2^\varepsilon \sum_{(*)}^1 1 \,. \tag{63}$$

Hence, the corresponding contribution to (61) is bounded by

$$(N_2 N_3)^{-4} N_2^\varepsilon (\# S) \,, \tag{64}$$

where S stands for the systems $(n_1, n_2, n_3, n_2', n_3')$ such that

$$\begin{cases} n_1 \in J \,,\; |n_2| \sim N_2 \,,\; |n_3| \sim N_3 \,,\; |n_2'| \sim N_2 ,\; |n_3'| \sim N_3 \\ n_2 \neq n_1, n_3,\; n_2' \neq n_1, n_3' \\ \langle n_2 - n_1, n_2 - n_3\rangle = \mu ,\; \langle n_2' - n_1, n_2' - n_3'\rangle = \mu \,. \end{cases} \tag{65}$$

Hence

$$\# S \ll N_2^2 (N_3^2 N_2^\varepsilon)^2$$

fixing $n_1 \in J$ and applying the second estimate of Lemma 2, (i). Hence

$$(64) \ll N_2^{\varepsilon - 2} \,. \tag{66}$$

Case (ii). $n_2 = n_2' \Rightarrow n_3 \neq n_3'$. Denote $\sum^2_{(*)}$ the corresponding subsummation of $\sum_{(*)}$. Thus

$$\left| \sum_{(*)}^2 \overline{g_{n_2}}\, g_{n_3}\, g_{n_2'} \overline{g_{n_3'}} \right|^2 \ll N_2^\varepsilon \sum_{n_3, n_3'} (\# S(n, n', n_3, n_3'))^2 \,, \tag{67}$$

where

$$S(n, n', n_3, n_3') = \{(n_1, n_2) \,|\, (n_1, n_2, n_3, n_2, n_3') \text{ satisfies } (*)\} \,. \tag{68}$$

Thus

$$\# S(n, n', n_3, n_3') \leq \#\{(n_1, n_2) \,|\, n_1 \in J,\; n = n_1 - n_2 + n_3, \langle n - n_1, n - n_3\rangle = \mu\} < N_2 \,, \tag{69}$$

and the contribution of $\sum^2_{(*)}$ to (61) is bounded by

$$(N_2N_3)^{-4}N_2^{1+\varepsilon}(\#S)\,, \tag{70}$$

where now S consists of the systems (n_1, n_2, n_3, n_3') such that $(n_1, n_2, n_3, n_2, n_3')$ fulfills (65). Hence clearly

$$\#S \ll N_2^2 N_3^2 N_2^\varepsilon N_3 \tag{71}$$

and

$$(70) \ll N_2^{\varepsilon-1}N_3^{-1}\,. \tag{72}$$

Case (iii). $n_3 = n_3' \Rightarrow n_2 \neq n_2'$. Denoting $\sum^3_{(*)}$ the corresponding subsummation of $\sum_{(*)}$, we have

$$\left|\sum^3_{(*)} \overline{g_{n_2}}\, g_{n_3}\, g_{n_2'}\, \overline{g_{n_3'}}\right|^2 \ll N_2^\varepsilon \sum_{n_2, n_2'} (\#S(n, n', n_2 n_2'))^2\,, \tag{73}$$

where

$$S(n, n', n_2, n_2') = \{(n_1, n_3)\,|\,(n_1, n_2, n_3, n_2', n_3) \text{ satisfies } (*)\}\,, \tag{74}$$

and thus

$$\#S(n, n', n_2, n_2') \leqq \#\{(n_1, n_3)\,|\,n = n_1 - n_2 + n_3, \langle n_3 - n_2, n - n_3\rangle = \mu\} \ll N_3^\varepsilon\,. \tag{75}$$

