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The Initial Value Problem for the Whitham Averaged System

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Abstract: We study the initial value problem for the Whitham averaged system which is important in determining the KdV zero dispersion limit. We use the hodograph method to show that, for a generic non-trivial monotone initial data, the Whitham averaged system has a solution within a region in the x-t plane for all time bigger than a large time. Furthermore, the Whitham solution matches the Burgers solution on the boundaries of the region. For hump-like initial data, the hodograph method is modified to solve the non-monotone (in x) solutions of the Whitham averaged system. In this way, we show that, for a hump-like initial data, the Whitham averaged system has a solution within a cusp for a short time after the increasing and decreasing parts of the initial data begin to interact. On the cusp, the Whitham and Burgers solutions are matched.

1. Introduction

In this paper, we study the Whitham averaged system:

$$\beta_{it} + \lambda_i(\beta_1, \beta_2, \beta_3)\beta_{ix} = 0, \quad i = 1, 2, 3 , \tag{1.1}$$

where

$$\begin{aligned}
\lambda_1(\beta_1,\beta_2,\beta_3) &= 2(\beta_1+\beta_2+\beta_3) + 4(\beta_1-\beta_2)\frac{K(s)}{E(s)}, \\
\lambda_2(\beta_1,\beta_2,\beta_3) &= 2(\beta_1+\beta_2+\beta_3) + 4(\beta_2-\beta_1)\frac{sK(s)}{E(s)-(1-s)K(s)}, \\
\lambda_3(\beta_1,\beta_2,\beta_3) &= 2(\beta_1+\beta_2+\beta_3) + 4(\beta_2-\beta_3)\frac{K(s)}{E(s)-K(s)},
\end{aligned} \tag{1.2}$$

and

$$s = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3} .$$

$K(s)$ and $E(s)$ are complete elliptic integrals of the first and second kind. Equation (1.1) was first found by Whitham [17], and its hierarchy was found independently by Lax and Levermore [7, 8], and Flaschka, Forest and McLaughlin [3].

The zero dispersion limit of the KdV equation can be determined by an initial value problem of the Whitham averaged system (1.1) and its hierarchy [7, 8, 16]. This initial data is the same as the KdV initial data. Solutions of the different systems of the hierarchy are matched naturally on the phase transition boundaries. In particular, the Whitham solution of (1.1) would match the solution of the Burgers equation:

$$\beta_t + 6\beta\beta_x = 0$$

on the boundaries separating the Whitham and Burgers solutions. The Burgers equation and the Whitham averaged system (1.1) are the first and second members of the Whitham hierarchy, respectively.

The investigation of the initial value problem of the Whitham averaged system began with Gurevich and Pitaevskii [4]. They solved the initial value problem of system (1.1) for step initial data, and studied numerically the case of cubic initial data. However, the structure of system (1.1) and its hierarchy was understood only during the last decade. Dubrovin and Novikov [1, 2] developed a geometric-Hamiltonian theory for the hierarchy. Based on this theory, Tsarev [15] was able to prove that each member of the hierarchy can be solved by a hodograph method. This method was put into an algebro-geometric setting by Krichever [5]. Using the Tsarev–Krichever approach, Potemin [10] and Wright [18] managed to solve the initial value problem of system (1.1) for cubic and cubic like initial data, respectively.

Another way to make use of Tsarev's hodograph method is to further transform system (1.1) into a linear overdetermined system of Euler–Poisson–Darboux type [11, 12, 13],

$$\begin{aligned} 2(\beta_i - \beta_j)\frac{\partial^2 q}{\partial\beta_i\partial\beta_j} &= \frac{\partial q}{\partial\beta_i} - \frac{\partial q}{\partial\beta_j}, \quad i,j = 1,2,3\,, \\ q(\beta,\beta,\beta) &= f(\beta), \quad i \neq j\,, \end{aligned} \tag{1.3}$$

where $x = f(u)$ is the KdV initial data. Part of this result was also obtained by Kudashev and Sharapov [6]. All the other members of the Whitham hierarchy are also connected with higher dimensional linear overdetermined systems of Euler–Poisson–Darboux type [14].

System (1.3) has a unique solution, and its solution can be written down explicitly. This explicit expression of solution to system (1.3) enabled the author [11, 12] to solve the initial value problem for decreasing initial data with only one inflection point.

In this paper, we consider the initial value problem for the Whitham averaged system for generic decreasing and hump-like initial data. We show that for a generic decreasing initial data, the initial value problem for system (1.1) has a solution for t bigger than a large time. Tsarev's hodograph method is modified to solve system (1.1) for hump-like initial data. We show that the Whitham averaged system has a solution for a short time after the increasing and decreasing parts of the hump-like initial data begin to interact.

This paper is organized as follows. In Sect. 2, we describe in detail the initial value problem and the hodograph method. The Whitham averaged system in the case of generic decreasing initial data is solved in Sect. 3 for large time. In the last section, we solve the initial value problem for hump-like initial data.

2. A Hodograph Method

In this section, we describe the initial value problem of the Whitham averaged system, and introduce the hodograph method. Properties of system (1.1) will be discussed, and some known results will be presented.

Consider a horizontal motion of an initial curve $u = u_0(x)$. Each point on the curve has a different speed. Initially, the curve is expressed by a single valued function $u = \beta(x,t)$, and the motion of each point is given by the Burgers equation:

$$\beta_t + 6\beta\beta_x = 0 , \quad \beta(x,0) = u_0(x) . \tag{2.1}$$

At a later time, the evolving curve can only, in general, be given by a multi-valued function with an odd number of branches: $u = \beta_k(x,t), k = 1,2,\ldots,2g+1$, where $\beta_{2g+1} < \beta_{2g} < \cdots < \beta_1$. These branches move according to the $(g+1)$th system in the Whitham hierarchy. In this paper, we concentrate on the three branch case. Therefore, the motion of β_1, β_2 and β_3 is governed by the Whitham averaged system (1.1).

Within the multivalued region, β_1, β_2 and β_3 satisfy system (1.1) while outside it, the single branch β is given by the Burgers equation (2.1). The Whitham and Burgers solutions are matched naturally on the boundaries.

a) At the trailing edge:

$$\beta_1 = \text{the Burgers solution defined outside the region} . \quad \beta_2 = \beta_3 . \tag{2.2}$$

b) At the leading edge:

$$\beta_1 = \beta_2 . \quad \beta_3 = \text{the Burgers solution defined outside the region} . \tag{2.3}$$

The initial value problem of the Whitham averaged system is to determine the multibranches β_1, β_2 and β_3 with boundary conditions (2.2) and (2.3) from the initial curve $u = u_0(x)$.

Complete elliptic integrals $K(s)$ and $E(s)$ have some well-known properties. As $-1 < s < 1$, we have:

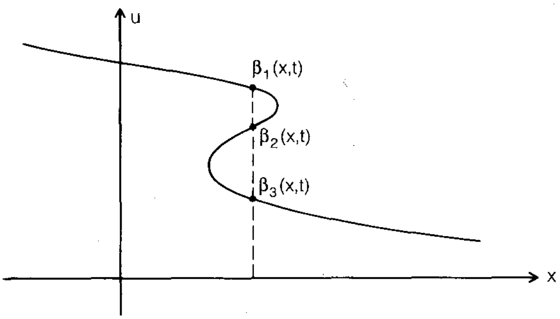

Fig. 1.

$$K(s) = \frac{\pi}{2}\left[1 + \frac{s}{4} + \frac{9}{64}s^2 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2 s^n + \cdots\right] \tag{2.4}$$

$$E(s) = \frac{\pi}{2}\left[1 - \frac{s}{4} - \frac{3}{64}s^2 - \cdots - \frac{1}{2n-1}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2 s^n - \cdots\right] \tag{2.5}$$

while, as $1 - s \ll 1$, we have:

$$K(s) \approx \frac{1}{2}\log\frac{16}{1-s}, \tag{2.6}$$

$$E(s) \approx 1 + \frac{1}{4}(1-s)\left[\log\frac{16}{1-s} - 1\right]. \tag{2.7}$$

Furthermore,

$$\frac{dK(s)}{ds} = \frac{E(s) - (1-s)K(s)}{2s(1-s)}, \tag{2.8}$$

$$\frac{dE(s)}{ds} = \frac{E(s) - K(s)}{2s}. \tag{2.9}$$

It immediately follows from (2.4) and (2.5) that:

$$\frac{1}{1-\frac{s}{2}} < \frac{K(s)}{E(s)} < \frac{1-\frac{s}{2}}{1-s}, \quad \text{for } 0 < s < 1. \tag{2.10}$$

Using (2.10) in (1.2), we obtain: for $\beta_1 > \beta_2 > \beta_3$,

$$\begin{aligned} &\lambda_1 - 2(\beta_1 + \beta_2 + \beta_3) > 0, \\ &\lambda_2 - 2(\beta_1 + \beta_2 + \beta_3) < 0, \\ &\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3) < 0. \end{aligned} \tag{2.11}$$

By (1.2) and (2.4)–(2.7), we find that λ_1, λ_2 and λ_3 have behavior:

1) At $\beta_2 = \beta_3$:

$$\begin{aligned} &\lambda_1(\beta_1, \beta_3, \beta_3) = 6\beta_1, \\ &\lambda_2(\beta_1, \beta_3, \beta_3) = \lambda_3(\beta_1, \beta_3, \beta_3) = 12\beta_3 - 6\beta_1. \end{aligned} \tag{2.12}$$

2) At $\beta_1 = \beta_2$:

$$\lambda_1(\beta_1, \beta_1, \beta_3) = \lambda_2(\beta_1, \beta_1, \beta_3) = 4\beta_1 + 2\beta_3, \tag{2.13}$$

$$\lambda_1(\beta_1, \beta_1, \beta_3) = 6\beta_3. \tag{2.13}$$

The Whitham averaged system (1.1) is a strictly hyperbolic and genuinely nonlinear system. In fact, we have [9]:

Lemma 2.1. *For* $\beta_1 > \beta_2 > \beta_3$,

i) *Strict hyperbolicity*:

$$\lambda_1(\beta_1,\beta_2,\beta_3) > \lambda_2(\beta_1,\beta_2,\beta_3) > \lambda_3(\beta_1,\beta_2,\beta_3)\,. \tag{2.14}$$

ii) *Genuine nonlinearity*:

$$\frac{\partial\lambda_i(\beta_1,\beta_2,\beta_3)}{\partial\beta_i} > 0, \quad i = 1,2,3\,. \tag{2.15}$$

Other results are given in the next lemma [11, 12].

Lemma 2.2. *For* $\beta_1 > \beta_2 > \beta_3$,

$$\frac{\partial\lambda_3}{\partial\beta_3} < \frac{3}{2}\frac{\lambda_2-\lambda_3}{\beta_2-\beta_3} < \frac{\partial\lambda_2}{\partial\beta_2}\,.$$

The most remarkable feature about the Whitham averaged system is that it can be solved by a hodograph method. More precisely, we have [15]:

Theorem 2.3. *If* $w_i(\beta_1,\beta_2,\beta_3)$*'s solve the following linear overdetermined system*:

$$\frac{\partial w_i}{\partial\beta_j} = a_{ij}(\beta_1,\beta_2,\beta_3)\left[w_i - w_j\right], \quad i,j = 1,2,3 \quad i \neq j\,, \tag{2.16}$$

where

$$a_{ij}(\beta_1,\beta_2,\beta_3) = \frac{\frac{\partial\lambda_i}{\partial\beta_j}}{\lambda_i-\lambda_j}, \quad i,j = 1,2,3 \quad i\neq j\,, \tag{2.17}$$

then the solution $(\beta_1(x,t),\beta_2(x,t),\beta_3(x,t))$ *of the hodograph transformation*:

$$x = \lambda_i(\beta_1,\beta_2,\beta_3)t + w_i(\beta_1,\beta_2,\beta_3), \quad i = 1,2,3 \tag{2.18}$$

satisfies system (1.1). *Conversely, any solution* $(\beta_1,\beta_2,\beta_3)$ *of system* (1.1) *can be obtained in this way in the neighborhood of* (x_0,t_0) *at which* β_{ix}*'s are not vanishing.*

We shall use the hodograph transform (2.18) to construct the Whitham solution satisfying boundary conditions (2.2) and (2.3). First, system (2.16) needs to be solved for $w_i(\beta_1,\beta_2,\beta_3)$'s. In this respect, we want to understand what kinds of boundary conditions should be imposed on $w_i(\beta_1,\beta_2,\beta_3)$'s.

Clearly, the Burgers solution of (2.1) outside the multivalued region satisfies the characteristics equation:

$$x = 6\beta t + f(\beta)\,, \tag{2.19}$$

where $f(u)$ is the inverse function of the decreasing initial data $u = u_0(x)$.

By (2.2), (2.3), (2.12), (2.13), (2.18) and (2.19), we see:

At the trailing edge:

$$\begin{aligned} w_1(\beta_1,\beta_3,\beta_3) &= f(\beta_1)\,,\\ w_2(\beta_1,\beta_3,\beta_3) &= w_3(\beta_1,\beta_3,\beta_3)\,. \end{aligned} \tag{2.20}$$

Similar conditions hold at the leading edge:

$$\begin{aligned} w_1(\beta_1,\beta_1,\beta_3) &= w_2(\beta_1,\beta_1,\beta_3)\,,\\ w_3(\beta_1,\beta_1,\beta_3) &= f(\beta_3)\,. \end{aligned} \tag{2.21}$$

Motivated by the above observation, we consider system (2.16) with boundary conditions (2.20) and (2.21). We shall explicitly construct all the solutions to this boundary value problem. This is carried out in the next two theorems [11, 12].

