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On the support of the Ashtekar-Lewandowski Measure

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Abstract: We show that the Ashtekar-Isham extension $\overline{\mathscr{A}/\mathscr{G}}$ of the configuration space of Yang-Mills theories $\mathscr{A}/\mathscr{G}$ is (topologically and measure-theoretically) the projective limit of a family of finite dimensional spaces associated with arbitrary finite lattices.

These results are then used to prove that $\mathscr{A}/\mathscr{G}$ is contained in a zero measure subset of $\overline{\mathscr{A}/\mathscr{G}}$ with respect to the diffeomorphism invariant Ashtekar-Lewandowski measure on $\overline{\mathscr{A}/\mathscr{G}}$. Much as in scalar field theory, this implies that states in the quantum theory associated with this measure can be realized as functions on the "extended" configuration space $\overline{\mathscr{A}/\mathscr{G}}$.

1. Introduction

The usual canonical approach to quantization of a (finite dimensional) system defines states as functions on a configuration space and defines an inner product of two such functions ψ and ϕ through

$$(\psi,\phi) = \int_{\mathscr{Q}} d\mu\ \psi^* \phi,$$

where μ is some measure on the configuration space $\mathscr{Q}$. Naively applying this procedure to Yang-Mills theories produces a "connection representation" with states that are functions of the Yang-Mills connection. In particular, these states are functions on the quotient space $\mathscr{A}/\mathscr{G}$, where $\mathscr{A}$ is the space of (C^1-)connections and $\mathscr{G}$ is the group of (C^2-)gauge transformations. The same is true for gravity formulated in terms of Ashtekar variables before one imposes the diffeomorphism and hamiltonian constraints [1,2].

A more sophisticated analysis of examples, such as scalar field theory [3-5], shows that the domain space of the wave functions may not be exactly the classical configuration space. Instead, some extension of $\mathscr{Q}$ is required.

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In order to define an inner product for a connection representation, one expects to give $\mathcal{A}/\mathcal{G}$, or some suitable extension, the structure of a measurable space (by choosing the measurable sets) and to define appropriate measures. Ashtekar and Isham described an algebraic program to construct such measures in [2]. They proposed, for a compact gauge group G, a compact extension $\overline{\mathcal{A}/\mathcal{G}}$ of $\mathcal{A}/\mathcal{G}$ on which regular Borel measures are well defined and are in one-to-one correspondence with positive continuous linear functionals on a certain C^*-algebra of connection observables known as the holonomy algebra $\overline{\mathcal{H}A}$. In [6], Ashtekar and Lewandowski constructed such a Borel measure μ_{AL} on $\overline{\mathcal{A}/\mathcal{G}}$ that is both diffeomorphism invariant and strictly positive on continuous cylindrical functions. To do so they, and independently Baez in [7], introduced the concepts of "cylindrical sets" and "cylindrical functions" on $\overline{\mathcal{A}/\mathcal{G}}$. Baez then generalized the Ashtekar-Lewandowski measure by finding an infinite dimensional space of diffeomorphism invariant measures. In [6] it was also shown that the Ashtekar-Isham space $\overline{\mathcal{A}/\mathcal{G}}$ is in one-to-one correspondence with the set of homomorphisms from the group of piecewise analytic hoops (i.e. based loops modulo an equivalence relation defined by the holonomies) $\mathcal{H}G_{x_0}$ to the gauge group K, modulo conjugation.

In what follows, we reinterpret some of the results of [2, 6] in terms of the theory of projective limits. In particular, we consider projective limits of infinite families of finite dimensional topological and measurable spaces associated with *arbitrary* finite lattices. This theory provides an appropriate framework for studying different properties of $\overline{\mathcal{A}/\mathcal{G}}$, both from the topological and measure theoretical points of view. Our main result is the use of this formalism to prove that the space $\mathcal{A}/\mathcal{G}$ is contained in a zero measure subset of $\overline{\mathcal{A}/\mathcal{G}}$ (with respect to the Ashtekar-Lewandowski measure).

The present work is organized as follows. In Sect. 2 we recall (mainly from [8]) some aspects of the theory of projective limits of infinite families of measurable spaces. Section 3 is devoted to reinterpreting some results of [2, 6] in the language of projective limits. In particular we show that $\overline{\mathcal{A}/\mathcal{G}}$ is a projective limit of a family of finite-dimensional spaces and that the Gel'fand topology on the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ coincides with the Tychonov topology on the projective limit. While the Ashtekar-Isham space $\overline{\mathcal{A}/\mathcal{G}}$ is defined only for compact gauge groups G, the projective limit is defined for the noncompact case as well. On the measure theoretical side we show that the measurable space $(\overline{\mathcal{A}/\mathcal{G}}, \mathcal{B}(\overline{\mathcal{C}}))$ (where $\mathcal{B}(\overline{\mathcal{C}})$ denotes the minimal σ-algebra containing the cylindrical sets $\overline{\mathcal{C}}$) is isomorphic to the projective limit. In Sect. 4 we prove the main result of the paper stated above. Section 5 is devoted to the study of the additive, but not σ-additive, measure $\hat{\mu}_{AL}$ induced by μ_{AL} on the (finite) algebra $\mathcal{C}$ of cylindrical sets of $\mathcal{A}/\mathcal{G}$,

$$\mathcal{C} = \{\bar{C} \cap \mathcal{A}/\mathcal{G};\ \bar{C} \subset \overline{\mathcal{C}}\},$$

where $\overline{\mathcal{C}}$ denotes the algebra of cylindrical sets on $\overline{\mathcal{A}/\mathcal{G}}$. We show that $\hat{\mu}_{AL}$ cannot be extended to a σ-additive measure on $\mathcal{A}/\mathcal{G}$ and that the space of square integrable (cylindrical) functions on $\mathcal{A}/\mathcal{G}$ is not complete. We also prove that the Cauchy completion of this space is $L^2\left(\overline{\mathcal{A}/\mathcal{G}}, \mu_{AL}, \mathcal{B}(\overline{\mathcal{C}})\right)$, justifying the use of the "generalized connections" in $\overline{\mathcal{A}/\mathcal{G}}$.

2. Projective Limit Measurable Spaces

In the present section we recall, mainly from [8], the relevant aspects of a class of measures on infinite dimensional spaces which are obtained as rigorously defined limits of measures on finite dimensional spaces. This class contains the direct product measures (on $\mathbb{R}^\infty$ for example) and the projective limit measures. First, however, we introduce some more terminology and notation that will prove useful.

The pair $(X, \mathscr{B})$ (or $(X, \mathscr{F})$), where X is a set and $\mathscr{B}(\mathscr{F})$ is a σ-algebra (algebra) of subsets of X, will be called a σ-measurable (measurable) space. In the mathematical literature, definitions of a measurable space have been given both that require $\mathscr{B}$ to be a σ-algebra and that require only that $\mathscr{B}$ be closed under finite operations. As we will be interested in a comparison of these two cases it will be convenient to use the above terminology to distinguish between them.

We will be interested in $\sigma - additive$ probability measures on $\mathscr{B}$, which are, by definition non-negative, normalized and σ-additive functions on the σ-algebra $\mathscr{B}$. That is, such a measure μ satisfies:

$$\mu(B) \geqq 0 \ , \quad B \in \mathscr{B}, \tag{2.1a}$$

$$\mu(X) = 1 \ , \tag{2.1b}$$

$$\mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i), \quad B_i \in \mathscr{B}, \quad B_i \cap B_j = \emptyset, \quad i \neq j \ . \tag{2.1c}$$

Additive measures on an algebra $\mathscr{F}$ satisfy (2.1) with $\mathscr{B}$ replaced by $\mathscr{F}$ and with only finite unions and sums in (2.1c). For a given measure μ on $\mathscr{F}$, an important question is whether or not it can be extended to a σ-additive measure on $\mathscr{B}(\mathscr{F})$, the minimal σ-algebra that contains $\mathscr{F}$. A necessary and sufficient condition for extendibility is given by the Hopf theorem [8]:

Theorem 2.1 (*Hopf Theorem*). *A measure μ on $\mathscr{F}$ can be extended to a σ-additive measure on $\mathscr{B}(\mathscr{F})$ if and only if for every decreasing sequence $\{F_i\}$ such that $F_i \in \mathscr{F}$, $F_1 \supset \cdots \supset F_n \supset \cdots$ with $\bigcap_{i=1}^{\infty} F_i = \emptyset$, we have*

$$\lim_{i \to \infty} \mu(F_i) = 0 \ . \tag{2.2}$$

Essentially, the condition (2.2) allows an extension $\tilde{\mu}$ to be consistently defined on elements of $\mathscr{B}(\mathscr{F})$ as limits of μ-measures of sets in $\mathscr{F}$. The triplet $\{(X, \mathscr{B}), \mu\}$ ($\{(X, \mathscr{F}), \mu\}$), where $\mathscr{B}(\mathscr{F})$ is a σ-algebra (algebra) and μ is σ-additive (additive) is called a σ-measure (measure) space.

