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Braid Group Action and Quantum Affine Algebras

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Abstract: We lift the lattice of translations in the extended affine Weyl group to a braid group action on the quantum affine algebra. This action fixes the Heisenberg subalgebra pointwise. Loop-like generators of the algebra are obtained which satisfy the relations of Drinfel'd's new realization. Coproduct formulas are given and a PBW type basis is constructed.

0. Introduction

The purpose of this paper is to establish explicitly the isomorphism between the quantum enveloping algebra $U_q(\hat{\mathfrak{g}})$ of Drinfel'd and Jimbo ($\hat{\mathfrak{g}}$ an untwisted affine Kac–Moody algebra) and the "new realization" [D2] of Drinfel'd. This is done using the braid group action defined on $U_q(\hat{\mathfrak{g}})$ by Lusztig. In particular, we consider a group of operators $\mathscr{P}$ arising from the lattice of translations in the extended affine Weyl group.

Drinfel'd found that the study of finite dimensional representations of $U_q(\hat{\mathfrak{g}})$ is made easier by the use of a "new realization" on a set of loop algebra-like generators over $\mathbb{C}[[h]]$. He gives (the proof is unpublished) an isomorphism to the usual presentation, although from his methods there is no explicit correspondence between the two sets of generators. Here we find the new Drinfel'd generators in $U_q(\hat{\mathfrak{g}})$ and prove a version of [D2] which sits inside the Lusztig form over $\mathbb{Q}[q, q^{-1}]$. We also give formulas for the coproduct of the Drinfel'd generators.

The method is to show that $U_q(\hat{\mathfrak{g}})$ contains n (= rank $\mathfrak{g}$) "vertex" subalgebras U_i, each isomorphic to $U_q(\widehat{\mathfrak{sl}_2})$. Applying work of Damiani [Da], it follows that $U_q(\hat{\mathfrak{g}})$ contains a Heisenberg subalgebra which is pointwise fixed by the group of translations $\mathscr{P}$. This subalgebra contains the purely imaginary Drinfel'd generators. We find the remaining generators as $\mathscr{P}$ translations of the usual Drinfel'd–Jimbo generators.

Having found expressions for imaginary root vectors in the usual presentation of $U_q(\hat{\mathfrak{g}})$, it is a straightforward application to define a basis of Poincaré–Birkhoff–Witt type (with the method of [L5]).

1. Notation

1.1 We review the following standard notation (see [K]). Let (a_{ij}), $i,j \in I=\{0,\dots,n\}$ be the $(n+1)\times(n+1)$ Cartan matrix of $\hat{\mathfrak{g}}$ so that (a_{ij}), $1\leqq i,j\leqq n$ is the Cartan matrix of the simple Lie algebra $\mathfrak{g}$. Let d_i be relatively prime positive integers such that $(d_i a_{ij})$ is a symmetric matrix. Let $P^\vee$ be a lattice over $\mathbb{Z}$ with basis $\omega_i^\vee$, $1\leqq i\leqq n$. Let $\alpha_j^\vee=\sum_{i=1}^n a_{ji}\omega_i^\vee$, $1\leqq j\leqq n$ and let $Q^\vee=\sum_i \mathbb{Z}\alpha_i^\vee \subset P^\vee$. Then $P^\vee, Q^\vee$ are called respectively the coweight and coroot lattices of $\mathfrak{g}$. Let $Q_+^\vee=\sum_i \mathbb{Z}_+\alpha_i^\vee$, $P_+^\vee=\sum_i \mathbb{Z}_+\omega_i^\vee$.

Define the root lattice $Q=\mathrm{Hom}(P^\vee,\mathbb{Z})$ with basis given by α_i such that $\langle\alpha_i,\omega_j^\vee\rangle=\delta_{ij}$. For $1\leqq i\leqq n$ define the reflection s_i acting on $P^\vee$ by $s_i(x)=x-\langle\alpha_i,x\rangle\alpha_i^\vee$. Additionally, s_i acts on Q by $s_i(y)=y-\langle y,\alpha_i^\vee\rangle\alpha_i$ for $y\in Q$. Let W_0 be the subgroup of $\mathrm{Aut}(P^\vee)$ generated by $s_1,\dots,s_n$. Let $\Pi=\{\alpha_1,\alpha_2,\dots,\alpha_n\}$, $\Pi^\vee=\{\alpha_1^\vee,\alpha_2^\vee,\dots,\alpha_n^\vee\}$. Define the root system (resp. coroot system) $R=W_0\Pi$ (resp. $R^\vee=W_0\Pi^\vee$), then the correspondence $\alpha_i\leftrightarrow\alpha_i^\vee$ extends to $R\leftrightarrow R^\vee$ and for $\alpha\in R$, $\langle\alpha,\alpha^\vee\rangle=2$.

1.2 Using the W_0 action on $P^\vee$ define $W=W_0 \ltimes P^\vee$, where the product is given by $(s,x)(s',y)=(ss',s'^{-1}(x)+y)$. $P^\vee$ is characterized as the subgroup of W consisting of elements with finitely many conjugates. For $s\in W_0$ write s for $(s,0)$. Similarly for $x\in P^\vee$ write x for $(1,x)$.

Let θ be the highest root of R. Then writing s_0 for $(s_\theta,\theta^\vee)$, the set $\{s_0,\dots,s_n\}$ generates a normal Coxeter subgroup $\tilde{W}$ of W with defining relations determined by (a_{ij}). $\mathcal{T}=W/\tilde{W}$ is a finite group in correspondence with a certain subgroup of diagram automorphisms of the Dynkin diagram of $\hat{\mathfrak{g}}$ (see [B]). $\mathcal{T}$ acts on $\tilde{W}$ by $\tau s_i\tau^{-1}=s_{\tau(i)}$, for $\tau\in\mathcal{T}$, $0\leqq i\leqq n$. Forming $\mathcal{T}\ltimes\tilde{W}$ we have $\mathcal{T}\ltimes\tilde{W}\cong W$. The length function of $\tilde{W}$ extends to W by setting $l_W(\tau w)=l_{\tilde{W}}(w)$, for $\tau\in\mathcal{T}$, $w\in\tilde{W}$. The semigroup $P_+^\vee$ has the properties:

$$l(s_i x)=l(x)+1,\quad 1\leqq i\leqq n\,,$$

$$l(xy)=l(x)+l(y),\ x,y\in P_+^\vee\,.$$

Extend Q to the affine root lattice $\tilde{Q}=\mathbb{Z}\alpha_0\oplus Q$ and set $\delta=\alpha_0+\theta$. Then W acts as an affine transformation group on $\tilde{Q}$. In particular, for $x\in P^\vee$, $1\leqq j\leqq n$, $x(\alpha_j)=\alpha_j-\langle\alpha_j,x\rangle\delta$. Introduce the symmetric bilinear form $(\cdot|\cdot)\colon\tilde{Q}\times\tilde{Q}\to\mathbb{Z}$ determined by $(\alpha_i|\alpha_j)=d_i a_{ij}$.