Theorem 2.4. *If $q(\beta_1, \beta_2, \beta_3)$ is a solution of:*

$$2(\beta_i - \beta_j)\frac{\partial^2 q}{\partial\beta_i \partial\beta_j} = \frac{\partial q}{\partial\beta_i} - \frac{\partial q}{\partial\beta_j}, \quad i,j = 1,2,3\,, \tag{2.22}$$

$$q(\beta,\beta,\beta) = f(\beta), \quad i \neq j\,, \tag{2.23}$$

then (w_1, w_2, w_3) given by:

$$w_i(\beta_1,\beta_2,\beta_3) = \frac{1}{2}\left[\lambda_i - 2(\beta_1+\beta_2+\beta_3)\right]\frac{\partial q}{\partial\beta_i} + q, \quad i = 1,2,3 \tag{2.24}$$

solves the boundary value problem (2.16), (2.20) *and* (2.21). *Conversely, every solution of* (2.16), (2.20) *and* (2.21) *can be obtained in this way.*

Theorem 2.5. *The boundary value problem* (2.22) *and* (2.23) *has a unique solution. This solution is symmetric, and is given by:*

$$q(\beta_1,\beta_2,\beta_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f\left(\frac{1+\mu}{2}\frac{1+\nu}{2}\beta_1 + \frac{1+\mu}{2}\frac{1-\nu}{2}\beta_2 + \frac{1-\mu}{2}\beta_3\right)}{\sqrt{(1-\mu)(1-\nu^2)}}\,d\mu d\nu\,. \tag{2.25}$$

The hodograph transform (2.18) with w_i's given by (2.24) and (2.25) needs to be solved to produce the solution to system (1.1). More precisely, we have [11, 12]:

Theorem 2.6. *Consider a decreasing initial data $x = f(u)$. Suppose that $f(u)$ has only one inflection point and that $f'''(u) < 0$ beyond this inflection point. Then the hodograph transform* (2.18) *with w_i's given by* (2.24) *and* (2.25) *can be solved for β_1, β_2 and β_3 within a cusp in the x-t plane for all time after the breaking time of the Burgers solution of* (2.1). *Furthermore, these β_1, β_2 and β_3 satisfy boundary conditions* (2.2) *and* (2.3) *on the cusp.*

Theorems 2.3 and 2.6 immediately establish [11, 12]:

Theorem 2.7. *Under the conditions of Theorem* 2.6, *the Whitham averaged system* (1.1) *has a solution $(\beta_1, \beta_2, \beta_3)$ within a cusp in the x-t plane for all time after the breaking time of the Burgers solution of* (2.1). *Furthermore, this solution satisfies boundary conditions* (2.2) *and* (2.3) *on the cusp.*

Local conditions on $f(u)$ will give short time results [11, 12].

Theorem 2.8. *Consider a decreasing initial data $x = f(u)$. Suppose that u^* is the inflection point that causes the breaking in the Burgers solution of* (2.1), *and that $f'''(u) < 0$ locally in a deleted neighborhood of $u = u^*$. Then the Whitham averaged system* (1.1) *has a solution $(\beta_1, \beta_2, \beta_3)$ within a cusp in the x-t plane for a short time after the breaking time of the Burgers solution of* (2.1). *Furthermore, this solution satisfies boundary conditions* (2.2) *and* (2.3) *on the cusp.*

A hump-like initial data can be decomposed into a decreasing and an increasing data. It is known that the decreasing part causes the Burgers solution of (2.1) to blow up, while the increasing one does not. These two data would not interact with each other for a short time after the breaking of the Burgers solution. As a consequence, a short time result similar to Theorem 2.8 holds for a hump-like initial data [11, 12].

Theorem 2.9. *For a hump-like initial data whose decreasing part satisfies conditions of Theorem* 2.8, *the Whitham averaged system* (1.1) *has a solution* $(\beta_1, \beta_2, \beta_3)$ *within a cusp in the x-t plane for a short time after the breaking time of the Burgers solution of* (2.1). *Furthermore, this solution satisfies boundary conditions* (2.2) *and* (2.3) *on the cusp.*

3. Large Time Results for the Whitham Averaged System

In this section, we study the initial value problem for the Whitham averaged system for large time. We shall show that for generic decreasing initial data, the Whitham averaged system has solutions after some large time. The main idea is to use the hodograph method to solve the Whitham averaged system for large time.

For convenience, we consider a smooth decreasing initial data $u = u_0(x)$ which is bounded at the infinity:

$$\lim_{x \to -\infty} u_0(x) = a, \qquad \lim_{x \to +\infty} u_0(x) = b\,.$$

Other types of decreasing initial data will be considered later in this section. The inverse function $x = f(u)$ of the initial data is defined over (b, a), and behaves as:

$$\lim_{u \to a} f(u) = -\infty, \qquad \lim_{u \to b} f(u) = +\infty\,. \tag{3.1}$$

First, we have:

Lemma 3.1. *Consider a decreasing initial data* $x = f(u)$ *defined over* (b, a). *Suppose that in addition to* (3.1), $f(u)$ *satisfies*:

$$f'''(u) < 0$$

in the neighborhood of $u = a$ *and* $u = b$. *Then there exists a* $\delta > 0$ *such that* $q(\beta_1, \beta_2, \beta_3)$ *of* (2.25) *satisfies*:

$$\frac{\partial^3}{\partial\beta_1^i \partial\beta_2^j \partial\beta_3^k} q(\beta_1, \beta_2, \beta_3) < 0, \quad k = 1, 2 \ \textit{and} \ i + j + k = 3$$

for $a > \beta_1 \geqq \beta_2 \geqq \beta_3 > b$ *and* $\delta > \beta_3 - b > 0$.

Proof. We first claim that

$$\begin{aligned} f''(u) < 0, &\quad \text{in a neighborhood of } u = a\,; \\ f''(u) > 0, &\quad \text{in a neighborhood of } u = b\,. \end{aligned} \tag{3.2}$$

We shall prove the first inequality by contradiction, and the second one can be shown in the same way. Suppose that the first inequality of our claim does not hold. Since $f'''(u) < 0$ in the neighborhood of $u = a$, we must have $f''(u) > 0$ near $u = a$. This implies that $f'(u)$ is increasing in the neighborhood of $u = a$, and that therefore, $f'(u)$ is bigger than a constant when u is near a. A simple integration would prove that $f(u)$ is bounded from below in the neighborhood of $u = a$. This contradicts the assumption (3.1), and the claim is justified.

It immediately follows from (3.1) and (3.2) that

$$\lim_{u \to a} f'(u) = -\infty, \quad \lim_{u \to b} f'(u) = -\infty . \tag{3.3}$$

Choose a_1 and b_1 such that $a > a_1 > b_1 > b$, and that

$$f'''(u) < 0, \text{ outside } (b_1, a_1) .$$

By (2.25), we obtain:

$$\frac{\partial^3}{\partial \beta_1^i \partial \beta_2^j \partial \beta_3^k} q(\beta_1, \beta_2, \beta_3)$$

$$= C \int_{\beta_2}^{\beta_1} \frac{\int_{\beta_3}^{\eta} f'''(\xi)(\xi - \beta_3)^{i+j}(\eta - \xi)^{k-\frac{1}{2}} d\xi}{(\eta - \beta_3)^{\frac{7}{2}}} (\beta_1 - \eta)^{j-\frac{1}{2}} (\eta - \beta_2)^{i-\frac{1}{2}} d\eta$$

for $a > \beta_1 > \beta_2 \geqq \beta_3 > b$, where

$$C = \frac{1}{2\pi(\beta_1 - \beta_2)^{i+j}} .$$

Since $f'''(u) < 0$ outside (b_1, a_1), it suffices to show that there exists a $\delta > 0$ such that

$$\int_{\beta_3}^{\eta} f'''(\xi)(\xi - \beta_3)^{i+j}(\eta - \xi)^{k-\frac{1}{2}} d\xi < 0, \quad \text{for all } \eta \geqq b_1 \tag{3.4}$$

when $0 < \beta_3 - b < \delta$.

For a fixed small $\varepsilon_0 > 0$, we have:

$$\int_{\beta_3}^{\eta} f'''(\xi)(\xi - \beta_3)^{i+j}(\eta - \xi)^{k-\frac{1}{2}} d\xi$$

$$= \int_{b_1 - \varepsilon_0}^{\eta} f'''(\xi)(\xi - \beta_3)^{i+j}(\eta - \xi)^{k-\frac{1}{2}} d\xi + \int_{\beta_3}^{b_1 - \varepsilon_0} f'''(\xi)(\xi - \beta_3)^{i+j}(\eta - \xi)^{k-\frac{1}{2}} d\xi$$

$$\leqq \int_{b_1 - \varepsilon_0}^{a_1} |f'''(\xi)|(\xi - \beta_3)^{i+j}(\eta - \xi)^{k-\frac{1}{2}} d\xi + \varepsilon_0^{k-\frac{1}{2}} \int_{\beta_3}^{b_1 - \varepsilon_0} f'''(\xi)(\xi - \beta_3)^{i+j} d\xi . \tag{3.5}$$

The first term is uniformly bounded for all $\beta_3 \in (b, a)$ and $\eta \in [b_1, a)$, and the second one can be decomposed into:

$$\int_{\beta_3}^{b_1 - \varepsilon_0} f'''(\xi)(\xi - \beta_3)^{i+j} d\xi$$

$$= (b_1 - \varepsilon_0 - \beta_3) f''(b_1 - \varepsilon_0) - f'(b_1 - \varepsilon_0) + f'(\beta_3), \quad \text{when } i + j = 1$$

or

$$= (b_1 - \varepsilon_0 - \beta_3)^2 f''(b_1 - \varepsilon_0) - 2(b_1 - \varepsilon_0 - \beta_3) f'(b_1 - \varepsilon_0)$$

$$+ 2f(b_1 - \varepsilon_0) - 2f(\beta_3), \quad \text{when } i + j = 2 .$$

This when combined with (3.1) and (3.3) proves:

$$\int_{\beta_3}^{b_1-\varepsilon_0} f'''(u)(\xi-\beta_3)^{i+j}d\xi \to -\infty, \quad \text{as } \beta_3 \to b\,,$$

which together with (3.5) implies (3.4). Therefore, Lemma 3.1 is proved for the case $\beta_1 > \beta_2 \geqq \beta_3$. The rest of Lemma 3.1 will be shown as follows.

At $\beta_1 = \beta_2$, by (2.25) we have

$$\frac{\partial^3}{\partial\beta_1^i\partial\beta_2^j\partial\beta_3^k}q(\beta_1,\beta_2,\beta_3) = C_1\int_{\beta_3}^{\beta_1} f'''(\xi)(\xi-\beta_3)^{i+j}(\beta_1-\xi)^{k-\frac{1}{2}}d\xi\,,$$

where

$$C_1 = \frac{1}{2^{1+i+j}\pi(\beta_1-\beta_3)^{\frac{7}{2}}}\int_{-1}^{1}(1+v)^{i-\frac{1}{2}}(1-v)^{j-\frac{1}{2}}dv\,.$$

This and (3.4) prove Lemma 3.1 at $\beta_1 = \beta_2$. The proof of Lemma 3.1 is completed.

We need the next two lemmas.

Lemma 3.2. *Under the conditions of Lemma* 3.1, *we have*:

$$\lim_{\beta_1\to a}\int_{\beta_3}^{\beta_1}\frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi = -\infty, \quad \textit{for each } \beta_3 \in (b,a)\,,$$

$$\lim_{\beta_3\to b}\int_{\beta_3}^{\beta_1}\frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi = +\infty \quad \textit{for each } \beta_1 \in (b,a)\,.$$

Proof. By (3.2), we can choose a_2 and b_2 such that $a > a_2 > b_2 > b$, and that

$$f''(u) < 0, \quad \text{for } u > a_2\,,$$
$$f''(u) > 0, \quad \text{for } u < b_2\,.$$

For each $\beta_3 \in (b,a)$, we can choose $\bar{a}_2$ such that $\max\{\beta_3,a_2\} < \bar{a}_2 < a$. Thus,

$$\begin{aligned}
&\int_{\beta_3}^{\beta_1}\frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi \\
&= \int_{\bar{a}_2}^{\beta_1} + \int_{\beta_3}^{\bar{a}_2} \\
&\leqq \frac{1}{\sqrt{\beta_1-\beta_3}}\int_{\bar{a}_2}^{\beta_1} f''(\xi)(\beta_1-\xi)d\xi + \int_{\beta_3}^{\bar{a}_2}\frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi \\
&= \frac{1}{\sqrt{\beta_1-\beta_3}}[-f'(\bar{a}_2)(\beta_1-\bar{a}_2)+f(\beta_1)-f(\bar{a}_2)] \\
&\quad + \int_{\beta_3}^{\bar{a}_2}\frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi
\end{aligned}$$

for $\beta_1 > \bar{a}_2$. This and (3.1) prove the first limit of Lemma 3.2.

As to the second limit, for each $\beta_1 \in (b,a)$ we choose $\bar{b}_2$ such that $b < \bar{b}_2 < \min\{\beta_1,b_2\}$. Therefore,

$$\int_{\beta_3}^{\beta_1} \frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi$$
$$= \int_{\bar{b}_2}^{\beta_1} + \int_{\beta_3}^{\bar{b}_2}$$
$$\geqq \frac{\beta_1-\bar{b}_2}{\sqrt{\bar{b}_2-\beta_3}}[f'(\bar{b}_2)-f'(\beta_3)] + \int_{\bar{b}_2}^{\beta_1} \frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi$$

for $\beta_3 < \bar{b}_2$ which when combined with (3.3) proves the second part of Lemma 3.2.

In the same way, we can prove the next lemma.

Lemma 3.3. *Under the conditions of Lemma* 3.1, *we have*:

$$\lim_{\beta_1 \to a} \int_{\beta_3}^{\beta_1} f''(\xi)(\xi-\beta_3)\sqrt{\beta_1-\xi}\,d\xi = -\infty, \quad \textit{for each } \beta_3 \in (b,a)\,,$$
$$\lim_{\beta_3 \to b} \int_{\beta_3}^{\beta_1} f''(\xi)(\xi-\beta_3)\sqrt{\beta_1-\xi}\,d\xi = +\infty, \quad \textit{for each } \beta_1 \in (b,a)\,.$$

We are now ready to use the scheme of Sect. 2 to solve the Whitham averaged system for large time. We need to solve system (2.18) with w_i's given by (2.24) and (2.25) for β_1, β_2 and β_3 as functions of (x,t).

System (2.18) is simplified as follows. Eliminating x from (2.18) yields:

$$F(t,\beta_1,\beta_2,\beta_3) = 0\,,$$
$$G(t,\beta_1,\beta_2,\beta_3) = 0\,, \tag{3.6}$$

where

$$F = (\lambda_1 t + w_1) - (\lambda_2 t + w_2)\,,$$
$$G = (\lambda_2 t + w_2) - (\lambda_3 t + w_3)\,.$$

Substituting (1.2) and (2.24) into system (3.6), we obtain:

$$\tilde{F}(t,\beta_1,\beta_2,\beta_3) = 0\,,$$
$$\tilde{G}(t,\beta_1,\beta_2,\beta_3) = 0\,, \tag{3.7}$$

where

$$\tilde{F}(t,\beta_1,\beta_2,\beta_3) = \frac{F(t,\beta_1,\beta_2,\beta_3)}{(\beta_1-\beta_2)K(s)}$$
$$= 4\left[\frac{1}{E(s)}\left(t+\frac{1}{2}\frac{\partial q}{\partial \beta_1}\right) + \frac{s}{E(s)-(1-s)K(s)}\left(t+\frac{1}{2}\frac{\partial q}{\partial \beta_2}\right)\right],$$

$$\tilde{G}(t,\beta_1,\beta_2,\beta_3) = \frac{G(t,\beta_1,\beta_2,\beta_3)}{(\beta_2-\beta_3)}$$
$$= -4\left[\frac{(1-s)K(s)}{E(s)-(1-s)K(s)}\left(t+\frac{1}{2}\frac{\partial q}{\partial \beta_2}\right)\right.$$
$$\left. + \frac{K(s)}{E(s)-K(s)}\left(t+\frac{1}{2}\frac{\partial q}{\partial \beta_3}\right)\right]. \tag{3.8}$$

Clearly, system (3.6) is equivalent to system (3.7) in the region $\beta_1 > \beta_2 > \beta_3$. The reason to consider system (3.7) is obvious from the fact that system (3.6) is degenerate at the trailing edge ($\beta_2 = \beta_3$) and leading edge ($\beta_1 = \beta_2$), while system (3.7) is not. We shall first solve system (3.7) at both the trailing and leading edges.