The possibility of extending a measure μ on $\mathscr{F}$ to a σ-additive measure $\tilde{\mu}$ on $\mathscr{B}(\mathscr{F})$ is in particular relevant to physical applications in quantum mechanics. Recall that quantum mechanical systems are often defined by first giving a linear pre-Hilbert space and then completing this space with respect to an inner product. In general, if μ is cylindrical but not σ-additive, the space $\mathscr{H}$ of μ-square integrable cylindrical functions on X (denoted through $\mathscr{C}L^2(X, \mathscr{F}, \mu)$) is only a pre-Hilbert space. Such spaces will be discussed in Sect. 5. However, if μ is extendible to a σ-additive measure $\tilde{\mu}$ on $(X, \mathscr{B}(\mathscr{F}))$ then the Cauchy completion of $\mathscr{H}$ leads to the space $\tilde{\mathscr{H}} = L^2(X, \mathscr{B}(\mathscr{F}), \tilde{\mu})$ (see Sect. 5). On the other hand if μ is not extendible then the Cauchy completion of $\mathscr{C}L^2(X,\mathscr{F},\mu)$ leads in general to a space with state-vectors which cannot be expressed as functions on the initial space X. This is the case in scalar field theory if one considers $X=\mathscr{S}(\mathbb{R}^3)$ (the Schwarz space of rapidly decreasing smooth C^∞ functions on $\mathbb{R}^3$) and μ is a cylindrical measure defined with the help of a positive definite function on $\mathscr{S}(\mathbb{R}^3)$, continuous in the nuclear space topology (see [3, 5, 8]). As we shall see in Sect. 5 this is also the case in Yang-Mills theory if we take $\mathscr{H}=\mathscr{C}L^2(\mathscr{A}/\mathscr{G},\mathscr{F}=\mathscr{C},\hat{\mu}_{AL})$, where $\hat{\mu}_{AL}$ is the Ashtekar-Lewandowski measure on $\mathscr{A}/\mathscr{G}$. In the scalar field case the Cauchy completion of $\mathscr{C}L^2(\mathscr{S}(\mathbb{R}^3),\mathscr{F},\mu)$ gives the space of square integrable functions on $\mathscr{S}'(\mathbb{R}^3)$ (the space of tempered distributions), while in the Yang-Mills case the completion of $\mathscr{C}L^2(\mathscr{A}/\mathscr{G},\mathscr{C},\hat{\mu}_{AL})$ gives the space $L^2(\overline{\mathscr{A}/\mathscr{G}},\mathscr{B}(\overline{\mathscr{C}}),\mu_{AL})$ of square integrable functions on the Ashtekar-Isham space $\overline{\mathscr{A}/\mathscr{G}}$ of generalized "distributional" connections modulo gauge transformations.

Let $\{(X,\mathscr{B}),\mu\}$ be a σ-measure space. The subset $Y\subset X$ is said to be μ-thick in X if for every $B\in\mathscr{B}$ such that $B\cap Y=\emptyset$, $\mu(B)=0$. If Y is μ-thick in X then μ induces a σ-additive measure μ_Y on the σ-measurable space

$$(Y,\ \mathscr{B}_Y)\ , \tag{2.3a}$$

where $\mathscr{B}_Y=\{B\cap Y,\ B\in\mathscr{B}\}$, through

$$\mu_Y(B\cap Y)=\mu(B),\quad \forall B\in\mathscr{B}\ . \tag{2.3b}$$

The measure μ_Y is called the trace of the measure μ on Y [8]. If Y is not μ-thick on X then (2.3b) is not well defined.

Note that if Y is μ-thick in X then

$$L^p(X,\mu,\mathscr{B})\cong L^p(Y,\mu_Y,\mathscr{B}_Y)\ ,$$

so that if we are concerned only with such spaces we can restrict ourselves to Y and μ_Y. This is particularly convenient when the set Y has advantages (for instance from the "differentiable" point of view) over X. When a set Y is μ-thick in X we say that the support of the measure μ is contained in Y. An illustrative example is the one given by the Wiener measure on $\mathbb{R}^{[0,1]}$ used in the (euclidean) path integral formulation of quantum mechanics. In this case the support of the measure is contained in the space $Y=C^0([0,1])$ of continuous functions on the interval [3].

The inclusion map from Y to X above is referred to as measurable. In general, a map between σ-measurable (measurable) spaces

$$\phi:\ X_1\to X_2 \tag{2.4}$$

is called measurable if for every measurable set $B_2\in\mathscr{B}_2$ the set $\phi^{-1}(B_2)$ is measurable, i.e. $\phi^{-1}(B_2)\in\mathscr{B}_1$. ϕ in (2.4) is called an isomorphism of σ-measurable (measurable) spaces if it is bijective and if both ϕ and ϕ^{-1} are measurable.

Let us now briefly review (see [8]) the construction of infinite products of σ-measurable spaces and of (projective) limits of infinite projective families of σ-measurable spaces. Let

$$\{(X^{(\lambda)},\mathscr{B}^{(\lambda)})\}_{\lambda\in\Lambda} \tag{2.5}$$

be an indexed family of σ-measurable spaces. The product σ-measurable space $(X^{(\Lambda)}, \mathscr{B}^{(\Lambda)})$ is, by definition, given by

$$X^{(\Lambda)} = \prod_{\lambda \in \Lambda} X^{(\lambda)} \tag{2.6}$$

with $\mathscr{B}^{(\Lambda)}$ being the minimal σ-algebra for which all the projections

$$\begin{aligned} p_{\lambda_0}\colon X^{(\Lambda)} &\to X^{(\lambda_0)}, \\ (x_\lambda)_{\lambda \in \Lambda} &\mapsto x_{\lambda_0} \end{aligned} \tag{2.7}$$

are measurable. That is, $\mathscr{B}^{(\Lambda)}$ is the σ-algebra generated by the inverse images of measurable sets in $X^{(\lambda)}$ under the projections p_λ. If all the $X^{(\lambda)}$, $\lambda \in \Lambda$ are different copies of the same set Y with the same σ-algebras $\mathscr{B}^{(\lambda)} = \mathscr{B}$, then the points of $X^{(\Lambda)} = Y^\Lambda$, $x \in Y^\Lambda$ are (arbitrary) maps from Λ to Y:

$$\begin{aligned} x \in X^{(\Lambda)} = Y^\Lambda &\Leftrightarrow x\colon \Lambda \to Y \ , \\ x = (x_\lambda)_{\lambda \in \Lambda}\, ; &\quad x_\lambda \in Y \ . \end{aligned} \tag{2.8}$$

Examples are the set of all sequences of real numbers

$$\mathbb{R}^\infty = \prod_{j \in \mathbb{N}} \mathbb{R}_{(j)}\, , \qquad (\mathbb{R}_{(j)} = \mathbb{R}) \tag{2.9}$$

and the set of all real valued functions on the interval $[0, 1]$,

$$\mathbb{R}^{[0,1]} = \prod_{t \in [0,1]} \mathbb{R}_t\, , \qquad (\mathbb{R}_t = \mathbb{R})\ . \tag{2.10}$$

Suppose we have a σ-additive measure μ on $X^{(\Lambda)}$ in (2.6). Let $\mathscr{L}$ be the family of all finite subsets of Λ and for $L \in \mathscr{L}$ let $(X^{(L)}, \mathscr{B}^{(L)})$ be the partial products of σ-measurable spaces with

$$X^{(L)} = \prod_{\lambda \in L} X^{(\lambda)} \tag{2.11}$$

and the corresponding $\mathscr{B}^{(L)}$. Then all the projections

$$\begin{aligned} p_L\colon X^{(\Lambda)} &\to X^{(L)}\ , \\ p_L((x_\lambda)_{\lambda \in \Lambda}) &= (x_\lambda)_{\lambda \in L} \end{aligned} \tag{2.12}$$

are measurable. Consider the family $\{\mu_L\}_{L \in \mathscr{L}}$ of σ-additive measures on $X^{(L)}$ defined by the pushforwards of the measure μ,

$$\mu_L(B) = \mu(p_L^{-1}(B)) \tag{2.13}$$

for $B \in \mathscr{B}_L$, which, in the notation of measure theory is written as

$$\mu_L = (p_L) * \mu\ .$$

This family satisfies a self consistency condition:

$$L \subset L' \ \Rightarrow\ \mu_L = (p_{LL'}) * \mu_{L'}\ , \tag{2.14}$$

where $p_{LL'}$ denote the measurable projections from $X^{(L')}$ to $X^{(L)}$. In [8] (see the corollary to Theorem 10.1) are found conditions for which the converse is also true:

Proposition 2.2. *For a family of σ-compact or complete and separable, metric spaces every family of Borel measures that is consistent in the sense of* (2.14) *can be extended to a σ-additive measure on the product σ-measurable space.*

Such a measure is in fact defined by (2.13), i.e. for $B_L \in \mathscr{B}_L$, $\mu(p_L^{-1}(B_L))$ is defined to be just $\mu_L(B_L)$. Recall that a topological space (X, τ) is said to be σ-compact if X can be represented as a countable union of compact sets.

Notice ([8]) that a measure μ satisfying (2.13) and given on the algebra

$$\mathscr{F}^{(\mathscr{L})} = \cup_{L \in \mathscr{L}} \; p_L^{-1}(\mathscr{B}_L) \tag{2.15}$$

always exists. The only question is whether μ can be extended to a σ-additive measure $\tilde{\mu}$ on $\mathscr{B}^{(\mathscr{L})}$, which is the minimal σ-algebra that contains (2.15),

$$\mathscr{B}^{(\mathscr{L})} = \mathscr{B}(\cup_{L \in \mathscr{L}} \; p_L^{-1}(\mathscr{B}_L)) \; . \tag{2.16}$$

For instance, in the example of $X^{(\Lambda)} = \mathbb{R}^{[0,1]}$ the question is to know when a self-consistent family of measures $\{\mu_{t_1,\dots,t_n}\}_{t_1,\dots,t_n \in [0,1]}$ on the finite dimensional spaces

$$\mathbb{R}^n = \prod_{i=1}^{n} \mathbb{R}_{t_i} \tag{2.17}$$

defines a σ-additive measure on the infinite dimensional space

$$\mathbb{R}^{[0,1]} \; . \tag{2.18}$$

Quite remarkably, in this example and in many others relevant to quantum field theory the answer is affirmative, as indicated by Proposition 2.2.

