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$gl(\infty)$ and Geometric Quantization

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Abstract. An axiomatic approach to the approximation of infinite dimensional algebras is presented; examples illustrating the need for a rigorous treatment of this subject. Geometric quantization is employed to construct systematically $su(N)$ approximations of diffeomorphism algebras which first appeared in the theory of relativistic membranes.

1. Introduction

Over the past years several authors [1] have studied (and used) the approximation of diffeomorphism groups by $SU(N)$. They started from the observation made in the context of membrane theories [2, 3], that in a specific basis of $su(N)$ the corresponding structure constants converge to those of $\mathrm{diff}_A S^2$ (the Lie algebra of infinitesimal area preserving diffeomorphisms of the 2-sphere) in the limit $N \to \infty$. Later it was found [4–7] that the same holds for the Lie algebra of infinitesimal (nonconstant) diffeomorphisms of the 2-torus $\mathrm{diff}_A T^2$.

A naive identification of $\mathrm{diff}_A S^2$ and $\mathrm{diff}_A T^2$, however, with the well known $su_{(+)}(\infty)$ [8] would be false. Although the three algebras (or rather certain subalgebras) may all be approximated by $su(N)$ in the above sense, they are pairwise non-isomorphic. This we will show in Appendix A. Moreover, in [9] the members of an infinite family of algebras (including $\mathrm{diff}_A T^2$) have been proven to be pairwise non-isomorphic although all of them can be approximated by $su(N)$, $N \to \infty$.

This ambiguity clearly shows the need for an additional concept, and appears to be worthwhile studying without reference to membrane theory. Interesting questions arising from the subject are its relation to $\hbar \to 0$ limits of quantum theories on compact phase spaces and its role in the construction and classification of infinite dimensional Lie algebras.

In Sect. 2 of the present paper we discuss the approximation scheme using $gl(N)$ as an example. Starting from a particular basis one obtains (by the standard embedding) $gl_{+}(\infty)$ [8] for $N \to \infty$. Allowing for arbitrary base transformations $C^{(N)}$ non-isomorphic infinite dimensional algebras may be obtained, if $C^{(N)}$ at $N = \infty$ does not exist as a transformation between infinite dimensional vector spaces. Several examples are given.

In an attempt to get this ambiguity under control we introduce in Sect. 3 a rigorous concept for the approximation of algebras (L_α-approximations), based on three axioms, and formulate a weak uniqueness theorem for the limit algebras (called quasilimits). The examples of the preceding section are used to illustrate this concept.

In Sect. 4 we systematically construct $su(n)$-approximations for algebras of infinitesimal area preserving diffeomorphisms of compact Kähler manifolds: Regarding the manifold as a classical phase space, a geometric quantization scheme is applied in order to approximate the algebra of infinitesimal canonical transformations by a sequence of (finite dimensional) algebras of quantum operators. In an addendum to this section we compare our approach to techniques involving F. A. Berezin's coherent states [29–31] that have recently been mentioned in the context of symplectic geometry and membranes by A. S. Schwarz [32]. In this comparison we use a global formulation due to J. H. Rawnsley et al. [33, 35, 36].

The calculus developed in Sect. 4 fits into the framework of L_α-approximations and is carried out explicitly for the $2n$-dimensional torus in Sect. 5. The result generalizes the sine-algebra which was used in [4–7] to approximate $\mathrm{diff}_A T^2$.

In Appendix A we present the proofs of some statements contained in Sects. 1–3 concerning the non-isomorphy of certain infinite dimensional algebras. Appendix B contains technical details of the calculations in Sect. 5, involving theta functions.

2. Remarks on $gl(N \to \infty)$

Here and in the following sections we will replace the real algebras $u(N)$ and $su(N)$ by their complexifications $gl(n)$, respectively $sl(n)$. To fix notation and for further reference let us give a definition of the infinite dimensional Lie algebras

$$gl(\infty), gl_+(\infty), L_\Lambda, \mathrm{diff}'_A T^2, \mathrm{diff}'_A S^2$$

and certain related algebras.

$gl(\infty)$ is the Lie algebra of complex ∞-dimensional matrices with finite support (see [8]), i.e.

$$gl(\infty) := \{(a_{ij})_{i,j\in\mathbb{Z}} | a_{ij} \in \mathbb{C}, \text{ all but a finite number of the } a_{ij} = 0\}. \tag{2-1}$$

The Lie bracket is the usual matrix commutator. A basis is given by the elementary matrices E_{ij}. The matrix E_{ij} has 1 as the $(i, j)^{\text{th}}$ entry and 0 as all other entries. Here (i, j) ranges over $\mathbb{Z} \times \mathbb{Z}$. The commutator of the basis elements is

$$[E_{ij}, E_{kl}] = \delta_{j,k} E_{il} - \delta_{i,l} E_{kj}. \tag{2-2}$$

If we replace $\mathbb{Z}$ by $\mathbb{N}$ we obtain the algebra $gl_+(\infty)$ by the analogous definitions. Any bijective map $\mathbb{N} \cong \mathbb{Z}$ induces an isomorphism of $gl_+(\infty)$ with $gl(\infty)$.

(Nevertheless, we will distinguish them, because there exists no canonical isomorphism between them.) Due to the finite support of the matrices the trace is well-defined and the subalgebras $sl(\infty)$, respectively $sl_+(\infty)$ can be obtained by restricting oneself to matrices A with trace $A=0$.

Let

$$V := \langle T_{\vec{m}} | \vec{m} \in \mathbb{Z}^2 \rangle_{\mathbb{C}}$$

be the $\mathbb{C}$-vector space generated by the basis $T_{\vec{m}}$. This vector space carries different Lie algebra structures, e.g. the family of sine-algebras [4]. They are defined as follows: for $\Lambda \in \mathbb{R}$ with $\Lambda \neq 0$ we set

$$[T_{\vec{m}}, T_{\vec{n}}]^{\Lambda} = \left(\frac{1}{2\pi\Lambda} \sin 2\pi\Lambda (\vec{m} \times \vec{n}) \right) T_{\vec{m}+\vec{n}}, \tag{2-3}$$

for $\Lambda = 0$ we set

$$[T_{\vec{m}}, T_{\vec{n}}] := [T_{\vec{m}}, T_{\vec{n}}]^0 := (\vec{m} \times \vec{n}) T_{\vec{m}+\vec{n}}. \tag{2-4}$$

Here we use the notation $\vec{m} \times \vec{n} = \vec{m} \wedge \vec{n} = m_1 n_2 - m_2 n_1$, where $\vec{m} = (m_1, m_2)$, $\vec{n} = (n_1, n_2)$. We denote these Lie algebras by $\tilde{L}_\Lambda = (V, [\ldots, \ldots]^\Lambda)$. Obviously, the algebras $\tilde{L}_\Lambda$ are direct sums (of Lie algebras)

$$\tilde{L}_\Lambda = \langle T_{(0,0)} \rangle \oplus \langle T_{\vec{m}} | \vec{m} \in \mathbb{Z}^2 \setminus \{(0,0)\} \rangle.$$

The first summand consists of multiples of the central element $T_{(0,0)}$. The second summand we call L_Λ. The Lie algebra L_0 is (by some abuse of notation) also called $\mathrm{diff}'_A T^2$, due to its relation with the complexified Lie algebra of the area preserving diffeomorphisms of T^2. Of course, $\mathrm{diff}'_A T^2$ is only the subalgebra of nonconstant vector fields generated[1] by finite linear combination of the generators $T_{\vec{m}}, (\vec{m} \neq 0)$. The element $T_{(0,0)}$ in $\tilde{L}_0$ does not correspond to a vector field, rather to a constant function in the Poisson algebra (see Sect. 4).

Our last infinite dimensional Lie algebra shall be the algebra generated by the elements

$$Y_{lm}, \quad \text{with} \quad l \in \mathbb{N}, \quad m = -l, \ldots, 0, \ldots, +l \tag{2-5}$$

with the Lie bracket[2]

$$[Y_{lm}, Y_{l'm'}] = g_{lm,l'm'}^{l''m''} Y_{l''m''}, \tag{2-6}$$

where the structure constants are given by

$$g_{lm,l'm'}^{l''m''} = \mathrm{i}(-1)^{m''} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \, \mathring{Y}_{l'',-m''} \left(\frac{\partial \mathring{Y}_{lm}}{\partial \theta} \frac{\partial \mathring{Y}_{l'm'}}{\partial \varphi} - (lm \leftrightarrow l'm') \right). \tag{2-7}$$

(Here $\mathring{Y}_{l,m}(\theta, \varphi)$ are the usual spherical harmonics [10].) This Lie algebra we will call $\mathrm{diff}'_A S^2$. Again, $\mathrm{diff}'_A S^2$ is only the (complexified) subalgebra of $\mathrm{diff}_A S^2$ generated by finite linear combinations of the vector fields corresponding to Y_{lm} (see [3]). In the following, it will sometimes be convenient to consider also the trivial central extension $\mathrm{diff}'_A S^2 \oplus \mathbb{C} \cdot Y_{00}$ by an additional element Y_{00}. Again, this

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[1] We use the word "generated" in the sense of generated as a vector space

[2] Here and in the following summation convention is assumed

element represents the constant function in the Poisson algebra. Note that the structure constants (2-7) will be nonvanishing only for

$$m'' = m + m' \quad \text{and} \quad |l - l'| \leqq l'' \leqq l + l' - 1. \tag{2-8}$$

We will need this later on.

Let us now study how to obtain the above infinite dimensional Lie algebras by some limit process from finite dimensional Lie algebras. In this sense we call them just "limits" of these finite dimensional algebras. In this section we do not want to give an exact definition of a "limit," rather we want to show some interesting observation in connection with this limit process.

Let us start with $gl_+(\infty)$. Induced by a numbering of the basis of the vector space on which $gl_+(\infty)$ is operating we get an embedding of the algebra $gl(N)$ into $gl_+(\infty)$ by considering the operations involving only the first N basis elements. This embedding we call the standard embedding. By increasing N one obtains a chain of subalgebras

$$gl(N) \subset gl(N+1) \subset gl(N+2) \subset \cdots.$$

As every element of $gl_+(\infty)$ lies in some $gl(N)$ we can call $gl_+(\infty)$ a "$gl(N), N \to \infty$ limit." In fact, $gl_+(\infty)$ is the "direct limit" of the standard embedding in the sense of the language of categories.

Let us now consider an arbitrary basis $\{T_a^N | a = 1, \ldots, N^2\}$ of $gl(N)$ with the corresponding structure constants $f_{ab}^{c,N}$. Let C^N be the $N^2 \times N^2$-matrix describing the base change, i.e.

$$T_a^N = C_{a,ij}^N E_{ij}, \quad a = 1, 2, \ldots, N^2, \tag{2-9}$$

where the E_{ij} are the generators introduced above (of course, now $i, j = 1, \ldots, N$). By definition, C^N is invertible, and the structure constants can be expressed in terms of C^N as follows:

$$f_{ab}^{c,N} = C_{a,ij}^N C_{b,jk}^N ((C^N)^{-1})_{ik,c} - (a \leftrightarrow b). \tag{2-10}$$

We consider the family of $gl(N)$ given by the generators T_a^N. If we assume that $f_{ab}^{c,N}$ has a well defined limit

$$f_{a,b}^c := \lim_{N \to \infty} f_{a,b}^{c,N} \tag{2-11}$$

for all a, b, c and that for fixed a and b the set

$$\{c \in \mathbb{N} | \text{there exists a } N \text{ such that } f_{a,b}^{c,N} \neq 0\}$$

is finite then we can define a Lie algebra generated by elements $\{T_a | a \in \mathbb{N}\}$ with the bracket

$$[T_a, T_b] = f_{a,b}^c T_c. \tag{2-12}$$

Clearly, this Lie algebra might also be viewed as a $gl(N), N \to \infty$ limit.

Nevertheless the above condition does not imply that the family of base transformations $C^{(N)}$ (2-9) has to define a base transformation C also in the limit. For this we would have to additionally require that (for fixed i, j) the element E_{ij} is only a finite(!) linear combination of the T_a^N and that the number of elements is bounded independent of N, and vice versa! Of course, if this condition is fulfilled, the limit (2-12) will be isomorphic to $gl_+(\infty)$. However, in most of the interesting examples this will not be the case. Hence, we take the convergence of the structure constants (2-11) as the starting point. The resulting algebra (2-12) will then in general not be isomorphic to $gl_+(\infty)$ as we will see. Note however that the existence of C as a transformation between vector spaces is a sufficient but not necessary condition for the isomorphy of the resulting algebra with the $gl_+(\infty)$ given by (2-2).

Now let us study specific examples of the relations (2-9)–(2-12).

Example 1. Let $N, M \in \mathbb{N}, N$ odd, $1 \leqq M < N$ with M and N relatively prime and $\omega := e^{4\pi i M/N}$ be a primitive N^{th} root of unity. Define the $N^2 \times N^2$-matrix[3] C with indices $a = \vec{m} = (m_1, m_2), m_1, m_2 = -\frac{N-1}{2}, \ldots, +\frac{N-1}{2}, i, j = 1, \ldots, N$,

$$C_{\vec{m},ij} := \frac{iN}{4\pi M} \omega^{(1/2)m_1 m_2 + (i-1)m_1} \delta_{i+m_2, j \bmod N}, \tag{2-13}$$

where $\delta_{k,l \bmod N}$ is equal to 1 if $k \equiv l \bmod N$ and 0 otherwise.

The inverse matrix can be calculated to be

$$(C^{-1})_{ij,\vec{m}} = \frac{-4\pi i M^2}{N^2} \omega^{m_1(m_2/2 - j + 1)} \delta_{m_2, j-i \bmod N} \tag{2-14}$$

which yields the structure constants

$$f^{\vec{a}}_{\vec{m},\vec{n}} = \frac{N}{2\pi M} \sin \frac{2\pi M}{N} (\vec{m} \times \vec{n}) \delta_{\vec{m}+\vec{n},\vec{a} \bmod N}. \tag{2-15}$$

($\delta_{\vec{m},\vec{n} \bmod N}$ is the obvious generalization of $\delta_{m,n \bmod N}$ to two dimensions.)

Let $M, N \to \infty$ in such a way that $M/N \to \Lambda \in \mathbb{R}$. Then we obtain as "limit" the infinite dimensional algebra $\tilde{L}_\Lambda$. For $M = 1$ and $N \to \infty$ we get as a special case the "limit" $\tilde{L}_0 = \text{diff}'_A T^2 \oplus \mathbb{C}$. This example shows that the "limit" is not unique. To see this, let Λ and Λ' be two different irrational numbers with $0 < \Lambda, \Lambda' < \frac{1}{4}$. We can approximate them by two sequences

$$\frac{M_k}{N_k} \quad \text{respectively} \quad \frac{M'_k}{N'_k} \quad \text{with} \quad N_k = N'_k.$$

Hence we have as the k^{th} element of the "sequence" the algebra $gl(N_k)$. But it was proven in [9] that $\tilde{L}_\Lambda$ is not isomorphic to $\tilde{L}_{\Lambda'}$ for $\Lambda \neq \Lambda'$.