Lemma 3.4. *Under conditions of Lemma* 3.1, *there exists a* $t^- > 0$ *such that system* (3.7) *has a unique solution* $(\beta_1^-(t), \beta_2^-(t), \beta_3^-(t))$ *with* $\beta_1^-(t) > \beta_2^-(t) = \beta_3^-(t)$ *for all* $t > t^-$.

Proof. Using (2.4) and (2.5) in (3.8), we find:

$$\tilde{F}(t,\beta_1,\beta_2,\beta_3) = \frac{8}{\pi}\left[\left(1+\frac{1}{4}s+\frac{7}{64}s^2+\cdots\right)\left(t+\frac{1}{2}\frac{\partial q}{\partial \beta_1}\right) + 2\left(1-\frac{1}{8}s-\frac{1}{32}s^2+\cdots\right)\left(t+\frac{1}{2}\frac{\partial q}{\partial \beta_2}\right)\right],$$

$$\tilde{G}(t,\beta_1,\beta_2,\beta_3) = -4\left[\left(1-\frac{7}{8}s+\cdots\right)2(\beta_1-\beta_3)\frac{\partial^2 q}{\partial\beta_2\partial\beta_3} + \left(-\frac{3}{2}+\cdots\right)\left(t+\frac{1}{2}\frac{\partial q}{\partial\beta_3}\right)\right]. \tag{3.9}$$

Therefore, at the trailing edge $\beta_2 = \beta_3$ where $s = 0$, (3.7) becomes:

$$6t + \frac{\partial q}{\partial\beta_1} + 2\frac{\partial q}{\partial\beta_3} = 0\,,$$

$$2(\beta_1-\beta_3)\frac{\partial^2 q}{\partial\beta_2\partial\beta_3} - \frac{3}{2}\left[t+\frac{1}{2}\frac{\partial q}{\partial\beta_3}\right] = 0\,.$$

Substituting the first equation into the second one, and using (2.22), we obtain:

$$U(t,\beta_1,\beta_3) = 6t + \frac{\partial q}{\partial\beta_1}(\beta_1,\beta_3,\beta_3) + 2\frac{\partial q}{\partial\beta_3}(\beta_1,\beta_3,\beta_3) = 0\,, \tag{3.10}$$

$$V(\beta_1,\beta_3) = \frac{\partial^2 q}{\partial\beta_1\partial\beta_3}(\beta_1,\beta_3,\beta_3) + 4\frac{\partial^2 q}{\partial\beta_2\partial\beta_3}(\beta_1,\beta_3,\beta_3) = 0\,. \tag{3.11}$$

By Theorem 2.5, (2.25) can be rewritten as:

$$q(\beta_1,\beta_2,\beta_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f\left(\frac{1+\mu}{2}\frac{1+\nu}{2}\beta_3 + \frac{1+\mu}{2}\frac{1-\nu}{2}\beta_2 + \frac{1-\mu}{2}\beta_1\right)}{\sqrt{(1-\mu)(1-\nu^2)}}d\mu d\nu\,.$$

Substituting this into (3.11), we have:

$$V(\beta_1,\beta_3) = \frac{\sqrt{2}}{16}\int_{-1}^{1}\frac{f''\left(\frac{1+\mu}{2}\beta_3+\frac{1-\mu}{2}\beta_1\right)}{\sqrt{1-\mu}}(1+\mu)d\mu$$

$$= \frac{1}{4(\beta_1-\beta_3)^{\frac{3}{2}}}\int_{\beta_3}^{\beta_1}\frac{f''(\xi)}{\sqrt{\xi-\beta_3}}(\beta_1-\xi)d\xi\,.$$

First we want to solve (3.11) when β_3 is close to b. Choose a fixed $\tilde{\beta}_1 \in (b+\delta, a)$, and it follows from the second limit of Lemma 3.2 that we can find a $\beta_3^* \in (b, b+\delta)$ such that

$$V(\tilde{\beta}_1, \beta_3^*) > 0\,,$$

where δ is given in Lemma 3.1. On the other hand, by the first limit of Lemma 3.2 we can also find a $\bar{\beta}_1 > \tilde{\beta}_1$ such that

$$V(\bar{\beta}_1, \beta_3^*) < 0\,.$$

These two inequalities show that there exists a $\beta_1^* > \beta_3^*$ such that

$$V(\beta_1^*, \beta_3^*) = 0\,.$$

Denote t^- by

$$t^- = -\frac{1}{6}\left[\frac{\partial q}{\partial \beta_1}(\beta_1^*, \beta_3^*, \beta_3^*) + 2\frac{\partial q}{\partial \beta_3}(\beta_1^*, \beta_3^*, \beta_3^*)\right] > 0\,. \qquad (3.12)$$

where we have used (2.25) and the assumption that $f^{+'} < 0$ in the inequality. Hence, $(t^-, \beta_1^*, \beta_3^*)$ satisfies (3.10) and (3.11), and $\beta_3^* \in (b, b+\delta)$. Before we proceed, we need the following lemma.

Lemma 3.5. *Under the conditions of Lemma* 3.1, *we have*:

$$\frac{\partial^2 q}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} < 0, \quad \frac{\partial^2 q}{\partial \beta_1^2} < 0\,,$$

$$\frac{\partial^2 q}{\partial \beta_2^2} = \frac{\partial^2 q}{\partial \beta_3^2} = 3\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} > 0$$

on the solution $(\beta_1, \beta_3, \beta_3)$ *of* (3.11), *where* $a > \beta_1 > \beta_3 > b$ *and* $\delta > \beta_3 - b > 0$.

Proof. By (2.22), we have:

$$2(\beta_1 - \beta_2)\frac{\partial^2 q}{\partial \beta_1 \partial \beta_2} = \frac{\partial q}{\partial \beta_1} - \frac{\partial q}{\partial \beta_2}\,.$$

Taking derivative with respect to β_3 yields

$$\frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} - \frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} = 2(\beta_1 - \beta_2)\frac{\partial^3 q}{\partial \beta_1 \partial \beta_2 \partial \beta_3}\,.$$

This and Lemma 3.1 imply

$$\frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} < \frac{\partial^2 q}{\partial \beta_2 \partial \beta_3}\,,$$

which together with (3.11) proves:

$$\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} > 0, \quad \frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} < 0\,.$$

The rest of Lemma 3.5 can be shown in the same way.

We now continue the proof of Lemma 3.4. Using (3.11), Lemma 3.1 and Lemma 3.5, we calculate partial derivatives of U and V on the solution $(\beta_1, \beta_3, \beta_3)$ of (3.11), where $\beta_3 \in (b, b+\delta)$:

$$\frac{\partial U}{\partial \beta_1} = \frac{\partial^2 q}{\partial \beta_1^2} + 2\frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} < 0\,,$$
$$\frac{\partial U}{\partial \beta_3} = 2\frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} + 8\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} = 0\,,$$
$$\frac{\partial V}{\partial \beta_1} = \frac{\partial^3 q}{\partial \beta_1^2 \partial \beta_3} + 4\frac{\partial^3 q}{\partial \beta_1 \partial \beta_2 \partial \beta_3} < 0\,,$$
$$\frac{\partial V}{\partial \beta_3} = \frac{\partial^3 q}{\partial \beta_1 \partial \beta_2 \partial \beta_3} + \frac{\partial^3 q}{\partial \beta_1 \partial \beta_3^2} + 4\frac{\partial^3 q}{\partial \beta_2^2 \partial \beta_3} + 4\frac{\partial^3 q}{\partial \beta_2 \partial \beta_3^2} < 0\,.$$

Therefore, by the Implicit Function Theorem, (3.10) and (3.11) can be solved for:

$$\beta_1^-(t) = A(t), \quad \beta_3^-(t) = B(t) \tag{3.13}$$

in the neighborhood of t^-, where t^- is given by (3.12). It can easily be checked that $A(t)$ and $B(t)$ are increasing and decreasing with time, respectively. Therefore, $\beta_3^-(t)$ keeps closer to b as t increases. Repeat the Implicit Function Theorem; we see that (3.13) are defined for all $t > t^-$. This proves Lemma 3.4.

At the leading edge $\beta_1 = \beta_2$ where $s = 1$, it follows from (2.6), (2.7), (2.22) and (3.8) that system (3.7) turns out to be:

$$t + \frac{1}{2}\frac{\partial q}{\partial \beta_1}(\beta_1, \beta_1, \beta_3) = 0\,,$$
$$t + \frac{1}{2}\frac{\partial q}{\partial \beta_3}(\beta_1, \beta_1, \beta_3) = 0\,. \tag{3.14}$$

In the same way as we handle Lemma 3.4, we can use Lemma 3.3 to solve the above system for β_1 and β_3 as functions of t. Therefore, we have:

Lemma 3.6. *Under conditions of Lemma* 3.1, *there exists a* $t^+ > 0$ *such that system* (3.7) *has a unique solution* $(\beta_1^+(t), \beta_2^+(t), \beta_3^+(t))$ *with* $\beta_1^+(t) = \beta_2^+(t) > \beta_3^+(t)$ *for all* $t > t^+$.

The following lemma is obvious.

Lemma 3.7. *On the solution* $(t, \beta_1, \beta_2, \beta_3)$ *of* (3.6) [*or equivalently* (3.7)] *in the region* $\beta_1 > \beta_2 > \beta_3$, *we have*:

$$\frac{\partial(\lambda_i t + w_i)}{\partial \beta_j} = 0, \quad \textit{for} \quad i,j = 1,2,3 \quad i \neq j\,.$$

Proof.

$$\begin{aligned}\frac{\partial(\lambda_i t + w_i)}{\partial \beta_j} &= \frac{\partial \lambda_i}{\partial \beta_j} t + \frac{\partial w_i}{\partial \beta_j} \\ &= a_{ij}(\beta_1, \beta_2, \beta_3)[(\lambda_i t + w_i) - (\lambda_j t + w_j)] \\ &= 0\,,\end{aligned}$$

where we have used (2.16) and (2.17) in the second equality, and (3.7) in the last one. This proves Lemma 3.7.

By Lemma 3.4, $(\beta_1^-(t), \beta_2^-(t), \beta_3^-(t))$ satisfies system (3.7). For each fixed $t > t^-$, we need to solve (3.7) for β_1 and β_3 as functions of β_2 in the neighborhood of $\beta_2^-(t)$. This is carried out in:

Lemma 3.8. *For each $t > \max\{t^-, t^+\}$, system (3.7) can be solved for β_1 and β_3 in terms of β_2 in the neighborhood of $(\beta_1^-(t), \beta_2^-(t), \beta_3^-(t))$:*

$$\begin{cases} \beta_1 = M(\beta_2) \\ \beta_3 = N(\beta_2) \end{cases} \tag{3.15}$$

such that $\beta_1^-(t) = M(\beta_2^-(t))$ and $\beta_3^-(t) = N(\beta_2^-(t))$. Moreover, for $\beta_2 > \beta_2^-(t)$,

$$N(\beta_2) < \beta_2 < M(\beta_2) . \tag{3.16}$$

Proof. Calculating first partial derivatives of (3.9) at $(\beta_1^-(t), \beta_2^-(t), \beta_3^-(t))$ and using (2.22), we find:

$$\begin{aligned}
\frac{\partial \tilde{F}}{\partial \beta_1} &= \frac{8}{\pi}\left[\frac{1}{2}\frac{\partial^2 q}{\partial \beta_1^2} + \frac{\partial^2 q}{\partial \beta_1 \partial \beta_2}\right] < 0 , \\
\frac{\partial \tilde{F}}{\partial \beta_2} &= \frac{8}{\pi}\left[\frac{3}{4}\frac{\partial^2 q}{\partial \beta_1 \partial \beta_2} + \frac{\partial^2 q}{\partial \beta_2^2}\right] = 0 , \\
\frac{\partial \tilde{F}}{\partial \beta_3} &= \frac{8}{\pi}\left[\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} + \frac{1}{2}\frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} - \frac{1}{4}\frac{\partial^2 q}{\partial \beta_2 \partial \beta_1}\right] = 0 , \\
\frac{\partial \tilde{G}}{\partial \beta_1} &= -4\left[\frac{\partial^2 q}{\partial \beta_1 \partial \beta_2} - \frac{3}{4}\frac{\partial^2 q}{\partial \beta_1 \partial \beta_3} + \frac{\partial^2 q}{\partial \beta_2 \partial \beta_3}\right] = 0 , \\
\frac{\partial \tilde{G}}{\partial \beta_2} &= -4\left[-\frac{5}{2}\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} + 2(\beta_1 - \beta_3)\frac{\partial^3 q}{\partial \beta_2^2 \partial \beta_3}\right] > 0 , \\
\frac{\partial \tilde{G}}{\partial \beta_3} &= -4\left[-\frac{1}{4}\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3} - \frac{3}{4}\frac{\partial^2 q}{\partial \beta_3^2} + 2(\beta_1 - \beta_3)\frac{\partial^3 q}{\partial \beta_2 \partial \beta_3^2}\right] > 0 ,
\end{aligned} \tag{3.17}$$

where we have used Lemma 3.5 in the first inequality, (3.11) and Lemma 3.5 in the second, third and fourth equations, and Lemma 3.1 and Lemma 3.5 in the last two inequalities.

It follows from (3.17) that the Jacobian:

$$\frac{\partial(\tilde{F}, \tilde{G})}{\partial(\beta_1, \beta_3)}$$

is not vanishing at $(\beta_1^-(t), \beta_2^-(t), \beta_3^-(t))$. Hence, (3.7) can be solved for:

$$\beta_1 = M(\beta_2), \quad \beta_3 = N(\beta_2)$$

in a neighborhood of $\beta_2^-(t)$ such that $\beta_1^-(t) = M(\beta_2^-(t))$ and $\beta_3^-(t) = N(\beta_2^-(t))$ for $t > \max\{t^-, t^+\}$.

It follows from (3.7) and (3.17) that

$$N'(\beta_2^-(t)) < 0 ,$$

which implies (3.16). The proof of Lemma 3.8 is completed.

Later, we shall show that, for each fixed $t > \max\{t^-, t^+\}$, solutions (3.15) of system (3.7) can be further extended whenever: $N(\beta_2) < \beta_2 < M(\beta_2)$. The Jacobian of system (3.7) with respect to (β_1, β_3) has to be estimated along the extension. This is carried out in the next lemma.