An infinite product σ-measure space can also be realized as a "projective limit" (which we will define next). However, the product space $X^{(\Lambda)}$ is a projective limit not of the family of spaces $X^{(\lambda)}$ labelled by $\lambda \in \Lambda$ but rather of the family of spaces $X_L = X^{(L)}$ labelled by $L \in \mathscr{L}$, the set of all finite subsets of Λ. In general, a projective limit space can be defined for any "projective family" of σ-measurable spaces; that is, for any family

$$\{(X_L, \mathscr{B}_L), p_{LL'}\}_{L, L' \in \mathscr{L}} \; , \tag{2.19}$$

of the following form. The set $\mathscr{L}$ is taken to be directed, i.e. partially ordered and such that for any two elements $L_1, L_2 \in \mathscr{L}$, there is some L such that $L_1 \leqq L$ and $L_2 \leqq L$. We will also assume that $\mathscr{L}$ does not have a maximum. Here $p_{LL'}$ are measurable projections, i.e. surjective mappings

$$p_{LL'} : X_{L'} \to X_L \quad L < L' \tag{2.20a}$$

satisfying

$$p_{LL'} \circ p_{L'L''} = p_{LL''} \quad \text{for } L < L' < L'' \; . \tag{2.20b}$$

Now, let $(X^{(\mathscr{L})}, \mathscr{B}^{(\mathscr{L})})$ denote the direct product of the family $\{(X_L, \mathscr{B}_L)\}_{L \in \mathscr{L}}$

$$X^{(\mathscr{L})} = \prod_{L \in \mathscr{L}} X_L \; .$$

Then the projective limit of the family (2.19) is by definition the σ-measurable space $(X_{\mathscr{L}}, \mathscr{B}_{\mathscr{L}})$, where

$$X_{\mathscr{L}} \subset X^{(\mathscr{L})}; \quad X_{\mathscr{L}} = \{(x_L)_{L\in\mathscr{L}} \in X^{(\mathscr{L})}: \ L < L' \Rightarrow x_L = p_{LL'}(x_{L'})\} \qquad (2.21a)$$

and

$$\mathscr{B}_{\mathscr{L}} = \{B \cap X_{\mathscr{L}}: \ B \in \mathscr{B}^{(\mathscr{L})}\} \ . \qquad (2.21b)$$

That is, $X_{\mathscr{L}}$ is the subset of $X^{(\mathscr{L})}$ that is consistent with the projections $p_{LL'}$. Note that a direct product space can also be thought of as a projective limit of the spaces formed by taking arbitrary finite products of the factors. A family of measures $(\mu_L)_{L\in\mathscr{L}}$ is said to be self-consistent if it satisfies (2.14) with $L \subset L'$ replaced by $L < L'$. A measure μ on $\mathscr{B}_{\mathscr{L}}$ always defines a self consistent family of measures $(\mu_L)_{L\in\mathscr{L}}$ through (2.13) and a consistent family $(\mu_L)_{L\in\mathscr{L}}$ defines a finitely additive measure on $X_{\mathscr{L}}$ through (2.13) as well. A measure on $X_{\mathscr{L}}$ defined by such a family is called cylindrical. An important result is (see [8] Corollary to Theorem 10.1):

Proposition 2.3. *Under the same conditions as in Proposition 2.2, a self-consistent family of Borel measures on a projective family (2.19) defines a cylindrical measure that can be extended to a σ-additive measure in the projective limit σ-measurable space (2.21) if for every increasing sequence*

$$\mathscr{M} = \{L_i\}_{i=1}^{\infty} \subset \mathscr{L} : \ L_1 < L_2 < \cdots < L_n < \cdots$$

with projective limit $(X_{\mathscr{M}}, \mathscr{B}_{\mathscr{M}})$ the projection

$$p_{\mathscr{M}} : \ X_{\mathscr{L}} \to X_{\mathscr{M}} \ ,$$

$$p_{\mathscr{M}}((x_L)_{L\in\mathscr{L}}) = (x_{L_n})_{L_n\in\mathscr{M}}$$

is surjective.

3. Ashtekar-Isham Space $\overline{\mathscr{A}/\mathscr{G}}$ as a Projective Limit

Let $\mathscr{A}/\mathscr{G}$ denote the space $\mathscr{A}$ of smooth C^1 G-connections modulo the group $\mathscr{G}$ of gauge transformations on a three dimensional analytic manifold Σ where, as in [6], the gauge group G is assumed to be $U(N)$ or $SU(N)$. Following [6], we consider the G-hoop group $\mathscr{H}G_{x_0} = \mathscr{L}\Sigma_{x_0}/\sim$, where $\mathscr{L}\Sigma_{x_0}$ is the space of piecewise analytic loops based at x_0 (see [6]) and the equivalence relation $\sim$ is

$$\alpha, \beta \in \mathscr{L}\Sigma_{x_0}, \ \alpha \sim \beta \ \text{ if and only if } \ H(\alpha, A) = H(\beta, A), \quad \forall A \in \mathscr{A} \ . \qquad (3.1)$$

Here, $H(\alpha, A)$ denotes the holonomy corresponding to the connection A and the loop α. The Ashtekar-Isham space $\overline{\mathscr{A}/\mathscr{G}}$ is a "compactification" of $\mathscr{A}/\mathscr{G}$ obtained as follows (see [2]). Let $T_\alpha, \alpha \in \mathscr{L}\Sigma_{x_0}$ denote the Wilson loop function on $\mathscr{A}/\mathscr{G}$ defined by

$$T_\alpha(A) = T_{[\alpha]}([A]) \equiv \frac{1}{N} Tr H(\alpha, A) \ . \qquad (3.2)$$

where $[\alpha]$ denotes the equivalence class of α in $\mathscr{H}G_{x_0}$, $[A]$ denotes the equivalence class of A in $\mathscr{A}/\mathscr{G}$ and the trace is taken in the fundamental representation of the gauge group. In the following, for simplicity, α, β will denote hoops. The holonomy algebra $\widetilde{\mathscr{H}A}$ is the commutative C^*-algebra generated by the Wilson loop functions. The Ashtekar-Isham space $\overline{\mathscr{A}/\mathscr{G}}$ is the compact Hausdorff space that is the spectrum [2] of $\widetilde{\mathscr{H}A}$ in which $\mathscr{A}/\mathscr{G}$ is densely embedded [2, 6, 8].

Ashtekar and Lewandowski [6] obtained a useful algebraic characterization of the space $\overline{\mathscr{A}/\mathscr{G}}$. They proved that there is a one-to-one correspondence between $\overline{\mathscr{A}/\mathscr{G}}$ and the space of *all* homomorphisms from the hoop group $\mathscr{H}G_{x_0}$ to the gauge group G, modulo conjugation. We will therefore identify these two sets and write

$$\begin{aligned} &\tilde{h} = [h_0] \in \overline{\mathscr{A}/\mathscr{G}} \Leftrightarrow \\ &[h_0] = \{h \in Hom(\mathscr{H}G_{x_0}, G) : (h(\alpha))_{\alpha \in \mathscr{H}G_{x_0}} = (gh_0(\alpha)g^{-1})_{\alpha \in \mathscr{H}G_{x_0}}, \\ &\qquad \text{for some } g \in G\} \ , \end{aligned} \tag{3.3}$$

where g above does not depend on the hoop α. Notice that no continuity condition has been imposed on the homomorphisms h in (3.3). This will allow us to interpret $\overline{\mathscr{A}/\mathscr{G}}$ (both topologically and measure theoretically) as a projective limit of finite dimensional spaces.

Let $\mathscr{L}$ denote the set of all subgroups of $\mathscr{H}G_{x_0}$ generated by a finite number of hoops $\beta_1, \ldots, \beta_n$ that are strongly independent in the sense of [6], i.e. such that loop representatives of the hoop equivalence classes β_i can be chosen in such a way that each contains an open segment which is traced exactly once and which intersects any of the other representative loops at most at a finite number of points. Then

$$\begin{aligned} &S^* \in \mathscr{L} \Leftrightarrow \\ &S^* = \{\text{group generated by } \beta_1, \ldots, \beta_n\} \subset \mathscr{H}G_{x_0} \ , \end{aligned} \tag{3.4}$$

and we write $S^* = S^*[\beta_1, \ldots, \beta_n]$. Now let $H_{S^*} = Hom(S^*, G)/Ad$ be the set of equivalence classes of homomorphisms from S^* to G under conjugation. If $S^* = S^*[\beta_1, \ldots, \beta_n]$ then, as shown in [6], a homomorphism from S^* to G is known if and only if we know it on the hoops $\beta_1, \ldots, \beta_n$ so that we have the one-to-one correspondence

$$H_{S^*} \to G^n/Ad \ , \tag{3.5a}$$

$$[h] \mapsto [h(\beta_1), \ldots, h(\beta_n)] \ . \tag{3.5b}$$

Consider now the following projective family of finite dimensional spaces

$$\{(H_{S^*}), p_{S^*S^{*\prime}}\}_{S^*, S^{*\prime} \in \mathscr{L}} \ , \tag{3.6}$$

where $p_{S^*S^{*\prime}}$, $S^* \subset S^{*\prime}$, denotes the mapping

$$p_{S^*S^{*\prime}} \colon H_{S^{*\prime}} \to H_{S^*} \ , \tag{3.7a}$$

$$p_{S^*S^{*\prime}}([h_{S^{*\prime}}]) = [h_{S^{*\prime}}|_{S^*}] \ , \tag{3.7b}$$

and $h_{S^{*\prime}}|_{S^*}$ denotes the restriction of $h_{S^{*\prime}}$ to the subgroup S^* of $S^{*'}$. From [6] we see that these projections are surjective. According to (2.21) the projective limit $H_{\mathscr{L}}$ of the family (3.6) is given by

$$H_{\mathscr{L}} \subset H^{(\mathscr{L})} = \prod_{S^* \in \mathscr{L}} H_{S^*} \ ,$$
$$H_{\mathscr{L}} = \{([h_{S^*}])_{S^* \in \mathscr{L}} \in H^{(\mathscr{L})} : \ S^* \subset S^{*'} \Rightarrow [h_{S^*}] = p_{S^* S^{*\prime}}([h_{S^{*\prime}}])\} \ . \tag{3.8}$$

We will now show that this is just the Ashtekar-Isham space $\overline{\mathscr{A}/\mathscr{G}}$.

Proposition 3.1. *There is a bijective map* ϕ

$$\overline{\mathscr{A}/\mathscr{G}} \xrightarrow{\phi} H_{\mathscr{L}} \tag{3.9a}$$

defined by

$$[h] \mapsto ([h_{S^*}])_{S^* \in \mathscr{L}} \, ; \ h_{S^*} = h\,|_{S^*} \ . \tag{3.9b}$$

Proof. Consider the space $Hom(\mathscr{H}G_{x_0}, G)$ of all homomorphisms from $\mathscr{H}G_{x_0}$ to G and the projective family $\{Hom(S^*, G), \tilde{p}_{S^* S^{*\prime}}\}_{S^*, S^{*\prime} \in \mathscr{L}}$, where $\tilde{p}_{S^* S^{*\prime}}$: $\tilde{p}_{S^* S^{*\prime}}(h_{S^{*\prime}}) = h_{S^{*\prime}}|_{S^*}$, $S^* \subset S^{*'}$ are surjective maps from $Hom(S^{*'}, G)$ to $Hom(S^*, G)$. Let $K^{(\mathscr{L})}$ be the infinite product space and $K_{\mathscr{L}}$ be the projective limit space of this family

$$K_{\mathscr{L}} = \{(h_{S^*})_{S^* \in \mathscr{L}} \in K^{(\mathscr{L})} : \ S^* \subset S^{*'} \Rightarrow h_{S^*} = \tilde{p}_{S^* S^{*\prime}}(h_{S^{*\prime}})\} \ . \tag{3.10}$$

We will need the following lemmas.