Example 2. Again let N be an odd positive integer. Let us define the matrix $C = (C_{lm,ij})$: The range of the first index pair shall be

$$l = 0, 1, \ldots, N-1, \quad m = -l, \ldots, 0, \ldots, +l. \tag{2-16}$$

For the second pair it is $i, j = 1, \ldots, N$. The matrix element is defined by

$$C_{lm,ij} = (-1)^{N-i} \begin{pmatrix} \frac{N-1}{2} & l & \frac{N-1}{2} \\ -i + \frac{N+1}{2} & m & j - \frac{N+1}{2} \end{pmatrix} \cdot \delta_{i-m,j} \cdot R_N(l), \tag{2-17}$$

[3] To avoid cumbersome notation we drop the superscript N in the following

where

$$R_N(l) = \sqrt{\frac{2l+1}{16\pi}} \sqrt{\frac{(N+l)!}{(N-l-1)!}} \cdot (N^2-1)^{(1-l)/2}, \tag{2-18}$$

and $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the $3j$-symbol [10]. The structure constants $f^{l''m''}_{lm,l'm'}$ with respect to the basis T_{lm} have been calculated in [3] (compare also [11]). They are nonvanishing only in the range given by (2-8). Taking their limit for $N \to \infty$ yields the structure constants (2-7) (with the trivial extension).

Hence the algebra $\mathrm{diff}'_A S^2 \oplus \mathbb{C}$ is another $gl(N), N \to \infty$ limit. In Appendix A we will prove, that $gl_+(\infty), \mathrm{diff}'_A S^2$ and $\mathrm{diff}'_A T^2$ (respectively their trivial central extensions) are pairwise non-isomorphic.

Note that what (2-13) and (2-17) have in common (despite their different appearance) is the factor $\delta_{i+\cdot,j}$ (mod N, in the case of (2-13)). This means that all matrices C_a, defined by $(C_a)_{ij} = C_{a,ij}$ with $a = \vec{m}$ or $a = (l, m)$ have non-zero elements only at one (mod N)-diagonal. If we rewrite (2-10) as

$$f^{c,N}_{ab} = \mathrm{Tr}([C_a, C_b], \tilde{C}_c) \tag{2-19}$$

with $\tilde{C} = (C^{-1})^{\mathrm{tr}}$ one can easily see that in this case (2-19) is well defined, as $N \to \infty$. One could take (2-19) as a starting point and guess some other structures which could leave $f^{c,N}_{ab}$ finite, as $N \to \infty$.

The simplest "new" solutions of (2-19) however, can be found by a direct product Ansatz for C:

$$C_{a_1 a_2, ij} = R_{a_1 i} \cdot S_{a_2 j} \tag{2-20}$$

yielding

$$f^{\vec{c}}_{\vec{a}\vec{b}} = \delta_{c_1,a_1} \delta_{c_2,b_2} (RS^{\mathrm{tr}})_{b_1 a_2} - (\vec{a} \leftrightarrow \vec{b}). \tag{2-21}$$

As long as $X^{(N)} = X = SR^{\mathrm{tr}}$ has a well defined limit in $gl_+(\infty)$ as $N \to \infty$, the structure constants (2-21) will lead to a well defined $gl(N), N \to \infty$ limit with the Lie bracket

$$[T_{ij}, T_{kl}] = X_{jk} T_{il} - T_{kj} X_{li} \quad i, j, k, l \in \mathbb{N} \tag{2-22}$$

which is in general not isomorphic to the usual $gl_+(\infty)$.

3. L_α-Approximations

In the last section we have shown that the concept of "$gl(\infty)$" or "$su(\infty)$" becomes troublesome if one simply assumes that there is some sort of "limit" at work: First of all, to define a limit of algebras in the strong sense one would need a concept of "measuring the distance" between two algebras L_α and L_β in a limit sequence to know under which conditions this sequence converges. But this could be too restrictive, since in practice the interest often lies in the approximation of structure constants. Also, a "true limit" should preferably be unique (at least up to isomorphism). But, as some of the preceding examples have shown, the same sequence ($gl(N)$) can give nonisomorphic "limits." Finally, the well-known mathematical procedures for the "approximation of algebraic structures" like direct or projective limits (see for example [12]) in pure algebra or the approximation of C^*-algebra by finite dimensional matrix subalgebras [13, Chap. 12] do not apply in the preceding examples, because there exist no typical homomorphisms (like for instance subalgebra relations) between the approximating $gl(N)$-algebras and the "limit algebra."

Therefore we are going to develop a mathematical notion of what could be meant by an approximation of the structure constants of a given Lie algebra. This concept which to the best of our knowledge is new, covers and generalizes the results which have been summarized in Sect. 2. We shall first give the definitions and theorems and then discuss some examples.

We start with a given family of real or complex Lie algebras $(L_\alpha, \alpha \in I)$ where the Lie brackets in each L_α is denoted by $[..,..]_\alpha$ and the index set is either $\mathbb{N}$ or $\mathbb{R}$. In addition, we require that each L_α carries a metric d_α. Now let $(L, [..,..])$ be another (real or complex) Lie algebra satisfying the following

Axiom 3.1. (i) *There exists a surjective map* $p_\alpha: L \to L_\alpha$ *for every* $\alpha \in I$.

(ii) *For each* $x, y \in L$ *the following holds: If* $d_\alpha(p_\alpha(x), p_\alpha(y)) \to 0$ *for* $\alpha \to \infty$ *then* $x = y$.

We call $(L_\alpha, [..,..]_\alpha, d_\alpha, \alpha \in I)$ an approximating sequence for $(L, [..,..])$ induced by $(p_\alpha, \alpha \in I)$ and L an L_α-quasilimit if the following axiom is also valid

Axiom 3.2. *For each* $x, y \in L$,

$$d_\alpha(p_\alpha[x, y], [p_\alpha x, p_\alpha y]_\alpha) \to 0 \quad (\alpha \to \infty).$$

A few remarks are to be made:

(a) If we set $y = 0$ in 3.1 (ii) and assume $p_\alpha x = 0$ for all $\alpha \in I$, we get $d_\alpha(p_\alpha x, p_\alpha y) = 0$, hence $x = y = 0$. In particular, for $x \in L$, $x \neq 0$ there exists always $\alpha \in I$ with $p_\alpha x \neq 0$. By this the vector space L can be considered as vector subspace of $\prod_{\alpha \in I} L_\alpha$.

(b) The above definitions depend on the metrics d_α chosen. However, as can be easily checked slight deformations of the sequence of metrics $(d_\alpha)_{\alpha \in I}$ into a new one $(d'_\alpha)_{\alpha \in I}$ in such a way that there exist positive $a, b \in \mathbb{R}$ such that

$$a \cdot d_\alpha(x_\alpha, y_\alpha) \leqq d'_\alpha(x_\alpha, y_\alpha) \leqq b \cdot d_\alpha(x_\alpha, y_\alpha) \quad \forall \alpha \in I, \quad \forall x_\alpha, y_\alpha \in L_\alpha$$

do not change the validity of Axioms 3.1 and 3.2. In these cases for all $x, y \in L$ $(d'_\alpha(p_\alpha x, p_\alpha y))$ is a zero sequence if and only if $(d_\alpha(p_\alpha x, p_\alpha y))$ is a zero sequence. Hence Axiom 3.1(ii) (and analogously Axiom 3.2) will be satisfied for (d_α) if and only if it is satisfied for (d'_α). In most of the cases we are interested in metrics which will come from a norm $\| \cdots \|_\alpha$ on L_α, i.e. $d_\alpha(x_\alpha, y_\alpha) = \| x_\alpha - y_\alpha \|_\alpha$.

(c) Of course, the concept of approximating sequences is by no means restricted to Lie algebras. The Lie structure can easily be replaced by other algebraic structures, like super algebras, associative algebras, ...

It is shown in the Examples 2 and 3 below, that the same sequence of algebras L_α could approximate non-isomorphic algebras. However, we have the following

Proposition 3.3 (*weak uniqueness*). *Let* $(L_\alpha, [..,..]_\alpha, d_\alpha, \alpha \in I)$ *be an approximating sequence for the Lie algebra* $(L, [..,..])$ *induced by* $(p_\alpha, \alpha \in I)$. *Furthermore, let* L' *be* *a linear subspace of L carrying a Lie product* $[\ldots,\ldots]'$ *and projecting onto each* L_α. *Then:* $(L',[\ldots,\ldots]')$ *is a Lie subalgebra of* $(L,[\ldots,\ldots])$, i.e. $[\ldots,\ldots]'$ *is the restriction of* $[\ldots,\ldots]$ *to* L' *if and only if the approximating sequence for L is by restriction also an approximating sequence for* L' *induced by the restriction of the* p_α.

Proof. Clearly, L' fulfills the Axiom 3.1 as a subspace of L, respectively by assumption. If L' is a subalgebra then Axiom 3.2 is trivially valid, hence $\Rightarrow$. Conversely, we obtain by the triangle inequality for $x, y \in L'$,

$$d_\alpha(p_\alpha[x,y], p_\alpha[x,y]') \leqq d_\alpha(p_\alpha[x,y], [p_\alpha x, p_\alpha y]_\alpha) + d_\alpha(p_\alpha[x,y]', [p_\alpha x, p_\alpha y]_\alpha).$$

Because this is a L_α-approximation for L and L', we see that on the right-hand side we have two zero sequences for $\alpha \to \infty$. Hence, we have also a zero sequence on the left-hand side. By Axiom 3.2(ii) (applying it for L) it follows $[x,y] = [x,y]'$, hence $\Leftarrow$. ■

In particular, setting $L' = L$ we see that the Lie structure $[\ldots,\ldots]$ on the underlying vector space L of an L_α-quasilimit is unique once the linear maps $(p_\alpha, \alpha \in I)$ are specified.

By a standard Zorn's lemma argument, using the weak uniqueness from above the following proposition can now be shown

Proposition 3.4. *Every* L_α*-quasilimit is a subalgebra of a maximal* L_α*-quasilimit* $(L,[\ldots,\ldots])$.

We will not need this later on.

In many examples the approximating sequence fulfills also the following

Axiom 3.5. *There exists a family of linear maps* $(i_\alpha : L_\alpha \to L, \alpha \in I)$ *and* $\alpha_0 \in I$ *such that for* $\alpha \geqq \alpha_0$,

$$p_\alpha \circ i_\alpha = \mathrm{id}_\alpha \quad \textit{and} \quad i_\alpha(L_\alpha) \subseteq i_\beta(L_\beta) \quad \beta \geqq \alpha.$$

If an approximating sequence L_α fulfills also Axiom 3.5 we call (L_α) a *splitting* L_α-approximation.

Example 1. Let L and $(L_\alpha, \alpha \in I)$ be different Lie algebras with the same underlying vector space V. If we choose as p_α and as i_α the identity map and as d_α a fixed metric d on V then Axioms 3.1 and 3.5 are clearly fulfilled. Axiom 3.2 reads as

$$d([x,y],[x,y]_\alpha) \to 0 \quad (\alpha \to \infty). \tag{3-1}$$

This reflects the approximation of the structure constants. To make the example more concrete let L (and hence all L_α) be generated by T_n with $n \in \mathbb{N}$ (or $\in \mathbb{Z}$). By $\langle T_n, T_m \rangle = \delta_{n,m}$ we get a scalar product on V. If we choose $d(x,y) := \sqrt{\langle x-y, x-y \rangle}$ then (3-1) implies for the structure constants $f^k_{nm}, f^{k,\alpha}_{nm}$ defined by

$$[T_n, T_m] = f^k_{nm} T_k \quad \text{respectively} \quad [T_n, T_m]_\alpha = f^{k,\alpha}_{nm} T_k \tag{3-2}$$

convergency

$$\lim_{\alpha \to \infty} f^{k,\alpha}_{nm} = f^k_{nm}. \tag{3-3}$$

Conversely, if for fixed n and m the set

$$\{k \in \mathbb{N} \text{ (respectively } \mathbb{Z}) \mid \text{there exists a } \alpha \text{ such that } f_{n,m}^{k,\alpha} \neq 0\}$$

is finite then (3-3) implies (3-1). ■

Example 2. $\operatorname{diff}'_\Lambda T^2$, the torus algebra. We start with the algebras $\tilde{L}_\Lambda$ introduced in Sect. 2. Here we are especially interested in $\Lambda = 0$ (the torus algebra) and $\Lambda = \frac{1}{N}$. We use L for $\tilde{L}_0$ and L^N for $\tilde{L}_{1/N}$. The subspace

$$J^N := \langle T_{\vec{m}} - T_{\vec{m}+N\cdot\vec{a}} \mid \vec{m}, \vec{a} \in \mathbb{Z}^2 \rangle_{\mathbb{K}} \tag{3-4}$$

is an ideal in L^N. Hence we can define the factor algebra $L^{(N)} := L^N/J^N$ with $\varphi_N : L^N \to L^{(N)}$ the canonical projection map. This Lie algebra has dimension N^2. A basis is given by the N^2 elements

$$\varphi_N(T_{\vec{m}}), \quad \vec{m} = (p, q) \quad 0 \leqq p, q < N.$$

By definition of the factor algebra, the Lie product is given as

$$[\varphi_N(T_{\vec{m}}), \varphi_N(T_{\vec{n}})]^{(N)} = \frac{N}{2\pi} \sin \frac{2\pi}{N} (\vec{m} \times \vec{n}) \varphi_N(T_{\vec{m}+\vec{n} \bmod N}). \tag{3-5}$$

If we compare (3-5) with (2-15) we see that for N odd $L^{(N)}$ is exactly the Lie algebra $gl(N)$ written in the basis as introduced in Sect. 2, Example 1.