Lemma 3.9. *Under conditions of Lemma* 3.1, *the following inequalities*:

$$\frac{\partial(\lambda_1 t + w_1)}{\partial\beta_1} < 0, \quad \frac{\partial(\lambda_2 t + w_2)}{\partial\beta_2} > 0, \quad \frac{\partial(\lambda_3 t + w_3)}{\partial\beta_3} < 0$$

hold on the solution $(\beta_1, \beta_2, \beta_3)$ *of* (3.6) [*or equivalently* (3.7)] *in the region* $\beta_3 < \beta_2 < \beta_1$, *where* $0 < \beta_3 - b < \delta$.

Proof. Using (2.24), we see that (3.6) is equivalent to:

$$[\lambda_1 - 2(\beta_1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial q}{\partial\beta_1}\right) = [\lambda_2 - 2(\beta_1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial q}{\partial\beta_2}\right), \tag{3.18}$$

$$[\lambda_2 - 2(\beta_1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial q}{\partial\beta_2}\right) = [\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial q}{\partial\beta_3}\right). \tag{3.19}$$

By Lemma 3.5,

$$\frac{\partial^2 q}{\partial\beta_1\partial\beta_2} < 0, \quad \frac{\partial^2 q}{\partial\beta_1\partial\beta_3} < 0, \quad \frac{\partial^2 q}{\partial\beta_2\partial\beta_3} > 0 \tag{3.20}$$

at the trailing edge.

We claim that (3.20) hold for all the solutions of (3.6) with $\beta_3 < \beta_2 < \beta_1$ and $0 < \beta_3 - b < \delta$. We justify the claim by contradiction. Suppose, otherwise, for instance, at some point $(\bar\beta_1, \bar\beta_2, \bar\beta_3)$ on the solution of (3.6) with $\bar\beta_3 < \bar\beta_2 < \bar\beta_1$ and $0 < \bar\beta_3 - b < \delta$:

$$\frac{\partial^2 q}{\partial\beta_2\partial\beta_3} = 0$$

which together with (2.22) gives:

$$\frac{\partial q}{\partial\beta_2} = \frac{\partial q}{\partial\beta_3} \quad \text{at } (\bar\beta_1, \bar\beta_2, \bar\beta_3).$$

This when combined with (2.14) and (3.19) implies:

$$t + \frac{1}{2}\frac{\partial q}{\partial\beta_2} = t + \frac{1}{2}\frac{\partial q}{\partial\beta_3} = 0. \tag{3.21}$$

By (2.11), (3.18) and (3.21), we obtain:

$$t + \frac{1}{2}\frac{\partial q}{\partial\beta_1} = 0$$

which together with (2.22) and (3.21) gives:

$$\frac{\partial^2 q}{\partial\beta_1\partial\beta_2} = \frac{\partial^2 q}{\partial\beta_1\partial\beta_3} = \frac{\partial^2 q}{\partial\beta_2\partial\beta_3} = 0 \tag{3.22}$$

at $(\bar\beta_1, \bar\beta_2, \bar\beta_3)$ where $\bar\beta_3 \in (b, b+\delta)$.

On the other hand, by (2.22) and Lemma 3.1:

$$\frac{\partial^2 q}{\partial\beta_1\partial\beta_3} - \frac{\partial^2 q}{\partial\beta_2\partial\beta_3} = 2(\beta_1 - \beta_2)\frac{\partial^3 q}{\partial\beta_1\partial\beta_2\partial\beta_3} < 0 \tag{3.23}$$

at $(\bar\beta_1, \bar\beta_2, \bar\beta_3)$.

Equations (3.22) and (3.23) contradict each other. This proves the claim.

By (2.22), we have:

$$2(\beta_1 - \beta_3)\frac{\partial^2 q}{\partial\beta_1\partial\beta_3} = \frac{\partial q}{\partial\beta_1} - \frac{\partial q}{\partial\beta_3} .$$

Differentiating this with respect to β_1 yields:

$$\frac{\partial^2 q}{\partial\beta_1^2} = 3\frac{\partial^2 q}{\partial\beta_1\partial\beta_3} + 2(\beta_1 - \beta_3)\frac{\partial^3 q}{\partial\beta_1^2\partial\beta_3} < 0 , \tag{3.24}$$

where we have used (3.20) and Lemma 3.1 in the last step.

It follows from (3.20) and (2.22) that

$$\frac{\partial q}{\partial\beta_1} < \frac{\partial q}{\partial\beta_3} < \frac{\partial q}{\partial\beta_2} \tag{3.25}$$

which when combined with (2.11), (3.18) and (3.19) gives:

$$t + \frac{1}{2}\frac{\partial q}{\partial\beta_1} < 0, \quad t + \frac{1}{2}\frac{\partial q}{\partial\beta_2} > 0, \quad t + \frac{1}{2}\frac{\partial q}{\partial\beta_3} > 0 \tag{3.26}$$

on the solution $(\beta_1, \beta_2, \beta_3)$ of (3.6) in the region $\beta_3 < \beta_2 < \beta_1$ where $\beta_3 \in (b, b+\delta)$.

Therefore, by (2.24):

$$\begin{aligned}\frac{\partial(\lambda_1 t + w_1)}{\partial\beta_1} &= \frac{\partial\lambda_1}{\partial\beta_1}\left(t + \frac{1}{2}\frac{\partial q}{\partial\beta_1}\right) + \frac{1}{2}[\lambda_1 - 2(\beta_1 + \beta_2 + \beta_3)]\frac{\partial^2 q}{\partial\beta_1^2} \\ &< 0 ,\end{aligned}$$

where in the last inequality we use (2.11), (2.15), (3.24) and (3.26). This proves the first inequality of Lemma 3.8.

Next we shall prove the rest of Lemma 3.8. By (2.22), we have:

$$2(\beta_2 - \beta_3)\frac{\partial^2 q}{\partial\beta_2\partial\beta_3} = \frac{\partial q}{\partial\beta_2} - \frac{\partial q}{\partial\beta_3} .$$

Differentiating this with respect to β_3 yields:

$$\frac{\partial^2 q}{\partial\beta_3^2} = 3\frac{\partial^2 q}{\partial\beta_2\partial\beta_3} - 2(\beta_2 - \beta_3)\frac{\partial^3 q}{\partial\beta_2\partial\beta_3^2} . \tag{3.27}$$

From (2.22) and (3.19), we obtain:

$$(\lambda_2 - \lambda_3)\left[t + \frac{1}{2}\frac{\partial q}{\partial \beta_2}\right] + [\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3)]\frac{\partial^2 q}{\partial \beta_2 \partial \beta_3}(\beta_2 - \beta_3) = 0$$

which together with (3.27) gives:

$$3\frac{\lambda_2 - \lambda_3}{\beta_2 - \beta_3}\left(t + \frac{1}{2}\frac{\partial q}{\partial \beta_2}\right) + [\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3)]\frac{\partial^2 q}{\partial \beta_3^2}$$

$$= -2[\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3)](\beta_2 - \beta_3)\frac{\partial^3 q}{\partial \beta_2 \partial \beta_3^2}$$

$$< 0\,, \tag{3.28}$$

where we have used (2.11) and Lemma 3.1 in the last inequality.

It follows from (2.24) that:

$$\frac{\partial(\lambda_3 t + w_3)}{\partial \beta_3} = \frac{\partial \lambda_3}{\partial \beta_3}\left(t + \frac{1}{2}\frac{\partial q}{\partial \beta_3}\right) + \frac{1}{2}[\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3)]\frac{\partial^2 q}{\partial \beta_3^2}$$

$$< \frac{3}{2}\frac{\lambda_2 - \lambda_3}{\beta_2 - \beta_3}\left(t + \frac{1}{2}\frac{\partial q}{\partial \beta_2}\right) + \frac{1}{2}[\lambda_3 - 2(\beta_1 + \beta_2 + \beta_3)]\frac{\partial^2 q}{\partial \beta_3^2}$$

$$< 0\,,$$

where we have used Lemma 2.2, (3.25) and (3.26) in the first inequality, and (3.28) in the last one.

This proves the third inequality of Lemma 3.9. In the same way, we can prove the second one. The proof of Lemma 3.9 is completed.

We are ready to solve (3.6) for β_1 and β_3 as functions of β_2 for $t > \max\{t^-, t^+\}$.

By Lemma 3.8, system (3.6) can be solved for:

$$\begin{cases} \beta_1 = M(\beta_2) \\ \beta_3 = N(\beta_2) \end{cases}$$

in the neighborhood of $(\beta_1^-(t), \beta_2^-(t), \beta_3^-(t))$, where $\beta_3^-(t) \in (b, b+\delta)$ for each $t > \max\{t^-, t^+\}$. Furthermore, (3.16) holds if $\beta_2 > \beta_2^-(t)$. We shall extend functions (3.15) in the positive β_2 direction as far as possible. It follows from Lemma 3.7 and Lemma 3.9 that, along the extension of (3.15) in the region $\beta_1 > \beta_2 > \beta_3$, where $\beta_3 \in (b, b+\delta)$, the Jacobian matrix:

$$\begin{pmatrix} \dfrac{\partial \tilde{F}}{\partial \beta_1} & \dfrac{\partial \tilde{F}}{\partial \beta_3} \\[2ex] \dfrac{\partial \tilde{G}}{\partial \beta_1} & \dfrac{\partial \tilde{G}}{\partial \beta_3} \end{pmatrix}$$

is diagonal and that therefore, nonsingular. Furthermore, system (3.6) determines (3.15) as two decreasing functions of β_2, and therefore, $N(\beta_2) \in (b, b+\delta)$ as β_2 increases.

This immediately guarantees that (3.15) can be extended as far as possible in the region: $\beta_1 > \beta_2 > \beta_3$ with $\beta_3 \in (b, b+\delta)$. Since $M(\beta_2)$ is decreasing, (3.15) stops at some point $\beta_2^+(t)$ where, obviously, $M(\beta_2^+(t)) = \beta_2^+(t)$. Therefore, we have shown that (3.7) determines β_1 and β_3 as decreasing functions of β_2 over interval $[\beta_2^-(t), \beta_2^+(t)]$.

Let

$$\beta_1^+(t) = M(\beta_2^+(t)), \quad \beta_3^+(t) = N(\beta_2^+(t)).$$

Clearly, $(\beta_1^+(t), \beta_2^+(t), \beta_3^+(t))$ solves system (3.7) at the leading edge $\beta_1 = \beta_2$. Hence, these $\beta_1^+(t), \beta_2^+(t)$ and $\beta_3^+(t)$ are exactly the ones appearing in Lemma 3.6.

Substituting (3.15) into (2.18), we obtain:

$$x = \lambda_2(M(\beta_2), \beta_2, N(\beta_2))t + w_2(M(\beta_2), \beta_2, N(\beta_2)),$$

which by Lemma 3.7 and Lemma 3.9 clearly determines x as an increasing function of β_2 over interval $[\beta_2^-(t), \beta_2^+(t)]$. It follows that, for each fixed $t > \max\{t^-, t^+\}$, β_2 is a function of x over the interval $[x^-(t), x^+(t)]$, and that so, therefore, are β_1 and β_3, where:

$$x^\pm(t) = \lambda_2(\beta_1^\pm(t), \beta_2^\pm(t), \beta_3^\pm(t))t + w_2(\beta_1^\pm(t), \beta_2^\pm(t), \beta_2^\pm(t)). \tag{3.29}$$

Thus, (2.18) can be solved for:

$$\beta_1 = \beta_1(x,t), \quad \beta_2 = \beta_2(x,t), \quad \beta_3 = \beta_3(x,t)$$

within a region:

$$x^-(t) < x < x^+(t), \quad \text{for } t > \max\{t^-, t^+\}, \tag{3.30}$$

where x^- and x^+ are given by (3.29).

Boundary conditions (2.2) and (2.3) can be checked easily. Therefore, we have proved:

Theorem 3.10. *Under the conditions of Lemma* 3.1, *the hodograph transform* (2.18) *with w_i's given by* (2.24) *and* (2.25) *can be solved for β_1, β_2 and β_3 as functions of* (x, t) *within region* (3.30) *for all* $t > \max\{t^-, t^+\}$. *Furthermore, these β_1, β_2 and β_3 satisfy boundary conditions* (2.2) *and* (2.3).

Theorem 2.3 and Theorem 3.10 immediately give:

Theorem 3.11. *Under the conditions of Lemma* 3.1, *the Whitham averaged system has a solution* $(\beta_1(x,t), \beta_2(x,t), \beta_3(x,t))$ *within region* (3.30) *for all* $t > \max\{t^-, t^+\}$, *and this solution satisfies boundary conditions* (2.2) *and* (2.3) *on the boundaries of the region.*

Remark. Conditions of Lemma 3.1 are quite generic. For instance, it is easy to check that these conditions are satisfied by decreasing initial data $u_0(x)$ with the following asymptotes at the infinity:

$$u_0(x) \approx b + \frac{1}{|x|^\alpha}, \quad \text{as } x \to +\infty,$$

$$u_0(x) \approx a - \frac{1}{|x|^\beta}, \quad \text{as } x \to -\infty,$$

where $\alpha, \beta > 0$.

We conclude this section by considering the case when one or both of a and b are infinite. In addition to the assumption that $f'''(u) < 0$, we need to put extra conditions at $a = +\infty$ or/and $b = -\infty$. More precisely, we suppose

$$f''(u) < 0, \quad \text{in the neighborhood of } u = a \text{ if } a = +\infty\,,$$

$$\lim_{u \to b} f''(u) = +\infty, \quad \text{if } b = -\infty\,. \tag{3.31}$$

Under these conditions, it is easy to check that Lemma 3.1, 3.2 and 3.3 with slightly different wording still hold when $a = +\infty$ or/and $b = -\infty$. Obviously, the proof of Lemma 3.4–3.11 do not or only superficially depend on whether a or/and b is infinite. Therefore, we have similar results in the case when the decreasing initial data is not bounded at $x = -\infty$ or/and $x = \infty$.

Theorem 3.12. *Consider a decreasing initial data $x = f(u)$ defined over (b, a), where $a = +\infty$ or/and $b = -\infty$. Suppose that in addition to* (3.1) *and the assumption that $f'''(u) < 0$ in the neighborhood of $u = a$ and $u = b$, $f(u)$ satisfies* (3.31). *Then there exists a $t^* > 0$ such that the Whitham averaged system has a solution $(\beta_1(x,t), \beta_2(x,t), \beta_3(x,t))$ within region* (3.30) *for all $t > t^*$, and this solution satisfies boundary conditions* (2.2) *and* (2.3) *on the boundaries of the region.*

As in Theorem 3.11, it is easy to see that the conditions of Theorem 3.12 are also quite generic.