Let $q_i=q^{d_i}$. Introduce the q-integer notation in $\mathbb{C}(q)$ by:

$$[n]_i=\frac{q_i^n-q_i^{-n}}{q_i-q_i^{-1}},\quad [n]_i!=\prod_{k=1}^{n}[k]_i\,.$$

1.3 One defines the *quantum affine algebra* $U_q(\hat{\mathfrak{g}})$ $(=U_q)$ of Drinfeld and Jimbo as an algebra over $\mathbb{C}(q)$ on generators E_i, F_i $(i\in I)$, K_α $(\alpha\in\tilde{Q})$, $C^{\pm1/2}$, $D^{\pm1}$ subject to the following relations:

$$[K_\alpha,K_\beta]=[K_\alpha,D]=0,\quad K_\alpha K_\beta=K_{\alpha+\beta},\quad K_0=1\,,$$

$$C^{\pm1/2}\text{ is central, }(C^{\pm1/2})^2=K_\delta^{\pm1}\,,$$

$$K_\alpha E_j K_\alpha^{-1}=q^{(\alpha|\alpha_j)}E_j,\quad DE_jD^{-1}=q^{\delta_{0j}}E_j\,,$$

$$K_\alpha F_j K_\alpha^{-1}=q^{-(\alpha|\alpha_j)}F_j,\quad DF_jD^{-1}=q^{-\delta_{0j}}F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)} E_j E_i^{(s)} = 0, \quad \sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)} F_j F_i^{(s)} = 0 .$$

Here $K_i = K_{\alpha_i}$ and $E_i^{(s)} = E_i/[s]_i!$. We have added the square root of the canonical central element K_δ for later notational convenience.

Introduce the $\mathbb{C}$-algebra automorphism Φ, and anti-automorphism Ω of U_q, defined by:

$$\Phi(E_i) = F_i, \quad \Phi(F_i) = E_i, \quad \Phi(K_\alpha) = K_\alpha, \quad \Phi(D) = D, \quad \Phi(q) = q^{-1},$$

$$\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_\alpha) = K_{-\alpha}, \quad \Omega(D) = D^{-1}, \quad \Omega(q) = q^{-1}.$$

As usual let U_q^+ (resp. U_q^-) denote the span of monomials in E_i (resp. F_i) and T the span of monomials in K_α, $C^{\pm 1/2}$ and $D^{\pm 1}$. Then $U_q = U_q^- \otimes T \otimes U_q^+$ [L, Ro]. U_q^+ is graded by $\tilde{Q}_+$ in the usual way and $U_q^+ = \oplus_\nu (U_q^+)_\nu$, where $\nu \in \tilde{Q}_+$. An element $x \in U_q^+$ is called homogeneous if $x \in (U_q^+)_\nu$ for some ν. In this case let [cf. L 1.1.1] $|x| = \nu$. Note that $|0| = \nu$ for all ν. For $i \in I$ introduce the twisted derivations r_i, ${}_ir$ of U_q^+ [cf. L 1.2.13] defined uniquely with the properties: $r_i(1) = {}_ir(1) = 0$, $r_i(E_j) = {}_ir(E_j) = \delta_{ij}$, and ${}_ir(xy) = {}_ir(x)y + q^{(|x|, \alpha_i)} x\, {}_ir(y)$, $r_i(xy) = q^{(|y|, \alpha_i)} r_i(x) y + x r_i(y)$ for x, y homogeneous.

The Braid group $\mathscr{B}$ associated to W is the group on generators T_w ($w \in W$) with the relation $T_w T_{w'} = T_{ww'}$ if $l(w) + l(w') = l(ww')$. A reduced presentation of $w \in W$ is an expression $w = \tau s_{i_1} \ldots s_{i_n}$, where $l(w) = n$, $\tau \in \mathscr{T}$.

Recall that the braid group associated to $\tilde{W}$, whose canonical generators one denotes by $T_i = T_{s_i}$, $i \in I$, acts as a group of automorphisms of the algebra U_q ([L]):

$$T_i E_i = -F_i K_i, \quad T_i E_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \quad \text{if } i \neq j,$$

$$T_i F_i = -K_i^{-1} E_i, \quad T_i F_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \quad \text{if } i \neq j,$$

$$T_i K_\beta = K_{s_i \beta}, \quad \beta \in \tilde{Q}, \quad T_i(D) = D K_i^{-\delta_{i0}}.$$

Then $\Omega T_i = T_i \Omega$, and $\Phi T_i = T_i^{-1} \Phi$. We extend this action to W by defining T_τ by $T_\tau(E_i) = E_{\tau(i)}$, $T_\tau(F_i) = F_{\tau(i)}$, $T_\tau(K_i) = K_{\tau(i)}$. Write τ for T_τ. Denote by $\mathscr{P}$ the group generated by the operators $T_{\omega_i^\vee}$ ($1 \leqq i \leqq n$) and their inverses. From now on, for notational convenience refer to $\omega_i^\vee$ by ω_i.

2. Some Background Material

2.1 We review the following method (cf. [DC–K, L4, L5]) of recovering the usual affine algebra through specialization at 1. Let $\mathscr{A}$ be the ring $\mathbb{C}[q, q^{-1}]$ localized at $(q-1)$. Let $U_{\mathscr{A}}$ be the $\mathscr{A}$ subalgebra of U_q generated by the elements E_i, F_i, $K_i^{\pm 1}$, $D^{\pm 1}$, $C^{\pm 1/2}$, and:

$$H_i = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad c = \frac{C - C^{-1}}{q - q^{-1}}, \quad d = \frac{D - D^{-1}}{q - q^{-1}}.$$

$U_{\mathscr{A}}$ includes the elements: $[K_i; d_i n] = H_i q_i^{-n} + K_i[n]_i$ and $[D; n] = dq^{-n} + D[n]$. Note the identities:

$$E_j H_i = [K_i; -d_i a_{ij}] E_j, \quad F_j H_i = [K_i; d_i a_{ij}] F_j$$

$$E_0 d = [D; -1] E_0, \quad F_0 d = [D; 1] F_0 .$$

Let $(q-1)U_{\mathscr{A}}$ be the left ideal generated by $(q-1)$ in $U_{\mathscr{A}}$. Define the algebra $\hat{U}_1$, the specialization of U_q at 1, by $\hat{U}_1 = U_{\mathscr{A}}/(q-1)U_{\mathscr{A}}$. We obtain the following:

Proposition [cf. DC-K 1.5]. *$\hat{U}_1$ is an associative algebra over $\mathbb{C}$ on the above generators with relations*:

$$c \text{ is central}$$

$$[E_i, F_j] = \delta_{ij} H_i, \quad [H_i, E_j] = a_{ij} K_i E_j, \quad [H_i, F_j] = -a_{ij} K_i F_j ,$$

$$[d, E_j] = \delta_{j0} D E_j, \quad [d, F_j] = \delta_{j0} D F_j, \quad K_i^2 = 1, \quad D^2 = 1, \quad C^2 = 1 ,$$

$$\mathrm{ad}^{(1-a_{ij})} E_i(E_j) = 0, \quad \mathrm{ad}^{(1-a_{ij})} F_i(F_j) = 0, \quad i \neq j .$$

In particular, $U_1 = \hat{U}_1/(K_i - 1, D - 1, C^{1/2} - 1)$ is isomorphic to the universal enveloping algebra of the affine Kac–Moody Lie algebra.

The following is due to Iwahori, Matsumoto and Tits.

Proposition. *Let $w \in W$ and let $\tau s_{i_1} s_{i_2} \ldots s_{i_n}$ be a reduced expression of w. Then the automorphism $T_w = \tau T_{i_1} T_{i_2} \ldots T_{i_n}$ of U_q depends only w and not on the reduced expression chosen. In particular, one reduced expression can be transformed to another by a finite sequence of braid relations.*

We recall the following from [L]. The notation is adapted to this paper.

Lemma [L 1.2.15]. *Let $\nu \in \tilde{Q}_+$, $\nu \neq 0$. Let $x \in (U_q)_\nu$:*

(a) *If $r_i(x) = 0$ for all $i \in I$ then $x = 0$.*
(b) *If ${}_i r(x) = 0$ for all $i \in I$ then $x = 0$.*

Proposition [L 3.1.6]. *Let $x \in U_q^+$, then*:

$$[x, F_i] = \frac{r_i(x) K_i - K_i^{-1} {}_i r(x)}{q_i - q_i^{-1}} .$$

Proposition [L 38.1.6].