Now we define an L_α approximation for our Lie algebra L. As index set I we take the natural numbers. We use as L_αs the algebras $L^{(N)}$, as p_α the canonical map φ_N and as metric on $L^{(N)}$ the norm induced by the standard scalar product

$$\langle \varphi_N(T_{\vec{m}}), \varphi_N(T_{\vec{n}}) \rangle = \delta_{m_1,n_1} \cdot \delta_{m_2,n_2}. \tag{3-6}$$

By setting $i_N(\varphi_N(T_{\vec{n}}))) := T_{\vec{n} \bmod N}$ we obtain a linear map $L^{(N)} \to L$ which obeys $\varphi_N \circ i_N = \mathrm{id}$. Axiom 3.5 is obvious. Axioms 3.1 and 3.2 are also fulfilled as will be shown in the following. Hence, this defines a splitting L_α-approximation. By definition φ_N is surjective. For 3.1 (ii): Let $x, y \in L$. We can write them as a finite sum,

$$x = \sum_{\vec{m} \in \mathbb{Z}^2} r_{\vec{m}} T_{\vec{m}} = \sum_{m_1=a}^{b} \sum_{m_2=c}^{d} r_{(m_1,m_2)} T_{(m_1,m_2)},$$
$$y = \sum_{\vec{m} \in \mathbb{Z}^2} s_{\vec{m}} T_{\vec{m}}. \tag{3-7}$$

Without restriction, we can assume the same range for the summands in the representation of x and y. If $N > 2 \cdot \max(|a|, |b|, |c|, |d|)$ then the $\varphi_N(T_{\vec{m}})$ for the $T_{\vec{m}}$ involved will be pairwise distinct. Hence they form a subset of the basis in $T^{(N)}$ and we obtain

$$\varphi_N(x) - \varphi_N(y) = \sum_{\vec{m} \in \mathbb{Z}^2} (r_{\vec{m}} - s_{\vec{m}}) \varphi_N(T_{\vec{m}}). \tag{3-8}$$

We calculate

$$d_N(\varphi_N(x), \varphi_N(y)) = \sqrt{\sum_{\vec{m} \in \mathbb{Z}^2} |r_{\vec{m}} - s_{\vec{m}}|^2}. \tag{3-9}$$

Obviously, this expression is independent of N, hence we get $\lim_{N\to\infty}(\varphi_N(x),\varphi_N(y))=0$ if and only if $x=y$.
Axiom 3.2: We consider first the case $x=T_{\vec{m}}$ and $y=T_{\vec{n}}$. We calculate

$$B_N(\vec{m},\vec{n}):=[\varphi_N(T_{\vec{m}}),\varphi_N(T_{\vec{n}})]-\varphi_N([T_{\vec{m}},T_{\vec{n}}])$$
$$=\left(\frac{N}{2\pi}\sin\frac{2\pi}{N}(\vec{m}\times\vec{n})-(\vec{m}\times\vec{n})\right)\varphi_N(T_{\vec{m}+\vec{n}\bmod N}). \qquad (3\text{-}10)$$

For N big enough we obtain

$$\|B_N(\vec{m},\vec{n})\|=\left|\frac{N}{2\pi}\sin\frac{2\pi}{N}(\vec{m}\times\vec{n})-(\vec{m}\times\vec{n})\right|, \qquad (3\text{-}11)$$

and hence $\lim_{N\to\infty}d_N(B_N(\vec{m},\vec{n}))=0$. Because arbitrary x and y are finite sums of such $T_{\vec{m}}$ the claim is also valid in these cases. ■

Example 3. $\mathrm{diff}'_A S^2$. Let L be the algebra $\mathrm{diff}'_A S^2\oplus\mathbb{C}\cdot Y_{00}$ introduced in Sect. 2. Take $I=\mathbb{N}$ and as L_α the algebras $L_{(N)}$ introduced in Example 2 of Sect. 2. We have again $L_{(N)}=gl(N)$ for N odd. Let us denote the generators of this algebra by T^N_{lm}. We use as linear map $p_N: L\to L_{(N)}$ the map induced by the map

$$p_N(Y_{lm})=\begin{cases}T^N_{lm}, & \text{if } l<N\\ 0, & \text{if } l\geqq N\end{cases} \qquad (3\text{-}12)$$

on the basis. As $i_N: L_{(N)}\to L$ we take the linear map induced by the inverse of (3-12). As metric d_N in $L_{(N)}$ we take the metric induced by the scalar product

$$\langle T^N_{lm},T^N_{l'm'}\rangle=\delta_{l,l'}\cdot\delta_{m,m'}. \qquad (3\text{-}13)$$

Again, Axiom 3.1(i) and Axiom 3.5 are obvious. Axiom 3.1(ii) can be shown in exactly the same way as above (replacing $\vec{m}$ by (m,l). To show Axiom 3.2 we have to make some minor modifications. We start with $x=Y_{lm}$ and $y=Y_{l'm'}$. Similar to (3-10) we get for

$$B_N(lm,l'm'):=[T^N_{lm},T^N_{l'm'}]-p_N([Y_{lm},Y_{l'm'}])$$

by using the result (2-8) on the range of the indices (with $m''=m+m'$)

$$\|(B_N(lm,l'm')\|^2=\sum_{l''=|l-l'|}^{l+l'-1}(f^{l''m'',N}_{lm,l'm'}-g^{l''m''}_{lm,l'm'})^2. \qquad (3\text{-}14)$$

Now the range of the summation is independent of N. Because every summand vanishes for $N\to\infty$ [3], the same is true for (3-14). Hence, we can conclude the argument as in Example 2. ■

In Examples 2 and 3 we showed (using as index set only the odd numbers) that both $\mathrm{diff}'_A T^2\oplus\mathbb{C}\cdot T_{00}$ and $\mathrm{diff}'_A S^2\oplus\mathbb{C}\cdot Y_{00}$ have as a L_α-quasilimit the same sequence of $gl(N)$. Nevertheless, as will be shown in Appendix A they are nonisomorphic.

Example 4. $gl_+(\infty)$. We take as L the algebra $gl_+(\infty)$ as L_α the $gl(N)$, considered as subalgebras of $gl_+(\infty)$ via the standard embedding, and as p_α the linear maps induced by

$$p_N(E_{ij}) = \begin{cases} E_{ij}, & 1 \leqq i,j \leqq N \\ 0, & \text{otherwise.} \end{cases}$$

The map p_N is the projection onto $gl(N)$. As $i_\alpha = i_N$ we take the obvious inverse linear map. As $d_\alpha = d_N$ we take the norm induced by the scalar product

$$\langle E_{ij}, E_{kl} \rangle = \delta_{i,k} \cdot \delta_{j,l} = \mathrm{Tr}(E_{ij}^{tr} \cdot E_{kl}).$$

Axiom 3.1(i) and Axiom 3.5 are valid by definition. Let x and y be 2 elements of $gl_+(\infty)$. If we choose N big enough (depending on the range of the nonvanishing coefficients of x and y with respect to the basis E_{ij}) we see that x and y are elements of

$$gl(N) \subset gl(N+1) \subset gl(N+2) \subset \cdots. \tag{3-15}$$

Because we have $[\ldots,\ldots]_N = [\ldots,\ldots]_{|gl(N)}$ Axiom 3.2 is now immediate. The embedding (3-15) is an isometric embedding, i.e. $d_{N+k_{|gl(N)}} = d_N$ for $k \geqq 0$. Hence,

$$d_{N+k}(x,y) = d_N(x,y).$$

This shows Axiom 3.1(ii). Hence, the above data defines a L_α-approximation. The base change (2-9) in $gl(N)$ can be described as an isomorphism φ_N. This gives again maps of $gl_+(\infty)$ to $gl(N)$ defined as the composition

$$gl_+(\infty) \xrightarrow{p_N} gl(N) \xrightarrow{\varphi_N} gl(N).$$

Now we choose a metric d_N on the second copy of $gl(N)$. For example, the metric induced by $\langle T_a, T_b \rangle = \delta_{a,b}$ might be a standard choice with respect to the new basis. The above axioms with respect to the maps $p'_N = \varphi_N \circ p_N$ are also valid with the exception of Axioms 3.1(ii). In general, the chain (3-15) will not be an isometric embedding anymore. Hence, we cannot conclude as above that the Axiom 3.1(ii) is necessarily valid. In fact, if we apply the above to Example 1 of Sect. 2 then

$$\lim_{N \to \infty} d_N(p'_N(E_{00}), p'_N(E_{11})) = 0$$

if we choose the metric induced by $\langle T_{\vec{m}}, T_{\vec{n}} \rangle = \delta_{\vec{m},\vec{n}}$.

Of course, if we choose as metric the pullback metric $(\varphi_N^{-1})^* d_N$ on the second copy everythings works again. ■

4. Geometric Quantization and $\mathrm{diff}_V M$

In this section we consider compact Kähler manifolds which in the context of geometric quantization seems to be the natural generalization of compact two-dimensional manifolds (like S^2 and T^2) to higher dimensions. Indeed, well-known theorems [15, 16] state that every orientable two-dimensional manifold carries a complex structure and a Hermitian structure whose real part is a Riemannian metric g and whose imaginary part is a nondegenerate closed volume form $\omega = \sqrt{\det g}\, dx^1 \wedge dx^2$.

In general, any manifold carrying a nondegenerate closed 2-form ω (a symplectic form) is the differential geometric arena for classical mechanics [17]. Such manifolds

(M, ω) are called symplectic manifolds. Necessarily, they are orientable and even dimensional, i.e. $\dim M = 2n$ and

$$\Omega := (-1)^{\binom{n}{2}} \frac{1}{n!} \omega^n$$

defines a volume form.

For the following, let (M, ω) be a symplectic manifold. By $\mathrm{diff}_V M$ we denote the Lie algebra of all divergence-free vector fields on M (with the usual Lie bracket of the vector fields). In the case of $\dim M = 2$ we use also the symbol $\mathrm{diff}_A M$.[4] The elements are characterized by

$$X \in \mathrm{diff}_V M \quad \text{if and only if} \quad L_X \Omega = 0. \tag{4-1}$$

where L_X is the Lie derivative with respect to X. (In the case of surfaces $\Omega = \omega$.) Equivalently, $\mathrm{diff}_V M$ can be given as the set of vector fields which correspond to the volume-preserving diffeomorphisms of M [14].

On p-forms we have [17]

$$L_X = i_X \circ d + d \circ i_X \tag{4-2}$$

(here i_X is the interior product, i.e. $i_X \alpha(\cdots) = \alpha(X, \ldots)$) and thus $L_X \omega = d i_X \omega$. To each smooth real valued function H on M one assigns its Hamiltonian vector field X_H defined by $i_{X_H} \omega = dH$. In certain local coordinates $(q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n)$ (which in general have nothing to do with complex coordinates) one has $\omega_| = \sum_i dq^i \wedge dp_i$. In these coordinates X_H can be expressed as

$$X_{H|} = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i=1}^{n} \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}.$$

Obviously, Hamiltonian vector fields are divergence-free, moreover

$$L_{X_H} \omega = 0 \quad \text{for all smooth functions } H \text{ on } M. \tag{4-3}$$

Any vector field X obeying $L_X \omega = 0$ is called locally Hamiltonian.

The Poisson bracket $\{f, g\}$ of two smooth real valued functions f and g on M is defined as

$$\{f, g\} := df(X_g). \tag{4-4}$$

It establishes a Lie structure on the space of all smooth real valued functions. This Lie algebra is called the Poisson algebra $\mathscr{P}(M)$. One has the important relation

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{4-5}$$

More generally, for two vector fields X, Y on M obeying $L_X \omega = L_Y \omega = 0$ one has

$$[X, Y] = -X_{\omega(X,Y)}.$$

(See [17] for proofs.) These relations show that the space of locally Hamiltonian vector fields is a subalgebra of $\mathrm{diff}_V M$ denoted by $L\,\mathrm{Ham}\, M$. The space of

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[4] The algebras $\mathrm{diff}'_A S^2$ and $\mathrm{diff}'_A T^2$ which were introduced in Sect. 2 are certain subalgebras of $\mathrm{diff}_A S^2$, respectively $\mathrm{diff}_A T^2$

Hamiltonian vector fields $\mathrm{Ham}\, M$, is an ideal of it. If $\dim M = 2$, then $\mathrm{diff}_A M$ is identical to $L\,\mathrm{Ham}\, M$ and the quotient $\mathrm{diff}_A M/\mathrm{Ham}\, M$ can be identified with the first de-Rham cohomology class $H^1(M, \mathbb{R})$ of M via $X \mapsto i_X\omega$.

Furthermore, the map

$$\mathscr{P}(M) \to \mathrm{Ham}\, M, \quad f \mapsto -X_f \tag{4-6}$$

is a surjective homomorphism of Lie algebras having kernel equal to the constant functions, i.e. $\mathscr{P}(M)$ is a central extension of $\mathrm{Ham}\, M$. In case M is compact, which we will assume in the following, this extension is trivial as can be seen by the Lie isomorphism ($\dim M = 2n$)

$$\mathscr{P}(M) \to \mathbb{R} \oplus \mathrm{Ham}\, M, \quad f \mapsto \left(\int_M \Omega f, -X_f \right). \tag{4-7}$$

Note, if we use (4-4) and the identity

$$\int_M df(X) \cdot \Omega = -\int_M f \,\mathrm{div}\, X \cdot \Omega \tag{4-8}$$

which is valid for arbitrary vector fields X and functions f [17, p. 153] we see that $\int_M \Omega\{f, g\}$ vanishes. For noncompact M, (4-7) in general is false as is best illustrated by $M = \mathbb{R}^2$ and $\{q, p\} = 1$. But for compact M all these arguments show that one can investigate the Poisson algebra $\mathscr{P}(M)$ in order to study an essential part of $L\,\mathrm{Ham}\, M$ ($= \mathrm{diff}_A M$ for $\dim M = 2$) and simply "omit the constants at the end."

We shall now relate the Lie algebra $\mathscr{P}(M)$ to a geometric quantization scheme. We assume M to be a compact Kähler manifold of arbitrary (real) dimension $2n$. First we recall some basic facts about this procedure (cf. [18, 19, 20], $\cdots$ for details). We write g for the Kähler metric and $I \in \mathrm{End}(TM)$ ($I^2 = -1_{TM}$) for the complex structure on M which form together the symplectic form ω

$$\omega(X, Y) = g(IX, Y). \tag{4-9}$$

Here X and Y are vector fields on M. One then needs a complex line bundle L over M, a sesquilinear fibre metric h in L and a covariant derivative ∇ in L. These data have to be compatible among themselves and with the symplectic form ω in the following sense. For two smooth sections s_1 and s_2 of L and two vector fields X and Y on M the following should hold:

$$h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2) = d(h(s_1, s_2))(X), \tag{4-10}$$

$$F(X, Y)s_1 := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s_1 = -\mathrm{i}\omega(X, Y)s_1. \tag{4-11}$$

F is the curvature 2-form of the covariant derivative and (4-11) is called the pre-quantum condition.

For every smooth real (or complex) valued function f on M the following prequantum operator P_f acting on the complex vector space $\Gamma(M, L)$ of all smooth sections of L is formed

$$P_f := -\nabla_{X_f} + \mathrm{i} f \cdot 1. \tag{4-12}$$

This defines a map

$$P: \mathscr{P}(M) \to Op(\Gamma(M, L)), \quad f \mapsto P_f.$$

The prequantum condition (4-11) guarantees that P is an injective Lie algebra homomorphism

$$P_{\{f,g\}} = [P_f, P_g]. \tag{4-3}$$

Defining a scalar product $\langle ..|..\rangle$ in $\Gamma(M,L)$ by

$$\langle s_1 | s_2 \rangle := \int_M \Omega h(s_1, s_2), \tag{4-14}$$

we see by using (4-3), (4-8) and condition (4-10) that P_f becomes an antihermitian operator in $\Gamma(M,L)$ for real valued f. Our unphysical convention to have the P_f antihermitian rather than hermitian is more advantageous for the formulation of (4-13) where else one would have a factor of i. The prequantum Hilbert space $\mathscr{H}$ is then defined to be the completion of $\Gamma(M,L)$ with respect to $\langle ..|..\rangle$. Note that the prequantum condition (4-11) strongly restricts the possible symplectic forms on M: Since for each complex line bundle over any manifold the Chern form $c := \frac{\mathrm{i}}{2\pi} F$ is integral [21, p. 99] (i.e. gives integers when integrated over any closed 2-surface in M) ω must be a "$2\pi \times$ integral" form.