Lemma 3.2. *The map*

$$\tilde{\phi} : \ Hom(\mathscr{H}G_{x_0}, G) \to K_{\mathscr{L}} \ ,$$
$$\tilde{\phi}(h) = (h\,|_{S^*})_{S^* \in \mathscr{L}} \tag{3.11}$$

is bijective and Ad-equivariant, i.e. $Ad_g \circ \tilde{\phi} = \tilde{\phi} \circ Ad_g$ *for every* $g \in G$.

Proof of Lemma 3.2. The injectivity of $\tilde{\phi}$ is trivial. Let us prove that $\tilde{\phi}$ is surjective. Fix an arbitrary element $(h^0_{S^*})_{S^* \in \mathscr{L}} \in K_{\mathscr{L}}$. Let us construct the homomorphism h^0 which is the pre-image of this element. Let α be an arbitrary hoop and $S_1^* \in \mathscr{L}$ such that $\alpha \in S_1^*$ (S_1^* always exists for a piecewise analytic hoop α [2]). Then choose $h^0(\alpha) = h^0_{S_1^*}(\alpha)$. To see that $h^0(\alpha)$ does not depend on the choice of the finitely generated group $S_1^* \ni \alpha$, let $\alpha \in \tilde{S}_1^*$ and S_2^* be a subgroup which contains both S_1^* and $\tilde{S}_1^*$. Then, according to the definition of $K_{\mathscr{L}}$, we have $h^0_{S_1^*}(\alpha) = h^0_{S_2^*}(\alpha)$ and $h^0_{\tilde{S}_1^*}(\alpha) = h^0_{S_2^*}(\alpha)$ which implies that $h^0_{\tilde{S}_1^*}(\alpha) = h^0_{S_1^*}(\alpha)$. We can easily show that h^0 constructed in this way is an homomorphism and that the map $\tilde{\phi}$ is equivariant. *Q.E.D.*

Lemma 3.2 implies that the map $\tilde{\phi}$ induces a bijective map ϕ_1,

$$\begin{aligned} \phi_1 &: Hom(\mathscr{H}G_{x_0}, G)/Ad \;\to\; K_{\mathscr{L}}/Ad \;, \\ \phi_1([h]) &= [(h\,|_{S^*})_{S^* \in \mathscr{L}}] \;. \end{aligned} \tag{3.12}$$

Lemma 3.3. *The map*

$$\begin{aligned} \phi_2 &: K_{\mathscr{L}}/Ad \;\to\; H_{\mathscr{L}} \;, \\ \phi_2 &([(h_{S^*})_{S^* \in \mathscr{L}}]) \\ &= ([h_{S^*}])_{S^* \in \mathscr{L}} \end{aligned} \tag{3.13}$$

is bijective.

Proof of Lemma 3.3. We will first show that ϕ_2 is surjective. To do so, recall that any element of $H_{\mathscr{L}}$ is a family $([h_{S^*}])_{S^*}$ of consistent equivalence classes in the sense of (3.7b). Now, choose a representative $h^0_{S^*}$ from each $[h_{S^*}]$ and construct the subgroup $C^0_{S^*}$ of G that commutes with $h^0_{S^*}$; that is, let

$$C^0_{S^*} = \{g \in G\colon\; \forall \alpha \in S^*,\; g h^0_{S^*}(\alpha) g^{-1} = h^0_{S^*}(\alpha)\} \;. \tag{3.14}$$

Note that $C^0_{S^*}$ is closed in G. Any closed subgroup of a Lie group is a Lie group and any closed subset of a compact space is compact, so that $C^0_{S^*}$ is again a compact Lie group. Thus, $C^0_{S^*}$ has some dimension $d_{S^*} \geqq 0$ and, by compactness, some finite number $m_{S^*} \geqq 1$ of connected components. There is then some least value d_0 of d_{S^*} ($d_0 = \min_{S^* \in \mathscr{L}} d_{S^*}$) and some m_0 that is the least value of m_{S^*} for which the dimension of $C^0_{S^*}$ is d_0 (i.e. $m_0 = \min_{d_{S^*} = d_0} m_{S^*}$). Choose some S^*_0 with $d_{S^*_0} = d_0$ and $m_{S^*_0} = m_0$.

Now, for every $S^* \supset S^*_0$, choose another representative $h^1_{S^*}$ of $[h_{S^*}]$ such that

$$h^1_{S^*}\,|_{S^*_0} = h^0_{S^*_0} \;, \tag{3.15}$$

and construct the corresponding $C^1_{S^*}$:

$$C^1_{S^*} = \{g \in G\colon\; \forall \alpha \in S_*\; g h^1_{S^*}(\alpha) g^{-1} = h^1_{S^*}(\alpha)\} \;. \tag{3.16}$$

Note that $C^1_{S^*} \subset C^0_{S^*_0}$ and that $C^1_{S^*}$ differs from $C^0_{S^*}$ only by conjugation. Thus, $C^1_{S^*}$ has dimension $d_{S^*} \geqq d_{S^*_0}$ and m_{S^*} connected components. But, since $C^1_{S^*}$ is contained in $C^0_{S^*_0}$, $d_{S^*_0} \geqq d_{S^*}$ so that $C^1_{S^*}$ and $C^0_{S^*_0}$ are of the same dimension. It follows that they agree in some neighborhood of the identity and thus on the entire component connected to the identity. Since $C^0_{S^*_0} \supset C^1_{S^*}$ is a disjoint union of $m_{S^*_0}$ copies of this component, $m_{S^*} \leqq m_{S^*_0}$. But, since $C^1_{S^*}$ has dimension d_0, we have $m_{S^*} \geqq m_{S^*_0}$ and in fact $m_{S^*} = m_{S^*_0}$. We thus conclude that $C^1_{S^*} = C^0_{S^*_0}$.

This means that $h^1_{S^*}$ is unique, since any g that commutes with $h^0_{S^*_0}(\alpha) = h^1_{S^*}(\alpha)$ for all $\alpha \in S^*_0$ lies in $C^0_{S^*_0} = C^1_{S^*}$ and commutes with $h^1_{S^*}(\alpha)$ for all $\alpha \in S^*$.

Thus, no other representative of $[h_{S^*}]$ satisfies (3.15). It now follows that for any $S^{*\prime} \supset S^* \supset S_0^*$,

$$h^1_{S^{*\prime}}|_{S^*} = h^1_{S^*} \ , \tag{3.17}$$

since $h^1_{S^{*\prime}}|_{S^*}$ is the unique representative of $[h^1_{S^*}]$ that satisfies

$$(h^1_{S^{*\prime}}\ |_{S^*})\big|_{S_0^*} = h^1_{S^{*\prime}}|_{S_0^*} = h^0_{S_0^*} \ . \tag{3.18}$$

Finally, for any S^* that does not contain S_0^*, let $S^{*\prime}$ be any subgroup of $\mathscr{H}G_{x_0}$ generated by a finite number of independent hoops that contains S^* and S_0^* (we see from [6] that such a group exists) and let

$$h^1_{S^*} = h^1_{S^{*\prime}}|_{S^*} \ . \tag{3.19}$$

Then the representatives $(h^1_{S^*})_{S^*\in\mathscr{L}} \in ([h_{S^*}])_{S^*\in\mathscr{L}}$ form a consistent family of homomorphisms in $K^{(\mathscr{L})}$ and the equivalence class of this family under the adjoint action is a member of $K_{\mathscr{L}}/Ad$ that maps to $([h_{S^*}])_{S^*\in\mathscr{L}}$ under the map ϕ_2. We conclude that ϕ_2 is surjective.

Now, injectivity of ϕ_2 follows in a straightforward fashion. Consider any other equivalence class of families $[(h'_{S^*})_{S^*\in\mathscr{L}}] \in K_{\mathscr{L}}/Ad$ that maps to the family $([h_{S^*}])_{S^*\in\mathscr{L}}$ chosen above under ϕ_2. As with the family constructed above, $h^1_{S_0^*}$ must be a representative of $[h'_{S_0^*}]$. Let $(h^2_{S^*})_{S^*\in\mathscr{L}}$ be any family in $[(h'_{S^*})_{S^*\in\mathscr{L}}]$ such that $h^2_{S_0^*} = h^1_{S_0^*}$. We have just seen that $(h^1_{S_0^*})_{S^*\in\mathscr{L}}$ is the unique self-consistent family of homomorphisms that includes $h^1_{S^*}$ and satisfies $[h^1_{S^*}] = [h_{S^*}]$. Therefore, $h^2_{S^*} = h^1_{S^*}$ and the families $[(h^2_{S^*})_{S^*\in\mathscr{L}}]$ and $[(h^1_{S^*})_{S^*\in\mathscr{L}}]$ coincide, showing that ϕ_2 is also injective. *Q.E.D.*

We complete the proof of the proposition by noticing that the bijective map ϕ is given by

$$\phi = \phi_2 \circ \phi_1 \ . \tag{3.20}$$

Q.E.D.