(a) $\{x \in U_q^+ \mid {}_i r(x) = 0\} = \{x \in U_q^+ \mid T_i(x) \in U_q^+\}$.
(b) $\{x \in U_q^+ \mid r_i(x) = 0\} = \{x \in U_q^+ \mid T_i^{-1}(x) \in U_q^+\}$.

Proposition [L 40.1.2]. *Let $w \in W$, $i \in I$ be such that $l(ws_i) = l(w) + 1$. If $w = s_{i_1} \ldots s_{i_r}$ is a reduced presentation then $T_{i_1} \ldots T_{i_r}(E_i) \in U_q^+$.*

Lemma [L2 2.7]. *Let $x \in P^\vee$, $i = 1, \ldots, n$, $s_i \in S$.*

(a) *If $s_i x = x s_i$, then $T_i T_x = T_x T_i$.*
(b) *If $s_i x s_i^{-1} = \alpha_i^{-1} x = \prod_j \omega_j^{a_j}$, then $T_i^{-1} T_x T_i^{-1} = \prod_j T_{\omega_j}^{a_j}$, in particular $T_i^{-1} T_{\omega_i} T_i^{-1} = T_{\omega_i}^{-1} \prod_{j \neq i} T_{\omega_j}^{-a_{ij}}$.*

Remark. $\Phi T_{\omega_i^{-1}} = T_{\omega_i}^{-1} \Phi$.

3. Subalgebras of $U_q(\hat{\mathfrak{g}})$

In this section we find certain subalgebras of $U_q(\hat{\mathfrak{g}})$ which are isomorphic to $U_q(\widehat{\mathfrak{sl}_2})$.

3.1 Lemma. *Let $\omega_i \in P^\vee$, $1 \leqq i \leqq n$.*

(a) *Any reduced presentation of ω_i starts with τs_j where $\tau \in \mathcal{T}$ and $\tau s_j = s_0 \tau$.*
(b) *Any reduced presentation of ω_i ends with s_i.*

Proof. For $j \neq 0$, $l(s_j \omega_i) = l(\omega_i) + 1$ and this implies (a). For (b), if $l(\omega_i s_j) < l(\omega_i)$ then $\omega_i(\alpha_j) < 0$ which is only the case when $i = j$.

Definition. *For $1 \leqq i \leqq n$ let $\omega_i' = \omega_i s_i$. Then $l(\omega_i') = l(\omega_i) - 1$.*

Remark. $T_{\omega_i'} = T_{\omega_i} T_i^{-1}$.

Definition. *For $1 \leqq i \leqq n$ let $U_i \subset U_q(\hat{\mathfrak{g}})$ be the subalgebra generated over $\mathbb{C}(q_i)$ by*

$$E_i,\ F_i,\ K_i^{\pm 1},\ T_{\omega_i'}(E_i),\ T_{\omega_i'}(F_i),\ T_{\omega_i'}(K_i^{\pm 1}),\ C^{\pm 1/2},\ D^{\pm d_i}.$$

It is clear $T_{\omega_i'} E_i \in U_q^+$, $T_{\omega_i'} F_i \in U_q^-$ since $l(\omega_i' s_i) = l(\omega_i) = l(\omega_i') + 1$.

The following is proved as in [L5 1.8]:

3.2 Lemma. *Let $i, j \in I$ and let $w \in W$ be such that $w(\alpha_i) = \alpha_j$. Then $T_w(E_i) = E_j$.*

Corollary. *Let $1 \leqq i \neq j \leqq n$. Then for $x \in U_j$, $T_{\omega_i}(x) = x$.*

Proof. $\omega_i(\alpha_j) = \alpha_j$.

Definition. *For $1 \leqq i \neq j \leqq n$, $a_{ij} \leqq 0$ introduce the elements:*

$$F_{ij} = -F_j F_i + q^{-(\alpha_i | \alpha_j)} F_i F_j\,,$$

$$E_{ij} = -E_i E_j + q^{(\alpha_i | \alpha_j)} E_j E_i\,.$$

3.3 Lemma. *Let $1 \leqq i \neq j \leqq n$.*

(a) $T_{\omega_i}(F_{ji}) = T_{\omega_j}(F_{ij})$,
(b) $T_{\omega_i}(E_{ji}) = T_{\omega_j}(E_{ij})$.

Proof. For (a) if $a_{ij} = 0$ then both sides of the equation equal 0. Otherwise, since the statement is symmetric in i and j we may assume $a_{ji} = -1$. Then:

$$T_{\omega_j}(F_{ij}) = T_{\omega_j}(T_j^{-1} F_i) = T_j T_{\omega_j}^{-1} T_{\omega_i}(F_i)$$

$$= T_{\omega_i} T_j(F_i) = T_{\omega_i}(F_{ji})$$

which implies (a). (b) follows by applying Ω.

3.4 Lemma. *Let $1 \leqq i \leqq n$, $[F_i, T_{\omega_i'}(E_i)] = 0$.*

Proof. By [L 3.1.6, 38.1.6] it suffices to check that both $T_i T_{\omega_i'}(E_i) \in U_q^+$ and $T_i^{-1} T_{\omega_i'}(E_i) \in U_q^+$. Since $\omega_i \in P_+^\vee$, $l(s_i \omega_i' s_i) = l(\omega_i) + 1 = l(s_i \omega_i') + 1$ so that $T_i T_{\omega_i'}(E_i) \in U_q^+$. Now $T_i^{-1} T_{\omega_i'}(E_i) = T_i^{-1} T_{\omega_i} T_i^{-1}(E_i) = T_{\omega_i}^{-1}(E_i)$. Since $\Phi \circ \Omega(U_q^+) = U_q^+$ and $\Phi \circ \Omega(T_{\omega_i}^{-1}(E_i)) = T_{\omega_i^{-1}}(E_i)$ it is enough to check $T_{\omega_i^{-1}}(E_i) \in U_q^+$. This follows because $\omega_i \in P_+^\vee$ and $l(\omega_i^{-1} s_i) = l(\omega_i^{-1}) + 1$.

3.5 Lemma. *Let $1 \leqq i \leqq n$, $j \neq i, 0$, $[F_j, T_{\omega_i'}(E_i)] = -C K_i^{-1} T_{\omega_j}(F_{ij})$.*

Proof. $[F_j, T_{\omega_i'}(E_i)]=[F_j, T_{\omega_i}(-K_i^{-1}F_i)]=-T_{\omega_i}([F_j, K_i^{-1}F_i])=-T_{\omega_i}(K_i^{-1})T_{\omega_i}$ $(q^{-(\alpha_i|\alpha_j)}F_jF_i-F_iF_j)=-CK_i^{-1}T_{\omega_i}(F_{ji})=-CK_i^{-1}T_{\omega_j}(F_{ij}))$

3.6 Lemma. *Let* $1\leqq i\leqq n$, *if* $\hat{\mathfrak{g}}\neq\widehat{\mathfrak{sl}}_2$ *then* $r_0(T_{\omega_i'}(E_i))=0$.

Proof. Since $l(\omega_i)>1$ and $T_{\omega_i'}(E_i)\in U_q^+$ it follows $T_0^{-1}T_{\omega_i'}(E_i)\in U_q^+$.