A second step in a geometric quantization scheme is the choice of a polarization. I.e. one would like to have only those wave functions in the prequantum Hilbert space $\mathscr{H}$ that depend on "only one (certain) half of the phase space variables." For Kähler manifolds there is a canonical concept. L should be a holomorphic line bundle. One then has for each fibre metric h in L a unique covariant derivative ∇ in L which is compatible with h in the sense of (4-10) and obeys the following additional condition: for each holomorphic section s of L and each complex vector field X on M of type (0, 1) [21, p. 78]

$$\nabla_X s = 0. \tag{4-15}$$

In other words, holomorphic sections become covariantly constant in antiholomorphic directions. In a local holomorphic chart $(z^1, \ldots, z^n)$ this means the following: if the holomorphic section s is represented by a holomorphic function $\hat{s}$ and the fibre metric h by a positive smooth real function $\hat{h}$, then ∇ can be expressed in the following way [21, p. 78]

$$\nabla \hat{s} = \partial \hat{s} + \bar{\partial} \hat{s} + \partial \log \hat{h} \cdot \hat{s}. \tag{4-16}$$

The above $\log \hat{h}$ is often denoted as (local) Kähler potential. Starting with a Kähler form ω which is a $2\pi \times$ integral form there exists always a holomorphic line bundle with connection ∇ and metric h such that (4-10), (4-11) and (4-15) are fulfilled.

The quantum Hilbert space is then defined to be the subspace $\Gamma_{\mathrm{hol}}(M,L)$ of all holomorphic sections in $\mathscr{H}$. For compact manifolds it is always finite dimensional [21, p. 147]. Hence, $\Gamma_{\mathrm{hol}}(M,L)$ is a closed subspace and it follows that the orthogonal projection

$$\rho : \mathscr{H} \to \Gamma_{\mathrm{hol}}(M,L)$$

is a bounded Hermitian operator. In order to define quantum observables or quantum operators Q_f acting on $\Gamma_{\mathrm{hol}}(M,L)$ one simply takes "the holomorphic part" of the prequantum operators P_f

$$Q_f := \rho \circ P_f \circ \rho. \tag{4-17}$$

Q_f clearly is an antihermitian operator for real valued smooth functions f but in general

$$Q_{\{f,g\}} \neq [Q_f, Q_g]. \tag{4-18}$$

To get an explicit expression for Q_f one can choose any orthonormal basis $|s_1\rangle, \ldots, |s_d\rangle$ ($d = \dim \Gamma_{\text{hol}}(M,L)$) of $\Gamma_{\text{hol}}(M,L)$ and set

$$Q_f := \sum_{a,b=1}^{d} |s_a\rangle\langle s_a|P_f|s_b\rangle\langle s_b|. \tag{4-19}$$

Hence it suffices to compute the matrix elements $\langle s_a|P_f|s_b\rangle$ of P_f. ρ is sometimes called "generalized Bergman kernel" [19]. To calculate the matrix elements the following result by Tuynman [20] is quite useful. We shall give a coordinate free proof.

Proposition 4.1. (*Tuynman*) *Let* (M,ω) *be a compact Kähler manifold, L a holomorphic prequantum line bundle over M, h a fibre metric in L,* ∇ *the associated compatible connection in L and* s_1 *and* s_2 *two holomorphic sections of L then the following equation holds*:

$$\langle s_1|P_f|s_2\rangle = \mathbf{i}\langle s_1|f - \tfrac{1}{2}\Delta f|s_2\rangle. \tag{4-20}$$

Proof. Because $P_f = \mathbf{i}f\cdot 1 - \nabla_{X_f}$ it suffices to compute the term containing the covariant derivative. Let I be the complex structure of M. For any vector field X on M $\frac{1}{2}(X \mp \mathbf{i}IX)$ is the holomorphic respectively antiholomorphic part of X. Hence (with condition (4-15))

$$\nabla_{1/2(X+\mathbf{i}IX)}s_2 = 0.$$

It follows that $\nabla_{IX}s_2 = \mathbf{i}\nabla_X s_2$. Furthermore, from Eq. (4-9) we get for the Hamiltonian vector field $X_f = -I \operatorname{grad} f$. It follows that

$$h(s_1, \nabla_{X_f}s_2) = -h(s_1, \nabla_{I\operatorname{grad} f}s_2) = -\mathbf{i}h(s_1, \nabla_{\operatorname{grad} f}s_2) \tag{4-21}$$

and from (4-10)

$$d(h(s_1,s_2))(X_f) - h(s_1, \nabla_{X_f}s_2) = h(\nabla_{X_f}s_1, s_2) = +\mathbf{i}h(\nabla_{\operatorname{grad} f}s_1, s_2). \tag{4-22}$$

Subtracting (4-22) from (4-21) we get

$$h(s_1, \nabla_{X_f}s_2) = \frac{1}{2}d(h(s_1,s_2))(X_f) - \frac{\mathbf{i}}{2}d(h(s_1,s_2))(\operatorname{grad} f).$$

Integrating this identity over M and using (4-8) we see with $\operatorname{div} X_f = 0$ and $\operatorname{div}\operatorname{grad} f = \Delta f$ that

$$\langle s_1|\nabla_{X_f}|s_2\rangle = \frac{i}{2}\langle s_1|\Delta f|s_2\rangle.$$

Here the laplacian has to be calculated with respect to g. ■

As explained in [18] one should add a "half-form correction" to the above quantization scheme to obtain the correct physical values. Because this correction would not change anything essential in the following we decided to ignore it here.

In order to achieve an L_α-approximation for Ham M we would like to have the afore-mentioned geometric quantization scheme dependent on a parameter α. This can be done by fixing a holomorphic line bundle L, a fibre metric h and a covariant derivative ∇ which fulfills the compatibility by Eqs. (4-10), (4-11) and (4-15) and then considering arbitrary m-fold tensor powers of L

$$L^m := L^{\otimes m} := L \otimes \cdots \otimes L \quad (m \text{ factors}). \tag{4-23}$$

For the holomorphic line bundle L^m one can now construct a canonical fibre metric $h^{(m)}$ with compatible covariant derivative $\nabla^{(m)}$ by

$$h^{(m)} := h \otimes \cdots \otimes h \quad m \text{ factors}. \tag{4-24}$$

$$\nabla^{(m)} := \sum_{k=1}^{m} 1 \otimes \cdots \otimes (\nabla)_k \otimes \cdots \otimes 1, \tag{4-25}$$

where in the k^{th} summand the ∇ is at the k^{th} position. If L is given by transition functions $c_{\sigma\tau}$ with respect to a trivializing covering, then L^m can be given by the transition functions $(c_{\sigma\tau})^m$ and the same trivialization. In this trivialization one has

$$\hat{h}^{(m)} = (\hat{h})^m, \tag{4-26}$$

$$\nabla^{(m)} = \partial + \bar{\partial} + m\,\partial \log \hat{h}. \tag{4-27}$$

The role of the exponent m becomes clear when we check the prequantum condition (4-11) for the bundles L^m

$$F^{(m)}(X, Y) = mF(X, Y) = -\mathbf{i} m\omega(X, Y). \tag{4-28}$$

Now, $m\omega$ is also a symplectic form on M being clearly $2\pi \times$ integral, and one can compare the formulae for Hamiltonian vector fields and Poisson brackets

$$X_f^{(m)} = \frac{1}{m} X_f, \quad f \in \mathscr{P}(M), \tag{4-29}$$

$$\{f, g\}^{(m)} = \frac{1}{m}\{f, g\}, \quad f, g \in \mathscr{P}(M). \tag{4-30}$$

If we now took the usual prequantum operators

$$P_f^{(m)} := -\nabla^{(m)}_{X_f^{(m)}} + \mathbf{i} f \cdot 1,$$

we would have

$$[P_f^{(m)}, P_g^{(m)}] = P^{(m)}_{\{f,g\}^{(m)}} = \frac{1}{m} P^{(m)}_{\{f,g\}} \tag{4-31}$$

and $\frac{1}{m}$ can be interpreted as $\hbar$. But since we are looking for a representation of $\mathscr{P}(M)$, i.e. the Poisson algebra w.r.t. ω and not w.r.t. $m\omega$ we have to rescale the prequantum operators as follows

$$\hat{P}_f^{(m)} := mP_f^{(m)} = -\nabla^{(m)}_{X_f} + \mathbf{i} m f \cdot 1 \tag{4-32}$$

which yields

$$[\hat{P}_f^{(m)}, \hat{P}_g^{(m)}] = \hat{P}^{(m)}_{\{f,g\}}. \tag{4-33}$$

If we denote by $\mathscr{H}^{(m)}$ (respectively $\Gamma_{\text{hol}}(M, L^{(m)})$, respectively $\rho^{(m)}$) the Hilbert space generated by all smooth sections of L^m (where we choose the volume form on M to be equal to Ω and not $m^n\Omega$) (respectively the subspace of the global holomorphic sections of L^m, respectively the orthogonal projection on this subspace) we can form the (rescaled) quantum operators in $\Gamma_{\text{hol}}(M, L^{(m)})$

$$\hat{Q}_f^{(m)} := \rho^{(m)} \circ \hat{P}_f^{(m)} \circ \rho^{(m)}. \tag{4-34}$$

Now we set

$$L_m := \{\text{antihermitian linear operators in } \Gamma_{\text{hol}}(M, L^{(m)})\}, \tag{4-35}$$

$$p_m : \mathscr{P}(M) \to L_m, \quad f \to \hat{Q}_f^{(m)}, \tag{4-36}$$

$$d_m : L_m \times L_m \to \mathbb{R}, \quad (A, B) \mapsto r_m \cdot \sqrt{\operatorname{Tr}(A-B)^+ \cdot (A-B)}, \tag{4-37}$$

where the r_m are positive real numbers. We formulate the following

Conjecture. *Let (M, ω) be a compact Kähler manifold with symplectic form ω. Then there is a ω-compatible complex structure I in M, with respect to which M is also a Kähler manifold, a holomorphic prequantum line bundle L compatible with I, a fibre metric h with compatible covariant derivative ∇ and a sequence of positive real numbers r_m, $m \in \mathbb{N}$ such that the Poisson algebra $\mathscr{P}(M)$ admits a (L_m, d_m) approximation induced by p_m. Here L_m, p_m and d_m are defined as in (4-35)–(4-37).*

If one thinks of m as $1/\hbar$ this concept can be interpreted as $\hbar \to 0$ limit. Note that we leave the complex structure to be adjustable because the main interest lies in the symplectic structure of M.

For technical reasons which will become clear in the forthcoming example it is more convenient to work with

$$\mathscr{P}^c(M) := \mathscr{P}(M) + \mathbf{i}\mathscr{P}(M),$$

the complexification of the Poisson algebra. Since for each

$$f = f_1 + \mathbf{i} f_2 \in \mathscr{P}^c(M), \quad f_1, f_2 \in \mathscr{P}(M)$$

we clearly have $P_f = P_{f_1} + \mathbf{i}P_{f_2}$ and thus $Q_f = Q_{f_1} + \mathbf{i}Q_{f_2}$ the above conjecture can be extended to $\mathscr{P}^c(M)$ being a (L_m^c, d_m) approximation (quasilimit) induced by p_m^c. Here L_m^c is the complexification of L_m which is isomorphic to $gl(n, \mathbb{C})$ (n depending on m) and p_m^c is the complexification of p_m.

The above conjecture can be related to work of F. A. Berezin concerning the concept of quantization [29–31][5]. In the following addendum we will therefore give an overview of his techniques in the more general formulation due to J. H. Rawnsley et al. [33, 35, 36].

Addendum on Berezin's Coherent States. An interesting approach to quantization where $\hbar \to 0$ limits can be dealt with is a sort of $*$-product quantization based on F. A. Berezin's coherent states. This concept was invented and outlined in general terms by F. A. Berezin in [29] and applied mainly to symmetric bounded domains

[5] Note that refering to Berezin's work A. S. Schwarz [32] also points out a connection between the quantization of symplectic manifolds and '$u(n)$-limits'

in [30]. The basic idea is to relate the classical phase space to a quantum Hilbert space by an overcomplete system of states in that Hilbert space (the so-called system of coherent states) which is parametrized by the phase space[6]. Taking expectation values of a bounded operator with respect to the coherent states leads to a complex function on phase space and the associative noncommutative product of operators can thus be transferred to a subspace of classical observables where it becomes a $*$-product. In the case of Kähler manifolds F. A. Berezin was able to introduce a parameter $\hbar$ in order to get a family of Hilbert spaces and coherent states parametrized by $\hbar$ such that the $*$-product of two functions is the ordinary (pointwise) product up to $O(\hbar)$ and the $*$-commutator times $1/\hbar$ is the Poisson bracket up to $O(\hbar)$ which reflects the correspondence principle of quantum theory.

What makes it a little difficult to compare Berezin's approach to the method of geometric quantization is the fact that he always works in one holomorphic chart and constructs everything in local terms. For such Kähler manifolds having global holomorphic charts like $\mathbb{C}^n$ or bounded (symmetric) domains this is perfectly suitable. For more general Kähler manifolds (like higher genus compact Riemann surfaces) he does not give a general recipe how to obtain the correct Hilbert spaces. In some examples he uses the following method: He removes a divisor[7] D from the manifold M and considers as Hilbert space the space of holomorphic functions on the open dense subset $M \backslash D$ which are integrable with respect to some $\hbar$ depending metric. In this context he mentions some global features related to compact Kähler manifolds, like the fact that the space of such admissible functions is finite dimensional and that $\hbar$ is quantized (i.e. $\hbar$ takes only a discrete set $\{\hbar_n | n \in \mathbb{Z}\}$ of values of $\mathbb{R}^+$ and $\lim_{n\to\infty} \hbar_n = 0$). From the global viewpoint of geometric quantization the above space of functions can be related to the space of holomorphic sections of a suitable ($\hbar$ depending) line bundle. The quantized nature of $\hbar$ can be interpreted as the fact that one uses tensor powers of just one fixed line bundle.

Of course, Berezin's procedure of removing a divisor D is not unique. As we want to avoid the examination under which conditions the derived objects are invariant under different choices of D, we prefer to sketch Berezin's idea in a global formulation due to J. H. Rawnsley et al. (cf. [33, 35, 36]). We use the same notation as in Sect. 4. Let L be a holomorphic prequantum line bundle (with hermitian metric h), which we assume to be very ample[8], M the compact Kähler manifold and $\chi: L \to M$ the bundle projection. Let L_0 be L minus the image of the zero section. Now for each $q \in L_0$ the "evaluation" of a holomorphic section s

$$s \mapsto s(\chi(q)) = \hat{q}(s) \cdot q, \quad \hat{q}(s) \in \mathbb{C} \tag{4-38}$$

defines a linear form $s \mapsto \hat{q}(s)$ on $\Gamma_{\mathrm{hol}}(M, L)$, and hence by Riesz's theorem[9] one and only one holomorphic section $e_q \in \Gamma_{\mathrm{hol}}(M, L)$ such that

$$\langle e_q | s \rangle = \hat{q}(s). \tag{4-39}$$

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[6] See [34] for group theoretical and physical applications.

