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Renormalization Group Fixed Points in the Local Potential Approximation for $d \geqq 3$

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Abstract: In the Local Potential Approximation, renormalization group equations reduce to a semilinear parabolic partial differential equation. Felder [8] has derived this equation and has constructed a family of non-trivial fixed points u^*_{2n} $(n = 2, 3, 4, \ldots)$ which have the form of n-well potentials and exist in the ranges of dimensions $2 < d < 2 + \frac{2}{n-1}$. In this paper we show that if $d \geqq 4$, then these non-trivial fixed points disappear, and if $3 \leqq d < 4$ then we have only the u^*_4 fixed point.

1. Introduction

Non-trivial fixed points of the renormalization group (RG) play a crucial role in the understanding of statistical mechanical systems in the vicinity of the critical point [1, 2]. In the case of a symmetric scalar field the non-trivial fixed points are expected to appear as bifurcating from the trivial massless fixed point as one varies continuously the dimension d of the space [3]. These bifurcations occur at thresholds $d_n = 2 + \frac{2}{n-1}$, $n = 2, 3, \ldots$, where the linearized RG acquires a zero mode (the fixed point which appears at $d = d_n$ is called the ϕ^{2n}_d fixed point and it looks like a $n - well$ potential). This pattern is not well understood from the mathematical point of view, but some pieces of it were established in toy models like Dyson's one [4]: Bleher and Sinai [5] proved the existence of a non-trivial fixed point if $d = d_n - \varepsilon$, where $\varepsilon > 0$ is small enough. Felder [8] showed that in the Local Potential Approximation the ϕ^{2n}_d fixed point exists for $2 < d < d_n$.

It is believed that the ϕ^{2n}_d fixed points disappear for $d \geqq d_n$. Gawedzki and Kupiainen [6] showed that if $d \geqq 4$ and the potential is even and "small" (weak coupling) then, under the flow of the RG transformation, it is driven to the Gaussian fixed point. Aizenman [7] showed in $d > 4$ dimensions that the (even) ϕ^d_4 Euclidean field theory, with a cut-off, is inevitably free in the continuum limit. In this paper we show that in the Local Potential Approximation if $d \geqq 4$ there is no non-trivial fixed point and if $3 \leqq d < 4$ the only non-trivial fixed point is the one which appears at $d = 4$, the ϕ^4_d fixed point.

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The Local Potential Approximation (LPA) is a continuous scale version of the hierarchical model first developed by Felder [8], where the flow of the effective potential $u(x,t)$ on momentum scale e^{-t} as a function of the field $x \in R$ is given by the partial differential equation

$$u_t = \frac{1}{2}u_{xx} - \frac{(d-2)}{2}xu_x + du - \frac{1}{2}u_x^2\,, \tag{1}$$

where $d > 2$.

In the LPA the fixed points of the RG must be identified with global stationary solutions of (1); therefore, the problem of searching for fixed points in the LPA reduces to the study of global (defined for all x) solutions of the following $O.D.E$:

$$u'' - (d-2)xu' + 2du - u'^2 = 0\,. \tag{2}$$

We consider only even solutions of (2) so that we may assume $u'(0) = 0$. Now for each $u(0)$ there is a local solution which is $\mathscr{C}^\infty$; however, most of these solutions blow up at a finite x.

Felder [8] showed that a fixed $n = 2, 3, \ldots$ and each $d \in (2, d_n)$, besides $u(x) = 0$ and $u(x) = x^2 - \frac{1}{d}$, the Gaussian and the high-temperature fixed points, respectively, Eq. (2) has a global even solution, u_{2n}^*, having $2n - 1$ critical points, the "ϕ_d^{2n}" fixed point.

Now we state our results [9].

Theorem 1.1. *If* $d \geqq 4$, *then the only global even solutions of* (2) *are the trivial ones,* $u(x) = 0$ *and* $u(x) = x^2 - \frac{1}{d}$.