3.7 Proposition. *Let* $1\leqq i\leqq n$.

(a) $E_i^{(3)}T_{\omega_i'}(E_i)-E_i^{(2)}T_{\omega_i'}(E_i)E_i+E_iT_{\omega_i'}(E_i)E_i^{(2)}-T_{\omega_i'}(E_i)E_i^{(3)}=0$,
(b) $T_{\omega_i'}(E_i)^{(3)}(E_i)-T_{\omega_i'}(E_i)^{(2)}E_iT_{\omega_i'}(E_i)+T_{\omega_i'}(E_i)E_iT_{\omega_i'}(E_i)^{(2)}-E_iT_{\omega_i'}(E_i)^{(3)}=0$.

Proof. (b) follows from (a) by applying $T_{\omega_i'}$. Denote the expression in (a) by x_i. To check $x_i=0$ it suffices [L 1.2.15] to check $r_j(x_i)=0$ for $j\in I$. For $j=0$ this is by the preceding lemma. Since for $x\in U_q^+$ $[F_j, x]=0$ for $x\in U_q^+$ implies $r_j(x)=0$, we can check $[F_j, x_i]=0$. This is straightforward using the expressions for $[F_j, T_{\omega_i'}(E_i)]$ in 3.4 and 3.5.

3.8 Proposition. *For each* $1\leqq i\leqq n$ *there is an algebra isomorphism* h_i: $U_q(\widehat{\mathfrak{sl}}_2)\to U_i$ *given by* $h_i(E_1)=E_i$, $h_i(E_0)=T_{\omega_i'}(E_i)$, $h_i(K_1^{\pm 1})=K_i^{\pm 1}$, $h_i(K_0^{\pm 1})=T_{\omega_i'}(K_i^{\pm 1})$, $h_i(F_1)=F_i$, $h_i(F_0)=T_{\omega_i'}(F_i)$, $h_i(C^{\pm 1/2})=C^{\pm 1/2}$, $h_i(D^{\pm 1})=D^{\pm d_i}$, $h_i(q)=q_i$.

Proof. Consider the defining relations of $U_q(\widehat{\mathfrak{sl}}_2)$. By the previous proposition and some simple checks they hold in U_i, where q is replaced by q_i. Therefore h_i is surjective. For $v\in\tilde{Q}(\hat{\mathfrak{g}})$, let $U_{i,v}^{\pm}=U_i\cap U_v^{\pm}$, then $h_{i|U_i^-}$ is homogeneous with respect to this grading. Therefore if $x\in\operatorname{Ker} h_{i|U_i^-}$, writing $x=\sum_j b_jx_j$ in terms of homogeneous components $h_{i|U_i^-}(x_j)=0$ for each j. Fix some x_j. By [L4 Prop. 2.6] (see also Remark 4.14) for $\beta\in\tilde{Q}$ there is a unique irreducible highest weight module M of $U_q(\widehat{\mathfrak{sl}}_2)$ with highest weight vector v such that $K_iv=q^{(\alpha_i|\beta)}v$ for $i=0,1$ and $Dv=q^{d_\beta}v$. Further we can pick β so that x_j acts non-trivially on M. The root system of $U_q(\widehat{\mathfrak{sl}}_2)$ imbeds into that of $U_q(\hat{\mathfrak{g}})$ via h_i and we can fix a $\beta'\in\tilde{Q}(\hat{\mathfrak{g}})$ so that pulling back the highest weight module M' with weight β' through h_i we have K_0, K_1, and D acting as on M. Now as a $U_q(\widehat{\mathfrak{sl}}_2)$ module M' has an irreducible quotient which is isomorphic to M. In particular, x_j must act non-trivially in M' which is a contradiction. Therefore $\operatorname{Ker} h_{i|U_i^-}=0$. Since multiplication induces a vector space isomorphism $U^-\otimes T\otimes U^+\overset{m}{\to}U$ both in U_i and $U_q(\widehat{\mathfrak{sl}}_2)$ it follows that h_i factors through this decomposition. Therefore $\operatorname{Ker} h_i=0$.

Corollary. *For* $1\leqq i\leqq n$,

(a) $T_{i|U_i}=h_i\circ T_1\circ h_i^{-1}$,
(b) $T_{\omega_i|U_i}=h_i\circ T_{\omega_1}\circ h_i^{-1}$.

Proof. Let M be an integrable U_q module. Decompose M into weight spaces with respect to the action of K_i, $M=\bigoplus_j M^j$. Let $u\in U_i$, $m\in M^n$ for a particular n. From the defining properties of the braid group action it follows:

$$T_1(h_i^{-1}(u))\cdot\sum_{a,b,c;\,-a+b-c=n}(-1)^bq^{-ac+b}E_1^{(a)}F_1^{(b)}E_1^{(c)}m$$
$$=\sum_{a,b,c}(-1)^bq^{-ac+b}E_1^{(a)}F_1^{(b)}E_1^{(c)}(h_i^{-1}u)m$$
$$=h_i^{-1}\left(\sum_{a,b,c}(-1)^bq_i^{-ac+b}E_i^{(a)}F_i^{(b)}E_i^{(c)}um\right)$$
$$=h_i^{-1}\left(T_i(u)\sum_{a,b,c}(-1)^bq_i^{-ac+b}E_i^{(a)}F_i^{(b)}E_i^{(c)}m\right)\Rightarrow h_i\circ T_1\circ h_i^{-1}=T_{i|U_i}.$$

This implies (a). $T_{\omega_i}^{-1}(E_i)=T_i^{-1}T_{\omega_i'}(E_i)=q_i^{-2}E_i^{(2)}T_{\omega_i'}(E_i)-q_i^{-1}E_iT_{\omega_i'}(E_i)E_i+T_{\omega_i'}(E_i)E_i^{(2)}$ and $T_{\omega_i}^{-1}T_{\omega_i'}(E_i)=-K_i^{-1}F_i$ so that $T_{\omega_i|U_i}$ acts on the generators of U_i as does $h_i\circ T_{\omega_1}\circ h_i^{-1}$. (b) follows.

3.9 Definition. *For* $1\leqq i\leqq n$, $k>0$, *let* $\bar{\psi}_{ik}=C^{-k/2}(q_i^{-2}E_iT_{\omega_i}^k(K_i^{-1}F_i)-T_{\omega_i}^k(K_i^{-1}F_i)E_i)$. *Note that* $\bar{\psi}_{ik}\in U_i$.

Versions of the next two propositions appear in the work of [Da Sect. 4] for $U_q(\widehat{\mathfrak{sl}}_2)$.

3.10 Proposition 1. *Let* $c=(q_i^2C^{1/2})$, $d=(q_i^2C^{-1/2})$, $r>0$, $m\in\mathbb{Z}$ *then:*

$$[\bar{\psi}_{ir},T_{\omega_i}^m(F_i)]=-C^{1/2}[2]_i\left(\sum_{k=1}^{r-1}c^{(k-1)}(q_i-q_i^{-1})\bar{\psi}_{i,r-k}T_{\omega_i}^{m+k}(F_i)+c^{(r-1)}T_{\omega_i}^{m+r}(F_i)\right),$$

$$[\bar{\psi}_{ir},T_{\omega_i}^m(E_i)]=C^{-1/2}[2]_i\left(\sum_{k=1}^{r-1}d^{(k-1)}(q_i-q_i^{-1})T_{\omega_i}^{m-k}(E_i)\bar{\psi}_{i,r-k}+d^{(r-1)}T_{\omega_i}^{m-r}(E_i)\right).$$

Proposition 2. *Let* $r>0$, $1\leqq i\leqq n$.

(a) $[\bar{\psi}_{i1},\bar{\psi}_{ir}]=0$,
(b) $T_{\omega_i}(\bar{\psi}_{ir})=\bar{\psi}_{ir}$.

Proof. It is sufficient to prove the previous two statements for $U_q(\widehat{\mathfrak{sl}}_2)$. Here $i=1$ and $\omega_1=\tau s_1$, where τ is the non-trivial Dynkin diagram automorphism. This follows because $l(\omega_1)=1$, $l(s_1\omega_1)=l(\omega_1)+1$ and ω_1 has only finitely many conjugates in W.