The contribution of $\sum^3_{(*)}$ to (61) is bounded by

$$(N_2N_3)^{-4}N_3^\varepsilon(\#S)\,, \tag{76}$$

where S consists of the (n_1, n_2, n_3, n_2') such that $(n_1, n_2, n_3, n_2', n_3)$ fulfills (65). Thus

$$\#S \ll N_2^2 N_3^2 N_2^\varepsilon \tag{77}$$

and

$$(76) \ll N_2^{-2+\varepsilon}N_3^{-2}\,. \tag{78}$$

Case (iv). $n_2 = n_3'$, $n_3 \neq n_2'$. Denoting $\sum^4_{(*)}$ the corresponding subsummation, we have

$$\left|\sum^4_{(*)} \overline{g_{n_2}}\, g_{n_3}\, g_{n_2'}\, \overline{g_{n_3'}}\right|^2 \ll N_2^\varepsilon \sum_{n_3, n_2'} (\#S(n, n', n_3, n_2'))^2\,, \tag{79}$$

where

$$S(n, n', n_3, n_2') = \{(n_1, n_2)\,|\,(n_1, n_2, n_3, n_2', n_2) \text{ satisfies } (*)\}\,, \tag{80}$$

and thus

$$\#S(n, n', n_3, n_2') \leqq \#\{(n_1, n_2)\,|\,n_1 \in J, n = n_1 - n_2 + n_3, \langle n - n_1, n - n_3\rangle = \mu\} < N_2\,. \tag{81}$$

The contribution of $\sum^4_{(*)}$ to (61) is bounded by

$$(N_2N_3)^{-4}N_2^{1+\varepsilon}(\#S)\,, \tag{82}$$

where S consists now of the (n_1,n_2,n_3,n_2') such that (n_1,n_2,n_3,n_2',n_2) fulfills (65). Hence

$$\#S \ll N_2^{2+\varepsilon} N_3^2\,, \tag{83}$$

by Lemma 2 (i) and Lemma (2) (iv) and

$$(82) \ll N_2^{-1+\varepsilon} N_3^{-2}\,. \tag{84}$$

Case (v). $n_2 \neq n_3'$, $n_3 = n_2'$: Same as (iv).

Case (vi). $n_2 = n_3'$, $n_3 = n_2'$. Denoting $\sum_{(*)}^{(6)}$ the corresponding subsummation, we have

$$\left|\sum_{(*)}^{(6)} \overline{g_{n_2}}\, g_{n_3}\, g_{n_2'}\, \overline{g_{n_3'}}\right|^2 \ll N_2^{\varepsilon}(\#S(n,n'))^2\,, \tag{85}$$

where

$$S(n,n') = \{(n_1,n_2,n_3)\,|\,(n_1,n_2,n_3,n_3,n_2) \text{ satisfies } (*)\}\,, \tag{86}$$

meaning that n_1,n_2,n_3 are different and

$$\begin{cases} n = n_1 - n_2 + n_3 & \langle n-n_1, n-n_3\rangle = \mu \\ n' = n_1 - n_3 + n_2 & \langle n'-n_1, n'-n_2\rangle = \mu\,. \end{cases} \tag{87}$$

Thus $n + n' = 2n_1$ and $\#S(n,n') < N_3$.

The contribution to (61) is thus

$$N_2^{\varepsilon}(N_2N_3)^{-4} \sum_{n \neq n'} (\#S(n,n'))^2 \ll N_2^{-4+\varepsilon} N_3^{-3}(\#S)\,, \tag{88}$$

where S consists of the pairs (n_1,n_2,n_3) such that

$$\langle n_2-n_1, n_2-n_3\rangle = \mu, \quad \langle n_3-n_1, n_3-n_2\rangle = \mu\,, \tag{89}$$

hence

$$|n_2 - n_3|^2 = 2\mu\,. \tag{90}$$

Thus

$$\#S \ll N_3^2 N_2^{\varepsilon} N_2 \tag{91}$$

and

$$(88) \ll N_2^{-3+\varepsilon} N_3^{-1}\,. \tag{92}$$

Collecting the various bounds (66), (72), (78), (84), (92), it follows that

$$(61) \ll N_2^{\varepsilon}(N_2N_3)^{-1} < N_2^{-1+\varepsilon}\,. \tag{93}$$

From (59), (93),

$$\|\mathcal{G}\mathcal{G}^*\| \ll N_2^{-1/2+\varepsilon}\,. \tag{94}$$

Hence

$$(56) \ll N_2^{-1/4+\varepsilon}\,, \tag{95}$$

which is the bound on (56) and thus for Case (e).