Theorem 1.2. *If* $3 \leqq d < 4$, *then the only non-trivial global even solutions of* (2) *are those having three critical points, namely, the* u_4^* *fixed points.*

This paper is organized as follows: In Sect. 2 we introduce the function v, which, up to rescaling, is the derivative of u, and the function $w = v'$, and we reduce our problem to a dynamical system on the phase plane vw. In these new variables, (v, w), the initial condition is of the form $(0, w_o)$, where $w_o \in R$. The initial conditions corresponding to the trivial fixed points are $(0,0)$ and $(0, c_d)$, where $c_d = \frac{2}{d+2}$. Following Felder [8], we divide the phase plane into six regions $(I, \ldots, VI)$, and we conclude that a solution $(v(x), w(x))$ is global if and only if it does not enter region II at finite x. In Proposition 2.1 we give sufficient conditions for the solution to enter region II at a finite x. In Sect. 3, we prove Theorem 1.1. To prove this theorem we show that any solution having initial condition w_o different from 0 or c_d must enter region II at a finite x and for that we need to show that the hypothesis of Proposition 2.1 is satisfied and the a priori bounds needed for that are provided by Propositions 3.1, 3.2 and 3.3. The estimates in Propositions 3.1 and 3.2 follow Felder's Liapunov function and the estimates in Proposition 3.3 follow comparison theorems.

The ϕ_d^{2n} fixed point corresponds to a global odd solution $v(x)$ having $2n - 1$ zeros. To prove Theorem 1.2 we show that any global odd solution $v(x)$ having more than three zeros must enter region II at a finite x and this is done following the same ideas in the proof of Theorem 1.1 and will be omitted [9].

Finally, we have the following conjectures:

1. (*Non-existence*) *The u_{2n}^* fixed points disappear if $d \geqq d_n$, i.e., if $d \geqq d_n$, then any even solution $u(x)$ of* (2) *having $2n-1$ critical points must blow-up at a finite x.*

2. (*Uniqueness*) *For each $d \in (2, d_n)$ there is a unique u_{2n}^* fixed point, i.e., for each $d \in (2, d_n)$ there is an initial condition $u_o^*(d) \in R$ such that any even solution $u(x)$ of* (2) *having $2n-1$ critical points must blow-up at a finite x, unless $u(0) = u_o^*(d)$.*

Theorems 1.1 and 1.2 prove the first conjecture for $n = 2$ and $n = 3$. The only limitation of the approach of this paper concerning the first conjecture for $d < 3$ is a technical one – as d decreases, we have to calculate zeros of polynomials of increasing degrees. For $d \geqq 3$ we need to calculate the zeros of polynomials of degrees at most three, which is straightforward. This approach can be used to prove the first conjecture for $n = 4$, namely, to show that for $d \in [\frac{8}{3}, 3)$ the only non-trivial global even solutions of (2) are those having three and five critical points, namely, the u_4^* and u_6^* fixed point, respectively. In this case we would have to calculate the zeros of polynomials of degrees at most 4. As we go to the next thresholds we have to calculate the zeros of polynomials of degrees bigger than four which is algebraically impossible. We still do not know how to prove the second conjecture. The ε-expansion [5] gives a local (for small initial conditions) proof for the second conjecture for $d \in (d_n - \varepsilon, d_n)$.

2. Sufficient Conditions for Blow-up at a Finite x

If we take the derivative of (2) and make the change of variables

$$u(x) = \int_0^x \frac{d+2}{\sqrt{d-2}} v(\sqrt{d-2}\,\xi)\,d\xi - \frac{d+2}{2d} w_o, \quad w_o \in \mathbf{R}, \tag{3}$$

and $\sqrt{d-2}x \to x$, then our problem of searching for global even solutions of (2) reduces to the one of searching for global odd solutions $v(x)$ of the following dynamical system on the phase plane:

$$\begin{cases} v' = w \\ w' = xw - \sigma v + 2\sigma v w \end{cases} \tag{4}$$

with initial conditions $v(0) = 0$ and $w(0) = w_o \in \mathbf{R}$, where $\sigma = \frac{d+2}{d-2}$. Notice that $d \geqq 4$ is equivalent to $1 < \sigma \leqq 3$.