Endowed with the natural topology, the spaces H_{S^*} are compact topological spaces (see (3.6)). The Tychonov topology τ_T on the product space $H^{(\mathscr{L})}$ is the minimal topology for which all the projections

$$\begin{aligned} \pi_{S^*} :\ & H_{\mathscr{L}} = \overline{\mathscr{A}/\mathscr{G}} \to H_{S^*} \\ & \pi_{S^*}([h]) = [h\ |_{S^*}] \end{aligned} \tag{3.21}$$

are continuous. It coincides with the topology of pointwise convergence in $H^{(\mathscr{L})}$, i.e the net $[h]^{(\nu)} = ([h_{S^*}]^{(\nu)})_{S^*\in\mathscr{L}}$ is τ_T-convergent

$$[h]^{(\nu)} \xrightarrow{\tau_T} [h] \ ,$$

if and only if

$$[h_{S^*}]^{(\nu)} \to [h_{S^*}], \quad \forall S^* \in \mathscr{L} \ , \tag{3.22}$$

where the last convergence is with respect to the topology on $H_{S^*} = G^n/Ad$. In this topology, the space $H^{(\mathscr{L})}$ is compact (see [8, Tychonov theorem]). Let us also refer to the topology induced on the projective limit $H_{\mathscr{L}} \subset H^{(\mathscr{L})}$ from $H^{(\mathscr{L})}$ as the Tychonov topology τ_T. Then from the continuity of the projections $p_{S^*S^{*\prime}}$, $H_{\mathscr{L}}$, is closed in $H^{(\mathscr{L})}$, and therefore

$$(H_{\mathscr{L}}, \tau_T) \tag{3.23}$$

is also a compact topological space. Since $H_{\mathscr{L}}$ is compact in the Tychonov topology and $\overline{\mathscr{A}/\mathscr{G}}$ is compact in the Gel'fand topology τ_{Gd}, it is natural to expect that the bijective map ϕ in (3.9) is actually a homeomorphism. Indeed we have

Proposition 3.4. *The bijective map in (3.9) is a homeomorphism*

$$\phi : \ (\overline{\mathscr{A}/\mathscr{G}}\ , \tau_{Gd}) \to (H_{\mathscr{L}}, \tau_T) \ , \tag{3.24}$$

where τ_{Gd} and τ_T denote the Gel'fand and Tychonov topologies respectively.

Proof. First let us obtain a more convenient characterization of the topology on the spaces H_{S^*}. As mentioned above, H_{S^*} endowed with the standard topology induced from G^n is a compact Hausdorff space. Consider on H_{S^*} the continuous functions

$$T_\alpha^{S^*}([h_{S^*}]) = Tr(h_{S^*}(\alpha)) \ , \qquad \alpha \in S^* \ . \tag{3.25}$$

They separate the points in H_{S^*} for the same reason that the T_α, $\alpha \in \mathscr{H}G_{x_0}$, separate the points in $\overline{\mathscr{A}/\mathscr{G}}$ [2, 6]. Therefore, according to the Stone-Weierstrass theorem [9] the algebra $\mathscr{H}A_{S^*}$ obtained by taking finite linear combinations (with complex coefficients) and products of $T_\alpha^{S^*}$ is dense in the C^*-algebra $C(H_{S^*})$ of all continuous functions on H_{S^*}, i.e.

$$\widetilde{\mathscr{H}A}_{S^*} = C(H_{S^*}) \ . \tag{3.26}$$

Using the first Gel'fand-Naimark theorem [2, 9, 10] we then conclude that the spectrum of $\widetilde{\mathscr{H}A}_{S^*}$, endowed with the Gel'fand topology (see below) is homeomorphic to H_{S^*}. An equivalent description of the initial topology in H_{S^*} is therefore given by the Gel'fand topology, which is, by definition, the weakest for which all the functions $T_\alpha^{S^*}$, $\alpha \in S^*$ are continuous.

Returning to (3.24) we see that, in accordance with (3.21), the Tychonov topology on $H_{\mathscr{L}}$ is the weakest for which all the functions $T_\alpha^{S^*} \circ \pi_{S^*} : H_{\mathscr{L}} \to \mathbb{C}$ $\alpha \in S^*, S^* \in \mathscr{L}$ are continuous. On the other hand the Gel'fand topology on $\overline{\mathscr{A}/\mathscr{G}}$ is the weakest for which all the functions $T_\alpha, \alpha \in \mathscr{H}G_{x_0}$ are continuous. Since for all $\alpha \in \mathscr{H}G_{x_0}$,

$$T_\alpha \circ \phi^{-1} = T_\alpha^{S^*} \circ \pi_{S^*}, \quad \forall S^* : \alpha \in S^* \ , \tag{3.27}$$

we conclude that ϕ in (3.9) is a homeomorphism. Q.E.D.

We now proceed to derive a measure theoretic analog of Proposition 3.4. Let $\mathscr{B}_{S^*}$ denote the Borel σ-algebra on H_{S^*} so that, since the projections $p_{S^*S^{*\prime}}$ are measurable,

$$\{(H_{S^*}, \mathscr{B}_{S^*}), p_{S^*S^{*\prime}}\}_{S^*S^{*\prime} \in \mathscr{L}} \tag{3.28}$$

is a projective family of σ-measurable spaces (see (2.19)). Let

$$(H_{\mathscr{L}}, \mathscr{B}_{\mathscr{L}}) \tag{3.29}$$

denote the projective limit σ-measurable space. In $\overline{\mathscr{A}/\mathscr{G}}$ we take the measurable sets to be generated by the class $\overline{\mathscr{C}}$ of "cylindrical sets" used in [6, 7], i.e. the inverse images C_B of Borel sets B in G^n/Ad with respect to $\pi_{S^*} \circ \phi$,

$$C_B \in \overline{\mathscr{C}} \Leftrightarrow \tag{3.30a}$$

$$C_B = (\pi_{S^*} \circ \phi)^{-1}(B) = \{[h] \in \overline{\mathscr{A}/\mathscr{G}} : \; [h(\beta_1), \ldots, h(\beta_n)] \in B \subset G^n/Ad \;\} \; , \tag{3.30b}$$

where, as in (3.5), we have identified H_{S^*} with G^n/Ad with the help of the independent hoops

$$\beta_1, \ldots, \beta_n \in S^* \; .$$

Note that the complement of a cylindrical set is cylindrical, as are finite unions and intersections of cylindrical sets so that $\overline{\mathscr{C}}$ is in fact a (finite) algebra. Denoting the minimal σ-algebra algebra containing the cylindrical sets by $\mathscr{B}(\overline{\mathscr{C}})$, the space

$$(\overline{\mathscr{A}/\mathscr{G}}, \mathscr{B}(\overline{\mathscr{C}})) \tag{3.31}$$

becomes a σ-measurable space. From the definition of $\mathscr{B}_{\mathscr{L}}$ and $\mathscr{B}(\overline{\mathscr{C}})$, we see that

Proposition 3.5. *The map (3.9)*

$$(\overline{\mathscr{A}/\mathscr{G}}, \mathscr{B}(\overline{\mathscr{C}})) \rightarrow (H_{\mathscr{L}}, \mathscr{B}_{\mathscr{L}}) \tag{3.32}$$

is an isomorphism of σ-measurable spaces.

Corollary 3.6.

(i) *$\mathscr{B}(\overline{\mathscr{C}})$ and $\mathscr{B}_{\mathscr{L}}$ are contained in the Borel algebras corresponding to the Gel'fand and Tychonov topologies respectively. This follows from the fact that the cylindrical sets in $\mathscr{B}_{\mathscr{L}}$ with open "base" B in $\mathscr{B}_{S^*}$ form a base in the topology τ_T.*

(ii) *We call a function f on $\overline{\mathscr{A}/\mathscr{G}}$ cylindrical if there exists $S^* \in \mathscr{L}$ such that f is a pull back of a function $\tilde{f}$ on H_{S^*}*

$$f = (\pi_{S^*} \circ \phi)^* \tilde{f} \; , \tag{3.33a}$$

i.e.

$$f([h]) = \tilde{f}([h\,|_{S^*}]) \; , \tag{3.33b}$$

where $\tilde{f}$ is a measurable function on H_{S^*}. The Wilson loop functions $T_\alpha([h]) = \frac{1}{N} Tr h(\alpha)$ (for $G = SU(N)$ or $G = U(N)$) are continuous cylindrical functions ([6]).

(iii) *The projective limit $H_{\mathscr{L}}$ provides a generalization of the Ashtekar-Isham space $\overline{\mathscr{A}/\mathscr{G}}$ to the case where the gauge group G is not compact.*

(iv) *There is a one-to-one correspondence between cylindrical measures μ on $\overline{\mathscr{C}}$ (i.e. additive on $\overline{\mathscr{C}}$ but σ-additive on the σ-subalgebras $(\pi_{S^*} \circ \phi)^{-1}(\mathscr{B}_{S^*})$) and families of measures $\{(\mu_{S^*})_{S^* \in \mathscr{L}}\}$ (μ_{S^*} are Borel measures on the finite dimensional spaces H_{S^*}) satisfying the self-consistency condition*

$$S^* \subset S^{*'} \;\Rightarrow\; \mu_{S^*} = (p_{S^* S^{*'}})_* \mu_{S^{*'}} \; . \tag{3.34}$$

The correspondence is given by

$$\mu_{S^*} = (\pi_{S^*} \circ \phi)_* \mu \ . \tag{3.35}$$

Recall [11] that a Borel measure μ is called regular if for every Borel set E

$$\mu(E) = inf\{\mu(V) : \ E \subset V, V \ open\} \ ,$$

$$\mu(E) = sup\{\mu(K) : \ E \supset K, K \ compact\} \ .$$

Also from [11, Theorem 2.18] it follows that on the spaces H_{S^*} every Borel measure is regular. The following result (similar to [6, Theorem 4.4] and [7, Proposition 2]) holds.