For the sake of exposition, we sketch a proof by induction on r which appears in [Da Sect. 4]. For $r=1$ the statements are readily checked. A direct calculation shows $[\bar{\psi}_{11},\bar{\psi}_{1r}]=[2]\,((\tau T_1)^{-1}(\bar{\psi}_{1,r+1})-\bar{\psi}_{1,r+1})$. This implies that $2a)_r$ is equivalent to $2b)_{r+1}$. Here we denote by $2a)_{r'}$ the statement 2a) for all $r\leqq r'$.

Proposition $2b)_r$ implies $1)_r$. This follows from an inductive calculation using the identities:

$$[\bar{\psi}_{1r},F_1]=C^{1/2}(q^{+2}[\bar{\psi}_{1,r-1},\tau T_1(F_1)]-[2](q-q^{-1})\bar{\psi}_{1,r-1}\tau T_1(F_1))$$

$$[\bar{\psi}_{1r},E_1]=C^{-1/2}(q^2[\bar{\psi}_{1,r-1},T_1^{-1}\tau(E_1)]+[2](q-q^{-1})T_1^{-1}\tau(E_1)\bar{\psi}_{1,r-1})$$

To show 2a) it is sufficient to show $r_j(C^{(r+1)/2}[\bar{\psi}_{11},\bar{\psi}_{1r}])=0$ for $j=0,1,r>0$. For $j=0$ this is straightforward. For $j=1$ this follows from $[[\bar{\psi}_{11},\bar{\psi}_{1r}],F_1]=0$. This is shown by induction on r. Assuming $2a)_{r-1}$, $2b)_r$, and $1)_r$, a direct calculation gives

$$[[\bar{\psi}_{11},\bar{\psi}_{1r}],F_1]=-C^{1/2}[2]\sum_{s=1}^{r-1}d^{(1-k)}[\bar{\psi}_{11},\bar{\psi}_{1s}]T_{\omega_1}^kF_1]=0\,.$$

This implies $2a)_r$. As noted this now implies $2b)_{r+1}$ and $1)_{r+1}$. This completes the proof of Propositions 1 and 2.

Remark: Much of the calculation through the end of §3 is inspired by the work of [Da] for $U_q(\widehat{\mathfrak{sl}}_2)$. The statements of Proposition 2 also appear for $U_q(\widehat{\mathfrak{sl}}_2)$ in [LSS].

3.11 Define $\bar{\varphi}_{ik}=\Omega(\bar{\psi}_{ik})$. Applying the anti-automorphism Ω to the above propositions gives similar identities with $\bar{\psi}_{ik}$ replaced by $\bar{\varphi}_{ik}$ and F_i (resp. E_i) replaced by E_i (resp. F_i). Here and in the future we omit writing these identities down although we implicitly assume them.

Let H be the subalgebra of U_q generated by $\bar{\psi}_{ik}$, $\bar{\varphi}_{ik}$ *for* $1 \leqq i \leqq n$, then we have shown:

3.12 Proposition. *The group of translations P fixes H pointwise.*

3.13 Proposition. *Let* $1 \leqq i \leqq n$, $r \in \mathbb{Z}$.

$$T^r_{\omega_i}(F_i)F_i - q_i^{-2}F_iT^r_{\omega_i}(F_i) = q_i^{-2}T^{r-1}_{\omega_i}(F_i)T_{\omega_i}(F_i) - T_{\omega_i}(F_i)T^{r-1}_{\omega_i}(F_i)\,.$$

Proof. This is checked in $U_q(\widehat{\mathfrak{sl}_2})$ directly.

3.14 Lemma. *Let* $a_{ij} \leqq 0$, $m \in \mathbb{Z}$.

(a) $[\bar{\psi}_{i1}, T^m_{\omega_j}(F_j)] = C^{1/2}[a_{ij}]_i T^{m+1}_{\omega_j}(F_j)$,
(b) $[\bar{\psi}_{i1}, T^m_{\omega_j}(E_j)] = -C^{-1/2}[a_{ij}]_i T^{m-1}_{\omega_j}(E_j)$.

Proof. We check (a) for $a_{ij} \leqq 0$. Note that by previous lemmas $[T_{\omega_i}(K_i^{-1}F_i), F_j] = -K_i^{-1}CT_{\omega_i}(F_{ji})$ and $T_{\omega_i}(F_{ji}) = T_{\omega_j}(F_{ij})$. Then:

$$\begin{aligned}
[\bar{\psi}_{i1}, F_j] &= C^{-1/2}([q_i^{-2}E_iT_{\omega_i}(K_i^{-1}F_i), F_j] - [T_{\omega_i}(K_i^{-1}F_i)E_i, F_j]) \\
&= -C^{1/2}K_i^{-1}([E_i, T_{\omega_i}(F_{ji})]) = -C^{1/2}K_i^{-1}T_{\omega_j}([E_i, F_{ij}]) \\
&= -C^{1/2}K_i^{-1}T_{\omega_j}([E_i, -F_jF_i + q^{-(\alpha_i|\alpha_j)}F_iF_j]) = C^{1/2}[a_{ij}]_iT_{\omega_j}(F_j)\,.
\end{aligned}$$

Now (a) follows by applying $T^m_{\omega_j}$ to the above equality. Using $\psi_{i1} = T^{-1}_{\omega_i}\psi_{i1} = C^{-(1/2)}q_i^{-2}T^{-1}_{\omega_i}(E_i)(K_i^{-1}F_i) - (K_i^{-1}F_i)T^{-1}_{\omega_i}(E_i)$ (b) follows similarly.

3.15 Lemma. *Let* $a = a_{ij} \leqq 0$, $r > 0$, $m \in \mathbb{Z}$, $c = (-q_i^a c^{1/2})$, $d = (-q_i^a C^{-1/2})$,

$$[\bar{\psi}_{ir}, T^m_{\omega_j}(F_j)] = C^{1/2}[a]_i\left(\sum_{k=1}^{r-1} c^{(k-1)}(q_i - q_i^{-1})\bar{\psi}_{i,r-k}T^{m+k}_{\omega_j}(F_j) + c^{(r-1)}T^{m+r}_{\omega_j}(F_j)\right),$$

$$[\bar{\psi}_{ir}, T^m_{\omega_j}(E_j)] = -C^{-1/2}[a]_i\left(\sum_{k=1}^{r-1} d^{(k-1)}(q_i - q_i^{-1})T^{m-k}_{\omega_j}(E_j)\bar{\psi}_{i,r-k} + d^{(r-1)}T^{m-r}_{\omega_j}(E_j)\right).$$

Proof. We check the second equation.