Following Felder [8], we divide the phase plane into six regions $I, II, \ldots, VI$, which are defined by

$$\begin{aligned} I &= \{(v,w) | w \geqq 1/2, \quad v < 0\}, \\ II &= \{(v,w) | w \geqq 1/2, \quad v \geqq 0\}, \\ III &= \{(v,w) | 0 \leqq w < 1/2, \quad v < 0\}, \\ IV &= \{(v,w) | 0 < w < 1/2, \quad v \geqq 0\}, \\ V &= \{(v,w) | w < 0, \quad v \leqq 0\}, \\ VI &= \{(v,w) | w \leqq 0, \quad v > 0\}. \end{aligned}$$

The flow of the solutions of (4) in the phase plane is given in Proposition 4.1 of [8]. It is shown that a solution $(v(x), w(x))$ blows up at a finite x, if and only if, it enters in region II.

Definition 2.1. *Let* $C_1, C_2 : (1,\infty) \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ *be defined by*

$$C_1(\sigma,x,w) = x^2 - \sigma + 1 + 2\sigma w\,, \tag{5}$$

$$C_2(\sigma,x,w) = x^2 - 2\sigma + 3 + 8\sigma w\,. \tag{6}$$

We note the following relation between the C_1 and C_2:

$$C_2(\sigma,x,w) = C_1(\sigma,x,w) + \left(w - \frac{\sigma-2}{6\sigma}\right)6\sigma\,. \tag{7}$$

Proposition 2.1. *Let $(v(x), w(x))$ be a solution of* (4) *and $\bar{x} \equiv \bar{x}(\sigma, w_o) > 0$ be such that $v(\bar{x}) = 0$ and $w(\bar{x}) \equiv \bar{w} > 0$. If $\sigma \in (1,3]$ and $C_i(\sigma,\bar{x},\bar{w}) > 0$ $(i = 1,2)$, then $(v(x), w(x))$ blows up at a finite x.*

Proof. From Proposition 4.1 of [8], we may assume that $\bar{w} < \frac{1}{2}$ and as long as $w(x) < \frac{1}{2}$, $v \in \mathscr{C}^\infty$ and it is straightforward to get its derivatives. In fact, from (4), we have the following relations:

$$w' = (x + 2\sigma v)w - \sigma v\,, \tag{8}$$

$$w'' = (x + 2\sigma v)w' + (1 - \sigma + 2\sigma w)w\,, \tag{9}$$

$$w''' = (x + 2\sigma v)w'' + (2 - \sigma + 6\sigma w)w'\,, \tag{10}$$

$$w'''' = (x + 2\sigma v)w''' + (3 - \sigma + 8\sigma w)w'' + 6\sigma w'^2\,. \tag{11}$$

Since $1 < \sigma \leqq 3$, it follows from (11) that $v(x) \geqq 0$ and $w^{(k)}(x) > 0$ $(k = 0,1,2,3)$ implies $w^{(4)}(x) > 0$. Notice that by definition $v(\bar{x}) = 0$ and $\bar{w} > 0$, and so by relations (8)–(11) and the hypothesis of Proposition 2.1, $w'(\bar{x}) = \bar{x}\,\bar{w} > 0$, $w''(\bar{x}) = C_1(\sigma,\bar{x},\bar{w})\,\bar{w} > 0, w^{(3)}(\bar{x}) = C_2(\sigma,\bar{x},\bar{w})\bar{x}\bar{w} > 0$. Hence, for $x > \bar{x}$ as long as $(v(x), w(x))$ is in region IV, $w^{(k)}(x) > w^{(k)}(\bar{x})$. In particular, $w(x) > \bar{w} + \bar{x}\bar{w}$ $(x - \bar{x})$ and $(v(x), w(x))$ enters region II at some finite $x < \bar{x} + \frac{\frac{1}{2} - \bar{w}}{\bar{x}\bar{w}}$ and by Proposition 4.1 of [8], we are done. □

For $\sigma \in (3,5]$ or equivalently $d \in [3,4)$, the analogue of Proposition 2.1 requires two additional conditions [9].

In Proposition 4.2 of [8], it is shown that a solution of (4) with initial condition $w_o < 0$ and $|w_o|$ sufficiently large blows up at a finite x. This proposition is related to our estimates for large initial conditions given in Sect. 3 and Lemma 2.2 below.