Case (f). By Lemma 1 and Lemma 2, (i) we get applying first Hölder's inequality

$$(56) \ll N_3^\varepsilon N_2 \Bigg(\sum_{\substack{n_2 \neq n_1, n_3, n_1 \in J \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \mu}} |a_1(n_1)|^2 N_2^{-2} N_3^{-2} \Bigg)^{1/2}$$

$$\ll N_3^\varepsilon N_2 (N_1^{1+\varepsilon} N_3 N_2^{-2} N_3^{-2})^{1/2} = N_3^\varepsilon N_2^{1/2} N_3^{-1/2} . \tag{96}$$

Hence, if $N_2 < N_3^{1-\frac{1}{100}}$, we get an estimate $N_3^{-\frac{1}{300}}$, say. Otherwise the estimates made above in case (e) will yield a saving of $N_3^{-\frac{1}{300}}$ also.

Case (g). We use estimate (36)

$$N_1^{s+\varepsilon} \Bigg\{ \sum_n \Bigg| \sum_{\substack{n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3 \\ \langle n - n_1, n - n_3 \rangle = \mu}} \frac{g_{n_1}(\omega)}{|n_1|} \frac{\overline{g_{n_2}(\omega)}}{|n_2|} a_3(n_3) \Bigg|^2 \Bigg)^{1/2} \tag{97}$$

with $\sum |a_3(n_3)|^2 < N_3^{-2s}$. From Hölder's inequality and Lemma 1, we get

$$(97) \ll N_1^{s+\varepsilon} (N_2 N_3)^{1/2} \Bigg(\sum_{\substack{n_2 \neq n_1, n_3 \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \mu}} N_1^{-2} N_2^{-2} |a_3(n_3)|^2 \Bigg)^{1/2} , \tag{98}$$

and from Lemma 2 (i)

$$(98) \ll N_1^{s+\varepsilon} (N_2 N_3)^{1/2} (N_1^{-2} N_2^{-2} N_1 N_2^{1+\varepsilon})^{1/2} N_3^{-s} \ll N_1^s \left(\frac{N_3}{N_1} \right)^{\frac{1}{2} - s} . \tag{99}$$

Thus we may assume $N_3 > N_1^{1-\frac{1}{100}}$ and we can use then the estimates from cases (f), (e).

Case (h). Estimate (36) as in case (g) with the same result. Again if $N_3 > N_1^{1-\frac{1}{100}}$, the estimate in case (f) applies.

Case (i), (j). Estimate (36)

$$N_1^{s+\varepsilon} \Bigg\{ \sum_n \Bigg| \sum_{\substack{n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3 \\ \langle n - n_1, n - n_3 \rangle = \mu}} \frac{g_{n_1}(\omega)}{|n_1|} a_2(n_2) \frac{g_{n_3}(\omega)}{|n_3|} \Bigg|^2 \Bigg)^{1/2} . \tag{100}$$

$$\ll N_1^{s+\varepsilon} (N_2 N_3)^{1/2} \Bigg(\sum_{\substack{n_2 \neq n_1, n_3 \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \mu}} N_1^{-2} N_3^{-2} |a_2(n_2)|^2 \Bigg)^{1/2}$$

$$\ll N_1^{s+\varepsilon} (N_2 N_3)^{1/2} N_2^{-s} N_1^{-1} N_3^{-1} (N_1 N_3)^{1/2} \ll N_1^\varepsilon \left(\frac{N_2}{N_1} \right)^{\frac{1}{2} - s} . \tag{101}$$

We may also apply the estimate from case (e), introducing an extra factor N_3^s to control u^3 in H^s. This yields the bound

$$N_1^{\varepsilon-\frac{1}{4}} \cdot N_3 \tag{102}$$

from (55). Thus it follows from (101), (102) that we may assume $N_2 > N_1^{1-\frac{1}{100}}$, $N_3 > N_1^{1/5}$.