[7] For the definition see [22]

[8] For the definition see [22], e.g. L has enough sections to separate points of M.

[9] This also works if $\Gamma_{\mathrm{hol}}(M, L)$ is infinite-dimensional, i.e. M is non-compact cf. [29, 33]

Using a orthonormal base (s_α) of $\Gamma_{\text{hol}}(M,L)$ one has an equivalent formula

$$e_q = \sum_\alpha \overline{\hat{q}(s_\alpha)} \cdot s_\alpha, \tag{4-40}$$

where the $\overline{}$ denotes complex conjugation.

This shows that the map

$$L_0 \to \Gamma_{\text{hol}}(M,L), \quad q \mapsto e_q \tag{4-41}$$

is smooth. Also note the following transformation property of e_q under $\mathbb{C}^*$,

$$e_{cq} = \bar{c}^{-1} e_q, \quad \forall c \in \mathbb{C}^*. \tag{4-42}$$

The bundle L being very ample, there is no point in M where all holomorphic sections simultaneously vanish. Hence all the e_q are different from zero. Because of (4.42) the following operators in $\Gamma_{\text{hol}}(M,L)$ depend on the points $\chi(q) \in M$ only·

$$P_{\chi(q)} := \frac{|e_q\rangle\langle e_q|}{\langle e_q|e_q\rangle}, \quad P_{\chi(q),\chi(q')} := \frac{|e_q\rangle\langle e_{q'}|}{\langle e_{q'}|e_q\rangle}, \tag{4-43}$$

where the second operator is defined only on some open neighbourhood of the diagonal in $M \times M$. Furthermore, for two holomorphic sections, s_1 and s_2 one has

$$h(\chi(q))(s_1(\chi(q)), s_2(\chi(q))) = \langle s_1|e_q\rangle\langle e_q|s_2\rangle \cdot |q|^2$$

with $|q|^2 := h(\chi(q))(q,q)$. If one integrates this over M and notes that the function

$$\varepsilon(\chi(q)) := |q|^2 \langle e_q|e_q\rangle \tag{4-44}$$

is well-defined on M because of (4-42) one gets the "over completeness property"

$$\langle s_1|s_2\rangle = \int_M \Omega(x)\varepsilon(x)\langle s_1|P_x|s_2\rangle. \tag{4-45}$$

The sections e_q are called coherent vectors. Note that in contrast to Berezin's local theory the coherent vectors are parametrized by L_0 and not by M. The associated elements $\langle e_q\rangle$ in $\mathbb{P}(\Gamma_{\text{hol}}(M,L))$ are called coherent states. They and the coherent projectors P_x depend on M only.

In [33] and [36] J. H Rawnsley et al. showed that there are many situations where the function ε(4-44) is constant. This is for example the case if M is a homogeneous Kähler manifold and L is a homogeneous bundle. It is also true if M is embedded into some projective space $\mathbb{P}^N$ and the symplectic structure on M is equivalent to the pullback of the symplectic structure of $\mathbb{P}^N$. In these cases one gets by setting $s_1 = s_\alpha = s_2$ in (4-45) and summing over α the formula

$$\varepsilon = \varepsilon(x) = \frac{\dim \Gamma_{\text{hol}}(M,L)}{\operatorname{vol} M} \quad \text{with} \quad \operatorname{vol} M := \int_M \Omega. \tag{4-46}$$

Now Berezin's covariant symbol [29] $\sigma(B)$ of a (bounded) linear operator B in the Hilbert space $\Gamma_{\text{hol}}(M,L)$ is a well-defined smooth complex-valued function on M. Let $x \in M$ and take any $q \in \chi^{-1}(x) \cap L_0$ then it is defined by

$$\sigma(B)(x) := \operatorname{Tr} BP_x = \frac{\langle e_q|B|e_q\rangle}{\langle e_q|e_q\rangle}. \tag{4.47}$$

It can be shown that in our situation the map $B \mapsto \sigma(B)$ is injective (see [29, p. 1122, Remark 1] for the local case and [36] for the case of compact Kähler manifolds). Hence on the space of covariant symbols a star product can be introduced

$$\sigma(B)*\sigma(C):=\sigma(BC), \tag{4-48}$$

where B and C are (bounded) linear operators in $\Gamma_{\text{hol}}(M,L)$. This product is associative. If one writes out (4-48) with the help of the projectors P_x one will need the "two point covariant symbols" (which again are only defined in a neighbourhood of the diagonal)

$$\sigma(B)(\chi(q),\chi(q')):=\frac{\langle e_{q'}|B|e_q\rangle}{\langle e_{q'}|e_q\rangle} \tag{4-49}$$

(compare [29, p. 1118, Eq. 2.6]). It is shown in [36] that in case $\varepsilon=\text{const}$ and M is compact one has the relation

$$\sigma(Q_f)=\mathbf{i}f \tag{4-50}$$

for the symbols of the quantum operators (4-17) related to the so-called quantizable functions f on M, i.e. those functions for which the associated Hamiltonian vector fields preserve the Kähler polarization or equivalently for which the (1,0)-part of the Hamiltonian vector fields are holomorphic. In this way contact is made to geometric quantization.

In order to bring in an $\hbar$ dependence of the concept, Berezin considers the Hilbert space $F_\hbar$ of those holomorphic functions on a holomorphic chart which are square integrable with respect to a fibre metric $\exp\left(-\frac{1}{\hbar}\Phi(z)\right)$, where Φ is some fixed Kähler potential and $\hbar\in E\subseteq R^+$ with $0\in$closure of E (see [29]). All the concepts discussed above will depend on $\hbar$. In particular, one gets a space $A_\hbar$ of covariant symbols for each value of $\hbar$ and a $*$-product also depending on $\hbar$. Under some technical assumptions, Berezin is able to prove a correspondence principle in the following form: Let f be a function on $E\times\mathbb{C}^n$ which is given in the form

$$f(\hbar|z)=f(0|z)+\hbar f_1(z)+\hbar^2 f_2(\hbar|z)$$

with suitable smooth functions $f(0,.)$, f_1 and f_2 such that the map $z\mapsto f(\hbar|z)$ is in $A_\hbar$ for every $\hbar$. Let g be another such function. Then for $\hbar\to 0$,

$$\begin{gathered}(f*g)(\hbar|z)\to f(0|z)\cdot g(0|z),\\ \frac{1}{\hbar}(f*g-g*f)(\hbar|z)\to\frac{1}{\mathbf{i}}\{f,g\}(0|z)\end{gathered} \tag{4-51}$$

(cf. [29, Eqs. (2.38) and (2.39)]).

In [36] the situation is analysed for compact Kähler manifolds: Here for each tensor power m of the complex holomorphic line bundle L chosen at the begining one has coherent states (e_q), $q\in(L^{\otimes m})_0$ and a finite-dimensional space $A_{1/m}$ of covariant symbols on M. This sequence of spaces $(A_{1/m})$ is shown to be nested, i.e. $A_{1/m}\supseteqq A_{1/m'}$ for $m\geqq m'$ if ε is constant. On the union of all the $A_{1/m}$ a star product $*$ is defined with similar asymptotic properties as above (see [36] for details).

An important relationship to the L_α quasilimits described in Sect. 4 is the following: In the next section we shall calculate in detail that

$$\frac{1}{m^n}\frac{1}{m^2}\operatorname{Tr}\hat{Q}_f^{(m)+}\hat{Q}_g^{(m)}\to\frac{1}{(2\pi)^n}\int_{T^{2n}}\Omega f^+g \quad (m\to\infty) \tag{4-52}$$

for the $2n$-torus which will establish the validity of Axiom 3.1(ii). The calculation of the above trace can be alternatively be done using coherent states: ($\hat{P}_f^{(m)}$ is the prequantum operator (4-32))

$$\frac{1}{m^n}\frac{1}{m^2}\operatorname{Tr}\hat{Q}_f^{(m)+}\hat{Q}_g^{(m)}$$

$$=\frac{1}{m^n}\frac{1}{m^2}\sum_{\alpha,\beta}\overline{\langle s_\beta|\hat{P}_f^{(m)}|s_\alpha\rangle}\langle s_\alpha|\hat{P}_g^{(m)}|s_\beta\rangle$$

$$=\frac{1}{m^2}m^2\sum_{\alpha,\beta}\left(\int_M\Omega(x)\int_M\Omega(x')\left(\overline{f(x)}-\frac{1}{2m}\overline{\Delta_x f(x)}\right)\left(g(x')-\frac{1}{2m}\Delta_{x'}g(x')\right)\right.$$

$$\left.\cdot\frac{1}{m^n}h^{(m)}(x)(s_\beta(x),s_\alpha(x))h^{(m)}(x')(s_\alpha(x'),s_\beta(x'))\right).$$

Using the equation preceding (4-44) one has for $q\in\chi^{-1}(x)\cap(L^{\otimes m})_0$ and $q'\in\chi^{-1}(x')\cap(L^{\otimes m})_0$:

$$\frac{1}{m^n}\sum_{\alpha,\beta}h^{(m)}(x)(s_\beta^{(m)}(x),s_\alpha^{(m)}(x))\cdot h^{(m)}(x')(s_\alpha^{(m)}(x'),s_\beta^{(m)}(x'))$$

$$=\frac{1}{m^n}\sum_{\alpha,\beta}\frac{\langle s_\beta^{(m)}|e_q^{(m)}\rangle\langle e_q^{(m)}|s_\alpha^{(m)}\rangle}{\langle e_q^{(m)}|e_q^{(m)}\rangle}\varepsilon^{(m)}(x)\frac{\langle s_\alpha^{(m)}|e_{q'}^{(m)}\rangle\langle e_{q'}^{(m)}|s_\beta^{(m)}\rangle}{\langle e_{q'}^{(m)}|e_{q'}^{(m)}\rangle}\varepsilon^{(m)}(x')$$

$$=\frac{1}{m^n}\varepsilon^{(m)}(x)\varepsilon^{(m)}(x')\frac{\langle e_q^{(m)}|e_{q'}^{(m)}\rangle\langle e_{q'}^{(m)}|e_q^{(m)}\rangle}{\langle e_q^{(m)}|e_q^{(m)}\rangle\langle e_{q'}^{(m)}|e_{q'}^{(m)}\rangle}. \tag{4-53}$$

For the $2n-$torus being homogeneous, $\varepsilon^{(m)}$ is a constant function and equals $m^n/\mathrm{vol}(T^{2n})$. The remaining factor in (4-53) (which depends on M only) equals up to an m independent rescaling to Berezin's kernel function $G_\hbar$, $\hbar=\frac{1}{m}$ (cf. [29, p. 1119, p. 1128]). For $G_\hbar$ he derives $G_\hbar\to\delta(x,x')$ for $\hbar\to 0$ [29, p. 1131, Theorem 2.4]. Hence one gets in the limit (4-52).

The main difference between our approach of L_α-quasilimits and the Berezin–Rawnsley procedure (besides our different goals) is that at each value of $\hbar=\frac{1}{m}$ we quantize all smooth functions on M and not only the corresponding covariant symbols (which form a finite-dimensional vector space). Furthermore, the notion of $\hbar\to 0$ $\left(\text{respectively } m=\frac{1}{\hbar}\to\infty\right)$ limit in the correspondence principle of Berezin is that of pointwise convergence of the symbol functions as $\hbar$ goes to zero (compare the proof of Theorem 2.2 in [29, p. 1128]). In contrast to that we use the norm convergence for the quantum operators: Let (B_m) be a sequence of operators. (An example which occurs in our situation is $B_m=([\hat{Q}_f^{(m)},\hat{Q}_g^{(m)}]-\hat{Q}_{\{f,g\}}^{(m)})$ for two smooth complex valued functions f and g on M.) Then $B_m\to 0$ for $m\to\infty$ means that

$$\frac{1}{m^{n+2}} \operatorname{Tr} B_m^+ B_m \to 0 \quad (m \to \infty). \tag{4-54}$$

Using coherent states $e_q^{(m)}$ this can be written as

$$\begin{aligned}
\frac{1}{m^{n+2}} \operatorname{Tr} B_m^+ B_m &= \frac{1}{m^{n+2}} \sum_\alpha \langle s_\alpha^{(m)} | B_m^+ B_m | s_\alpha^{(m)} \rangle \\
&= \frac{1}{m^{n+2}} \sum_\alpha \int_M \Omega(x) \varepsilon^{(m)}(x) \frac{\langle s_\alpha^{(m)} | e_q^{(m)} \rangle \langle e_q^{(m)} | B_m^+ B_m | s_\alpha^{(m)} \rangle}{\langle e_q^{(m)} | e_q^{(m)} \rangle} \\
&= \frac{1}{m^{n+2}} \int_M \Omega(x) \varepsilon^{(m)}(x) \frac{\langle e_q^{(m)} | B_m^+ B_m | e_q^{(m)} \rangle}{\langle e_q^{(m)} | e_q^{(m)} \rangle} \\
&= \frac{1}{m^{n+2}} \int_M \Omega(x) \varepsilon^{(m)}(x) \int_M \Omega(x') \varepsilon^{(m)}(x') \frac{\langle e_q^{(m)} | B_m^+ | e_{q'}^{(m)} \rangle \langle e_{q'}^{(m)} | B_m | e_q^{(m)} \rangle}{\langle e_q^{(m)} | e_q^{(m)} \rangle \langle e_{q'}^{(m)} | e_{q'}^{(m)} \rangle} \\
&= \int_M \int_M \Omega(x) \varepsilon^{(m)}(x) \Omega(x') \varepsilon^{(m)}(x') \\
&\quad \cdot \frac{1}{m^n} \frac{\langle e_q^{(m)} | e_{q'}^{(m)} \rangle \langle e_{q'}^{(m)} | e_q^{(m)} \rangle}{\langle e_q^{(m)} | e_q^{(m)} \rangle \langle e_{q'}^{(m)} | e_{q'}^{(m)} \rangle} \frac{\overline{\sigma(B_m)(x,x')}}{m} \frac{\sigma(B_m)(x,x')}{m}.
\end{aligned}$$

Hence, up to multiplication with Berezin's kernel function (which will give a $\delta(x,x')$ in the limit $m \to \infty$ in some important cases anyway) this is a L^2-convergence which is in general different from pointwise convergence.