$$\begin{aligned}
[\bar{\psi}_{ir}, E_j] &= C^{-r/2}(q_i^{-2}T^{-1}_{\omega_i}(E_i)T^{r-1}_{\omega_i}(K_i^{-1}F_i)E_j - T^{r-1}_{\omega_i}(K_i^{-1}F_i)T^{-1}_{\omega_i}(E_i)E_j \\
&\quad - q_i^{-2}E_jT^{-1}_{\omega_i}(E_i)T^{r-1}_{\omega_i}(K_i^{-1}F_i) + E_jT^{r-1}_{\omega_i}(K_i^{-1}F_i)T^{-1}_{\omega_i}(E_i)) \\
&\quad \text{since: } T^{r-1}_{\omega_i}(K_i^{-1}F_i)E_j = q_i^{-a}E_jT^{r-1}_{\omega_i}(K_i^{-1}F_i) \\
&= C^{-r/2}(q_i^{-2-a}T^{-1}_{\omega_i}(E_i)E_jT^{r-1}_{\omega_i}(K_i^{-1}F_i) - T^{r-1}_{\omega_i}(K_i^{-1}F_i)T^{-1}_{\omega_i}(E_i)E_j \\
&\quad - q_i^{-2}E_jT^{-1}_{\omega_i}(E_i)T^{r-1}_{\omega_i}(K_i^{-1}F_i) + q_i^aT^{r-1}_{\omega_i}(K_i^{-1}F_i)E_jT^{-1}_{\omega_i}(E_i)) \\
&= C^{-r/2}(-q_i^{-2-a}T^{-1}_{\omega_i}(E_{ij})T^{r-1}_{\omega_i}(K_i^{-1}F_i) + T^{r-1}_{\omega_i}(K_i^{-1}F_i)T^{-1}_{\omega_i}(E_{ij})) \\
&\quad \text{now use: } T^{-1}_{\omega_i}(E_{ij}) = T^{-1}_{\omega_j}(E_{ji}) = T^{-1}_{\omega_j}(-E_jE_i + q_i^aE_iE_j), \\
&= C^{-r/2}(q_i^{-2-a}T^{-1}_{\omega_j}(E_j)E_iT^{r-1}_{\omega_i}(K_i^{-1}F_i) - q_i^{-2+a}E_iT^{r-1}_{\omega_i}(K_i^{-1}F_i)T^{-1}_{\omega_j}E_j \\
&\quad - q_i^{-a}T^{-1}_{\omega_j}(E_j)T^{r-1}_{\omega_i}(K_i^{-1}F_i)E_i + q_i^aT^{r-1}_{\omega_i}(K_i^{-1}F_i)E_iT^{-1}_{\omega_j}E_j) \\
&= C^{-1/2}(-q_i^a[\bar{\psi}_{i,r-1}, T^{-1}_{\omega_j}(E_j)] - [a]_i(q_i - q_i^{-1})T^{-1}_{\omega_j}(E_j)\bar{\psi}_{i,r-1})\,.
\end{aligned}$$

Now the second statement follows by induction and applying $T^m_{\omega_j}$. The first statement follows by a similar calculation.

3.16 Lemma. *Let* $r \in \mathbb{Z}$, $(\alpha_i|\alpha_j) \leqq 0$,

$$-T^r_{\omega_i}F_iF_j + q^{-(\alpha_i|\alpha_j)}F_jT^r_{\omega_i}F_i = q^{-(\alpha_i|\alpha_j)}T^{r-1}_{\omega_i}F_iT_{\omega_j}F_j - T_{\omega_j}F_jT^{r-1}_{\omega_i}F_i\,.$$

Proof. The left-hand side equals $T^r_{\omega_i}(F_{ji})$. The right-hand side equals:

$$q^{-(\alpha_i|\alpha_j)}T^{r-1}_{\omega_i}F_iT_{\omega_j}F_j - T_{\omega_j}F_jT^{r-1}_{\omega_i}F_i = T^{r-1}_{\omega_i}T_{\omega_j}(F_{ij}) = T^r_{\omega_i}(F_{ji})\,.$$

4. The Relations in Drinfel'd's Realization

Let Γ be the Dynkin diagram of $\mathfrak{g}$. Orient the vertices of Γ by defining $o\colon V \to \{\pm 1\}$ so that for i and j adjacent in Γ, $o(i) = -o(j)$. Now define $\hat{T}_{\omega_i} = o(i)T_{\omega_i}$, and modify all the definitions by replacing T_{ω_i} with $\hat{T}_{\omega_i}$.

4.1 Lemma. *Let* $a = a_{ij}$, $r > 0$, $m \in \mathbb{Z}$, $c = (q_i^a c^{1/2})$, $d = (q_i^a C^{-1/2})$,

$$[\bar{\psi}_{ir}, \hat{T}^m_{\omega_j}(F_j)] = -C^{1/2}[a]_i\left(\sum_{k=1}^{r-1} c^{(k-1)}(q_i - q_i^{-1})\bar{\psi}_{i,r-k}\hat{T}^{m+k}_{\omega_j}F_j + c^{(r-1)}\hat{T}^{m+r}_{\omega_j}F_j\right),$$

$$[\bar{\psi}_{ir}, \hat{T}^m_{\omega_j}(E_j)] = C^{-1/2}[a]_i\left(\sum_{k=1}^{r-1} d^{(k-1)}(q_i - q_i^{-1})\hat{T}^{m-k}_{\omega_j}E_j\bar{\psi}_{i,r-k} + d^{(r-1)}\hat{T}^{m-r}_{\omega_j}E_j\right).$$

Proof. This follows directly from Sect. 3.

Now for $k > 0$ introduce generators $h_{ik} \in H$ by the change of variables (cf. [D2, G]):

$$(q_i - q_i^{-1})\sum_{k>0} h_{ik}z^k = \log\left(1 + (q_i - q_i^{-1})\sum_{k'>0}\bar{\psi}_{i,k'}z^{k'}\right).$$

Differentiating both sides and considering the coefficient of z^r gives:

$$(*)\qquad rh_{ir} = r\bar{\psi}_{ir} - (q_i - q_i^{-1})\sum_{k=1}^{r-1} k\bar{\psi}_{i,r-k}h_{ik}\,.$$

Similarly introduce $h_{i,-k} = \Omega(h_{ik})$ so that:

$$(**)\qquad rh_{i,-r} = r\bar{\varphi}_{ir} - (q_i^{-1} - q_i)\sum_{k=1}^{r-1} kh_{i,-k}\bar{\varphi}_{i,r-k}\,.$$

4.2 Lemma. *Let* $1 \leqq i, j \leqq n$, $k > 0$.

(a) $[h_{ik}, \hat{T}^m_{\omega_j}F_j] = -\dfrac{1}{k}[ka_{ij}]_iC^{k/2}\hat{T}^{m+k}_{\omega_j}F_j$,

(b) $[h_{ik}, \hat{T}^m_{\omega_j}E_j] = \dfrac{1}{k}[ka_{ij}]_iC^{-k/2}\hat{T}^{m-k}_{\omega_j}E_j$,

Proof. Part (b) is an induction on k using the following identities:

$$[\bar{\psi}_{ik}, \hat{T}^m_{\omega_j}E_j] = C^{-1/2}(-q_i^{a_{ij}}[\bar{\psi}_{i,k-1}, \hat{T}^{m-1}_{\omega_j}E_j] - [a_{ij}]_i(q_i - q_i^{-1})\hat{T}^{m-1}_{\omega_j}E_j\bar{\psi}_{i,k-1}),$$

$$[h_{ik'}, \hat{T}^m_{\omega_j}E_j] = C^{-1/2}\frac{(k'-1)[k'a_{ij}]_i}{k'[(k'-1)a_{ij}]_i}[h_{i,k'-1}, \hat{T}^{m-1}_{\omega_j}E_j],\quad \text{where } k' < k\,,$$

and (a) is similar.

Remark. As before, we omit the identities obtained by applying Ω.

For $1\leqq i\leqq n$, $r>0$, introduce the elements $\psi_{ir}=(q_i-q_i^{-1})K_i\bar{\psi}_{ir}$, $\varphi_{ir}=\Omega(\psi_{ir})$. Then:

$$\psi_{ir}=(q_i-q_i^{-1})C^{r/2}[E_i,\hat{T}^r_{\omega_i}F_i]\,,$$
$$\varphi_{ir}=(q_i-q_i^{-1})C^{-r/2}[F_i,\hat{T}^r_{\omega_i}E_i]\,.$$

Set $\psi_{i,0}=K_i$, $\varphi_{i,0}=K_i^{-1}$.