Lemma 2.1. *Let $1 < \sigma < \infty$ and (v, w) be a solution of* (4) *with $w_o \in (0, c_d)$, where $c_d = \frac{2}{d+2} = \frac{\sigma-1}{2\sigma}$, then it enters in region VI at a finite x.*

Proof. From (4), we have

$$w' = xw - v + (w - c_d)2\sigma v\,, \tag{12}$$

$$w'' = (x + 2\sigma v)w' + (w - c_d)2\sigma w\,. \tag{13}$$

Since $v(0) = 0$ and $0 < w_o < c_d$, then (12) and (13) imply that $w''(0) < 0$. From (13), $w''(x) < 0$ whenever $v(x) \geqq 0, 0 < w(x) < c_d$ and $w'(x) \leqq 0$. Therefore, $w''(x) < 0$ for all x as long as $v(x) > 0$ and $0 < w(x) < c_d$, which implies that $w(x)$ is strictly concave and $(v(x), w(x))$ enters in region VI at a finite x. □

This lemma generalizes Proposition 4.3 of [8] in the following way: in this proposition it is assumed that $\sigma > 3$ and w_o small. Combining this lemma and Proposition 4.1 of [8] we conclude that any solution of (4) with initial conditions $w_o \in (0, c_d)$ has at least two positive zeros. Also, from Proposition 4.1 of [8], any solution of (4) with initial condition $w_o < 0$ has at least one positive zero.

Lemma 2.2. *Let $(v(x), w(x))$ be a solution of (4). If for some $x_o \geqq 0$ we have $w(x_o) > c_d$, then $(v(x), w(x))$ blows up at a finite x.*

Proof. Same arguments given in the proof of the previous lemma imply that $w(x)$ is strictly convex for all $x > x_o$, then $(v(x), w(x))$ enters region *II* at a finite x and by Proposition 4.1 of [8] we are done. □

From now on we may assume that $w(x) < c_d$.

3. Estimates

In this section, we prove Propositions 3.1, 3.2 and 3.3, which provide the bounds we need to prove Theorem 1.1.

Definition 3.1. *Given a solution $(v(x), w(x))$ of (4) with initial condition w_o, let $N = 1$ if $w_o < 0$ and $N = 2$ if $0 < w_o < c_d$. It follows from remarks in Sect. 2 that $v(x)$ has at least N zeros $0 < \bar{x}_1 < \dots \bar{x}_N \equiv \bar{x}$. Define*

$$w_{\min} = \min_{0 \leqq x \leqq \bar{x}} w(x) \tag{14}$$

and

$$\lambda = (1 + 2|w_{\min}|)\sigma\,. \tag{15}$$

Let $x_o, x_{\min} \in [0, \bar{x}]$ be defined such that $x_{\min} < x_o \leqq \bar{x}$, $w(x_{\min}) = w_{\min}$ and $v(x_o) = v(x_{\min})$.

It is clear from the above definitions that $v(\bar{x}) = 0$ and $w(\bar{x}) > 0$. We note from Lemmas 2.1 and 2.2 and Proposition 4.1 of [8], that any solution $(v(x), w(x))$ having nonzero initial condition $w_o < c_d$ enters in regions *VI* and *V*, hence $w_{\min} < 0$. On the other hand, since $w(x)$ is strictly decreasing in *VI* and is convex in *V*, we conclude that $w_{\min} \in V$. For $w_o < 0, \bar{x}$ will be the first zero of $v(x)$ and $w_{\min} = w_o$.

Definition 3.2. (*see Proposition* 4.2 *of* [8]) *Let*

$$J(v, w) = \left(w - \frac{1}{2}\right) e^{2(w - \sigma v^2)}\,. \tag{16}$$

Then $J(v, w)$ is a non-decreasing Liapunov function for all $x \geqq 0$.