Proposition 3.7. *There is a one-to-one correspondence between regular Borel measures μ on $\overline{\mathcal{A}/\mathcal{G}}$ and self-consistent families of measures $\{(\mu_{S^*})_{S^* \in \mathcal{L}}\}$.*

Proof. From (i) and (iv) we see that a regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ defines, by restriction, a σ-additive measure on $\mathcal{B}(\overline{\mathcal{C}})$ and therefore a consistent family of measures $\{(\mu_{S^*})_{S^* \in \mathcal{L}}\}$. Conversely let $\{(\mu_{S^*})_{S^* \in \mathcal{L}}\}$ be a consistent family of Borel measures on $\{H_{S^*}\}$ and μ_0 be the cylindrical measure on $\overline{\mathcal{C}}$ defined by this family. The family $\{(\mu_{S^*})_{S^* \in \mathcal{L}}\}$ (or equivalently the measure μ_0) defines a positive functional on the continuous cylindrical functions $f = (\pi_{S^*} \circ \phi)^* \tilde{f}$ on $\overline{\mathcal{A}/\mathcal{G}}$,

$$\Gamma_{\mu_0}(f) = \int_{H_{S^*}} \tilde{f} d\mu_{S^*} \ . \tag{3.36}$$

This functional is bounded with respect to the sup-norm

$$| \Gamma_{\mu_0}(f) | \leqq \| f \|_\infty \ , \tag{3.37}$$

where $\| f \|_\infty = \sup_{[h] \in \overline{\mathcal{A}/\mathcal{G}}} | f([h]) |$. Since the space of continuous cylindrical functions is dense in the C^*-algebra $C(\overline{\mathcal{A}/\mathcal{G}})$ of all continuous functions on $\overline{\mathcal{A}/\mathcal{G}}$ (see [6]) the functional Γ_{μ_0} can be extended in a unique way to a continuous positive (norm 1) functional on $C(\overline{\mathcal{A}/\mathcal{G}})$ (see [9]). But in accordance with the Riesz representation theorem (see [11]) there is then a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma_{\mu_0}(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu f \tag{3.38}$$

for every $f \in C(\overline{\mathcal{A}/\mathcal{G}})$, where we denoted the extension of Γ_{μ_0} to $C(\overline{\mathcal{A}/\mathcal{G}})$ with the same letter. Regular Borel measures are completely determined if the integral of continuous functions is known (see [11], p.41), which implies that μ and μ_0 coincide on $\overline{\mathcal{C}}$. Therefore μ is the unique (see [12]) extension of μ_0 to $\mathcal{B}(\overline{\mathcal{C}})$ and (as we have showed) the unique regular extension to a Borel measure. *Q.E.D.*

4. $\mathcal{A}/\mathcal{G}$ is Contained in a Zero Measure Subset of $\overline{\mathcal{A}/\mathcal{G}}$

The present section contains the main result of this paper. For simplicity we will use (3.32) to identify the σ-measurable spaces $(\overline{\mathcal{A}/\mathcal{G}}, \mathcal{B}(\overline{\mathcal{C}}))$ and $(H_{\mathcal{L}}, \mathcal{B}_{\mathcal{L}})$, so that we will consider $\overline{\mathscr{A}/\mathscr{G}}$ to be the projective limit of the projective family of finite dimensional spaces (3.6).

In [6] Ashtekar and Lewandowski introduced the following measure μ_{AL} on $(\overline{\mathscr{A}/\mathscr{G}}, \mathscr{B}(\overline{\mathscr{C}}))$. Let μ_H be the normalized Haar measure on G and μ_n^H and μ_{S*}^H the corresponding measures on G^n/Ad and H_{S*} (μ_{S*}^H is obtained from μ_n^H using (3.5)). Then the (uncountable) family $(\mu_{S*}^H)_{S*\in\mathscr{L}}$ satisfies the self-consistency conditions (2.20). The Ashtekar-Lewandowski measure μ_{AL} is the corresponding (unique) measure on $(\overline{\mathscr{A}/\mathscr{G}}, \mathscr{B}(\overline{\mathscr{C}}))$ satisfying

$$\mu_{S*}^H = (\pi_{S*})_* \mu_{AL} \ . \tag{4.1}$$

The measure μ_{AL} is σ-additive, *Diff*(Σ)-invariant, and strictly positive as a functional on the space continuous cylindrical functions on $\overline{\mathscr{A}/\mathscr{G}}$ (see [6]).

The space $\mathscr{A}/\mathscr{G}$ is canonically embedded in $\overline{\mathscr{A}/\mathscr{G}}$ [2] and is topologically dense there [6, 10]. It is interesting to find out whether $\mathscr{A}/\mathscr{G}$ is also μ_{AL}-thick in $\overline{\mathscr{A}/\mathscr{G}}$; that is, whether $\mathscr{A}/\mathscr{G}$ supports the measure μ_{AL}. We will in fact prove that this is far from being the case:

Theorem 4.1. *There exists a measurable set*

$$Z \in \mathscr{B}(\overline{\mathscr{C}}) \tag{4.2a}$$

such that

$$\mu_{AL}(Z) = 0 \tag{4.2b}$$

and

$$\mathscr{A}/\mathscr{G} \subset Z \ . \tag{4.2c}$$

Proof. We need the following lemma

Lemma 4.2. *For every $q \in (0, 1]$ there exists $Q^{(q)} \subset \overline{\mathscr{A}/\mathscr{G}}$ such that*

$$\mu_{AL}(Q^{(q)}) = q \tag{4.3}$$

and

$$\mathscr{A}/\mathscr{G} \subset Q^{(q)} \ . \tag{4.4}$$

Proof of Lemma. The complement $Q^{(q)^c}$ of $Q^{(q)}$ will be constructed essentially (i.e. modulo dividing by Ad) by taking an infinite product of sets consisting of copies of G with holes cut out around the identity such that the "diameter" of the holes decreases to zero. These copies of G are chosen to correspond to a certain "convergent" sequence of hoops. In order to do this explicitly, choose r_0 such that the exponential map is one-to-one in the subset $\overline{\mathscr{U}_{r_0}(0)}$ of $Lie(G)$, where

$$\overline{\mathscr{U}_{r_0}(0)} = \{v \in Lie(G) \colon \| v \| \leqq r_0\}$$

and

$$exp \colon \overline{\mathscr{U}_{r_0}(0)} \to \overline{\mathscr{O}_{r_0}(e)} \subset G \ , \tag{4.5}$$

that is, $\overline{\mathcal{O}_{r_0}(e)}$ is the image of $\overline{\mathcal{U}_{r_0}(0)}$ under the exponential map, where e is the identity of the group. Here $r_0 > 0$ and $\| \cdot \|$ denotes the norm induced by a bi-invariant inner product in $Lie(G)$ (the Killing form if G is semisimple).

Let us define a function on $\overline{\mathcal{O}_{r_0}(e)}$ that measures the "distance" to the identity e,

$$\begin{aligned} d_e : \ & \overline{\mathcal{O}_{r_0}(e)} \to \mathbb{R}^+ \cup \{0\} , \\ d_e(g) = & \| ln(g) \|, \end{aligned} \tag{4.6}$$

and denote by the same letter d_e the following extension to the whole group G:

$$d_e : G \to \mathbb{R}^+ \cup \{0\} , \tag{4.7a}$$

$$d_e(g) = \begin{cases} r_0 & g \in \mathcal{O}_{r_0}(e)^c \\ \| ln(g) \| & g \in \mathcal{O}_{r_0}(e) . \end{cases} \tag{4.7b}$$

The Ad-invariance of $\| \cdot \|$ on $Lie(G)$ implies that $d_e(\cdot)$ is Ad-invariant on G. Consider now the basic sets

$$\begin{aligned} & \Delta^\varepsilon \subset G , \\ & \Delta^\varepsilon = \{g \in G : \ d_e(g) \geqq \varepsilon\} \quad 0 \leqq \varepsilon \leqq r_0 . \end{aligned} \tag{4.8}$$

The function given by

$$\begin{aligned} s : \ & [0, r_0) \to \mathbb{R}^+, \\ s(\varepsilon) = & \mu_H(\Delta^\varepsilon) \end{aligned} \tag{4.9}$$

is continuous, monotonically decreasing and $s(0) = 1$. Now let $\Delta_n^{\{\varepsilon_i\}_{i=1}^n}$ be the subset of G^n given by

$$\Delta_n^{\{\varepsilon_i\}_{i=1}^n} = \{(g_1, \ldots, g_n) : \ d_e(g_i) \geqq \varepsilon_i\} = \prod_{i=1}^n \Delta^{\varepsilon_i} . \tag{4.10}$$

Clearly we have

$$\mu_n^H(\Delta_n^{\{\varepsilon_i\}_{i=1}^n}) = \prod_{i=1}^n s(\varepsilon_i) . \tag{4.11}$$

Notice that the set $\Delta_n^{\{\varepsilon_i\}}$ is an Ad-invariant subset of G^n. It is the inverse image of the set

$$\tilde{\Delta}_n^{\{\varepsilon_i\}_{i=1}^n} \subset G^n / Ad , \tag{4.12a}$$

$$\tilde{\Delta}_n^{\{\varepsilon_i\}_{i=1}^n} = \{[g_1, \ldots, g_n] : \ d_e(g_i) \geqq \varepsilon_i\} \tag{4.12b}$$

under the quotient map π: $G^n \to G^n/Ad$. By the definition of the measure μ_n^H on G^n/Ad we thus have

$$\mu_n^H(\tilde{\Delta}_n^{\{\varepsilon_i\}_{i=1}^n}) = \prod_{i=1}^n s(\varepsilon_i) . \tag{4.13}$$

Now, for each $q \in (0,1]$ choose a sequence

$$\{\varepsilon_i^{(q)}\}_{i=1}^{\infty} , \tag{4.14a}$$

such that $\varepsilon_i^{(q)} \neq 0$ but

$$\lim_{i\to\infty} \varepsilon_i^{(q)} = 0 \tag{4.14b}$$

and

$$1 - q = \lim_{n\to\infty} \prod_{i=1}^{n} s(\varepsilon_i^{(q)}) . \tag{4.14c}$$

Let $\{\beta_i\}_{i=1}^{\infty}$ be an arbitrary sequence of independent hoops. Then the sets

$$\begin{aligned} &\hat{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \subset \overline{\mathscr{A}/\mathscr{G}} , \\ &\hat{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} = (\pi_{S^*[\beta_1,\dots,\beta_n]}\circ\phi)^{-1}\left(\tilde{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}\right) , \end{aligned} \tag{4.15a}$$

where we used (3.5) to identify H_{S^*} and G^n/Ad, form a decreasing sequence

$$\hat{\Delta}_1^{\{\varepsilon_1^{(q)}\}} \supset \cdots \supset \hat{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \supset \cdots , \tag{4.15b}$$

such that

$$\mu_{AL}\left(\hat{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}\right) = \mu_n^H\left(\tilde{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}\right) . \tag{4.15c}$$