4.3 Lemma. *Let $k,l\geqq 1$. Then $[h_{ik},\psi_{jl}]=0$.*

Proof.

$$\begin{aligned}\frac{1}{q_j-q_j^{-1}}[h_{ik},\psi_{jl}]&=C^{l/2}[h_{ik},[E_j,\hat{T}^l_{\omega_j}F_j]]\\
&=C^{l/2}[[h_{ik},E_j],\hat{T}^l_{\omega_j}F_j]+[E_j,[h_{ik},\hat{T}^l_{\omega_j}F_j]]\\
&=C^{l/2}\left(\frac{[ka_{ij}]_i}{k}C^{-k/2}[\hat{T}^{-k}_{\omega_j}E_j,\hat{T}^l_{\omega_j}F_j]+\left[[E_j,-\frac{[ka_{ij}]_i}{k}C^{k/2}\hat{T}^{l+k}_{\omega_j}F_j]\right]\right)\\
&=C^{k+l/2}\frac{[ka_{ij}]_i}{k}(C^{-k}[\hat{T}^{-k}_{\omega_j}E_j,\hat{T}^l_{\omega_j}F_j]-[E_j,\hat{T}^{l+k}_{\omega_j}F_j])=0\,,\end{aligned}$$

since $\hat{T}^{-1}_{\omega_j}[E_j,\hat{T}^l_{\omega_j}F_j]=C[E_j,\hat{T}^l_{\omega_j}F_j]$.

Similarly:

4.4 Lemma. *Let $k,r>0$. Then*

$$[h_{ik},\varphi_{jr}]=\begin{cases}-\dfrac{[ka_{ij}]_i}{k}(C^k-C^{-k})\varphi_{j,r-k} & \text{if } r\geqq k\,,\\ 0 & \text{if } k>r\,.\end{cases}$$

Rewriting (**) in terms of the φ_{ir} we have:

(***) $$r(q_i^{-1}-q_i)h_{i,-r}=rK_i\varphi_{ir}+(q_i-q_i^{-1})K_i\sum_{k=1}^{r-1}k\varphi_{i,r-k}h_{i,-k}\,.$$

4.5 Lemma. *Let $k,l>0$. Then*

$$[h_{ik},h_{jl}]=\delta_{k,-l}\frac{1}{k}[ka_{ij}]_i\frac{C^k-C^{-k}}{q_j-q_j^{-1}}\,.$$

Proof. Induction using (***).

4.6 Definition. *For $1\leqq i\leqq n$, $k\in\mathbb{Z}$ define $x^-_{ik}=\hat{T}^k_{\omega_i}(F_i)$, $x^+_{ik}=\hat{T}^{-k}_{\omega_i}(E_i)$.*

We can now prove:

4.7 Theorem [cf. D2]. *$U_q(\hat{\mathfrak{g}})$ is generated over $\mathbb{C}(q)$ by the elements $x^\pm_{ij}$, $h^\pm_{ik}$, $K_i^{\pm 1}$, $C^{\pm 1/2}$, $D^{\pm 1}$, where $1\leqq i\leqq n$, $j\in\mathbb{Z}$, and $k\in\mathbb{Z}\setminus\{0\}$. The following are defining relations for $U_q(\hat{\mathfrak{g}})$:*

(1) $[C^{\pm 1/2},h_{ik}]=[C^{\pm 1/2},x^\pm_{ik}]=[K_j,h_{ik}]=[K_i,K_j]=0\,,$

$K_i x^\pm_{jk}K_i^{-1}=q^{\pm(\alpha_i,\alpha_j)}x^\pm_{jk}$, $\quad Dx^\pm_{jk}D^{-1}=q^k x^\pm_{jk}$, $\quad Dh_{ik}D^{-1}=q^k h_{ik}\,,$

(2) $[h_{ik}, h_{jl}] = \delta_{k,-l} \frac{1}{k}[ka_{ij}]_i \frac{C^k - C^{-k}}{q_j - q_j^{-1}}$,

(3) $[h_{ik}, x_{jl}^{\pm}] = \pm\frac{1}{k}[ka_{ij}]_i C^{\mp(|k|/2)} x_{j,k+l}^{\pm}$,

(4) $x_{i,k+1}^{\pm} x_{jl}^{\pm} - q^{\pm(\alpha_i|\alpha_j)} x_{jl}^{\pm} x_{i,k+1}^{\pm} = q^{\pm(\alpha_i|\alpha_j)} x_{ik}^{\pm} x_{j,l+1}^{\pm} - x_{j,l+1}^{\pm} x_{ik}^{\pm}$,

(5) $[x_{ik}^{+}, x_{jl}^{-}] = \delta_{ij} \frac{1}{q_i - q_i^{-1}} (C^{k-l/2} \psi_{i,k+l} - C^{l-k/2} \varphi_{i,k+l})$,

For $i \neq j$, $n = 1 - a_{ij}$,

(6) $\mathrm{Sym}_{k_1, k_2, \ldots, k_n} \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_i x_{i,k_1}^{\pm} \ldots x_{i,k_r}^{\pm} x_{jl}^{\pm} x_{i,k_{r+1}}^{\pm} \ldots x_{i,k_n}^{\pm} = 0$.

Sym denotes symmetrization with respect to the indices $k_1, k_2, \ldots, k_n$. Here ψ_{ik} and φ_{ik} are defined by the following functional equations:

$$\sum_{k \geq 0} \psi_{ik} u^k = K_i \exp\left((q_i - q_i^{-1}) \sum_{k=1}^{\infty} h_{ik} u^k\right),$$

$$\sum_{k \geq 0} \varphi_{ik} u^k = K_i^{-1} \exp\left((q_i^{-1} - q_i) \sum_{k=1}^{\infty} h_{i,-k} u^{-k}\right).$$

Proof. Relations (1)–(5) follow from the previous calculations. Relation (6) is obtained by applying $\hat{T}_{\omega_i}$, $i = 1, \ldots, n$ to the Chevalley relations and an induction on $\max\{|k_{i_r} - k_{i_s}|\}$. Let R be the algebra over $\mathbb{C}(q)$ on the above generators with defining relations (1)–(6). By the previous consideration there exists an algebra surjection F: $R \to U_q$. To check that F is an isomorphism we specialize at 1 as in Sect. 2. Let $R_{\mathscr{A}}$ be the $\mathscr{A}$ subalgebra of R generated by:

$$K_i^{\pm 1},\ C^{\pm 1/2},\ D^{\pm 1},\ h_{i,0} = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$c = \frac{C - C^{-1}}{q - q^{-1}},\quad d = \frac{D - D^{-1}}{q - q^{-1}},\quad h_{ik},\ x_{ik}^{\pm}.$$

Define $\hat{R}_1 = R_{\mathscr{A}}/(q-1)R_{\mathscr{A}}$. Then $\hat{R}_1$ is an associative algebra over $\mathbb{C}$ on the above generators with the defining relations:

(1) $[K_i, K_j] = [D, K_i] = 0, \quad C^2 = D^2 = K_i^2 = 1$,

$[d, h_{ik}] = k h_{ik}, \quad [d, x_{jk}^{\pm}] = kD x_{jk}^{\pm}$,

(2) $[h_{ik}, h_{jl}] = \delta_{k,-l} a_{ij} \frac{c}{d_j} (C^{k-1} + \cdots + C^{1-k})$,

(3) $[h_{ik}, x_{jl}^{\pm}] = \pm a_{ij} C^{\mp|k|/2} x_{j,k+l}^{\pm}, \quad [h_{i0}, x_{jl}^{\pm}] = \pm a_{ij} K_i x_{jl}^{\pm}$,

(4) $x_{i,k+1}^{\pm} x_{jl}^{\pm} - x_{jl}^{\pm} x_{i,k+1}^{\pm} = x_{ik}^{\pm} x_{j,l+1}^{\pm} - x_{j,l+1}^{\pm} x_{ik}^{\pm}$,

(5) $[x_{ik}^{+}, x_{jl}^{-}] = \delta_{ij} K_i C^{(k-l)/2} h_{i,k+l}$,

(6) $[x_{i,k_1}^{\pm}, [x_{i,k_2}^{\pm}, \ldots, [x_{i,k_n}^{\pm}, x_{jl}^{\pm} \ldots]]] = 0, \quad n = 1 - a_{ij}$.