4. The Whitham Solution for Hump-like Initial Data

In this section, we consider the case of hump-like initial data. For convenience, we assume the initial data to have a single extremum. We further normalize the initial data such that

$$\max_{-\infty < x < +\infty} u_0(x) = u_0(0) = 1\,.$$

We denote $f^+(u)$ and $f^-(u)$ as the inverse functions of the decreasing and increasing parts of $u = u_0(x)$, respectively.

As in Sect. 2, the initial value problem is to solve the Whitham averaged system for the multibranches β_1, β_2 and β_3 from the initial curve $u = u_0(x)$. Boundary conditions (2.2) and (2.3) should also be satisfied on the trailing and leading edges.

The hodograph transform (2.18) when $i = 1$ after differentiation with respect to x and use of Lemma 3.7 becomes:

$$1 = \frac{\partial(\lambda_1 t + w_1)}{\partial \beta_1} \beta_{1x}\,.$$

Since the maximum of the initial curve is preserved along the horizontal motion, the above equation indicates that (2.18) when $i = 1$ is singular at the maximum of β_1 (see Fig. 2.). A modification of the hodograph method is therefore necessary.

Instead of β_1, β_2 and β_3, we introduce $X_i(x,t)$'s:

$$\beta_i(x,t) = u_0(X_i(x,t))\,, \quad i = 1,2,3\,.$$

Later we will see that $X_i(x,t)$'s are monotone in x.

As a result, the Whitham averaged system (1.1) becomes:

$$X_{it} + \lambda_i(u_0(x_1), u_0(x_2), u_0(x_3))X_{ix} = 0, \quad i = 1,2,3\,. \tag{4.1}$$

The Burgers equation (2.1) becomes:

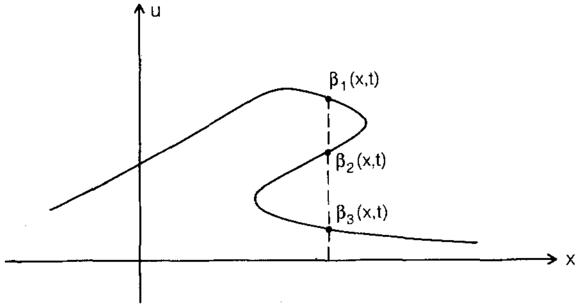

Fig. 2.

$$X_t + 6u_0(X)X_x = 0\,. \tag{4.2}$$

Figure 2 indicates that (X_1, X_2, X_3) should be restricted in the region:

$$f^-(u_0(X_2)) < X_1, \quad f^-(u_0(X_3)) < X_1, X_2 > 0, \quad \text{and } X_3 > 0\,. \tag{4.3}$$

Boundary conditions (2.2) and (2.3) are transformed to:

a) At the trailing edge:

$X_1 =$ the solution of (4.2) defined outside the multivalued region,

$$X_2 = X_3 > 0\,. \tag{4.4}$$

b) At the leading edge:

$X_1 = X_2 > 0$,

$X_3 =$ the solution of (4.2) defined outside the multivalued region. (4.5)

Therefore, the initial value problem of system (1.1) for the initial data $u = u_0(x)$ becomes the initial value problem of the modified Whitham averaged system (4.1) for the initial data $X = x$.

We want to use another version of Theorem 2.3 to solve system (4.1) for the initial data $X = x$. Consider the hodograph transform:

$$x = \lambda_i(u_0(X_1), u_0(X_2), u_0(X_3))t + W_i(X_1, X_2, X_3), \quad i = 1,2,3\,, \tag{4.6}$$

where $W_i(X_1, X_2, X_3)$'s are determined by the linear overdetermined system:

$$\frac{\partial W_i}{\partial X_j} = a_{ij}(u_0(X_1), u_0(X_2), u_0(X_3))u_0'(X_j)[W_i - W_j], \quad i,j = 1,2,3, i \neq j\,, \tag{4.7}$$

and a_{ij}'s are given in (2.17).

We need to understand what kinds of boundary conditions should be imposed on $W_i(X_1, X_2, X_3)$'s. The solution of (4.2) satisfies the characteristics equation:

$$x = 6u_0(X)t + X\,. \tag{4.8}$$

By (2.12), (2.13), (4.4), (4.5), (4.6) and (4.8), we see that at the trailing edge:

$$W_1(X_1, X_3, X_3) = X_1, \quad X_3 > 0\,,$$

$$W_2(X_1, X_3, X_3) = W_3(X_1, X_3, X_3)\,. \tag{4.9}$$

Similar conditions hold at the leading edge:

$$W_1(X_1,X_1,X_3) = W_2(X_1,X_1,X_3), \quad X_1 > 0\,,$$
$$W_3(X_1,X_1,X_3) = X_3\,. \tag{4.10}$$

As a consequence, it is natural to consider system (4.7) with boundary conditions (4.9) and (4.10). Similar to Theorem 2.4, we have:

Theorem 4.1. *If* $Q(X_1,X_2,X_3)$ *defined in region* (4.3) *is a solution of*:

$$2[u_0(X_i) - u_0(X_j)]\frac{\frac{\partial^2 Q}{\partial X_i \partial X_j}}{u_0'(X_i)u_0'(X_j)} = \frac{\frac{\partial Q}{\partial X_i}}{u_0'(X_i)} - \frac{\frac{\partial Q}{\partial X_j}}{u_0'(X_j)}, \quad i,j = 1,2,3 \;\; i \neq j \tag{4.11}$$

with boundary conditions:

$$Q(X,X,X) = X, \quad \textit{for } X > 0\,, \tag{4.12}$$

$$2[u_0(X_1) - u_0(X_3)]\frac{\frac{\partial Q(X_1,X_3,X_3)}{\partial X_1}}{u_0'(X_1)} + Q(X_1,X_3,X_3) = X_1, \quad \textit{for } X_1 < 0\,, \tag{4.13}$$

then (W_1, W_2, W_3) *defined by*:

$$\begin{aligned} &W_i(X_1,X_2,X_3) \\ &= \frac{1}{2}[\lambda_i(u_0(X_1), u_0(X_2), u_0(X_3)) - 2(u_0(X_1) + u_0(X_2) + u_0(X_3))]\frac{\frac{\partial Q}{\partial X_i}}{u_0'(X_i)} \\ &+ Q(X_1,X_2,X_3), \quad i = 1,2,3 \end{aligned} \tag{4.14}$$

solves the boundary value problem (4.7), (4.9) *and* (4.10).

Before we prove this theorem, we shall solve the boundary value problem (4.11)–(4.13). This is carried out in the following theorem:

Theorem 4.2. *System* (4.11)–(4.13) *has one and only one smooth solution in region* (4.3). *The solution is symmetric with respect to* X_2 *and* X_3, *and is given by*:

$$Q(X_1,X_2,X_3) = \frac{1}{\pi}\int_{-1}^{1}\frac{F\left(X_1, f^+\left(\frac{1-\nu}{2}u_0(X_2) + \frac{1+\nu}{2}u_0(X_3)\right)\right)}{\sqrt{1-\nu^2}}d\nu\,, \tag{4.15}$$

where

$$F(X,Y) = \frac{1}{\sqrt{u_0(X) - u_0(Y)}}\left[\int_0^X \frac{\xi u_0'(\xi)}{2\sqrt{u_0(\xi) - u_0(Y)}}d\xi + \frac{\sqrt{1 - u_0(Y)}}{2\sqrt{2}}\int_{-1}^{1}\frac{f^+\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(Y)\right)}{\sqrt{1-\mu}}d\mu\right].$$

Furthermore,

$$\frac{\frac{\partial Q}{\partial X_1}(X_1,X_2,X_3)}{u_0'(X_1)}$$

is also smooth in region (4.3).

Proof. Under a change of variable:

$$\xi = f^{+}\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(Y)\right),$$

we obtain:

$$\frac{\sqrt{1-u_0(Y)}}{2\sqrt{2}}\int_{-1}^{1}\frac{f^{+}\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(Y)\right)}{\sqrt{1-\mu}}d\mu = -\frac{1}{2}\int_{0}^{Y}\frac{\xi u_0'(\xi)}{\sqrt{u_0(\xi)-u_0(Y)}}d\xi$$

for $Y > 0$. This enables us to rewrite $F(X, Y)$ of (4.15) as:

$$F(X,Y) = -\frac{1}{2}\left[\int_{X}^{Y}\frac{\xi u_0'(\xi)}{\sqrt{u_0(\xi)-u_0(Y)}}d\xi\right]\frac{1}{\sqrt{u_0(X)-u_0(Y)}}$$

which under a new transform:

$$\xi = f^{+}\left(\frac{1-\mu}{2}u_0(X) + \frac{1+\mu}{2}u_0(Y)\right)$$

becomes:

$$F(X,Y) = \frac{1}{2\sqrt{2}}\int_{-1}^{1}\frac{f^{+}\left(\frac{1-\mu}{2}u_0(X) + \frac{1+\mu}{2}u_0(Y)\right)}{\sqrt{1-\mu}}d\mu$$

in the case that $X \geqq 0$ and $Y > 0$. This allows us to write $Q(X_1, X_2, X_3)$ as

$$Q(X_1,X_2,X_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f^{+}\left(\frac{1-\mu}{2}u_0(X_1) + \frac{1+\mu}{2}\frac{1-\nu}{2}u_0(X_2) + \frac{1+\mu}{2}\frac{1+\nu}{2}u_0(X_3)\right)}{\sqrt{(1-\mu)(1-\nu^2)}}d\mu d\nu \tag{4.16}$$

for $X_1 \geqq 0, X_2 > 0$ and $X_3 > 0$. Notice that (4.16) is exactly the same as (2.25) in view of the transform $\beta_i = u_0(X_i)$ and the symmetry as stated in Theorem 2.5. It immediately follows from (4.15) and (4.16) that $Q(X_1, X_2, X_3)$ is smooth in region (4.3). A simple calculation with (4.15) also shows that

$$\frac{\frac{\partial Q}{\partial X_1}(X_1,X_2,X_3)}{u_0'(X_1)}$$

is smooth at $X_1 = 0$, and therefore it is also smooth in region (4.3).

We next rewrite $Q(X_1, X_2, X_3)$ as:

$$Q(X_1,X_2,X_3) = \begin{cases} Q^{-}(u_0(X_1), u_0(X_2), u_0(X_3)), & X_1 < 0 \\ Q^{+}(u_0(X_1), u_0(X_2), u_0(X_3)), & X_1 > 0 \end{cases}$$

where

$$Q^{\lambda}(\beta_1,\beta_2,\beta_3) = \frac{1}{\pi}\int_{-1}^{1}\frac{G^{\lambda}\left(\beta_1, \frac{1-\nu}{2}\beta_2 + \frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}d\nu \tag{4.17}$$

and

$$G^{\lambda}(\alpha,\beta) = \frac{1}{\sqrt{\alpha-\beta}}\left[\int_0^{f^{\lambda}(\alpha)} \frac{\xi u_0'(\xi)}{2\sqrt{u_0(\xi)-\beta}}d\xi + \frac{\sqrt{1-\beta}}{2\sqrt{2}}\int_{-1}^{1}\frac{f^{+}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}\beta\right)}{\sqrt{1-\mu}}d\mu\right] \tag{4.18}$$

and $\lambda = +, -$.

To prove that $Q(X_1,X_2,X_3)$ satisfies (4.11), it suffices to show that:

$$2(\beta_i-\beta_j)\frac{\partial^2 Q^{\lambda}}{\partial\beta_i\partial\beta_j} = \frac{\partial Q^{\lambda}}{\partial\beta_i} - \frac{\partial Q^{\lambda}}{\partial\beta_j}\,.$$

It is straightforward to use (4.17) to check that

$$2(\beta_2-\beta_3)\frac{\partial^2 Q^{\lambda}}{\partial\beta_2\partial\beta_3} = \frac{\partial Q^{\lambda}}{\partial\beta_2} - \frac{\partial Q^{\lambda}}{\partial\beta_3}\,.$$

Since $Q^{\lambda}(\beta_1,\beta_2,\beta_3)$ is obviously symmetric with respect to β_2 and β_3, it is enough to show that

$$2(\beta_1-\beta_3)\frac{\partial^2 Q^{\lambda}}{\partial\beta_1\partial\beta_3} = \frac{\partial Q^{\lambda}}{\partial\beta_1} - \frac{\partial Q^{\lambda}}{\partial\beta_3}\,. \tag{4.19}$$

A simple calculation on (4.18) yields:

$$2(\alpha-\beta)\frac{\partial^2 G^{\lambda}}{\partial\alpha\partial\beta} = 2\frac{\partial G^{\lambda}}{\partial\alpha} - \frac{\partial G^{\lambda}}{\partial\beta}\,. \tag{4.20}$$

By (4.17), we have:

$$\begin{aligned}
2(\beta_1-\beta_3)\frac{\partial^2 Q^{\lambda}}{\partial\beta_1\partial\beta_3} &= \frac{1}{\pi}\int_{-1}^{1}\frac{2(\beta_1-\beta_3)\frac{\partial^2 G^{\lambda}}{\partial\alpha\partial\beta}\left(\beta_1,\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}\frac{1+\nu}{2}d\nu \\
&= \frac{2}{\pi}\int_{-1}^{1}\frac{\left[\beta_1-\left(\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)\right]\frac{\partial^2 G^{\lambda}}{\partial\alpha\partial\beta}\left(\beta_1,\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}\frac{1+\nu}{2}d\nu \\
&\quad + \frac{1}{\pi}\int_{-1}^{1}\frac{(\beta_2-\beta_3)\frac{\partial^2 G^{\lambda}}{\partial\alpha\partial\beta}\left(\beta_1,\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}\frac{1-\nu^2}{2}d\nu \\
&= \frac{1}{\pi}\int_{-1}^{1}\frac{\left(2\frac{\partial G^{\lambda}}{\partial\alpha}-\frac{\partial G^{\lambda}}{\partial\beta}\right)\left(\beta_1,\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}\frac{1+\nu}{2}d\nu \\
&\quad + \frac{1}{\pi}\int_{-1}^{1}\frac{(\beta_2-\beta_3)\frac{\partial^2 G^{\lambda}}{\partial\alpha\partial\beta}\left(\beta_1,\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}\frac{1-\nu^2}{2}d\nu \\
&= \frac{\partial Q^{\lambda}}{\partial\beta_1} - \frac{\partial Q^{\lambda}}{\partial\beta_3} + \frac{1}{\pi}\int_{-1}^{1}\frac{\frac{\partial G^{\lambda}}{\partial\alpha}}{\sqrt{1-\nu^2}}\nu d\nu \\
&\quad + \frac{1}{\pi}\left[\int_{-1}^{1}\frac{\frac{\partial^2 G^{\lambda}}{\partial\alpha\partial\beta}\left(\beta_1,\frac{1-\nu}{2}\beta_2+\frac{1+\nu}{2}\beta_3\right)}{\sqrt{1-\nu^2}}\frac{1-\nu^2}{2}d\nu\right](\beta_2-\beta_3)\,,
\end{aligned} \tag{4.21}$$

where we have used (4.20) in the third equality. Simply integrating by parts, we can check that the last two terms of (4.21) cancel each other. This proves (4.19), and therefore, system (4.11).