We next prove one more estimate, repeating mainly the argument from case (e). Define $\mathscr{G} = \mathscr{G}_\omega = (\sigma_{n,n_2})_{\substack{|n|<N_1 \\ |n_2|\sim N_2}}$ by

$$\sigma_{nn_2} = N_1^{-1} N_3^{-1} \sum_{\substack{n=n_1-n_2+n_3, n_2 \neq n_1, n_3 \\ \langle n-n_1, n-n_3\rangle = \mu}} g_{n_1}(\omega)\, g_{n_3}(\omega)\,. \tag{103}$$

Estimate (100) by

$$N_1^{s+\varepsilon} N_2^{-s} \|\mathscr{G}\mathscr{G}^*\|^{1/2}\,, \tag{104}$$

where

$$\|\mathscr{G}\mathscr{G}^*\| \leqq \max_n \Big(\sum_{n_2} |\sigma_{n,n_2}|^2\Big) + \Bigg(\sum_{n\neq n'} \Big|\sum_{n_2} \sigma_{nn_2}\, \overline{\sigma_{n',n_2}}\Big|^2\Bigg)^{1/2}\,. \tag{105}$$

By the condition $\langle n-n_1, n_1-n_2\rangle = \mu$, the number of summands in the definition of σ_{n,n_2} is at most N_1^ε. Hence the first term of (105) is bounded by

$$N_2^2 N_1^\varepsilon N_1^{-2} N_3^{-2} \ll N_1^{-1/3}\,, \tag{106}$$

since we assumed $N_3 > N_1^{1/5}$.

We analyze again the second term in (105). Write explicitly

$$\sum_{n\neq n'} \Big|\sum_{n_2} \sigma_{n,n_2} \overline{\sigma}_{n',n_2}\Big|^2 = (N_1 N_3)^{-4} \sum_{n \neq n'} \Big|\sum_{(*)} g_{n_1}\, g_{n_3}\, \overline{g_{n_1'}}\, \overline{g_{n_3'}}\Big|^2\,, \tag{107}$$

where $(*)$ refers to the set of $(n_1, n_2, n_3, n_1', n_3')$ such that

$$\begin{cases} n = n_1 - n_2 + n_3, \quad n_2 \neq n_1, n_3, \quad \langle n-n_1, n-n_3\rangle = \mu \\ n' = n_1' - n_2 + n_3', \quad n_2 \neq n_1', n_3', \quad \langle n'-n_1', n'-n_3'\rangle = \mu\,. \end{cases} \tag{108}$$

Consider the following cases:

$$\begin{cases} \text{case (i):} & \text{The indices } n_1, n_3, n_1', n_3' \text{ are different}\,. \\ \text{case (ii):} & n_1 = n_1' \ (n_3 \neq n_3')\,. \\ \text{case (iii):} & n_1 = n_3' \ (n_3 \neq n_1')\,. \end{cases}$$

There are the symmetric cases. Observe that if $n_1 = n_3$ say, we get $g_{n_1}^2$ and the $\{g_n^2\}$ are still independent of mean zero, since the g_n are complex Gaussians. Hence this case does not require a separate argument.

Case (*i*). Denote $\sum^1$ the corresponding subsummation. If the n_1, n_3, n_1', n_3' are all different, each of these order 4 Gaussian products only appears a bounded number of times in the summation and

$$\left|\sum_{(*)}^{1} g_{n_1} g_{n_3} \overline{g_{n_1'}}\,\overline{g_{n_3'}}\right|^2 \ll N_1^{\varepsilon} \sum_{(*)} 1 . \tag{109}$$

Thus

$$(107) \ll (N_1 N_3)^{-4} N_1^{\varepsilon} (\# S) , \tag{110}$$

where S stands for the systems $(n_1, n_2, n_3, n_1', n_3')$ such that

$$\begin{cases} n_2 \neq n_1, n_3, n_1', n_3' \\ \langle n_2 - n_1, n_2 - n_3 \rangle = \mu, \quad \langle n_2 - n_1', n_2 - n_3' \rangle = \mu . \end{cases} \tag{111}$$

Hence

$$\# S \ll N_1^{\varepsilon} N_2^2 (N_1 N_3)^2 \tag{112}$$

and

$$(110) \ll N_1^{\varepsilon} N_1^{-2} N_2^2 N_3^{-2} \ll N_1^{-1/3} . \tag{113}$$