Now, introducing $R^{(q)}(\{\beta_i\})$ whose complement in $\overline{\mathscr{A}/\mathscr{G}}$ is

$$R^{(q)}(\{\beta_i\})^c = \cap_{n=1}^{\infty} \hat{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} , \tag{4.16}$$

we conclude from (4.13) and the σ-additivity of μ_{AL} that

$$\mu_{AL}(R^{(q)}(\{\beta_i\})^c) = \lim_{n\to\infty} \mu_{AL}(\hat{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}) = 1 - q$$

and

$$\mu_{AL}(R^{(q)}(\{\beta_i\})) = q \in (0,1] . \tag{4.17}$$

Let us now turn to the second part of the lemma namely the choice of $Q^{(q)}$ satisfying (4.3) and (4.4). Take for $\hat{\beta}_i$ the hoops corresponding to coordinate squares (all parallel to a fixed coordinate plane) with a corner at x_0 and fix a metric. Choose $\hat{\beta}_i$ to have areas such that

$$Area(\hat{\beta}_i) = \varepsilon_i^{(q)}\delta_i , \tag{4.18}$$

where $\{\varepsilon_i^{(q)}\}_{i=1}^{\infty}$ is the same as in (4.14) and $\{\delta_i\}_{i=1}^{\infty}$ is any sequence with $\delta_i \to 0$. Let

$$Q^{(q)} = R^{(q)}(\{\hat{\beta}_i\}) .$$

Then, for every $A \in \mathscr{A}$ we have(from the smoothness of A)

$$H(\hat{\beta}_i, A) = 1 + F(A)\varepsilon_i^{(q)}\delta_i + O[(\varepsilon_i^{(q)})^2\delta_i^2] , \tag{4.19}$$

where $F(A)$ denotes the component of the curvature at x_0 in the plane of the squares $\hat{\beta}_i$. Then for every $[A] \in \mathscr{A}/\mathscr{G}$ there exists a constant $c([A]) > 0$ such that

$$d_e(H(\hat{\beta}_i, [A])) < c([A])\varepsilon_i^{(q)}\delta_i , \tag{4.20}$$

and, since $\delta_n \to 0$, for n large enough we have

$$d_e(H(\hat{\beta}_n, [A])) < \varepsilon_n^{(q)} .$$

Thus, for every $[A] \in \mathscr{A}/\mathscr{G}$, $[A] \in Q^{(q)}$.

We have therefore proved that with our choice (4.18) of $\hat{\beta}_i$ we have

$$\mathscr{A}/\mathscr{G} \subset Q^{(q)} . \tag{4.4}$$

Q.E.D.

Let us now prove the theorem. From (4.4) we conclude that for every $q > 0$,

$$\mathscr{A}/\mathscr{G} \subset Q^{(q)} \subset \overline{\mathscr{A}/\mathscr{G}} \tag{4.21a}$$

and

$$\mu_{AL}(Q^{(q)}) = q . \tag{4.21b}$$

Considering now the decreasing sequence $Q^{(1/n)}$. We have

$$\mathscr{A}/\mathscr{G} \subset Z \equiv \cap_{N=1}^{\infty} Q^{(1/n)} , \tag{4.22}$$

while the σ-additivity of μ_{AL} implies that

$$\mu_{AL}(Z) = \lim_{N\to\infty} \mu_{AL}(Q^{(1/n)}) = 0 . \tag{4.23}$$

Q.E.D.

5. Completion of the Space of Square Integrable Functions on $\mathscr{A}/\mathscr{G}$

Although $\mathscr{A}/\mathscr{G}$ is not a projective limit of the family (3.6) a procedure similar to that of (2.14), (2.19)-(2.21) can be used to define a measure $\hat{\mu}_{AL}$ on $\mathscr{A}/\mathscr{G}$ as was noted in [6]. This is done by returning to the notion of a cylindrical set (3.17) but now in $\mathscr{A}/\mathscr{G}$. That is, we introduce (surjective) projections

$$\begin{aligned} \hat{\pi}_{S^*} &: \ \mathscr{A}/\mathscr{G} \to H_{S^*[\beta_1,\ldots,\beta_n]} , \\ \pi_{S^*}([A]) &= [H(\beta_1, A), \ldots, H(\beta_n, A)] , \end{aligned} \tag{5.1}$$

where again we are identifying G^n/Ad with H_{S^*}, and take as measurable sets

$$\begin{aligned} C_B &\subset \mathscr{A}/\mathscr{G} , \\ C_B &= \hat{\pi}_{S^*}^{-1}(B) , \end{aligned} \tag{5.2}$$

for some $B \in \mathscr{B}_{S^*}$. Let $\mathscr{C}$ be the collection of such cylindrical sets in $\mathscr{A}/\mathscr{G}$. Note that $\mathscr{C}$ is closed under union, intersection and complementation (i.e. forms an algebra) so that the pair

$$(\mathscr{A}/\mathscr{G},\mathscr{C}) \tag{5.3}$$

is a measurable space. The measure $\hat{\mu}_{AL}$ is then defined by

$$\hat{\mu}_{AL}(\hat{\pi}_{S^*}^{-1}(B)) = \mu_{S^*}^H(B) \ . \tag{5.4}$$

The additivity of $\mu_{S^*}^H$ for every S^* implies additivity of $\hat{\mu}_{AL}$. However, the σ-additivity of the $\mu_{S^*}^H$ does not imply σ-additivity of $\hat{\mu}_{AL}$. Indeed, we have the following

Proposition 5.1. *The measure (5.4) on $\mathscr{A}/\mathscr{G}$ cannot be extended to a σ-additive measure on $\mathscr{B}(\mathscr{C})$.*

Proof. This theorem follows easily from Lemma 4.2. Indeed consider the same sets $\tilde{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \subset G^n/Ad$ as in (4.12)-(4.14) and define analogously to (4.15) the decreasing sequence

$$\dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \subset \mathscr{A}/\mathscr{G} \ , \tag{5.5a}$$

$$\dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} = \hat{\pi}_{S^*[\hat{\beta}_1,\dots,\hat{\beta}_n]}^{-1}\left(\tilde{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}\right), \tag{5.5b}$$

$$\dot{\Delta}_1^{\{\varepsilon_1^{(q)}\}} \supset \cdots \supset \dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \supset \cdots \ , \tag{5.5c}$$

where the sequence $\{\hat{\beta}_i\}_{i=1}^\infty$ is defined as in (4.18). Then for the same reason as in (4.20) there is not a single $[A]$ belonging to the intersection of all $\dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}$, i.e. now we have

$$\cap_{n=1}^\infty \dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} = \emptyset \tag{5.6}$$

even though

$$\lim_{n\to\infty} \hat{\mu}_{AL}\left(\dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}\right) = 1 - q \ . \tag{5.7}$$

Therefore, choosing q: $0 < q < 1$ we conclude from the Hopf theorem 2.1 that $\hat{\mu}_{AL}$ is not extendible to a σ-additive measure on $\mathscr{B}(\mathscr{C})$. *Q.E.D.*

Let us recall aspects of integration theory for the so called (non-σ) measurable spaces with limit structure (see [13] def. 1.5). The measurable space $(X,\mathscr{F}_X)$ is said to be a space with limit structure if

$$\mathscr{F}_X = \cup_{L\in\mathscr{L}}\mathscr{B}_L \ , \tag{5.8}$$

where for all $L \in \mathscr{L}$, $\mathscr{B}_L$ is a σ-algebra and for every $L_1,L_2 \in \mathscr{L}$, there exists a L_3 such that $\mathscr{B}_{L_1} \cup \mathscr{B}_{L_2} \subset \mathscr{B}_{L_3}$. If the family $\{\mathscr{B}_L\}_{L\in\mathscr{L}}$ does not have a maximal element then $\mathscr{F}_X$ is not a σ-algebra. Obviously every projective limit defined as in (2.19)-(2.21) is a measurable space with limit structure. The converse is also true as we can see by taking as projective family of σ-measurable spaces (see [8] p. 20)

$$\{(X_L, \mathscr{B}_L), p_{LL'}\}_{L,L' \in \mathscr{L}} = \{(X, \mathscr{B}_L), id\}_{L \in \mathscr{L}} \ . \tag{5.9}$$

Though this makes the class of projective limit spaces equivalent to that of measurable spaces with limit structure the latter is more "natural" for integration theory.

In a measurable space with limit structure $(X, \mathscr{F}_X)$ the sets $F \in \mathscr{F}_X$ are called cylindrical sets and the map f to a σ-measurable space $(Y, \mathscr{B})$ is called cylindrical if there is a $L \in \mathscr{L}$ such that

$$f\colon\ (X, \mathscr{B}_L) \to (Y, \mathscr{B}) \ ,$$

is measurable. A measure μ on $\mathscr{F}_X$ is called a quasi-σ-measure (quasi-measure in [13]) if its restriction $\mu_L = \mu\,|_{\mathscr{B}_L}$ to every $\mathscr{B}_L \subset \mathscr{F}_X$ is σ-additive. The triple $\{(X, \mathscr{F}_X), \mu_X\}$, where $(X, \mathscr{F}_X)$ is a measurable space with limit structure and μ_X is a quasi-measure is called a quasi-measure space. Let $L_0 \in \mathscr{L}$ be such that the (complex-valued) cylindrical function

$$f\colon\ (X, \mathscr{B}_{L_0}) \ \to \ (\mathbb{C}, \mathscr{B})$$

where $\mathscr{B}$ denotes the σ-algebra of the complex plane, is measurable. Then a function f on the quasi-measure space $\{(X, \mathscr{F}_X), \mu\}$ is said to be μ-integrable if it is μ_{L_0} integrable in the usual sense

$$\int_X f d\mu(x) = \int_{X_{L_0}} f d\mu_{L_0}(x) \ . \tag{5.10}$$

Definition 5.2. *The set of square-integrable cylindrical functions on the quasi-measure space* $\{(X, \mathscr{F}_X), \mu\}$ *will be denoted through* $\mathscr{C}L^2(X, \mathscr{F}_X, \mu)$.

It is easy to see that $\mathscr{C}L^2(X, \mathscr{F}_X, \mu)$ is a pre-Hilbert space with inner product given by

$$(f, g) = \int_X \overline{f(x)} g(x) d\mu(x) = \int_X \overline{f(x)} g(x) d\mu_{L_0}(x) \ , \tag{5.11}$$

where L_0 is such that both $f\colon (X, \mathscr{B}_{L_0}) \to (\mathbb{C}, \mathscr{B})$ and $g\colon (X, \mathscr{B}_{L_0}) \to (\mathbb{C}, \mathscr{B})$ are measurable.