Boundary conditions can be checked as follows. At $X_2 = X_3$, we have:

$$\frac{\partial Q}{\partial X_1} = \frac{\partial F}{\partial X}(X_1,X_3)$$

$$= - \frac{u_0'(X_1)}{2(u_0(X_1)-u_0(X_3))^{\frac{3}{2}}}\left[\int_0^{X_1} \frac{\xi u_0'(\xi)}{2\sqrt{u_0(\xi)-u_0(X_3)}}d\xi \right.$$

$$\left. + \frac{\sqrt{1-u_0(X_3)}}{2\sqrt{2}}\int_{-1}^{1}\frac{f^+\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}d\mu\right] + \frac{X_1 u_0'(X_1)}{2[u_0(X_1)-u_0(X_3)]},$$

which together with (4.15) yields (4.13). Boundary condition (4.12) immediately follows from (4.16).

Finally, we want to prove the uniqueness of solution to the boundary value problem (4.11)–(4.13). Suppose $Q(X_1,X_2,X_3)$ and $\tilde{Q}(X_1,X_2,X_3)$ are two solutions. Since the boundary value problem in question is a linear one, $\hat{Q} = Q - \tilde{Q}$ satisfies (4.11) and boundary conditions (4.12) and (4.13) with homogeneous terms. In the region $X_1 > 0$, $X_2 > 0$, $X_3 > 0$, if we let $X_1 = f^+(\beta_1)$, $X_2 = f^+(\beta_2)$ and $X_3 = f^+(\beta_3)$, we have:

$$2(\beta_i - \beta_j)\frac{\partial^2 \hat{Q}(f^+(\beta_1),f^+(\beta_2),f^+(\beta_3))}{\partial\beta_i\partial\beta_j} = \frac{\partial\hat{Q}}{\partial\beta_i} - \frac{\partial\hat{Q}}{\partial\beta_j}, \quad i,j = 1,2,3,$$

$$\hat{Q}(f^+(\beta),f^+(\beta),f^+(\beta)) = 0, \quad i \neq j,$$

which by Theorem 2.5 implies:

$$\hat{Q}(X_1,X_2,X_3) = 0, \quad \text{for } X_1,X_2,X_3 > 0. \tag{4.22}$$

On the other hand, for $X_1 < 0$, condition (4.13) with a homogeneous term can be rewritten as:

$$\frac{\partial}{\partial X_1}[\sqrt{u_0(X_1)-u_0(X_3)}\hat{Q}(X_1,X_3,X_3)] = 0.$$

This and (4.22) at $X_1 = 0$ prove

$$\hat{Q}(X_1,X_3,X_3) = 0, \quad \text{for } X_1 < 0.$$

Using the notation $X_2 = f^+(\beta_2)$ and $X_3 = f^+(\beta_3)$, by (4.11) we have:

$$2(\beta_2 - \beta_3)\frac{\partial^2\hat{Q}(X_1,f^+(\beta_2),f^+(\beta_3))}{\partial\beta_2\partial\beta_3} = \frac{\partial\hat{Q}}{\partial\beta_2} - \frac{\partial\hat{Q}}{\partial\beta_3},$$

$$\hat{Q}(X_1,f^+(\beta),f^+(\beta)) = 0, \quad \text{for } X_1 < 0,$$

which by Lemma 3.4 of [12] gives:

$$\hat{Q}(X_1,X_2,X_3) = 0, \quad \text{for } X_1 < 0.$$

This and (4.22) proves the uniqueness, and the proof of Theorem 4.2 is completed.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. For $X_1 \neq 0, \lambda = +, -$, let

$$X_1 = f^{\lambda}(\beta_1),\ X_2 = f^{+}(\beta_2),\ X_3 = f^{+}(\beta_3)\,. \tag{4.23}$$

In this way, we write W_i's of (4.14) as:

$$W_i(f^{\lambda}(\beta_1), f^{+}(\beta_2), f^{+}(\beta_3)) = \frac{1}{2}[\lambda_i(\beta_1,\beta_2,\beta_3) - 2(\beta_1+\beta_2+\beta_3)]\frac{\partial Q}{\partial \beta_i} + Q(f^{\lambda}(\beta_1), f^{+}(\beta_2), f^{+}(\beta_3))\,.$$

In view of (4.11) and (4.23), $Q(f^{\lambda}(\beta_1), f^{+}(\beta_2), f^{+}(\beta_3))$ satisfies (2.22). By Theorem 2.4, we obtain

$$\frac{\partial W_i}{\partial \beta_j} = a_{ij}(\beta_1,\beta_2,\beta_3)[W_i - W_j]\,,$$

which is equivalent to (4.7).

Boundary conditions can be checked as follows. The second condition of (4.9) at the trailing edge follows from (2.12), (4.14) and the fact that $Q(X_1,X_2,X_3)$ is symmetric with respect to X_2 and X_3. The first condition of (4.9) can be easily verified using (2.12) and (4.13) in the case of $X_1 < 0$. The part of the trailing edge for $X_1 > 0$ can be handled as follows.

It follows from (2.12) and (4.14) that

$$W_1(X_1,X_3,X_3) = 2[u_0(X_1) - u_0(X_3)]\frac{\frac{\partial Q}{\partial X_1}}{u_0'(X_1)} + Q(X_1,X_3,X_3)\,. \tag{4.24}$$

Differentiating this with respect to X_3 yields:

$$\begin{aligned}\frac{dW_1(X_1,X_3,X_3)}{dX_3} &= \frac{\partial Q}{\partial X_2} + \frac{\partial Q}{\partial X_3} - 2u_0'(X_3)\frac{\frac{\partial Q}{\partial X_1}}{u_0'(X_1)} + 2[u_0(X_1) - u_0(X_3)]\frac{\frac{\partial^2 Q}{\partial X_1 \partial X_2} + \frac{\partial^2 Q}{\partial X_1 \partial X_3}}{u_0'(X_1)}\\ &= 0\,,\end{aligned}$$

where we have used (4.11) in the last equation. $W_1(X_1,X_3,X_3)$ is independent of X_3, and therefore, the first condition of (4.9) follows by substituting $X_3 = X_1$ into (4.24) and using (4.12). Boundary condition (4.10) can be checked in the same way. This completes the proof of Theorem 4.1.

We now study the hodograph transform (4.6) with W_i's given by (4.14) and (4.15). We shall show that transform (4.6) can be solved for X_1, X_2 and X_3 as functions of (x,t) within a cusp in the x-t plane.

We make some assumption about the initial data. We suppose that the decreasing part of the initial curve $x = f^{+}(u)$ has only one inflection point at $u = u^*$, more precisely,

$$f^{+''}(u^*) = 0; \quad f^{+'''}(u) < 0, \quad \text{for } u \neq u^*\,. \tag{4.25}$$

Therefore, by Theorem 2.6 and 2.9, the hodograph transform (4.6) can be solved within a cusp until a finite time T when the maximum of the initial curve hits the trailing edge. The main purpose of this section is to solve the hodograph transform for time after T.

First, we simplify the hodograph transform. Eliminating x from (4.6), we obtain:

$$(\lambda_1 t + W_1) - (\lambda_2 t + W_2) = 0\,,$$
$$(\lambda_2 t + W_2) - (\lambda_3 t + W_3) = 0\,, \qquad (4.26)$$

which is equivalent to:

$$\frac{(\lambda_1 t + W_1) - (\lambda_2 t + W_2)}{[u_0(X_1) - u_0(X_2)]K(s)} = 0\,,$$
$$\frac{(\lambda_2 t + W_2) - (\lambda_3 t + W_3)}{u_0(X_2) - u_0(X_3)} = 0\,. \qquad (4.27)$$

Similar to the equations above (3.10), this system becomes:

$$6t + \frac{\frac{\partial Q(X_1,X_3,X_3)}{\partial X_1}}{u_0'(X_1)} + 2\frac{\frac{\partial Q(X_1,X_3,X_3)}{\partial X_3}}{u_0'(X_3)} = 0\,,$$
$$2[u_0(X_1) - u_0(X_3)]\frac{\frac{\partial^2 Q(X_1,X_3,X_3)}{\partial X_2 \partial X_3}}{[u_0'(X_3)]^2} - \frac{3}{2}\left[t + \frac{1}{2}\frac{\frac{\partial Q(X_1,X_3,X_3)}{\partial X_3}}{u_0'(X_3)}\right] = 0 \qquad (4.28)$$

at the trailing edge $X_2 = X_3$. Substituting the first equation into the second one, and using (4.11), we get

$$\tilde{U}(t,X_1,X_3) = 6t + \frac{\frac{\partial Q(X_1,X_3,X_3)}{\partial X_1}}{u_0'(X_1)} + 2\frac{\frac{\partial Q(X_1,X_3,X_3)}{\partial X_3}}{u_0'(X_3)} = 0\,, \qquad (4.29)$$

$$\tilde{V}(X_1,X_3) = \frac{\frac{\partial^2 Q(X_1,X_3,X_3)}{\partial X_1 \partial X_3}}{u_0'(X_1)u_0'(X_3)} + 4\frac{\frac{\partial^2 Q(X_1,X_3,X_3)}{\partial X_2 \partial X_3}}{[u_0'(X_3)]^2} = 0\,. \qquad (4.30)$$

In particular, as mentioned previously, when $t = T$ system (4.29) and (4.30) has a solution $(T,X_1^-(T),X_3^-(T))$ with $X_1^-(T) = 0$ and $X_3^-(T) > 0$. To solve (4.29) and (4.30) for X_1 and X_3 for $t > T$, we shall calculate partial derivatives of $\tilde{U}$ and $\tilde{V}$ on the solution (t,X_1,X_3) of (4.30), where $u_0(X_1) > u_0(X_3)$.

Integration by parts gives:

$$\int_0^{X_1} \frac{\xi u_0'(\xi)}{[u_0(\xi) - u_0(X_3)]^{\frac{3}{2}}} d\xi = -\frac{2X_1}{\sqrt{u_0(X_1) - u_0(X_3)}} + 2\int_0^{X_1} \frac{d\xi}{\sqrt{u_0(\xi) - u_0(X_3)}} \qquad (4.31)$$

and

$$\int_{-1}^{1} \frac{f^+\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}} d\mu = [u_0(X_3) - 1]\int_{-1}^{1} \frac{f^{+'}\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}(1-\mu)d\mu\,. \qquad (4.32)$$

Using (4.15) and the symmetry of Q about X_2 and X_3, we rewrite $\tilde{U}$ of (4.29) as:

$$\tilde{U}(t,X_1,X_3) = 6t + \frac{\frac{\partial F(X_1,X_3)}{\partial X_1}}{u_0'(X_1)} + \frac{\frac{\partial F(X_1,X_3)}{\partial X_3}}{u_0'(X_3)}$$

$$= 6t + \frac{1}{2}\int_0^{X_1} \frac{d\xi}{\sqrt{[u_0(\xi) - u_0(X_3)][u_0(X_1) - u_0(X_3)]}}$$

$$+ \frac{\sqrt{2}}{4}\sqrt{\frac{1-u_0(X_3)}{u_0(X_1)-u_0(X_3)}} \int_{-1}^{1} \frac{f^{+'}\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}} d\mu \,, \quad (4.33)$$

where we have used (4.31) and (4.32) in the second equality. This gives:

$$\frac{\partial \tilde{U}}{\partial X_1} = \frac{1}{2}\frac{1}{u_0(X_1) - u_0(X_3)} - \frac{u_0'(X_1)}{2[u_0(X_1) - u_0(X_3)]^{\frac{3}{2}}} \left\{ \frac{1}{2}\int_0^{X_1} \frac{d\xi}{\sqrt{u_0(\xi) - u_0(X_3)}} \right.$$

$$\left. + \frac{\sqrt{2}}{4}\sqrt{1-u_0(X_3)} \int_{-1}^{1} \frac{f^{+'}\left(\frac{1-\mu}{2} + \frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}} d\mu \right\}$$

$$> 0, \quad \text{for } X_1 \leqq 0 \,,$$

where we have used the assumption that $u_0(x)$ is increasing for $x < 0$ and decreasing for $x > 0$.

Differentiating (4.11) when $i = 2$ and $j = 3$ with respect to X_3 yields

$$\frac{\frac{\partial^2 Q}{\partial X_3^2}}{u_0'(X_3)} - \frac{u_0''(X_3)}{[u_0'(X_3)]^2}\frac{\partial Q}{\partial X_3} = 3\frac{\frac{\partial^2 Q}{\partial X_2 \partial X_3}}{u_0'(X_2)} - 2[u_0(X_2) - u_0(X_3)]\frac{\partial}{\partial X_3}\frac{\frac{\partial^2 Q}{\partial X_2 \partial X_3}}{u_0'(X_2)u_0'(X_3)} \,.$$

This becomes

$$\frac{\partial^2 Q}{\partial X_3^2} - \frac{u_0''(X_3)}{u_0'(X_3)}\frac{\partial Q}{\partial X_3} = 3\frac{\partial^2 Q}{\partial X_2 \partial X_3} \quad (4.34)$$

when $X_2 = X_3$. Since $Q(X_1,X_2,X_3)$ is symmetric with respect to X_2 and X_3, we get:

$$\frac{\partial^2 Q}{\partial X_2^2} - \frac{u_0''(X_2)}{u_0'(X_2)}\frac{\partial Q}{\partial X_2} = 3\frac{\partial^2 Q}{\partial X_2 \partial X_3}, \quad \frac{\partial^2 Q}{\partial X_1 \partial X_2} = \frac{\partial^2 Q}{\partial X_1 \partial X_3} \quad (4.35)$$

when $X_2 = X_3$.

By (4.29), (4.34) and (4.35), we obtain:

$$\frac{\partial \tilde{U}}{\partial X_3} = \frac{2\frac{\partial^2 Q}{\partial X_1 \partial X_3}}{u_0'(X_1)} + \frac{8\frac{\partial^2 Q}{\partial X_2 X_3}}{u_0'(X_3)}$$

$$= 0 \quad (4.36)$$

on the solution (X_1,X_3,X_3) of (4.30).