Case (*ii*). Denoting $\sum^2$ the corresponding subsummation,

$$\left|\sum_{(*)}^{2} g_{n_1} g_{n_3} \overline{g_{n_1'}}\,\overline{g_{n_3'}}\right|^2 \ll N_1^{\varepsilon} \sum_{n_3, n_3'} (\# S(n, n', n_3, n_3'))^2 , \tag{114}$$

where

$$S(n, n', n_3, n_3') = \{(n_1, n_2) \,|\, (n_1, n_2, n_3, n_1, n_3') \text{ satisfies } (*)\} .$$

Hence

$$\# S(n, n', n_3, n_3') \leqq \#\{n_1 \,|\, |n_1| \sim N_1, \quad \langle n - n_1, n - n_3 \rangle = \mu\} < N_1 , \tag{115}$$

and the contribution of $\sum^2$ to (107) is bounded by

$$(N_1 N_3)^{-4} N_1^{1+\varepsilon} (\# S) , \tag{116}$$

where S consists of the (n_1, n_2, n_3, n_3') such that $(n_1, n_2, n_3, n_1, n_3')$ satisfies (111). Thus

$$\# S \ll N_2^2 N_1^{\varepsilon} N_1 N_3 N_3$$

and

$$(116) \ll N_1^{\varepsilon} N_1^{-2} N_2^2 N_3^{-2} < N_1^{-1/3} . \tag{117}$$

Case (*iii*). Denoting $\sum^3$ the corresponding summation,

$$\left|\sum_{(*)}^{3} g_{n_1} g_{n_3} \overline{g_{n_1'}}\,\overline{g_{n_3'}}\right|^2 \ll N_1^{\varepsilon} \sum_{n_3, n_1'} (\# S(n, n', n_3, n_1'))^2, \tag{118}$$

where

$$S(n, n', n_3, n_1') = \{(n_1, n_2) \,|\, (n_1, n_2, n_3, n_1', n_1) \text{ satisfies } (*)\} .$$

Hence

$$\# S(n, n', n_3, n_1') < N_1 , \tag{119}$$

and the contribution of $\sum^3$ to (107) is bounded by

$$(N_1N_3)^{-4}N_1^{1+\varepsilon}(\#S)\,, \tag{120}$$

where S consists of the (n_1,n_2,n_3,n_1') such that (n_1,n_2,n_3,n_1',n_1) satisfies (111). Thus

$$\#S \ll N_1^2N_3^2N_2^\varepsilon N_1$$

and

$$(120) \ll N_1^\varepsilon N_3^{-2} < N_1^{-1/3}\,. \tag{121}$$

Summarizing, it follows from (113), (117), (121) that (107) $< N_1^{-1}$, thus by (106)

$$\| \mathscr{G}\mathscr{G}^* \| < N_1^{-1/6}\,. \tag{122}$$

Hence

$$(36),(100) \ll N_1^{\varepsilon-1/12}\left(\frac{N_1}{N_2}\right)^s \ll N_1^{\varepsilon-\frac{1}{12}+\frac{s}{100}}\,. \tag{123}$$

From the preceding, we get in case (i), (j) the estimate

$$N_1^{-\frac{1}{300}}\,. \tag{124}$$

This completes the analysis of the different cases (a)–(l).

It follows from this analysis that fixing an interval $[0,\tau]$, we have

$$(18) < c\tau^\delta \tag{125}$$

for some $\delta > 0$. Here $w = u_1\,\overline{u}_2\,u_3$ with $u_i \in S(t)\phi_\omega + (|||\quad|||_s\text{-ball})$, $\phi_\omega = \sum \frac{g_n(\omega)}{|n|}e^{i\langle n,x\rangle}$, and (125) will hold outside an ω-set Ω of measure $< e^{-1/\tau^{\delta'}}$, for some $\delta' > 0$.

Observe also that if for one of the u_i we consider $\sum_{|n|>M}\frac{g_n(\omega)}{|n|}e^{i(\langle n,x\rangle+|n|^2t)}$, there is an extra saving of $M^{-\delta}$, i.e.

$$(18) < c\tau^\delta M^{-\delta}\,. \tag{126}$$

The transformation T defined in (16) is a contraction, since

$$|||Tu - Tv|||_s \leqq c\tau^\delta|||u-v|||_s\,. \tag{127}$$

In this estimate, one of the u_i's equals $u - v \in H^s$. Hence, for $\omega \notin \Omega$, Picard's theorem gives a solution u to (11).