Definition 3.3. *Let $f\colon (0, \infty) \to (0, 1)$ and $g\colon \left(0, \frac{1}{2}\right) \to (0, 1)$ be the monotone decreasing functions*

$$f(z) = (1 + 2z)e^{-2z}\,, \tag{17}$$

$$g(z) = (1 - 2z)e^{2z}\,, \tag{18}$$

we have

$$g(w(x_o)) \leqq f(|w(x_{\min})|)\,. \tag{19}$$

Indeed, since $x_o > x_{\min}$ then from the non-decreasing property of J and the definitions of f and g, we get

$$-\frac{1}{2}g(w(x_o))e^{-2\sigma v^2(x_o)} = J(v(x_o), w(x_o))$$
$$\geqq J(v(x_{\min}), w(x_{\min})) = -\frac{1}{2}f(|w(x_{\min})|)e^{-2\sigma v^2(x_{\min})}$$

to obtain (19), we notice that $v(x_o) = v(x_{\min})$.

Proposition 3.1. *If* $|w_{\min}| < \frac{1}{2}$, *then*

$$w(\bar{x}) \geqq \left(\frac{1-2|w_{\min}|}{1+2|w_{\min}|}\right)|w_{\min}|. \tag{20}$$

Proof. We claim that

$$w(x_o) \geqq \left(\frac{1-2|w(x_{\min})|}{1+2|w(x_{\min})|}\right)|w(x_{\min})|, \tag{21}$$

and since $\bar{w} \geqq w(x_o)$, this implies (20). Next we prove (21). In fact since $x_o > x_{\min}$ and $v(x_o) = v(x_{\min})$, then from (16) and the non-decreasing property of the Liapunov function J, we have

$$\left(w(x_o) - \frac{1}{2}\right)e^{2w(x_o)} \geqq \left(w(x_{\min}) - \frac{1}{2}\right)e^{2w(x_{\min})} \tag{22}$$

multiplying (22) by $-2e^{-2w(x_o)}$, using the fact that $e^{-t} \leqq 1 - t + \frac{t^2}{2}$, with $t = 2(|w(x_{\min})| + w(x_o))$, subtracting $1 - 2w(x_o)$ from both sides of the inequality we get and dividing the result by $2(|w(x_{\min})| + w(x_o))$, we are done. □

Proposition 3.2. *Suppose* $\sigma \in (1,3]$ *and let*

$w_1 = \frac{\sigma-1}{2\sigma}$, $w_2 = \frac{\sigma-2}{2\sigma}$, $w_3 = \frac{\sigma-2}{6\sigma} > \frac{5\sigma-14}{10\sigma}$, $m_1 = \frac{2}{3}$, $m_2 = \frac{1}{3}$ *and* $m_3 = \frac{3}{10}$.

If $|w_{\min}| \geqq m_i$, *then* $w(\bar{x}) \geqq w_i (i = 1,2,3)$.

The next lemma implies Proposition 3.2.

Lemma 3.1. *Given any* $\beta \in \left(0, \frac{1}{2}\right)$, *let* $\alpha > 0$ *be the solution of* $f(z) = g(\beta)$. *If* $|w(x_{\min})| \geqq \alpha$ *then* $w(\bar{x}) \geqq \beta$.

Proof. Assume $|w(x_{\min})| \geqq \alpha$ then from (19), the monotonicity of $f(z)$ and the definition of β, we have $g(w(x_o)) \leqq f(|w(x_{\min})|) \leqq f(\alpha) = g(\beta)$. Since g is monotone decreasing, it follows that $w(x_o) \geqq \beta$. To conclude we note that $w(\bar{x}) \geqq w(x_0)$. □

Proof (*of Proposition 3.2*). Let $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{6}$, $\beta_3 = \frac{1}{18}$, then $w_i \leqq \beta_i$ and $f(m_i) < g(\beta_i)(i = 1,2,3)$, where w_i are defined in Proposition 3.2. Let α_i be the solution of $f(z) = g(\beta_i)$, then $f(m_i) < g(\beta_i) = f(\alpha_i)$, and since f is monotone decreasing, it follows that $m_i > \alpha_i$. By Lemma 3.1 if $|w(x_{\min})| \geqq \alpha_i$ then $w(\bar{x}) \geqq \beta_i$. Since $\beta_i \geqq w_i$ and $m_i > \alpha_i$, we conclude that if $|w(x_{\min})| \geqq m_i$ then $w(\bar{x}) \geqq w_i$. □