5. Complex Tori (an Example)

Every $2n$-dimensional real torus $T^{2n} \cong U(1) \times \cdots \times U(1)$ ($2n$ factors) carries symplectic forms ω which are invariant under T^{2n} translations. All these forms can be expressed by

$$\omega = \sum_{k,l=1}^{2n} \pi \beta_{kl} dx^k \wedge dx^l, \tag{5-1}$$

where the dx^k are (globally defined) constant 1-forms along the $U(1)$ factors and $\beta = (\beta_{kl})$ is any skewsymmetric nonsingular real matrix. To introduce a complex structure one alternatively describes T^{2n} as

$$T^{2n} = V/\Lambda, \quad \pi : V \to T^{2n}, \tag{5-2}$$

where V is a n-dimensional complex vector space, Λ a lattice

$$\Lambda = \langle \lambda_1, \lambda_2, \ldots, \lambda_{2n} \rangle_{\mathbb{Z}}, \quad \lambda_i \in V \tag{5-3}$$

with the λ_i linearly independent over $\mathbb{R}$ and π the quotient map. The map π carries the complex structure of V to T^{2n}. In order to have prequantum holomorphic line bundles over T^{2n} the form ω has to be a $2\pi \times$ integral 2-form. This is equivalent to demand that β is an integral matrix, see [22, 16] if we choose as basis

$x_1, x_2, \ldots, x_{2n}$ of V (over $\mathbb{R}$) the dual basis of the lattice basis given by

$$\int_{\lambda_j} dx_i = \delta_{i,j}, \quad i, j = 1, \ldots, 2n. \tag{5-4}$$

Furthermore, one can choose the basis $\lambda_1, \ldots, \lambda_{2n}$, of the lattice in such a way that the matrix β can be given as

$$\beta = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \quad D = \operatorname{diag}(d_1, d_2, \ldots, d_n), \tag{5-5}$$

where the d_i are integers, such that d_i divides d_{i+1}. If all the d_i are equal to 1 the torus T^{2n} is said to have principal polarization. By choosing the elements $e_k = (d_k)^{-1} \lambda_k$ as a complex basis of the vector space V the lattice can be given as

$$\lambda_k = d_k e_k, \quad \lambda_{n+k} = \sum_{l=1}^{n} Z_{kl} e_l, \quad 1 \leqq k \leqq n. \tag{5-6}$$

The $n \times n$ complex matrix $Z = (Z_{kl})$ is called the period matrix.

Obviously, all tori are Kähler manifolds. In the following, we take as symplectic forms (5-1) only those forms which are Kähler forms. Hence, they are positive (1, 1) forms with respect to the complex structure of V. In this case the period matrix Z is a complex, symmetric matrix with positive definite imaginary part [22, p. 305]. Conversely, any such matrix Z and integers $d_1, d_2, \ldots, d_n$ lead to those complex tori we are interested in. These tori can even be embedded into projective space (use Kodaira's embedding theorem), hence they are abelian varieties.

In order to perform the L_m-approximation scheme introduced in the preceding section it is advantageous to pull everything back to the complex vector space V. So, if we are given a holomorphic line bundle L over T^{2n} we get the pull-back bundle $\hat{L} = \pi^* L$ over V, where

$$\hat{L} = \{(v, l) \in V \times L \,|\, \pi(v) = \mu(l)\}$$

(μ denoting the bundle projection $L \to T^{2n}$). Because every line bundle over $V \cong \mathbb{C}^n$ is trivial we have a bundle isomorphism $\Phi: \hat{L} \to V \times \mathbb{C}$. If we define Φ_v by $\Phi(v, l) = (v, \Phi_v(l))$ then Φ_v is an isomorphism of the fibre of L over $\pi(v)$ to $\mathbb{C}$. Since $\pi(v + \lambda) = \pi(v)$ we see that $\Phi_{v+\lambda}$ is another such isomorphism. It follows that

$$e_\lambda(v) := \Phi_{v+\lambda} \circ \Phi_v^{-1} \tag{5-7}$$

is an isomorphism $\mathbb{C} \to \mathbb{C}$, hence a nonzero complex number. The e_λ considered as functions on V are nowhere vanishing holomorphic functions. The collection $\{e_\lambda, \lambda \in \Lambda\}$ is called a system of multipliers for the bundle L [22, p. 308]. The multipliers obey the following conditions:

$$e_{\lambda'}(v + \lambda) \cdot e_\lambda(v) = e_{\lambda+\lambda'}(v) = e_\lambda(v + \lambda') \cdot e_{\lambda'}(v), \quad \lambda, \lambda' \in \Lambda, \quad v \in V. \tag{5-8}$$

Equivalently, such a system of multipliers defines a complex line bundle.

In the same sense all structures we need can be pulled back to V. They give functions with certain transformation properties under the action of Λ. We have the following prescriptions:

(a) complex valued functions f on T^{2n} correspond to Λ-invariant functions $\hat{f}$ on V

$$\hat{f}(v + \lambda) = \hat{f}(v), \quad \lambda \in \Lambda. \tag{5-9}$$

(b) (holomorphic) sections s of L correspond to (holomorphic) functions $\hat{s}$ on V with the transformation property

$$\hat{s}(v+\lambda) = e_\lambda(v)\cdot\hat{s}(v), \quad \lambda\in\Lambda. \tag{5-10}$$

(c) Fibre metrics h in L correspond to positive real functions $\hat{h}$ on V with

$$\hat{h}(v+\lambda) = \frac{1}{|e_\lambda(v)|^2}\cdot\hat{h}(v), \quad \lambda\in\Lambda. \tag{5-11}$$

This is necessary for $\hat{s}_1^*\cdot\hat{s}_2\cdot\hat{h}$ to be a Λ-invariant function.
(d) m-fold tensor powers of L are constructed with the m^{th} powers of the multiplicators e_λ and the metric is the m^{th} power of the fibre metric $\hat{h}$.
(e) Integration over the torus T^{2n} corresponds to integration over the full unit cell spanned by the lattice vectors $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ with respect to the volume

$$\Omega = (2\pi)^n(d_1\cdot d_2\cdots d_n)dx^1\wedge dx^2\wedge\cdots\wedge dx^{2n}, \tag{5-12}$$

where the x^k are the real coordinates (5-4).

In the following we consider only principal polarization. Starting with a symplectic form ω coming from a Kähler structure we choose another complex structure compatible with ω which has as period matrix a diagonal purely imaginary one

$$Z = \mathbf{i}\cdot\operatorname{diag}(\tau_1, \tau_2, \ldots, \tau_n), \quad \tau_k > 0, \quad 1\leqq k\leqq n. \tag{5-13}$$

As a system of multiplier we choose

$$e_{\lambda_k}(v)\equiv 1, \quad e_{\lambda_{n+k}}(v) = \exp(\pi\tau_k - 2\pi\mathbf{i}v_k), \quad 1\leqq k\leqq n. \tag{5-14}$$

By (5-8), this fixes e_λ for all lattice vectors λ. In (5-14) are the v_k the k^{th} coordinate of v with respect to the basis $e_1, e_2, \ldots, e_n$. We set $v_k = x_k + \mathbf{i}y_k$. Because we have principal polarization this is not in conflict with the above use of $x_1, \ldots, x_n$. The other x coordinates are related to the imaginary part y_k by $y_k = \tau_k x_{n+k}$.

Let L be the holomorphic line bundle defined by these multipliers. It is known [22, p. 320] that the space $\Gamma_{\text{hol}}(T^{2n}, L)$ is 1-dimensional and is generated by the Riemann Θ-function

$$\Theta(v) = \sum_{l\in\mathbb{Z}^n}\exp(\pi\mathbf{i}l^{\text{tr}}\cdot Z\cdot l + 2\pi\mathbf{i}l^{\text{tr}}\cdot v). \tag{5-15}$$

As a fibre metric h we can take [22, p. 310]

$$\hat{h}(v) = \prod_{k=1}^{n}\exp\left(-\frac{2\pi}{\tau_k}y_k^2\right) \tag{5-16}$$

and obtain the curvature

$$F = -2\pi\mathbf{i}\sum_{k=1}^{n}dx_k\wedge\frac{1}{\tau_k}dy_k. \tag{5-17}$$

It fulfills the prequantum condition (4-11) because the symplectic form ω is given by

$$\omega = 2\pi\sum_{k=1}^{n}dx_k\wedge\frac{1}{\tau_k}dy_k. \tag{5-18}$$

From this the Kähler metric and the Laplacian Δ are easily computed

$$g = 2\pi \sum_{k=1}^{n} \frac{1}{\tau_k}(dx_k \otimes dx_k + dy_k \otimes dy_k), \tag{5-19}$$

$$\Delta = \frac{1}{2\pi} \sum_{k=1}^{n} \tau_k \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right). \tag{5-20}$$

Arbitrary tensor powers L^m of L are constructed as mentioned before by the m^{th} powers of the multipliers (5-15). They have a fibre metric $h^{(m)}$ equal to the m^{th} power of (5-16). The vector space $\Gamma_{\text{hol}}(T^{2n}, L^{(m)})$ is now m^n-dimensional and can be generated by certain theta functions with characteristics [23, p. 124][10]

$$\hat{f}_a(v)^{(m)} = \sum_{l \in \mathbb{Z}^n} \exp\left(\mathbf{i}\pi m \cdot \left(l + \frac{a}{m} \right)^{\text{tr}} \cdot Z \cdot \left(l + \frac{a}{m} \right) + 2\pi \mathbf{i} m \cdot \left(l + \frac{a}{m} \right)^{\text{tr}} \cdot v \right). \tag{5-21}$$

with $a = (a_1, a_2, \ldots, a_n)^{\text{tr}}$, $a_i \in \mathbb{Z}$ with $0 \leqq a_i < m$ for $1 \leqq i \leqq n$.

The following proposition (which no doubt is known) shows that the corresponding sections $f_a^{(m)}$ of L^m are up to a global factor orthonormal.

Proposition 5.1. *In the above notations we have*

$$\langle f_a^{(m)} | f_b^{(m)} \rangle = \left(\frac{2\pi}{\sqrt{2m}} \right)^n \frac{1}{\sqrt{\tau_1 \cdots \tau_n}} \delta_{a_1, b_1} \cdots \delta_{a_n, b_n}. \tag{5-22}$$

The proof consists mainly of calculations which will be done in Appendix B.

As a consequence, the following holomorphic sections are orthonormal

$$\hat{\Theta}_a^{(m)} := \left(\frac{2\pi}{\sqrt{2m}} \right)^{-n/2} (\tau_1 \cdots \tau_n)^{1/4} \hat{f}_a^{(m)}. \tag{5-23}$$

Next, we calculate the matrix elements (4-21) for the rescaled prequantum operators $\hat{P}_f^{(m)}$ of the Fourier modes

$$F_r(v) = \exp\left(2\pi \mathbf{i} \sum_{k=1}^{n} \left(r_k x_k + \frac{1}{\tau_k} r_{k+n} y_k\right) \right), \tag{5-24}$$

where $r = (r_1, \ldots, r_{2n})$ is a $2n$-tupel of integers.

Theorem 5.1. *In the above notation we have*

$$\begin{aligned} \langle \Theta_a^{(m)} | \hat{P}_{F_r}^{(m)} | \Theta_b^{(m)} \rangle = \mathbf{i} m & \left(\prod_{l=1}^{n} \exp\left(-\frac{\pi \mathbf{i}}{m} r_l r_{l+n} \right) \right) \\ & \cdot \left(1 + \frac{\pi}{m} \sum_{k=1}^{n} \tau_k \left(r_k^2 + \frac{r_{k+n}^2}{\tau_k^2} \right) \right) \left(\prod_{s=1}^{n} \exp\left(-\frac{\pi \tau_s}{2m} \left(r_s^2 + \frac{r_{s+n}^2}{\tau_s^2} \right) \right) \right) \\ & \cdot (S^{m-r_1} T^{r_{n+1}} \otimes \cdots \otimes S^{m-r_n} T^{r_{2n}})_{ab}, \end{aligned} \tag{5-25}$$

[10] See also the use of theta functions in [27]

where S and T denote the $m \times m$ matrices

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad T = \mathrm{diag}(1, q, q^2, \ldots, q^{m-1}), \quad q = \exp\left(-\frac{2\pi \mathbf{i}}{m}\right). \tag{5-26}$$

Again the proof of this and a more general formula for the matrix elements of arbitrary functions can be found in Appendix B. Using this formula, we can compare the Poisson bracket and the matrix commutators. The poisson bracket of F_r and F_s is easily calculated to be

$$\{F_r, F_s\} = -2\pi \sum_{k=1}^{n} (r_k s_{k+n} - r_{k+n} s_k) F_{r+s}. \tag{5-27}$$

We define the matrix

$$Q_r^{(m)} = (Q_r^{(m)}{}_{ab}), \quad Q_r^{(m)}{}_{ab} := \langle \Theta_a^{(m)} | \hat{P}_{F_r}^{(m)} | \Theta_b^{(m)} \rangle, \tag{5-28}$$

and use the abbreviations

$$\varphi_m(r) = \left(1 + \frac{\pi}{m} \sum_{k=1}^{n} \tau_k \left(r_k^2 + \frac{r_{k+n}^2}{\tau_k^2}\right)\right) \cdot \left(\prod_{s=1}^{n} \exp\left(-\frac{\pi \tau_s}{2m}\left(r_s^2 + \frac{r_{s+n}^2}{\tau_s^2}\right)\right)\right). \tag{5-29}$$

The proof of the following proposition can be found in Appendix B.

Proposition 5.2. *The rescaled operators*

$$T_r^{(m)} := -\frac{1}{2\pi \varphi_m(r)} Q_r^{(m)} \tag{5-30}$$

obey the commutation relations of the well-known sine algebra (2-3) *with parameter* $\Lambda = (2m)^{-1}$ (*more precisely, the tensor product of n copies of it*), i.e.

$$[T_r^{(m)}, T_s^{(m)}] = \frac{1}{2\pi\Lambda} \sin(2\pi\Lambda(r \times s)) T_{r+s}^{(m)}, \tag{5-31}$$

where

$$r \times s := \sum_{k=1}^{n} (r_k s_{k+n} - r_{k+n} s_k). \tag{5-32}$$

We are now able to formulate the main result of this section, namely that $\mathscr{P}(T^{2n})$ is in fact a $u(m^n)$-approximation in the sense made clear in Sect. 3. Again, the proof can be found in Appendix B.

As we explained in Sect. 4 $L\,\mathrm{Ham}\,T^{2n}$ consists of the noncentral part of $\mathscr{P}(T^{2n})$ and a (vector space) complement generated by $2n$ additional vector fields. Moreover, if $n = 1$ $L\,\mathrm{Hom}\,T^2$ equals $\mathrm{diff}_A\,T^2$.

Theorem 5.2. *We assume the above notation. We put the following norm on all complex $m^n \times m^n$ matrices*

$$\|A\|_m := m^{-n/2-1} \sqrt{\mathrm{Tr}(A^+ \cdot A)}. \tag{5-33}$$

Furthermore let

$$p_m:\mathscr{P}(T^{2n})\to u(m^n),\quad f\mapsto p_m f:=(\langle\Theta_a^{(m)}|\hat{P}_f^{(m)}|\Theta_b^{(m)}\rangle). \tag{5-34}$$

Then we have $(f,g\in\mathscr{P}(T^{2n}))$

(i) p_m *is surjective.*

(ii)

$$\lim_{m\to\infty}\|p_m f\|_m=\frac{1}{(\sqrt{2\pi})^n}\sqrt{\int_{T^{2n}}\Omega f^*\cdot f}. \tag{5-35}$$

In particular, $\lim_{m\to\infty}\|p_m f\|_m=0$ *implies* $f=0$.