Let $\phi = \phi_\omega$ be a "good data" as above with solution u, $u(0) = \phi$. Let $\psi \in H^s$, $\|\varphi - \psi\|_s < \frac{1}{10}$. Consider the map $T_1v : S(t)\psi + i\int_0^t S(t-\tau)[(v|v|^2 - 2v(|v|^2)(\tau)]d\tau$. Writing $T_1v = Tv + S(t)(\psi - \phi)$, it follows that

$$|||T_1v - S(t)\phi|||_s \leqq \|\phi-\psi\|_s + |||Tv - S(t)\phi|||_s < \frac{1}{10} + \tau^\delta < 1\,.$$

Hence T_1 maps $S(t)\phi + (|||\quad|||_s\text{-ball})$ to itself and is a contraction, since $T_1(v) - T_1(v') = T(v) - T(v')$. Thus (11) has also a solution v for initial data $v(0) = \psi$. Moreover

$$|||u - v|||_s \leqq 2\|\varphi - \psi\|_{H^s} \tag{127'}$$

and also (cf. [B1] or the discussion in [B2], Sect. 2).

$$\|u(t)-v(t)\|_{H^s} \leqq C\|\varphi-\psi\|_{H^s} \quad \text{for } |t|<\tau\,. \tag{127''}$$

5. Comparison and Convergence of Solutions

Let us compare next the solution of the truncated equation (10)

$$\begin{cases} iu_t^N - \Delta u^N + P_N(u^N|u^N|^2 - 2u^N \int |u^N|^2) = 0 \\ u^N = P_N u,\ u^N(0) = \phi_\omega^N(x) = \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i\langle n,x\rangle} \end{cases}, \tag{128}$$

and the solution u obtained above for

$$\begin{cases} iu_t - \Delta u + (u|u|^2 - 2u \int |u|^2) = 0 \\ u(0) = \phi_\omega(x) = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle}\,. \end{cases} \tag{129}$$

In (128), $u^N \in S(t)\phi^N + (|||\ \ |||_s\text{-ball})$ and in (129), $u \in S(t)\phi + (|||\ \ |||_s\text{-ball})$. Fix $0 \leqq s_1 < s$. Analyze the expression

$$u|u|^2 - 2u\int |u|^2 - P_N(u^N|u^N|^2 - 2u^N \int |u^N|^2)\,, \tag{130}$$

writing it as a sum of products $v_1\overline{v_2}v_3$ where for some i, either $P_{\frac{N}{3}} v_i = 0$ or $v_i = u - u^N$. Taking (126), (127) into account, we get

$$\begin{aligned} &|||[u-S(t)\phi]-[u^N-S(t)\phi_N]|||_{s_1} \\ &\qquad \leqq N^{s_1-s} + N^{-\delta} + c\tau^\delta |||[u-S(t)\phi]-[u^N-S(t)\phi_N]|||_{s_1}\,. \end{aligned} \tag{131}$$

Here we perform the analysis of the nonlinear term in $|||\ \ |||_{s_1}$. For $P_{\frac{N}{3}} v_i = 0$, either v_i appears in the $|||\ \ |||_s$-ball in which case there is an N^{s_1-s} bound in $|||\ \ |||_{s_1}$ or $v_i = \sum_{|n|>\frac{N}{3}} \frac{g_n(\omega)}{|n|} e^{i(\langle n,x\rangle+|n|^2 t)}$ in which case we invoke (126). Write

$$u-u^N = (S(t)\phi - S(t)\phi_N) + [(u-S(t)\phi)-(u^N-S(t)\phi_N)]\,,$$

and apply again (126) if one of the v_i's equals $S(t)\phi - S(t)\phi_N$.