Proposition 5.3. *Suppose that we are given two quasi-measure spaces* $\{(X, \mathscr{F}_X), \mu_X\}$ *and* $\{(Y, \mathscr{F}_Y), \mu_Y\}$, *where*

$$\mathscr{F}_X = \cup_{L \in \mathscr{L}} \mathscr{B}_L(X) \quad \textit{and} \quad \mathscr{F}_Y = \cup_{L \in \mathscr{L}} \mathscr{B}_L(Y) \ ,$$

and that $Y \subset X$. *Let* $\chi :\ \mathscr{F}_X \to \mathscr{F}_Y$ *be an isomorphism of set algebras given by* $\chi(B) = B \cap Y$ *for* $B \in \mathscr{F}_X$ *and such that the restriction to every* $\mathscr{B}_L(X)$ *is an isomorphism of σ-algebras* $\mathscr{B}_L(X) :\ \mathscr{B}_L(X) \to \mathscr{B}_L(Y)$. *Assume also that* $\mu_Y \circ \chi = \mu_X$. *Then if* μ_X *is extendible to a σ-additive measure* $\tilde{\mu}_X$ *on* $\mathscr{B}(\mathscr{F}_X)$, *the completion of* $\mathscr{C}L^2(Y, \mathscr{F}_Y, \mu_Y)$ *is* $L^2(X, \mathscr{B}(\mathscr{F}_X), \tilde{\mu}_X)$.

Proof. Note that the map $\chi\colon\ \mathscr{F}_X \to \mathscr{F}_Y$ induces a one-to-one correspondence between the sets $\mathscr{X}_Y$ of characteristic functions of sets in $\mathscr{F}_Y$ and $\mathscr{X}_X$ of characteristic functions of sets in $\mathscr{F}_X$. Further, since χ is an isomorphism of finite set algebras, this correspondence extends to an isomorphism over the linear spans of $\mathscr{X}_Y$ and $\mathscr{X}_X$. Finally, since χ preserves the measure of sets, this correspondence preserves the inner product in these linear spaces. We need the following lemma.

Lemma 5.4.
(i) *The completion of* $\mathscr{X}_Y$ $(\mathscr{X}_X)$ *is equal to the completion of* $\mathscr{C}L^2(Y, \mathscr{F}_Y, \mu_Y)$ $(\mathscr{C}L^2(X, \mathscr{F}_X, \mu_X))$.
(ii) *The space* $\mathscr{X}_X$ *is dense in* $L^2\left(X, \mathscr{B}(\mathscr{F}_X), \tilde{\mu}_X\right)$.

Proof of Lemma.
(i) Obviously $\mathscr{X}_Y$ is a subset of $\mathscr{C}L^2(Y, \mathscr{F}_Y, \mu_Y)$. It is sufficient to show that any $f \in \mathscr{C}L^2(Y, \mathscr{F}_Y, \mu_Y)$ can be represented as

$$f = \lim_{n \to \infty} \phi_n \ , \tag{5.12}$$

where $\phi_n \in \mathscr{X}_Y$ and the sequence converges in the norm of $\mathscr{C}L^2(Y, \mathscr{F}_Y, \mu_Y)$. But for $f \in \mathscr{C}L^2(Y, \mathscr{F}_Y, \mu_Y)$ there exists a $L_0 \in \mathscr{L}$ such that f belongs to the (complete) space $L^2(Y, \mathscr{B}_{L_0}(Y), \mu_Y |_{\mathscr{B}_{L_0}(Y)})$. Since $\mathscr{X}_Y |_{\mathscr{B}_{L_0}(Y)} \subset \mathscr{X}_Y$ is dense in $L^2(Y, \mathscr{B}_{L_0}(Y), \mu_Y |_{\mathscr{B}_{L_0}(Y)})$ (see [12]) f can be represented in the form (5.10).
(ii) For a quasi-measure space $\{(X, \mathscr{F}_X), \mu_X\}$ satisfying the conditions of Proposition 5.2 we have

$$\mathscr{X}_X \subset \mathscr{C}L^2(X, \mathscr{F}_X, \mu_X) \subset L^2\left(X, \mathscr{B}(\mathscr{F}_X), \tilde{\mu}_X\right) \ , \tag{5.13}$$

where clearly all the inclusions are isometric. It will be sufficient to prove that for every set $B \in \mathscr{B}(\mathscr{F}_X)$ its characteristic function χ_B is in the L^2-closure of $\mathscr{X}_X$. But this result follows easily from Theorem 3.3 in [8]. *Q.E.D.*

Proof of Proposition. We have an isometric isomorphism (i.e. one which preserves the inner product) between the spaces $\mathscr{X}_Y$ and $\mathscr{X}_X$, which are dense in $\widetilde{\mathscr{X}_Y} = \widetilde{\mathscr{C}L^2}(Y, \mathscr{F}_Y, \mu_Y)$ and $L^2\left(X, \mathscr{B}(\mathscr{F}_X), \tilde{\mu}_X\right)$ respectively. The isomorphism therefore extends to a natural isometric isomorphism

$$\eta \ : \ \widetilde{\mathscr{C}L^2}(Y, \mathscr{F}_Y, \mu_Y) \ \rightarrow \ L^2\left(X, \mathscr{B}(\mathscr{F}_X), \tilde{\mu}_X\right) \ . \tag{5.14}$$

Q.E.D.

In the case of $\mathscr{A}/\mathscr{G}$ since the projections $\hat{\pi}_{S^*}$ are surjective we have

$$\hat{\pi}_{S_1^*}^{-1}(B_1) = \hat{\pi}_{S_2^*}^{-1}(B_2) \tag{5.15}$$

if and only if there is some $S^* \subset S_1^* \cap S_2^*$ and some $B \subset H_{S^*}$ such that

$$B_1 = \pi_{S^* S_1^*}^{-1}(B) \ , \quad B_2 = \pi_{S^* S_2^*}^{-1}(B) \ . \tag{5.16}$$

Since the same is true for the algebra $\overline{\mathscr{C}}$ of cylindrical sets in $\overline{\mathscr{A}/\mathscr{G}}$, there is a one-to-one correspondence between $\mathscr{C}$ and $\overline{\mathscr{C}}$ given by

$$\hat{\pi}_{S^*}^{-1}(B) = \chi((\pi_{S^*} \circ \phi)^{-1}(B)) \ , \tag{5.17}$$

where ϕ and π_{S^*} have been defined in (3.9) and (3.11) respectively. Note that the map χ is an isomorphism of set algebras, and that it preserves measures in the sense that

$$\chi(\tilde{B}) = \mathscr{A}/\mathscr{G} \cap \tilde{B} \tag{5.18a}$$

and that

$$\hat{\mu}_{AL} \circ \chi = \mu_{AL} \mid_{\overline{\mathscr{C}}} , \tag{5.18b}$$

so that the conditions of Proposition 5.3 are satisfied for this case. In this way, the completion of $\mathscr{C}L^2 \left(\mathscr{A}/\mathscr{G}, \mathscr{C}, \hat{\mu}_{AL}\right)$ is $L^2 \left(\overline{\mathscr{A}/\mathscr{G}}, \mathscr{B}(\overline{\mathscr{C}}), \mu_{AL}\right)$ and we arrive at the space $\overline{\mathscr{A}/\mathscr{G}}$.

Let us also show that $\mathscr{C}L^2 \left(\mathscr{A}/\mathscr{G}, \hat{\mu}_{AL}, \mathscr{C}\right)$ (hereafter referred to as simply $\mathscr{C}L^2 (\mathscr{A}/\mathscr{G})$) is not complete. To see this, consider the sets

$$\tilde{\Delta}_n \equiv \tilde{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \subset G^n/Ad \tag{5.19a}$$

and

$$\dot{\Delta}_n \equiv \dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n} \subset \mathscr{A}/\mathscr{G} \tag{5.19b}$$

introduced above, for some $q < 1$, as well as the corresponding characteristic functions χ_n.

Since

$$\hat{\mu}_{AL}(\dot{\Delta}_n) \rightarrow 1 - q > 0, \tag{5.20}$$

given any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $\forall n \geqq m > N$,

$$\| \chi_n - \chi_m \|^2 = \int_{\mathscr{A}/\mathscr{G}} (\chi_n - \chi_m)^2 \, d\hat{\mu}_{AL} = \hat{\mu}_{AL}(\dot{\Delta}_m) - \hat{\mu}_{AL}(\dot{\Delta}_n) < \varepsilon \tag{5.21}$$

and the sequence $\{\chi_n\}_{n=1}^{\infty}$ is Cauchy. Suppose that it converges to some

$$f \in \mathscr{C}L^2 (\mathscr{A}/\mathscr{G}) ,$$

which implies that f is itself a cylindrical function, $f = \tilde{f} \circ \hat{\pi}_{S_0^*}$ for some function $\tilde{f}$ on some $H_{S_0^*}$.

Consider now the finitely generated subgroups $S_n^* = S^*[\hat{\beta}_1, \ldots, \hat{\beta}_n]$ used to define $\dot{\Delta}_n^{\{\varepsilon_i^{(q)}\}_{i=1}^n}$ and χ_n. For large enough N, no $\hat{\beta}_m$ for $m \geqq N$ lies in S_0^*. Thus, if ${S^*}'_m$, $m \geqq N$, is the subgroup generated by hoops in S_m^* and hoops in S_0^*, $\chi_m(h) = 0$ for any homomorphism h, $[h] \in H_{{S^*}'_n}$ such that $d_e(h(\hat{\beta}_N)) \leqq \varepsilon_N$. Let R_m be the set of all such $[h] \in H_{{S^*}'_n}$. Then

$$\begin{aligned} ||\chi_m - f||^2 &= \int_{H_{{S^*}'_n}} d\mu_{{S^*}'_n} |\chi_m - f|^2 \circ \pi_{{S^*}'_n}^{-1} \\ &\geqq \int_{R_m} d\mu_{{S^*}'_n} |f|^2 \circ \pi_{{S^*}'_n}^{-1} \\ &= s(\varepsilon_N) \int_{S_0^*} d\mu_{S_0^*} |\tilde{f}|^2 , \end{aligned} \tag{5.22}$$

so that $||\chi_m - f||^2$ is bounded away from zero unless $\tilde{f}$ is the zero function. However, if f is the zero function then

$$||\chi_m - f||^2 = ||\chi_m||^2 \geqq q, \tag{5.23}$$

so that the Cauchy sequence $\{\chi_n\}_{n=1}^{\infty}$ does not converge in $\mathscr{C}L^2(\mathscr{A}/\mathscr{G})$ and $\mathscr{C}L^2(\mathscr{A}/\mathscr{G})$ is incomplete.

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