(iii)

$$\|p_m\{f,g\}-[p_m f,p_m g]_m\|_m\to 0\quad (m\to\infty). \tag{5-36}$$

In other words, by setting

$$d_m:u(m^n)\times u(m^n)\to\mathbb{R},\quad (A,B)\mapsto\|A-B\|_m$$

the sequence $(u(m^n),[\ldots,\ldots]_m,d_m,m\in\mathbb{N})$ *is an approximating sequence for* $(\mathscr{P}(T^{2n}),\{\ldots,\ldots\})$ *induced by the* p_m.

1. Note that assertion (ii) of Theorem 5.2 implies that the sesquilinear form on $\mathscr{P}(T^{2n})$

$$\langle f|g\rangle_m:=m^{-n-2}\operatorname{Tr}\hat{Q}_f^{(m)+}\cdot\hat{Q}_g^{(m)} \tag{5-37}$$

converges for $m\to\infty$ to the classical scalar product

$$\langle f|g\rangle:=\frac{1}{(2\pi)^n}\int_{T^{2n}}\Omega f^+\cdot g \tag{5-38}$$

on the phase space.

2. If we had defined the quantum operators $\hat{Q}_f^{(m)}$ to be equal to

$$\tilde{Q}_f^{(m)}:=\mathrm{i}m\langle\Theta_a^{(m)}|\exp\left(-\frac{1}{4m}\Delta\right)f|\Theta_b^{(m)}\rangle \tag{5-39}$$

(following a proposal of Tuynman [20]) the factors $\varphi_m(r)$ (5-29) would have been equal to 1 thus giving us the "pure" sine algebra in (5-31) up to a factor of (-2π).

3. Note that for those functions $f\not\equiv 0$ on the torus for which

$$\int_{T^{2n}} f\Omega=0 \tag{5-40}$$

holds it is in general not true that $\operatorname{Tr}\hat{Q}_f^{(m)}=0$. For instance the Fourier modes F_r with $r\in\mathbb{Z}^{2n}$ have a $\hat{Q}_f^{(m)}$ proportional to the identity matrix giving nonzero trace, see (5-25). On the other hand $\int F_r\Omega=0$ because F_r is for $r\neq 0$ orthogonal to the constant functions. In order to achieve a $su(m^n)$-approximation of $\operatorname{Ham}T^{2n}$ $(\leq\operatorname{diff}_V T^{2n})$ one has to make all quantum operators traceless

$$\check{Q}_f^{(m)}:=\hat{Q}_f^{(m)}-\frac{1}{m^n}\operatorname{Tr}\hat{Q}_f^{(m)}. \tag{5-41}$$

Since for m large enough $\check{Q}_{F_r}^{(m)}$ is equal to $\hat{Q}_{F_r}^{(m)}$ (for fixed r) the reasoning of the above theorem can be carried through (see Appendix B for some more details).

Appendix A

In this appendix we prove that the algebras $gl_+(\infty)$ (2-2), $\mathrm{diff}'_A S^2$ (2-6) and $\mathrm{diff}'_A T^2$ (2-4) are pairwise nonisomorphic. In the proofs below we can always replace $gl_+(\infty)$ by its subalgebra $sl_+(\infty)$ without changing the argument. The same is true if we replace $\mathrm{diff}'_A T^2$, respectively $\mathrm{diff}'_A S^2$ by its trivial central extension $\mathrm{diff}'_A T^2 \oplus \mathbb{C}$, respectively $\mathrm{diff}'_A S^2 \oplus \mathbb{C}$.

Proposition A.1. *$gl_+(\infty)$ is not isomorphic to* $\mathrm{diff}'_A S^2$.

Proof. Assume the existence of a Lie algebra isomorphism

$$\Phi: \{Y_{lm}\} \to \{E_{ij}\}. \tag{A-1}$$

Let Φ_{lm} denote $\Phi(Y_{lm})$. To reach a contradiction it suffices to look at the relation (coming from $\mathrm{diff}'_A S^2$)

$$[\Phi_{10}, \Phi_{lm}] = m\sqrt{\frac{3}{4\pi}}\,\Phi_{lm}, \tag{A-2}$$

$$[\Phi_{11}, \Phi_{l,-1}] = -\sqrt{\frac{3}{8\pi}}\sqrt{l(l+1)}\,\Phi_{l0}. \tag{A-3}$$

Because all Φ_{lm} are finite linear combinations of the E_{ij} Φ_{10} and Φ_{11} will be zero outside some upper left block of size $J \times J$. Now $(a_{ij}) := [\Phi_{10}, E_{kl}]$ will have vanishing entries if both indices i and j are bigger than J. Hence, this will also be true for Φ_{lm} (for $m \neq 0$ use (A-2), for $m = 0$ use (A-3)). Hence Φ cannot be surjective. ■

Proposition A.2. *$gl_+(\infty)$ is not isomorphic to* $\mathrm{diff}'_A T^2$.

Proof. Assume the existence of a Lie algebra isomorphism

$$\Phi: \{E_{ij}\} \to \{T_{\vec{m}}\}. \tag{A-4}$$

For

$$\Phi_{ij} = \Phi(E_{ij}) = \sum_{\vec{m}} c_{\vec{m}}^{ij} T_{\vec{m}} \tag{A-5}$$

we define $\vec{m}_{ij}$ as the highest double index occurring in this finite sum using lexicographical order, i.e.

$$\vec{m} > \vec{n} \Leftrightarrow m_1 > n_1 \quad \text{or} \quad (m_1 = n_1 \text{ and } m_2 > n_2).$$

Due to the relations in $\mathrm{diff}'_A T^2$, the highest index of a Lie bracket is the sum of the highest indices of the factors if the indices are not proportional to each other. Proportionality we denote by $\vec{m} \propto \vec{n}$. It is equivalent to $m_1 n_2 - m_2 n_1 = 0$. Obviously, it is an equivalence relation. Using

$$[E_{12}, E_{ij}] = 0, \quad \text{if} \quad i \neq 2 \quad \text{and} \quad j \neq 1, \tag{A-6}$$

it follows that

$$\hat{m}_{12} \propto \hat{m}_{ij}, \quad \text{if} \quad i \neq 2 \quad \text{and} \quad j \neq 1. \tag{A-7}$$

In just the same way it follows from

$$[E_{34}, E_{ij}] = 0, \quad [E_{56}, E_{ij}] = 0, \tag{A-8}$$

which holds for $i \neq 4$ and $j \neq 3$, (respectively $i \neq 6$ and $j \neq 5$) that

$$\hat{m}_{ij} \propto \hat{m}_{34}, \quad \hat{m}_{ij} \propto \hat{m}_{56} \tag{A-9}$$

with the same restriction for i and j.

Every index pair fulfills at least one of this 3 conditions. Because $m_{12} \propto m_{34} \propto m_{56}$ all $\hat{m}_{ij}$ are therefore proportional to each other.

Choose $\vec{m} \not\propto \hat{m}_{ij}$ and consider

$$T_{\vec{m}} = \sum_{ij} c_{ij} \Phi_{ij}. \tag{A-10}$$

Let c_{kl} be a nonvanishing coefficient in this finite sum. We choose indices r and p in such a way that $r \neq l, p \neq k$ and $r \neq p$ and $c_{pk} = 0$. Using the commutator relations of the E_{ij} we calculate

$$[[T_{\vec{m}}, \Phi_{lp}], \Phi_{rk}] = -c_{kl} \Phi_{pr}. \tag{A-11}$$

We get

$$((\vec{m} + \hat{m}_{lp}) + \hat{m}_{rk}) \propto \hat{m}_{pr}. \tag{A-12}$$

This implies $\vec{m} \propto \hat{m}_{ij}$, which is in contradiction with the assumption. ■

Proposition A.3. $\mathrm{diff}'_A S^2$ *is not isomorphic to* $\mathrm{diff}'_A T^2$.

Proof. Assume the existence of a Lie algebra isomorphism

$$\Phi : \{Y_{lm}\} \to \{T_{\vec{m}}\}. \tag{A-13}$$

Using that for the structure constants of $\mathrm{diff}'_A S^2$ (2-6)

$$g_{lm,l'm'}^{l''m''} \neq 0 \quad \text{only if} \quad m'' = m + m', \quad |l - l'| \leqq l'' \leqq l + l' - 1 \tag{A-14}$$

we see that the adjoint action of Y_{11} given by ad $Y_{11} := [Y_{11}, ..]$ is locally nilpotent (i.e. for each finite linear combination A of Y_{lm}'s there exists an integer n such that $(\mathrm{ad}\, Y_{11})^n(A) = 0$). Clearly, $\Phi_{11} = \Phi(Y_{11})$ has to share this property. Now (using the notation introduced above) choose $\vec{m} \not\propto \hat{m}_{11}$. Using the additivity of the highest index (if not proportional) we see that

$$(\mathrm{ad}\, \Phi_{11})^n (T_{\vec{m}}) \neq 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{A-15}$$

This contradiction completes the proof. ■

Note however, that the above does not show that there is no embedding of $\mathrm{diff}'_A S^2$ or $\mathrm{diff}'_A T^2$ in the following algebras:

$$\overline{gl}(\infty) = \{(a_{ij})_{i,j \in \mathbb{Z}} \mid \text{there is an } r \text{ such that } a_{ij} = 0 \text{ if } |i - j| > r\},$$
$$\overline{gl_+}(\infty) = \{(a_{ij})_{i,j \in \mathbb{N}} \mid \text{there is an } r \text{ such that } a_{ij} = 0 \text{ if } |i - j| > r\}.$$

In this context it is interesting to note that Floratos [24][11] was able to show that

[11] Please note also several other contribution to the subject by this author [25–28]

L_Λ for $\Lambda \neq 0$ can be embedded into $\overline{gl}(\infty)$. The question whether this is true also for $L_0 = \mathrm{diff}'_A\, T^2$ remains still open.

Appendix B

In this appendix we supply the proofs of some claims in Sect. 5. We start with

Proof of Proposition 5.1. (Orthogonality of the theta functions with characteristics.) Since the volume ω^n, the fibre metric $\hat{h}^{(m)}$ and each section $f_a^{(m)}$ factorizes in n terms each depending on the coordinates of a 2-torus only one gets

$$\langle f_a^{(m)} | f_b^{(m)} \rangle = \prod_i^n \langle f_{a_i}^{(m)} | f_{b_i}^{(m)} \rangle$$

(the $f_{a_i}^{(m)}$ denote the obvious factors in (5-20)). We calculate a 2-torus integral: (using $a = a_k, b = b_k, \tau = \tau_k, x = x_k, y = y_k$)

$$\begin{aligned}\langle f_a^{(m)} | f_b^{(m)} \rangle &= \int_0^1 dx \int_0^\tau \frac{2\pi}{\tau} dy \exp\left(-\frac{2\pi m}{\tau} y^2 \right) \\ &\cdot \sum_{l,k \in \mathbb{Z}} \exp\left(-\pi m \tau \left(\left(l + \frac{a}{m} \right)^2 + \left(k + \frac{b}{m} \right)^2 \right) \right) \\ &\cdot \exp\left(-2\pi m y \left(l + k + \frac{a+b}{m} \right) \right) \exp\left(2\pi \mathrm{i} m x \left(k - l + \frac{b-a}{m} \right) \right). \qquad \text{(B-1)}\end{aligned}$$

The x integral gives a factor $\delta_{0,m(k-l)+b-a}$ which is equal to $\delta_{l,k}\delta_{a,b}$ because $b-a$ is divisible by m if and only if $b = a$. It follows that (B-1) is equal to

$$\begin{aligned}\delta_{a,b} \cdot \frac{2\pi}{\tau} \sum_{l \in \mathbb{Z}} \int_0^\tau dy \exp\left(-\frac{2\pi m}{\tau} \left(y + \tau \left(l + \frac{a}{m} \right) \right)^2 \right) &= \delta_{a,b} \frac{2\pi}{\tau} \int_{-\infty}^{\infty} dt \exp\left(-\frac{2\pi m}{\tau} t^2 \right) \\ &= \frac{2\pi}{\sqrt{2m\tau}} \delta_{a,b}.\end{aligned}$$

The multidimensional result is the product of n such terms. ■

In order to prove Theorem 5.1 we shall first give a general formula to calculate quantum operators for an arbitrary function f on the torus.

Lemma B.1. *In the notation of Sect. 5 we have*

$$\begin{aligned}\langle \Theta_a^{(m)} | \hat{P}_f^{(m)} | \Theta_b^{(m)} \rangle &= \mathrm{i} m \frac{(2m)^{n/2}}{\sqrt{\tau_1 \cdots \tau_k}} \sum_{l \in \mathbb{Z}^n} \int_0^1 dx_1 \cdots \int_0^1 dx_n \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \\ &\cdot \prod_{k=1}^n \exp\left(-\pi m \left(\frac{1}{\tau_k} \left(y_k + \tau_k \left(l_k + \frac{a_k}{m} \right) \right)^2 \right.\right. \\ &\left.\left. + \frac{1}{\tau_k} \left(y_k + \tau_k \frac{b_k}{m} \right)^2 + 2\mathrm{i} \left(l_k + \frac{a_k - b_k}{m} \right) x_k \right) \right) \left(1 - \frac{1}{2m} \Delta \right) \hat{f}(v).\end{aligned}$$

Proof.

$$\langle \Theta_a^{(m)} | \hat{P}_f^{(m)} | \Theta_b^{(m)} \rangle = \int_0^1 dx_1 \cdots \int_0^1 dx_n \int_0^{\tau_1} dy_1 \cdots \int_0^{\tau_n} dy_n \frac{(2\pi)^n}{\tau_1 \cdots \tau_n}$$

$$\cdot \exp\left(-2\pi m \left(\frac{y_1^2}{\tau_1} + \cdots + \frac{y_n^2}{\tau_n} \right) \right) \cdot \left(\frac{2\pi}{\sqrt{2m}} \right)^{-n} \sqrt{\tau_1 \cdots \tau_k} \sum_{l,l' \in \mathbb{Z}} \prod_{k=1}^{n}$$

$$\cdot \exp\left(-\pi m \left(\tau_k \left(\left(l_k + \frac{a_k}{m} \right)^2 + \left(l'_k + \frac{b_k}{m} \right)^2 \right) + 2\left(l_k + \frac{a_k}{m} + l'_k + \frac{b_k}{m} \right) y_k \right.\right.$$

$$\left.\left. + 2\mathbf{i} \left(l_k + \frac{a_k}{m} - l'_k - \frac{b_k}{m} \right) x_k \right) \right) \cdot \mathbf{i} m \left(1 - \frac{1}{2m} \Delta \right) \hat{f}(v)$$

$$= \mathbf{i} m \frac{(2m)^{n/2}}{\sqrt{\tau_1 \cdots \tau_k}} \sum_{l,l' \in \mathbb{Z}} \int_0^1 dx_1 \cdots \int_0^1 dx_n \int_0^{\tau_1} dy_1 \cdots \int_0^{\tau_n} dy_n$$

$$\cdot \prod_{k=1}^{n} \exp\left(-\pi m \left(\frac{1}{\tau_k} \left(y_k + \tau_k \left(l_k + \frac{a_k}{m} \right) \right)^2 + \frac{1}{\tau_k} \left(y_k + \tau_k \left(l'_k + \frac{b_k}{m} \right) \right)^2 \right) \right)$$

$$\cdot \left(\prod_{k=1}^{n} \exp\left(-2\pi \mathbf{i} m \left(l_k - l'_k + \frac{a_k - b_k}{m} \right) x_k \right) \right) \left(1 - \frac{1}{2m} \Delta \right) \hat{f}(v).$$