Proposition 3.3. *Let λ be given by* (15), *and $\bar{x}_1$ and $\bar{x}_2$ be the first and the second zeros of $v(\sigma, w_o, x)$, respectively.*

1. *If $\lambda \in (1,3]$ then $\bar{x}_1^2 \geqq 3$.*
2. *If $\lambda \in (3,5]$ then $\bar{x}_1^2 \geqq 5 - \sqrt{10} > \frac{9}{5}$.*
3. *If $\lambda \in (3,5]$ then $\bar{x}_1^2 > \frac{3!}{\lambda - 1}$.*
4. *If $\lambda \in (1,7)$ then $\bar{x}_1^2 > 1$ and $\bar{x}_2^2 > 3$.*

Before proving this proposition, we need some lemmas. By definition $\bar{x}(\sigma, w_o)$ is a positive zero of $v(x) = v(\sigma, w_o, x)$. In order to estimate $\bar{x}$, we compare the solution $v(x)$ of the nonlinear problem (4) to the linear one, $v_\lambda(x)$, defined below. In Lemma 3.5 we show that the zeros of v_λ give a lower bound for the zeros of v, and so need to estimate the zeros of v_λ. In the proposition of the appendix [8], is given the number of the zeros of $v_\lambda(x)$ as a function of λ and, in Lemma 3.4, we show that its zeros are monotone decreasing functions of λ. Moreover, if λ is an odd integer, then $v_\lambda(x)$ is a Hermite polynomial of degree λ. We take the advantage of the monotone decreasing dependence of the zeros of $v_\lambda(x)$ on λ and we bound its zeros by the zeros of a Hermite polynomial of degree k, where k is the smallest odd integer $\geqq \lambda$, and in Lemma 3.3 we give estimates for $v_k(x)$ for some values of k which we need. The general idea is to use the properties of $v_\lambda(x)$ as a function of λ to bound its zeros by the zeros of some polynomial.

Definition 3.4. *Let $u_\lambda(x)$ and $v_\lambda(x)$ be the solutions of the equation*

$$u'' - xu' + \lambda u = 0 \tag{23}$$

with initial conditions $v_\lambda(0) = 0$, $v'_\lambda(0) = 1$ and $u_\lambda(0) = 1$, $u'_\lambda(0) = 0$.

We have the following recursion relations:

$$u'_\lambda(x) = -\lambda v_{\lambda-1}(x), \quad v'_\lambda(x) = u_{\lambda-1}(x) \tag{24}$$

whose proof is based on the fact that if u is a solution of (23) then u' is a solution of (23) with λ replaced by $\lambda - 1$.

Lemma 3.2. *Let $v_\lambda(x)$ be defined by* (23) *with $3 < \lambda \leqq 5$ and $x_1(\lambda)$ be the first positive zero of $v_\lambda(x)$. Then*

$$x_1^2(\lambda) > \frac{3!}{\lambda - 1}.$$

Proof. From Taylor's Theorem, we have

$$v_\lambda(x) = x - \frac{(\lambda - 1)}{3!}x^3 + R_5(\lambda, x),$$

where $R_5(\lambda, x) = \frac{v_\lambda^{(5)}(\xi)}{5!}x^5$ for some $\xi \in (0, x)$. We claim that if $3 < \lambda \leqq 5$, then $R_5(\lambda, x) > 0$ for all x. In fact, from the recursion relation (24), we have

$$v_\lambda^{(5)}(x) = (\lambda - 1)(\lambda - 3)u_{\lambda-5}(x)$$

and from the Proposition of the appendix of [8], $u_{\lambda-5}(x) > 0$ for all x if $\lambda \leqq 5$. □