On the other hand, it follows from (4.30), (4.33) and (4.36) that

$$
\begin{aligned}
\tilde{V}(X_1,X_3) =& \frac{1}{2}\frac{\frac{\partial \tilde{U}}{\partial X_3}}{u_0'(X_3)} \\
=& \frac{1}{16\sqrt{u_0(X_1)-u_0(X_3)}}\left\{2\int_0^{X_1}\frac{d\xi}{[u_0(\xi)-u_0(X_3)]^{\frac{3}{2}}}\right. \\
&+2\int_0^{X_1}\frac{d\xi}{\sqrt{u_0(\xi)-u_0(X_3)}[u_0(X_1)-u_0(X_3)]} \\
&+\sqrt{2}\sqrt{1-u_0(X_3)}\int_{-1}^{1}\frac{f^{+''}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}(1+\mu)d\mu \\
&\left.+\sqrt{2}\frac{1-u_0(X_1)}{[u_0(X_1)-u_0(X_3)]\sqrt{1-u_0(X_3)}}\int_{-1}^{1}\frac{f^{+'}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}d\mu\right\}.
\end{aligned}
\tag{4.37}
$$

Therefore, we obtain:

$$
\begin{aligned}
\frac{\partial \tilde{V}}{\partial X_1} =& \frac{1}{16\sqrt{u_0(X_1)-u_0(X_3)}}\left\{4\frac{1}{[u_0(X_1)-u_0(X_3)]^{\frac{3}{2}}}\right. \\
&-2\int_0^{X_1}\frac{u_0'(X_1)d\xi}{\sqrt{u_0(\xi)-u_0(X_3)}[u_0(X_1)-u_0(X_3)]^2} \\
&-\frac{\sqrt{2}u_0'(X_1)}{[u_0(X_1)-u_0(X_3)]\sqrt{1-u_0(X_3)}}\int_{-1}^{1}\frac{f^{+'}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}d\mu \\
&\left.-\sqrt{2}\frac{[1-u_0(X_1)]u_0'(X_1)}{[u_0(X_1)-u_0(X_3)]^2\sqrt{1-u_0(X_3)}}\int_{-1}^{1}\frac{f^{+'}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}d\mu\right\} \\
&>0, \text{ for } X_1 \leqq 0
\end{aligned}
$$

on the solution (X_1,X_3) of (4.30) with $u_0(X_1) > u_0(X_3)$.

Similarly,

$$
\begin{aligned}
\frac{\partial \tilde{V}}{\partial X_3} \leqq& \frac{u_0'(X_3)}{16\sqrt{u_0(X_1)-u_0(X_3)}}\left\{\int_0^{X_1}\frac{3d\xi}{[u_0(\xi)-u_0(X_3)]^{\frac{5}{2}}}\right. \\
&+\int_0^{X_1}\frac{d\xi}{[u_0(\xi)-u_0(X_3)]^{\frac{3}{2}}[u_0(X_1)-u_0(X_3)]} \\
&+2\int_0^{X_1}\frac{d\xi}{\sqrt{u_0(\xi)-u_0(X_3)}[u_0(X_1)-u_0(X_3)]^2} \\
&+\frac{\sqrt{2}}{2}\frac{1-2u_0(X_1)+u_0(X_3)}{[u_0(X_1)-u_0(X_3)]\sqrt{1-u_0(X_3)}}\int_{-1}^{1}\frac{f^{+''}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}(1+\mu)d\mu \\
&\left.+\frac{\sqrt{2}}{2}\frac{[1-u_0(X_1)][2+u_0(X_1)-3u_0(X_3)]}{[u_0(X_1)-u_0(X_3)]^2[1-u_0(X_3)]^{\frac{3}{2}}}\int_{-1}^{1}\frac{f^{+'}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}}d\mu\right\}
\end{aligned}
$$

$$= \frac{u_0'(X_3)}{16\sqrt{u_0(X_1)-u_0(X_3)}} \left\{ \int_0^{X_1} \frac{3d\xi}{[u_0(\xi)-u_0(X_3)]^{\frac{5}{2}}} \right.$$

$$+ \int_0^{X_1} \frac{2\frac{u_0(X_1)-u_0(X_3)}{1-u_0(X_3)} d\xi}{[u_0(\xi)-u_0(X_3)]^{\frac{3}{2}}[u_0(X_1)-u_0(X_3)]}$$

$$+ \int_0^{X_1} \frac{\frac{1+2u_0(X_1)-3u_0(X_3)}{1-u_0(X_3)}}{\sqrt{u_0(\xi)-u_0(X_3)}[u_0(X_1)-u_0(X_3)]^2} d\xi$$

$$\left. + \frac{\sqrt{2}}{2} \frac{[1-u_0(X_1)][1+3u_0(X_1)-4u_0(X_3)]}{[u_0(X_1)-u_0(X_3)]^2[1-u_0(X_3)]^{\frac{3}{2}}} \int_{-1}^{1} \frac{f^{+'}\left(\frac{1-\mu}{2}+\frac{1+\mu}{2}u_0(X_3)\right)}{\sqrt{1-\mu}} d\mu \right\}$$

$$> 0, \quad \text{for } X_1 \leqq 0$$

on the solution (X_1, X_3) of (4.30) where we have used $f^{+'''} \leqq 0$ in the first inequality, and replaced the integral involving $f^{+''}$ by (4.30) and (4.37) in the equality.

All these imply that the Jacobian:

$$\frac{\partial(U,V)}{\partial(X_1,X_3)} \neq 0$$

at the solution (X_1, X_3) of (4.30), where $u_0(X_1) > u_0(X_3)$ for $X_1 \leqq 0$. Since system (4.29) and (4.30) hold for $(T, X_1^-(T), X_3^-(T))$, it follows from the Implicit Function Theorem that system (4.29) and (4.30) can be solved for X_1 and X_3 as functions of t for t in the neighborhood of T. Furthermore, it is easy to check that the solutions $X_1^-(t)$ and $X_3^-(t)$ are decreasing and increasing, respectively, as t increases. Since $X_1^-(T)=0$, we have $X_1(t) \leqq 0$ for $t \geqq T$. Equations (4.29) and (4.33) imply that $X_1^-(T)$ can not catch up with $f^-(u_0(X_3^-(t)))$ in finite time. Using the Implicit Function Theorem again, we see that $X_1^-(t)$ and $X_3^-(t)$ can be further extended for all $t \geqq T$. Therefore, we have established:

Lemma 4.3 *Consider a smooth initial data $u = u_0(x)$ with a single hump. Suppose that $u_0(x)$ reaches its only maximum at $x = 0$, where the maximum is normalized to be* 1. *If the inverse function $f^+(u)$ of the decreasing part of $u = u_0(x)$ satisfies* (4.25), *system* (4.27) *has a unique solution $(X_1^-(t), X_2^-(t), X_3^-(t))$ with $X_2^-(t) = X_3^-(t)$ for all $t \geqq T$. Furthermore, we have $u_0(X_1^-(t)) > u_0(X_3^-(t))$ for $t > T$.*

At the leading edge $X_1 = X_2 > 0$, system (4.27), similar to (3.14), is equivalent to:

$$t + \frac{1}{2} \frac{\frac{\partial}{\partial X_1} Q(X_1, X_1, X_3)}{u_0'(X_1)} = 0 ,$$

$$t + \frac{1}{2} \frac{\frac{\partial}{\partial X_3} Q(X_1, X_1, X_3)}{u_0'(X_3)} = 0$$

This system, under transform $\beta_i = u_0(X_i)$, becomes (3.14) with $q(\beta_1, \beta_2, \beta_3)$ replaced by $Q(f^+(\beta_1), f^+(\beta_2), f^+(\beta_3))$. Therefore, by Lemma 3.6 we have:

Lemma 4.4. *Under conditions of Lemma* 4.3, *system* (4.27) *has a unique solution $(X_1^+(t), X_2^+(t), X_3^+(t))$ with $X_1^+(t) = X_2^+(t)$ for all $t \geqq T$. Furthermore, we have $u_0(X_2^+(t)) > u_0(X_3^+(t))$.*

For each $t > T$, we want to solve (4.27) at $X_1 = 0$. Under a change of variable:

$$X_1 = f^+(\beta_1), \quad X_2 = f^+(\beta_2), \quad X_3 = f^+(\beta_3) \tag{4.38}$$

system (4.27) when $X_1 = 0$ becomes (3.7) with $\beta_1 = 1$ and $q(\beta_1, \beta_2, \beta_3)$ replaced by $Q(f^+(\beta_1), f^+(\beta_2), f^+(\beta_3))$. It follows from (3.9) and (3.17) that

$$\frac{\partial \tilde{F}}{\partial t} = \frac{24}{\pi}, \quad \frac{\partial \tilde{F}}{\partial \beta_2} = 0, \quad \frac{\partial \tilde{F}}{\partial \beta_3} = 0$$

$$\frac{\partial \tilde{G}}{\partial t} = 6, \quad \frac{\partial \tilde{G}}{\partial \beta_2} > 0, \quad \frac{\partial \tilde{G}}{\partial \beta_3} > 0$$

which implies the non-vanishing of the Jacobian:

$$\frac{\partial(\tilde{F}, \tilde{G})}{\partial(t, \beta_3)}$$

at $(T, 1, u_0(X_2^-(T)), u_0(X_3^-(T)))$. Hence, system (3.7) can be solved for β_3 and t as functions of β_2:

$$\beta_3 = H(\beta_2), \quad t = L(\beta_2)$$

in the neighborhood of $u_0(X_2^-(T))$.

Clearly, $H(\beta_2)$ decreases as β_2 increases, and therefore, $(t, \beta_1, \beta_2, \beta_3) = (L(\beta_2), 1, \beta_2, H(\beta_2))$ satisfies system (3.7) for β_2 in the neighborhood of $u_0(X_2^-(T))$ and $\beta_2 > u_0(X_2^-(T))$.

Before continuing, we need a lemma whose proof will be given later.

Lemma 4.5. *The following inequalities:*

$$\frac{\partial(\lambda_2 t + W_2)}{\partial \beta_2} > 0, \quad \frac{\partial(\lambda_3 t + W_3)}{\partial \beta_3} < 0$$

hold on the solution $(t, 1, \beta_2, \beta_3)$ *of* (3.6) *in the region* $\beta_2 > \beta_3$.

By (2.14), Lemma 3.7 and Lemma 4.5, system (3.7) determines $t = L(\beta_2)$ as an increasing functions of β_2. Accordingly, β_2 is an increasing function of t, and therefore, β_3 is a decreasing function of t for $t > T$ in the neighborhood of T. Using Lemma 3.7, Lemma 4.5 and Implicit Function Theorem again, we see that we can extend t for all $t > T$. This together with (4.38) establishes:

Lemma 4.6. *Under conditions of Lemma* 4.3, *system* (4.27) *when* $X_1 = 0$ *has a unique solution* $(t, 0, X_2^*(t), X_3^*(t))$ *for all* $t \geqq T$ *with* $X_2^*(T) = X_2^-(T)$ *and* $X_3^*(T) = X_3^-(T)$. *Furthermore,* $X_2^*(t)$ *and* $X_3^*(t)$ *are decreasing and increasing functions of t, respectively.*

We now come back to prove Lemma 4.6.

Proof of Lemma 4.6. By (4.11) and (4.38), we have:

$$2(\beta_1 - \beta_2)\frac{\partial^2 Q}{\partial \beta_1 \partial \beta_2} = \frac{\partial Q}{\partial \beta_1} - \frac{\partial Q}{\partial \beta_2}\,.$$

Taking derivative with respect to β_3 yields

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_3} - \frac{\partial^2 Q}{\partial\beta_2\partial\beta_3} = 2(\beta_1 - \beta_2)\frac{\partial^3 Q}{\partial\beta_1\partial\beta_2\partial\beta_3} . \tag{4.39}$$

Using (4.25) in (4.16), we find:

$$\frac{\partial^3 Q}{\partial\beta_1\partial\beta_2\partial\beta_3} < 0 .$$

These two equations and inequalities give:

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_3} < \frac{\partial^2 Q}{\partial\beta_2\partial\beta_3}$$

at $(T, 1, u_0(X_2^-(T)), u_0(X_3^-(T)))$.

This together with (4.30) implies:

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_3} < 0, \quad \frac{\partial^2 Q}{\partial\beta_2\partial\beta_3} > 0$$

at $(T, 1, u_0(X_2^-(T)), u_0(X_3^-(T)))$. The symmetry of Q with respect to β_2 and β_3 gives:

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_2} = \frac{\partial^2 Q}{\partial\beta_1\partial\beta_3} < 0$$

at $(T, 1, u_0(X_2^-(T)), u_0(X_3^-(T)))$.

We claim that

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_3} < 0, \quad \frac{\partial^2 Q}{\partial\beta_2\partial\beta_3} > 0, \quad \frac{\partial^2 Q}{\partial\beta_1\partial\beta_2} < 0 \tag{4.40}$$

on all the solutions $(t, 1, \beta_2, \beta_3)$ of (3.7) with $\beta_2 > \beta_3$.

Suppose otherwise, for instance,

$$\frac{\partial^2 Q}{\partial\beta_2\partial\beta_3} = 0$$

at some point $(1, \bar{\beta}_2, \bar{\beta}_3)$ on the solution of (3.7) where $\bar{\beta}_2 > \bar{\beta}_3$. This when combined with (4.11) gives:

$$\frac{\partial Q}{\partial\beta_2} = \frac{\partial Q}{\partial\beta_3} \tag{4.41}$$

at $(1, \bar{\beta}_2, \bar{\beta}_3)$ where $\bar{\beta}_2 > \bar{\beta}_3$.

Using (4.14) and (4.38), we write (4.26) as

$$[\lambda_1(1, \beta_2, \beta_3) - 2(1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial Q}{\partial\beta_1}\right)$$
$$= [\lambda_2(1, \beta_2, \beta_3) - 2(1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial Q}{\partial\beta_2}\right) , \tag{4.42}$$

$$[\lambda_2(1, \beta_2, \beta_3) - 2(1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial Q}{\partial\beta_2}\right)$$
$$= [\lambda_3(1, \beta_2, \beta_3) - 2(1 + \beta_2 + \beta_3)]\left(t + \frac{1}{2}\frac{\partial Q}{\partial\beta_3}\right) . \tag{4.43}$$

By (2.14), (4.41) and (4.43), we have:

$$t+\frac{1}{2}\frac{\partial Q}{\partial \beta_2}=t+\frac{1}{2}\frac{\partial Q}{\partial \beta_3}=0\,,$$

which together with (2.11) and (4.42) gives:

$$t+\frac{1}{2}\frac{\partial Q}{\partial \beta_1}=0\,.$$

Therefore, by (4.11) we have

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_3}=\frac{\partial^2 Q}{\partial\beta_2\partial\beta_3}=\frac{\partial^2 Q}{\partial\beta_1\partial\beta_2}=0 \tag{4.44}$$

at $(1,\bar{\beta}_2,\bar{\beta}_3)$, where $\bar{\beta}_2>\bar{\beta}_3$.