Making the substitution $\bar{y}_k := y_k + \tau_k l'_k$ and $\bar{l} := l_k - l'_k$ and using the Λ-invariance of $\hat{f}$ and Δ we get

$$\frac{\mathbf{i} m (2m)^{n/2}}{\sqrt{\tau_1 \cdots \tau_k}} \sum_{\bar{l} \in \mathbb{Z}^n} \int_0^1 dx_1 \cdots \int_0^1 dx_n \sum_{l' \in \mathbb{Z}^n} \int_{\tau_1 l'_1}^{\tau_1 (l'_1 + 1)} d\bar{y}_1 \cdots \int_{\tau_n l'_n}^{\tau_n (l'_n + 1)} d\bar{y}_n$$

$$\cdot \prod_{k=1}^{n} \exp\left(-\pi m \frac{1}{\tau_k} \left(\left(\bar{y}_k + \tau_k \left(\bar{l}_k + \frac{a_k}{m} \right) \right)^2 + \left(\bar{y}_k + \tau_k \frac{b_k}{m} \right)^2 \right) \right)$$

$$\cdot \left(\prod_{k=1}^{n} \exp\left(-2\pi \mathbf{i} m \left(\bar{l}_k + \frac{a_k - b_k}{m} \right) x_k \right) \right) \left(1 - \frac{1}{2m} \Delta \right) \hat{f}(v).$$

Now the sum over l' plus the $\bar{y}$-integration give $\bar{y}$-integrals from $-\infty$ to ∞ and the lemma is proved. ■

Proof of Theorem 5.1. If we use as f in the preceding lemma the Fourier mode

$$\exp\left(2\pi \mathbf{i} \sum_{k=1}^{n} \left(r_k x_k + \frac{1}{\tau_k} r_{k+n} y_k \right) \right),$$

then the operator $\left(1 - \frac{1}{2m} \Delta \right)$ produces the factor

$$1 + \frac{\pi}{m} \sum_{k=1}^{n} \tau_k \left(r_k^2 + \frac{r_{k+n}^2}{\tau_k^2} \right)$$

(see (5-20)). The x-integration can be performed yielding factors of

$$\delta_{l_1 m, b_1 - a_1 + r_1} \cdots \delta_{l_n m, b_n - a_n + r_n}.$$

Hence there remain terms in the l summation (i.e. just one term) if and only if $m|(b_k - a_k + r_k)$ for all k. In this case we can replace $l_k + \frac{a_k}{m}$ by $\frac{b_k + r_k}{m}$ and substitute the l summation by the factors

$$\delta_{a_1 - b_1, r_1 \bmod m} \cdots \delta_{a_n - b_n, r_n \bmod m}. \tag{B-2}$$

These are precisely the matrix elements of

$$S^{m-r_1} \otimes \cdots \otimes S^{m-r_n}. \tag{B-3}$$

After substituting

$$\bar{y}_k := y_k + \frac{(2b_k + r_k)\tau_k - \mathbf{i} r_{n+k}}{2m},$$

we can perform the Gaussian y-integrations and get the factors

$$\frac{\sqrt{\tau_1 \cdots \tau_k}}{(2m)^{n/2}} \prod_{k=1}^{n} \exp\left(-\frac{\pi \tau_k}{2m}\left(r_k^2 + \frac{r_{k+n}^2}{\tau_k} \right)\right) \cdot \exp\left(-\frac{2\pi \mathbf{i}}{m}\left(r_{k+n} b_k + \frac{r_k \tau_{k+n}}{2} \right)\right).$$

Here the factors

$$\prod_{k=1}^{n} \exp\left(-\frac{2\pi \mathbf{i}}{m} r_{k+n} b_k \right) \tag{B-4}$$

constitute the (diagonal) matrix elements of

$$T^{r_{n+1}} \otimes \cdots \otimes T^{r_{2n}}, \tag{B-5}$$

and the theorem is proved. ∎

Proof of Proposition 5.2. We calculate

$$Q_r^{(m)} Q_s^{(m)} = -m^2 \varphi_m(r) \varphi_m(s) \prod_{l=1}^{n} \exp\left(-\frac{\pi \mathbf{i}}{m}(r_l r_{l+n} + s_l s_{l+n}) \right)$$
$$\cdot S^{m-r_1} T^{r_{n+1}} S^{m-s_1} T^{s_{n+1}} \otimes \cdots \otimes S^{m-r_n} T^{r_{2n}} S^{m-s_n} T^{s_{2n}}.$$

Because of

$$TS = q^{-1} ST, \quad q^{-1} = \exp\left(\frac{2\pi i}{m} \right),$$

this is equal to

$$-m^2 \varphi_m(r) \varphi_m(s) \exp\left(-\frac{2\pi \mathbf{i}}{m} \sum_{l=1}^{n} \frac{r_l r_{l+n} + s_l s_{l+n}}{2} \right) q^{-((m-s_1)r_{n+1} + \cdots + (m-s_n)r_{2n})}$$
$$\cdot S^{2m-(r_1+s_1)} T^{r_{n+1}+s_{n+1}} \otimes \cdots \otimes S^{2m-(r_n+s_n)} T^{r_{2n}+s_{2n}}.$$

Since $q^m = 1$ and $S^m = 1$ this equals

$$\mathbf{i} m \frac{\varphi_m(r) \varphi_m(s)}{\varphi_m(r+s)} \exp\left(-\frac{2\pi \mathbf{i}}{m} \sum_{l=1}^{n} \frac{s_l r_{n+l} - r_l s_{n+l}}{2} \right) \cdot Q_{r+s}^{(m)}.$$

Hence

$$[Q_r^{(m)}, Q_s^{(m)}] = -2\pi \frac{\varphi_m(r) \varphi_m(s)}{\varphi_m(r+s)} \frac{m}{\pi} \sin\left(\frac{\pi}{m} \sum_{l=1}^{n} (r_l s_{n+l} - r_{n+l} s_l) \right) \cdot Q_{r+s}^{(m)}. \quad ∎$$

Proof of Theorem 5.2. (i) Since $p_m F_r = Q_r^{(m)}$ and by using Eq. (5-25), it suffices to show that the m^2 matrices $S^k T^l$ generate all complex $m \times m$-matrices. Indeed, take an arbitrary polynomial $\sum_{a=0}^{m-1} \alpha_a T^a$. It is a diagonal matrix with $\sum_{a=0}^{m-1} \alpha_a q^{ba}$ as the b^{th} diagonal element. Now the matrix $(q^{ba})_{a,b=0,\ldots,n-1}$ is non-singular (its determinant is a Vandermonde determinant). Hence the linear equation $\sum_{a=0}^{n-1} \alpha_a q^{ba} = \delta_{b,b_0}$ is solvable for all $0 \leqq b_0 < m$ and we get all diagonal matrices $E_{b_0 b_0}$. But since $S^k E_{b_0 b_0} = E_{b_0 - k, b_0}$ (where the indices should always be reduced mod m) we can thereby generate all $m \times m$-matrices. Now, taking antihermitian or hermitian part of $p_m f$ is equivalent to taking real or imaginary part of f. Hence, for real f, p_m is surjective on $u(m^n)$.

(ii) We need the following little lemma

Lemma B.2.

$$\operatorname{Tr} Q_r^{(m)*} Q_s^{(m)} = \begin{cases} 0, & r \not\equiv s \bmod m\mathbb{Z}^{2n} \\ m^{n+2} \varphi_m(r) \varphi_m(s) \varepsilon_m(r,s), & r \equiv s \bmod m\mathbb{Z}^{2n}, \end{cases}$$

where $\varepsilon_m(r,s)$ takes values $+1$ or -1 and is equal to 1 for $r = s$.

Proof of the Lemma. Since

$$\operatorname{Tr}(A_1 \otimes \cdots \otimes A_n) = \operatorname{Tr} A_1 \cdots \operatorname{Tr} A_n,$$

it suffices to calculate $\operatorname{Tr} S^k T^l$. Now $S^m = 1 = T^m$ and clearly $\operatorname{Tr} S^k T^l = 0$ for $m \nmid k$ because then $S^k T^l$ has zeros on the diagonal. If $m | k$ then $S^k = 1$ and

$$\operatorname{Tr} S^k T^l = \operatorname{Tr} T^l = \begin{cases} m, & \text{if} \quad m | l \\ \dfrac{1 - q^{lm}}{1 - q^l} = 0, & \text{if} \quad m \nmid l. \end{cases}$$

Hence

$$\operatorname{Tr} S^k T^l = \begin{cases} m, & \text{if} \quad (k,l) \equiv (0,0) \bmod m\mathbb{Z}^2 \\ 0, & \text{if} \quad (k,l) \not\equiv (0,0) \bmod m\mathbb{Z}^2. \end{cases}$$

Now

$$\operatorname{Tr} Q_r^* Q_s = m^2 \prod_{l=1}^{n} \exp\left(\frac{\pi \mathrm{i}}{m} (r_l r_{l+n} - s_l s_{l+n}) \right) \cdot \varphi_m(r) \varphi_m(s)$$
$$\cdot \prod_{k=1}^{n} \operatorname{Tr}((T^*)^{r_{k+n}} (S^*)^{m - r_k} S^{m - s_k} T^{s_{k+n}}).$$

Since S and T are unitary matrices we get the result using the cyclic property of the trace. ■

In particular, we get the formula

$$\| Q_r^{(m)} \|_m = \varphi_m(r). \tag{B-6}$$

Now we take an arbitrary $f \in \mathscr{P}(T^{2n})$ and expand it in a Fourier series

$$f = \sum_{r \in \mathbb{Z}^{2n}} \lambda_r F_r. \tag{B-7}$$

Because $\varphi_m(r)$ goes very fast to zero for increasing $r\in\mathbb{Z}^{2n}$ we have

$$\begin{aligned}\|p_m f\|_m^2 &= \left\|\sum_{r\in\mathbb{Z}^{2n}} \lambda_r Q_r^{(m)}\right\|^2 = m^{-n-2}\sum_{r,s\in\mathbb{Z}^{2n}} \lambda_r^*\lambda_s \operatorname{Tr}(Q_r^{(m)*}Q_s^{(m)})\\ &= \sum_{r,k\in\mathbb{Z}^{2n}} \lambda_r^*\lambda_{r+mk}\varphi_m(r)\varphi_m(r+mk)\varepsilon_m(r,r+mk)\\ &= \sum_{r\in\mathbb{Z}^{2n}} |\lambda_r|^2\varphi_m(r)^2 + \sum_{r,k\in\mathbb{Z}^{2n},k\neq 0} \lambda_r^*\lambda_{r+mk}\varphi_m(r)\varphi_m(r+mk)\varepsilon_m(r,r+mk). \qquad \text{(B-8)}\end{aligned}$$

In the limit $m\to\infty$ we may take the limit inside the sum because it converges uniformly. Since

$$\lim_{m\to\infty}\varphi_m(r)=1, \quad \lim_{m\to\infty}\lambda_{r+mk}=0,\ (k\neq 0), \quad \lim_{m\to\infty}\varphi_m(r+mk)=0$$

by Eq. (5.29), respectively the pointwise convergency of the Fourier series (B-7) we get

$$\lim_{m\to\infty}\|p_m f\|_m = \sqrt{\sum_{r\in\mathbb{Z}^{2n}}|\lambda_r|^2}.$$

On the other hand, since the Fourier modes are orthogonal

$$\int_{T^{2n}} \Omega F_r^* F_s = (2\pi)^n\delta_{r,s}, \quad r,s\in\mathbb{Z}^{2n},$$

we get the result (5-35). In particular, since f^*f is a nonnegative function, the zero sequence criterion is an obvious consequence.

(iii) Taking $f=F_r, g=F_s$ we get the equation

$$\|p_m\{F_r,F_s\}-[p_mF_r,p_mF_s]\|_m = \|p_m\{F_r,F_s\}-[Q_r^{(m)},Q_s^{(m)}]\|_m.$$

By (5-27) and Prop. 5.2 this is equal to

$$\begin{aligned}&\left\|-2\pi\left(\sum_{k=1}^{n}(r_ks_{k+n}-r_{k+n}s_k)-\frac{\varphi_m(r)\varphi_m(s)}{\varphi_m(r+s)}\frac{m}{\pi}\sin\left(\frac{\pi}{m}\sum_{k=1}^{n}(r_ks_{k+n}-r_{k+n}s_k)\right)\right)Q_{r+s}^{(m)}\right\|_m\\ &= 2\pi\left|(r\times s)-\frac{\varphi_m(r)\varphi_m(s)}{\varphi_m(r+s)}\frac{m}{\pi}\sin\left(\frac{\pi}{m}(r\times s)\right)\right|\varphi_m(r+s)\end{aligned}$$

(see (B-6)). Now, because

$$\lim_{\varepsilon\to\infty}\frac{\sin(\varepsilon(r\times s))}{\varepsilon}=r\times s \quad\text{and}\quad \lim_{m\to\infty}\varphi_m(t)=1,$$

the result follows for the Fourier modes and extends to all functions in $\mathscr{P}(T^{2n})$ by linearity and Fourier expansion. ■

Sketch of the proof of the $su(m^n)$-approximation for Ham T^{2n}: The commutator relations (the analog of Prop. 5.2) are valid because the multiples of the identity vanishes on the left-hand side of (5-31) whereas the sin-factor on the right-hand side equals zero for $(r+s)\in m\cdot\mathbb{Z}^{2n}$. In the analog of Lemma B.2 the factor $\varepsilon_m(r,s)$ is to be replaced by

$$\varepsilon_m(r,s)-\delta_{0,r\,\mathrm{mod}\,m}\cdot\delta_{0,s\,\mathrm{mod}\,m}.$$

Here $\delta_{a,b\,\mathrm{mod}\,m}$ denotes the obvious generalization of the Kronecker δ for $a, b \in \mathbb{Z}^{2n}$ On the right-hand side of Eq. (B-6) an additional factor of $\sqrt{1-\delta_{0,r\,\mathrm{mod}\,m}}$ appears. Finally Eq. (B-8) is modified by the factor $1\text{-}\delta_{0,r\,\mathrm{mod}\,m}$ in the first sum and by replacing $\varepsilon_m(r, r+mk)$ by

$$\varepsilon_m(r, r+mk) - \delta_{0,r\,\mathrm{mod}\,m}$$

in the second sum. Hence, in the limit $m \to \infty$ the second sum vanishes by the same argument as used above. The first sum may be broken up into a sum up to $r=(m,\ldots,m)$ and a remaining summand. This shows that in the limit $m \to \infty$ this will also converge to $\sum_r |\lambda_r|^2$. ■

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