From (131), we get an approximation

$$|||(u-S(t)\phi)-(u^N-S(t)\phi^N)|||_{s_1} \leqq N^{s_1-s}+N^{-\delta}\,, \tag{132}$$

and also

$$\|(u-S(t)\phi)-(u^N-S(t)\phi^N)\|_{L^\infty_{H^{s_1}(\mathbb{T}^2)}(0,\tau)} \leqq N^{s_1-s}+N^{-\delta}\,. \tag{133}$$

The conclusion is that for ω outside a set Ω of measure $< e^{-1/\tau^{\delta'}}$, $u_\omega^N - S(t)\phi_\omega^N$ will converge to $u_\omega - S(t)\phi_\omega$ in H^s for some $s>0$ and, more precisely

$$\|(u_\omega - S(t)\phi_\omega)-(u_\omega^N - S(t)\phi_\omega^N)\|_s < CN^{-\delta} \quad \text{for } t\in[0,\tau]\,. \tag{134}$$

Denote $S^N(t)$ the flow map associated to (128). Fix a large positive integer $\overline{N}$ and denote $u_{\overline{N}}$ the Gibbs measure $e^{-H_{\overline{N}}}\Pi d\phi$. Thus $\mu_{\overline{N}}$ is invariant under the flow of

(6) and hence of (10), thus $S^{\overline{N}}(t)$. The solutions are indeed related by $e^{2ic_N(\omega)t}$ multiplication, where $c_N(\omega) = \sum_{|n|\leqq N} \frac{|g_n(\omega)|^2-1}{|n|^2}$ and thus only depends on $|g_n(\omega)|$. It follows from (134) that

$$\left\|[S^{\overline{N}}(t) - S(t) - (S^N(t) - S(t))P_N]\phi\right\|_s < CN^{-\delta} \tag{135}$$

for $N < \overline{N}$, $t \in [0,\tau]$ and $\phi = P_{\overline{N}}\phi$ taken outside a set Λ of measure $\mu_{\overline{N}}(\Lambda) < e^{-1/\tau^{\delta'}}$.

Our next purpose is to extend (135) for t in an arbitrary interval. Consider say [0,1], fix a small number $\tau > 0$ and partition [0,1] in $1/\tau$ intervals I_α of size τ. We will mainly repeat the invariant measure consideration from [B2].

Thus for $\phi \notin \Lambda$, (135) holds

$$\left\|[S^{\overline{N}}(t) - S^N(t)P_N - S(t)(I - P_N)]\phi\right\|_s \leqq N^{-\delta} \quad (t \in [0,\tau])\,, \tag{136}$$

and thus, denoting $\phi_1 = S^{\overline{N}}(\tau)\phi$,

$$\|\phi_1 - [S^N(\tau)P_N + S(\tau)(I - P_N)]\phi\|_s \leqq N^{-\delta}\,. \tag{137}$$

Assume ϕ_1 is again a "good" data, thus $\phi_1 \notin \Lambda$, hence

$$\phi \notin \Lambda \cup S^{\overline{N}}(\tau)^{-1}(\Lambda)\,. \tag{138}$$

Repeating (136), one gets again for $N \leqq \overline{N}$, $t \in [0,\tau]$,

$$\left\|[S^{\overline{N}}(t) - S^N(t)P_N - S(t)(I - P_N)]\phi_1\right\|_s \leqq N^{-\delta}\,,$$

thus

$$\left\|\left|S^{\overline{N}}(\tau + t)\phi - [S^N(t)P_N + S(t)(I - P_N)]\phi_1\right\|\right|_s \leqq N^{-\delta}\,. \tag{139}$$

It follows from (137) that

$$\|S(t)(I - P_N)\phi_1 - S(\tau + t)(I - P_N)\phi\|_s \leqq N^{-\delta}\,, \tag{140}$$

$$\|P_N\phi_1 - S^N(\tau)P_N\phi\|_s \leqq N^{-\delta}\,. \tag{141}$$

Since ϕ_1 is a "good data", $S^N(t)$ acts in a Lipschitz way on $P_N\phi_1 + (H^s\text{-ball})$, $t \leqq \tau$, and (141), (127″) implies

$$\|S^N(t)P_N\phi_1 - S^N(\tau + t)P_N\phi\|_s \leqq N^{-\delta}\,. \tag{142}$$

Combining (139), (140), (142), it follows that for $t \in [0,\tau]$,

$$\left\|[S^{\overline{N}}(\tau + t) - S^N(\tau + t)P_N - S(\tau + t)(I - P_N)\phi]\right\|_s \leqq N^{-\delta}\,, \tag{143}$$

and thus (136) holds for $t \in [0,2\tau]$, provided (138).