On the other hand, we obtain from (4.16), (4.38) and (4.39) that

$$\frac{\partial^2 Q}{\partial\beta_1\partial\beta_3}-\frac{\partial^2 Q}{\partial\beta_2\partial\beta_3}=2(\beta_1-\beta_2)\frac{\partial^3 Q}{\partial\beta_1\partial\beta_2\partial\beta_3}<0 \tag{4.45}$$

at $(1,\bar{\beta}_2,\bar{\beta}_3)$ where $\bar{\beta}_2>\bar{\beta}_3$.

Equations (4.44) and (4.45) contradict each other, and the claim is justified.

It follows from (4.11), (4.38) and (4.40) that

$$\frac{\partial Q}{\partial\beta_1}<\frac{\partial Q}{\partial\beta_3}<\frac{\partial Q}{\partial\beta_2}\,,$$

which when combined with (2.11), (4.42) and (4.43) gives:

$$t+\frac{1}{2}\frac{\partial Q}{\partial \beta_2}>t+\frac{1}{2}\frac{\partial Q}{\partial \beta_3}>0>t+\frac{1}{2}\frac{\partial Q}{\partial \beta_1} \tag{4.46}$$

on the solution $(1,\beta_2,\beta_3)$ of (3.7) with $1>\beta_2>\beta_3$.

By (4.11) and (4.38), we have:

$$2(\beta_2-\beta_3)\frac{\partial^2 Q}{\partial\beta_2\partial\beta_3}=\frac{\partial Q}{\partial\beta_2}-\frac{\partial Q}{\partial\beta_3}\,.$$

Differentiating this with respect to β_2 yields:

$$\frac{\partial^2 Q}{\partial\beta_2^2}=3\frac{\partial^2 Q}{\partial\beta_2\partial\beta_3}+2(\beta_2-\beta_3)\frac{\partial^3 Q}{\partial\beta_2^2\partial\beta_3}\,. \tag{4.47}$$

Using (4.11) and (4.38) in (4.43), we obtain:

$$(\lambda_2-\lambda_3)\left[t+\frac{1}{2}\frac{\partial Q}{\partial\beta_3}\right]+[\lambda_2-2(\beta_1+\beta_2+\beta_3)]\frac{\partial^2 Q}{\partial\beta_2\partial\beta_3}(\beta_2-\beta_3)=0\,,$$

which together with (4.47) gives:

$$3\frac{\lambda_2-\lambda_3}{\beta_2-\beta_3}\left(t+\frac{1}{2}\frac{\partial Q}{\partial \beta_3}\right)+[\lambda_2-2(\beta_1+\beta_2+\beta_3)]\frac{\partial^2 Q}{\partial \beta_2^2}$$
$$=2[\lambda_2-2(\beta_1+\beta_2+\beta_3)](\beta_2-\beta_3)\frac{\partial^3 Q}{\partial \beta_2^2\partial \beta_3}$$
$$>0, \quad \text{when } \beta_2>\beta_3 \tag{4.48}$$

where we have used (2.11), (4.16), (4.25) and (4.38) in the last inequality.

It follows from (4.14) and (4.38) that

$$\frac{\partial(\lambda_2 t+W_2)}{\partial \beta_2}=\frac{\partial \lambda_2}{\partial \beta_2}\left(t+\frac{1}{2}\frac{\partial Q}{\partial \beta_2}\right)+\frac{1}{2}[\lambda_2-2(\beta_1+\beta_2+\beta_3)]\frac{\partial^2 Q}{\partial \beta_2^2}$$
$$>\frac{3}{2}\frac{\lambda_2-\lambda_3}{\beta_2-\beta_3}\left(t+\frac{1}{2}\frac{\partial Q}{\partial \beta_3}\right)+\frac{1}{2}[\lambda_2-2(\beta_1+\beta_2+\beta_3)]\frac{\partial^2 Q}{\partial \beta_2^2}$$
$$>0, \qquad \text{when } \beta_2>\beta_3\,,$$

where we have used Lemma 2.2 and (4.46) in the first inequality, and (4.48) in the last one. This proves:

$$\frac{\partial(\lambda_2 t+W_2)}{\partial \beta_2}>0\,.$$

In the same way, we can show that

$$\frac{\partial(\lambda_3 t+W_3)}{\partial \beta_3}<0\,.$$

This completes the proof of Lemma 4.6.

Next, we want to solve system (4.26) for X_1 and X_3 as functions of X_2 at each $t>T$ when (t,X_1,X_2,X_3) is in the neighborhood of $(T,0,X_2^*(T),X_3^*(T))$.

By (2.12) and (4.9), we have:

$$\frac{\partial(\lambda_1 t+W_1)}{\partial X_1}=1\,, \tag{4.49}$$

at $(T,0,X_2^-(T),X_3^-(T))$, where $u_0(x)$ reaches its maximum at $x=0$.

It follows from (1.2), (2.4), (2.5), (2.8) and (2.9) that

$$\frac{\partial}{\partial \beta_2}\lambda_2(\beta_1,\beta_2,\beta_3)|_{\beta_2=\beta_3}=9\,.$$

This together with (2.12), (4.14) and (4.35) gives:

$$\frac{\partial(\lambda_2 t+W_2)}{\partial X_2}$$
$$=\frac{\partial \lambda_2}{\partial X_2}\left[t+\frac{1}{2}\frac{\frac{\partial Q}{\partial X_2}}{u_0'(X_2)}\right]+4[u_0(X_3)-u_0(X_1)]\left\{\frac{\frac{\partial^2 Q}{\partial X_2^2}}{u_0'(X_2)}-\frac{u_0''(X_2)}{[u_0'(X_2)]^2}\frac{\partial Q}{\partial X_2}\right\}$$
$$=9u_0'(X_3)\left[t+\frac{1}{2}\frac{\frac{\partial Q}{\partial X_2}}{u_0'(X_2)}\right]+12[u_0(X_3)-u_0(X_1)]\frac{\frac{\partial^2 Q}{\partial X_2\partial X_3}}{u_0'(X_3)}$$

at $X_2 = X_3$. Using (4.28) and the symmetry of Q about X_2 and X_3 in the last equation, we obtain:

$$\frac{\partial(\lambda_2 t + W_2)}{\partial X_2} = 0 \tag{4.50}$$

at $(T,0,X_2^-(T),X_3^-(T))$.

Equation (4.16) when combined with (4.25) gives

$$\frac{\frac{\partial^3 Q}{\partial X_2^2 \partial X_3}}{[u_0'(X_2)]^2 u_0'(X_3)} < 0, \quad \frac{\frac{\partial^3 Q}{\partial X_2 \partial X_3^2}}{u_0'(X_2)[u_0'(X_3)]^2} < 0$$

at $(T,0,X_2^-(T),X_3^-(T))$. This together with (4.49) and (4.50) allows us to choose an $\varepsilon > 0$ such that

$$|X_2^*(t) - X_2^-(T)| < \varepsilon, \quad |X_3^*(t) - X_3^-(T)| < \varepsilon, \quad \text{when } t \in [T, T+\varepsilon) \tag{4.51}$$

and that the following inequalities

$$\frac{\partial(\lambda_1 t + W_1)}{\partial X_1} > 0, \quad \left|\frac{\partial(\lambda_2 t + W_2)}{\partial X_2}\right| < \frac{1}{2}\left|\frac{\partial(\lambda_1 t + W_1)}{\partial X_1}\right|$$

$$\frac{\frac{\partial^3 Q}{\partial X_2^2 \partial X_3}}{[u_0'(X_2)]^2 u_0'(X_3)} < 0, \quad \frac{\frac{\partial^3 Q}{\partial X_2 \partial X_3^2}}{[u_0'(X_2)[u_0'(X_3)]^2} < 0 \tag{4.52}$$

hold in a set S where

$$S = \{(t,X_1,X_2,X_3) \setminus T \leqq t < T+\varepsilon, |X_1| < \varepsilon, |X_2 - X_2^-(T)| < \varepsilon,$$

$$|X_3 - X_3^-(T)| < \varepsilon, \text{and } u_0(X_1) > u_0(X_2) \geqq u_0(X_3)\}.$$

Following the proof of Lemma 4.5, we can use (4.40) and (4.52) to show that

$$\frac{\partial(\lambda_2 t + W_2)}{\partial X_2} < 0, \quad \frac{\partial(\lambda_3 t + W_3)}{\partial X_3} > 0 \tag{4.53}$$

hold on the solution (t,X_1,X_2,X_3) of (4.27) whenever the solution is in S with $X_2 \neq X_3$.

Lemma 3.7, (4.52) and (4.53) enable us to solve (4.27) for

$$X_1 = m(X_2), \quad X_3 = n(X_2)$$

in the neighborhood of $X_2^*(t)$ for each $t \in (T, T+\varepsilon)$. Moreover, $m(X_2)$ and $n(X_2)$ are decreasing functions of X_2, and in particular, we have:

$$m'(X_2) = \frac{\frac{\partial}{\partial X_2}(\lambda_2 t + W_2)}{\frac{\partial}{\partial X_1}(\lambda_1 t + W_1)}. \tag{4.54}$$

Using the Implicit Function Theorem again, we can extend $m(X_2)$ and $n(X_2)$ in the positive X_2 direction so far as (t,X_1,X_2,X_3) is in S and $X_2 < X_3$.

By Lemma 4.6, $X_2^*(t)$ and $X_3^*(t)$ are decreasing and increasing functions of t, respectively. Hence, we have $X_2^*(t) < X_2^-(T) = X_3^-(T) < X_3^*(t)$ for $t > T$. Since $X_3 = n(X_2)$ decreases as X_2 increases, if we increase X_2 starting at $X_2^*(t)$, then by (4.51) we find

$$|X_2 - X_2^-(T)| < \varepsilon, \quad |X_3 - X_3^-(T)| < \varepsilon$$

for $X_2 \leqq X_3$. Thus, to prove that $(t,X_1,X_2,X_3) \in S$ before X_2 and $n(X_2)$ meet, it suffices to show that $-\varepsilon < X_1 \leqq 0$ during this extension.

It follows from $m(X_2^*(t)) = 0$ and the fact that $X_1 = m(X_2)$ is a decreasing function of X_2 that $X_1 \leqq 0$. Using $m(X_2^*) = 0$ again, we see

$$\begin{aligned} X_1 &= \int_{X_2^*(t)}^{X_2} m'(\xi)d\xi \\ &> -\frac{1}{2}[X_2 - X_2^*(t)] \\ &> -\varepsilon \end{aligned}$$

for $X_2 > X_2^*(t)$, where we have also used (4.52) and (4.54) in the first inequality, and (4.51) and $|X_2(t) - X_2^-(T)| < \varepsilon$ in the last one. Therefore, we have proved that $(t,X_1,X_2,X_3) \in S$ before $X_3 = n(X_2)$ and X_2 meet.

As a consequence, we can extend $m(X_2)$ and $n(X_2)$ so long as $m(X_2) < X_2 < n(X_3)$. Eventually, $n(X_2)$ and X_2 will meet at X_2^-. Denote,

$$X_1^- = m(X_2^-), \quad X_3^- = n(X_2^-)\,.$$

Obviously, $(t,X_1^-(t),X_2^-(t),X_3^-(t))$ satisfies (4.27) and therefore, $X_i^-(t)$'s are given in Lemma 4.3. Thus, we have proved:

Lemma 4.7. *Under the conditions of Lemma* 4.1, *there exists an* $\varepsilon > 0$ *such that* (4.27) *can be solved for*

$$\begin{aligned} X_1 &= m(X_2), \\ & \qquad\qquad X_2^*(t) \leqq X_2 \leqq X_2^-(t)\,, \\ X_3 &= n(X_2), \end{aligned}$$

for each $t \in [T, T+\varepsilon)$. *Furthermore,* $m(X_2)$ *and* $n(X_2)$ *are decreasing functions of* X_2.

We are now ready to solve the hodograph transform (4.6) for X_1,X_2 and X_3 as functions of (x,t) for a short time after T. By Lemma 4.7, (4.27) determines $X_1 = m(X_2)$ and $X_3 = n(X_2)$ for $X_2^-(t) \leqq X_2 \leqq X_2^*(t)$. We want to extend $m(X_1)$ and $n(X_2)$ for $X_2 < X_2^*(t)$. The change of variables (4.38) allows us to transform system (4.27) when $X_1 \geqq 0$ into system (3.7). The method of Sect. 3 can be used to show that (4.27) determines X_1 and X_3 as decreasing functions of X_2 for $X_2^+(t) \leqq X_2 \leqq X_2^*(t)$, where $X_2^+(t)$ is given in Lemma 4.4. Therefore, we have shown that (4.27) determines X_1 and X_3 as decreasing functions of X_2 over $[X_2^+(t),X_2^-(t)]$. Substituting $X_1 = m(X_2)$ and $X_3 = n(X_2)$ into the hodograph transform (4.6), we obtain

$$\begin{aligned} x = {}& \lambda_2(u_0(m(X_2)), u_0(X_2), u_0(n(X_3)))t \\ &+ W_2(u_0(m(X_2)), u_0(X_2), u_0(n(X_3)))t\,, \end{aligned}$$

which by Lemma 3.7, Lemma 3.9, (4.52) and (4.53) determines x as a decreasing function of X_2 over $[X_2^+(t),X_2^-(t)]$. This implies that X_2 is a function of (x,t) for $x^-(t) < x < x^+(t)$ and $t \geqq T$, where

$$\begin{aligned} x^\pm(t) = {}& \lambda_2(u_0(m(X_2^\pm(t))), u_0(X_2^\pm(t)), u_0(n(X_2^\pm(t))))t \\ &+ W_2(u_0(m(X_2^\pm(t))), u_0(X_2^\pm(t)), u_0(n(X_3^\pm(t))))\,. \end{aligned}$$

In the same region, X_2 and X_3 are, accordingly, functions of (x, t).

Thus, (4.6) can be solved for

$$X_1 = X_1(x,t), \quad X_2 = X_2(x,t), \quad X_3 = X_3(x,t)$$

within a cusp for a short time after T. Therefore, we have proved:

Theorem 4.8 *Under conditions of Lemma* 4.3, *the hodograph transform* (4.6) *with* W_i *given by* (4.14) *and* (4.15) *can be solved for* X_1,X_2 *and* X_3 *as functions of* (x,t) *within a cusp for a short time after T. Furthermore, boundary conditions* (4.4) *and* (4.5) *are satisfied on the cusp.*

Theorems 2.3, 4.8 and the transform $\beta_i = u_0(X_i)$ immediately establish the main theorem of this section.

Theorem 4.9 *Under conditions of Lemma* 4.3, *the Whitham averaged system has a solution* $(\beta_1(x,t), \beta_2(x,t), \beta_3(x,t))$ *within a cusp for a short time after T. Furthermore, the Whitham solution satisfies boundary conditions* (2.2) *and* (2.3) *on the cusp.*

Remark. Lemmas 4.1–4.6 are all time results. However, we did not succeed in proving Theorems 4.8 and 4.9 for all time $t \geqq T$.

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