It follows from the Gabber–Kac theorem [G-K] (see [G] for the relations in R_1 below) that:

$$R_1 = \hat{R}_1/(K_i - 1, C^{1/2} - 1, D - 1) \cong U(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d).$$

Now specialize $U_q(\hat{\mathfrak{g}})$ to $U_1(\hat{\mathfrak{g}})$ as in Sect. 2. Then F induces the isomorphism:

$$F: R_1 \cong U_1(\hat{\mathfrak{g}}),$$

Since specialization doesn't change the root multiplicities, $F: R \to U_q$ is an isomorphism.

Remark: Let $s_{\theta_i} \in W_0$ so that $s_{\theta_i}(\alpha_i) = \theta$. By Lemma 3.2 it follows $T_{\theta_i} T_{\omega_i}(-K_i^{-1} F_i) = E_0$. This gives the inverse to the isomorphism $F: R \to U_q$. In particular, $F^{-1}(E_0) = -o(i) C K_\theta^{-1} T_{\theta_i} x_{i1}^-$.

5. The Coproduct

Since the Drinfeld generators are now expressed in terms of the braid group, calculating their coproduct depends on how the coproduct commutes with the braid group.

Define for $1 \leqq i \leqq n$:

$$R_i = \sum_{k \geqq 0} (-1)^k q_i^{\frac{-k(k-1)}{2}} (q_i - q_i^{-1})^k [k]_i! \, T_i(F_i)^{(k)} \otimes T_i(E_i)^{(k)},$$

$$\bar{R}_i = (T_i^{-1} \otimes T_i^{-1}) R_i^{-1} = \sum_{k \geqq 0} q_i^{\frac{k(k-1)}{2}} (q_i - q_i^{-1})^k [k]_i! \, F_i^{(k)} \otimes E_i^{(k)}.$$

The following proposition is due in the finite type case to [K-R], [L-S]. The Kac–Moody case is due to [L 37.3.2].

5.1 Proposition. *Let* $S_i = T_i \otimes T_i$. *Let* $1 \leqq i \leqq n$, $x \in U_q$.

(a) $\Delta(T_i(x)) = R_i^{-1} \cdot S_i \Delta(x) \cdot R_i$,
(b) $\Delta(T_i^{-1}(x)) = \bar{R}_i^{-1} \cdot S_i^{-1} \Delta(x) \cdot \bar{R}_i$.

Let $\tau s_{i_1} \ldots s_{i_r}$ be a reduced presentation of w. Define

$$R_w = \tau(S_{i_1} S_{i_2} \ldots S_{i_{r-1}}(R_{i_r}) \ldots S_{i_1}(R_{i_2}) R_{i_1}),$$
$$\bar{R}_w = (S_{i_r}^{-1} \ldots S_{i_2}^{-1}(\bar{R}_{i_1}) \ldots S_{i_r}^{-1}(\bar{R}_{i_{r-1}}) \bar{R}_{i_r}.$$

5.2 Lemma. *Let* $w \in W$, R_w, $\bar{R}_w$ *are well defined.*

Proof. If W is the affine Weyl group of $\widehat{\mathfrak{sl}_2}$ any reduced presentation is unique. Otherwise, since any two reduced presentations differ by a finite sequence of braid relations it is enough to check the statement for the rank two case. Consider $R_{s_i s_j s_i}$, $R_{s_j s_i s_j}$ in the simply laced case. They are certainly equal since both (up to a torus element) are expressions for the rank 2 universal R-matrix (see [K-R], [L-S]).

5.3 Proposition. *Let* $1 \leqq i \leqq n$, $k \geqq 0$. *Let* $w = k\omega_i$.

(a) $\Delta(x_{ik}^-) = R_w^{-1}(x_{ik}^- \otimes K_{-\alpha_i + k\delta} + 1 \otimes x_{ik}^-) R_w$,

(b) $\Delta(x_{i,-k}^-) = \bar{R}_w^{-1}(x_{i,-k}^- \otimes K_{-\alpha_i - k\delta}^{-1} + 1 \otimes x_{i,-k}^-) \bar{R}_w$.

Proof. This follows inductively from the above formulas.

To obtain the coproduct on x_{ik}^{+} note that $\Omega(x_{i,-k}^{-})=x_{ik}^{+}$ and use $\Delta\circ\Omega=\Omega\otimes\Omega\circ\sigma\circ\Delta$ to the above formulas.

6. A PBW basis of U_q

For $1\leq i\leq n$ the elements $C^{k/2}\bar{\psi}_{ik}\in U_{\mathscr{A}}^{+}$. On specialization to $q=1$ they form a basis of the root space $k\delta$ of $\hat{\mathfrak{g}}$. This follows from the previous section since $\bar{\psi}_{ik}=h_{ik}\bmod(q-1)$, which implies their linear independence on specialization. Note that if $w(\alpha_i)=\beta$ (α_i simple, β positive, $w\in W$) then $T_w(E_i)$ specializes to a root vector of $\hat{\mathfrak{g}}$ of root β.

For $\beta\in\Delta_{+}^{\mathrm{re}}(\hat{\mathfrak{g}})$ choose $w_\beta\in\tilde{W}$ so that $w_\beta(\alpha_{i_\beta})=\beta$ for some $i_\beta\in I$. Define $E_\beta=T_{w_\beta}(E_{i_\beta})$. *For* $\kappa\colon\Delta_{+}^{\mathrm{re}}\to\mathbb{N}$, $\iota\colon\{1,\ldots,n\}\times\Delta_{+}^{\mathrm{im}}\to\mathbb{N}$ define

$$E^{\kappa,\iota}=\prod E_\beta^{\kappa(\beta)}(C^{k/2}\bar{\psi}_{ik})^{\iota(i,k\delta)},\quad F^{\kappa',\iota'}=\Omega(E^{\kappa',\iota'})\,,$$

where the product is in a predetermined total order over the positive roots counted with multiplicity.

6.1 Proposition. *The $E^{\kappa,\iota}$ form a basis of $U_q^{+}(\hat{\mathfrak{g}})$ as a $\mathbb{C}(q)$-vector space. The elements $F^{\kappa',\iota'}K_\alpha C^{r'+1/2}D^rE^{\kappa,\iota}$ ($\alpha\in\tilde{Q}$, $r,r'\in\mathbb{Z}$, κ,ι as above) form a basis of $U_q(\hat{\mathfrak{g}})$ as a $\mathbb{C}(q)$-vector space.*

Proof. The proof can be repeated almost word for word as found in [L5 Sect. 1]. In the proof of linear independence of the $E^{\kappa,\iota}$, a dominant integral highest weight should be chosen so that for $\kappa,\iota\in\mathfrak{G}$ (in the notation found there) the $\bar{E}^{\kappa,\iota}$ form a linearly independent set in $\bar{M}$.

Remark. The above basis is called of Poincaré–Birkhoff–Witt type because on specialization to 1 it degenerates to a PBW basis of the enveloping algebra $U(\hat{\mathfrak{g}})$.

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