The continuation of this process is clear. One gets eventually (136) on [0,1], provided

$$\phi \notin \Lambda \cup S^{\overline{N}}(\tau)^{-1}(\Lambda) \cup \cdots \cup S^{\overline{N}}(\tau)^{-k}(\Lambda) \quad (k \sim \tau^{-1})\,, \tag{144}$$

and since $S^{\overline{N}}(\tau)$ is $\mu_{\overline{N}}$-preserving, the set Λ_τ defined in (144) satisfies

$$\mu_{\overline{N}}(\Lambda_\tau) \leqq \frac{1}{\tau}e^{-1/\tau^{\delta'}} \underset{\tau\to 0}{\longrightarrow} 0\,. \tag{145}$$

It follows from (145) that given $\sigma > 0$, there is a set Λ_σ, $\mu_{\overline{N}}(\Lambda_\sigma) < \sigma$, such that for $\phi \notin \Lambda_\sigma$ and $t \in [0,1]$,

$$\left\|[S^{\overline{N}}(t) - S^N(t)P_N - S(t)(I - P_N)]\phi\right\|_s \leqq c(\sigma)N^{-\delta} \quad \text{for } N \leqq \overline{N}\,. \tag{146}$$

Since $\mu_{\overline{N}}$ converges to the normalized Gibbs measure μ defined in Sect. 1, letting $\overline{N} \to \infty$ in the preceding shows that

$$\|([S^{N_1}(t) - S(t)]P_{N_1} - [S^{N_2}(t) - S(t)]P_{N_2})\phi\|_s \leqq c(\sigma)(N_1 \wedge N_2)^{-\delta} \tag{147}$$

for all $t \in [0,1]$, $\phi \notin \Lambda_\sigma$ with $\mu(\Lambda_\sigma) < \sigma$ and any integers N_1, N_2.

We get in particular from (147) for $N_2 > N_1$, $\phi \notin \Lambda_\sigma$,

$$\|(P_{N_1}S^{N_2}(t)P_{N_2} - S^{N_1}(t)P_{N_1})\phi\|_\sigma < C(\sigma)N_1^{-\delta}\,. \tag{148}$$

Also $S^N(t)P_N\phi$ converges weakly to some $S^\infty(t)\phi \in S(t)\phi + B_{H^s}(C(\sigma))$ (take $N_2 = 0$ in (147) and let $N_1 \to \infty$). From (147), (148), for $t \in [0,1]$, $\phi \notin \Lambda_\sigma$,

$$\|([S^\infty(t) - S(t)] - [S^N(t) - S(t)]P_N)\phi\|_s \leqq C(\sigma)N^{-\delta} \tag{149}$$

and

$$\|(P_NS^\infty(t) - S^N(t)P_N)\phi\|_s < C(\sigma)N^{-\delta}\,. \tag{150}$$

$S^\infty(t)\phi$ is the solution of (11) obtained in Sect. 4 and from (150), it easily follows that μ is invariant under the flow $S^\infty(t)$ (using again the invariance of μ_N for $S^N(t)$).

Coming back to Eqs. (6),

$$u_t^N = i\frac{\partial H_N}{\partial \overline{u^N}}, \qquad u^N(0) = P_N\phi\,, \tag{151}$$

we have $u^N = e^{2ic_N(\phi)t}S^N(t)P_N\phi$, where $c_N(\phi) = c_N(\omega) = \sum_{|n| \leqq N} \frac{|g_n(\omega)|^2 - 1}{|n|^2}$ for $\phi = \sum \frac{g_n(\omega)}{|n|}e^{i\langle x,n\rangle}$. Thus $c_N(\phi)$ converge μ-almost surely to some $c_\infty(\phi)$, and hence the $u^N(t)$ converge weakly for $N \to \infty$ to $e^{2ic_\infty(\phi)t} \cdot S^\infty(t)\phi$ for all time, μ-almost surely in ϕ. In fact, from (150)

$$\left\|u^N(t) - e^{2ic_N(\phi)t}P_NS^\infty(t)\phi\right\|_s < c(\sigma, T)N^{-\delta} \tag{152}$$

for $\phi \in \Lambda_{\sigma,T}$, $\mu(\Lambda_{\sigma,T}) < \sigma$. In particular, (ii) converges in H^s for some $s > 0$.

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