Lemma 3.3. *Let $v_\lambda(x)$ be defined by* (23) *$x_1(\lambda)$ and $x_2(\lambda)$ be the first and the second zeros of $v_\lambda(x)$, respectively. Then*

$$\begin{aligned}
&x_1^2(3) = 3\,,\\
&x_1^2(5) = 5 - \sqrt{10}, \quad x_2^2(5) = 5 + \sqrt{10}\,,\\
&x_1^2(7) = 7 - 2\sqrt{14}\sin\left(\frac{\pi}{6} + \frac{1}{3}\arctan\left(\sqrt{\frac{5}{2}}\right)\right) \simeq 1.33\,,\\
&x_2^2(7) = 7 - 2\sqrt{14}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arctan\left(\sqrt{\frac{5}{2}}\right)\right) \simeq 5.6\,.
\end{aligned}$$

Proof. If λ is an odd integer, then v_λ is a Hermite polynomial of degree λ. In particular,

$$\begin{aligned}
v_3(x) &= x - \frac{x^3}{3}\,,\\
v_5(x) &= x - \frac{2}{3}x^3 + \frac{x^5}{15}\,,\\
v_7(x) &= x - x^3 + \frac{x^5}{5} - \frac{x^7}{105}\,;
\end{aligned}$$

from the above expressions this lemma is straightforward. □

Lemmas 3.4 and 3.5 below say that the (positive) zeros of v_λ are monotone decreasing functions of λ and that the zeros of v_λ give a lower bound for the zeros of v. Since the proofs of these lemmas are similar, we give only the proof of Lemma 3.5.

Lemma 3.4. *Let $v_\lambda(x)$ be defined by* (23) *with $\lambda > 1$, then the positive zeros of $v_\lambda(x)$ are monotone decreasing functions of λ.*

Lemma 3.5. *Let $v(x)$ be the solution of* (4) *satisfying $v(0) = 0$ and $v'(0) = w_o$, N and $\lambda(\sigma, w_o)$ be as in Definition* 3.1, *then*

$$\bar{x}_i > x_i, \quad i = 1, 2, \ldots, N\,, \tag{25}$$

where $\bar{x}_i$ and x_i are the i^{th} positive zero of v and v_λ, respectively.

Proof. We may assume that $\bar{x}_N < \sqrt{3}$ for $1 < \sigma \leqq 3$; otherwise, by Proposition 2.1, we are done. On the other hand, as long as $0 \leqq x \leqq \bar{x}_N$, $w(x)$ is $\mathscr{C}^\infty$, in particular, $g_1(x) \equiv (1 - 2w(x))\sigma e^{-\frac{x^2}{2}}$ is continuous for $0 \leqq x \leqq \bar{x}_N$. From the definitions of $w_{\min}(\sigma, w_o)$ and $\lambda(\sigma, w_o)$, it follows that $g_1(x) < \lambda e^{-\frac{x^2}{2}} \equiv g_2(x)$, for all $0 < x < \bar{x}_N$, but $x_{\min}$. Let $p(x) = e^{-\frac{x^2}{2}}$, then $p(x) > 0$ and p, p', g_1 and g_2 are continuous in $(0, \bar{x}_N)$.

Suppose φ and ψ are real solutions in $(0, \bar{x}_N)$ of $(pu')' + g_1(x)u = 0$ and $(pu')' + g_2(x)u = 0$, respectively. Since $g_2(x) > g_1(x)$ on $(0, \bar{x}_N)$, if $\bar{x}_i$ and $\bar{x}_{i+1}$ are successive zeros of φ on $(0, \bar{x}_N)$, then by [10] ψ must vanish at some point of $(\bar{x}_1, \bar{x}_2)$. To conclude we make $\varphi(x) = v(x)$ and $\psi(x) = v_\lambda(x)$. □

Proof (*Proposition 3.3*). Lemma 3.2 implies Item 3 and Lemmas 3.5, 3.4 and 3.3 imply Items 1, 2 and 4. □

4. The Proof of Theorem 1.1

The Gaussian and the high-temperature fixed *points* correspond to $v(x) = 0$ and $v(x) = c_d x$, respectively. Therefore, the following statement implies Theorem 1.1:

Let $\sigma \in (1,3]$, then any solution $(v(x), w(x))$ of (4) satisfying the initial condition $(v(0), w(0)) = (0, w_o)$ must blow-up at a finite x, unless, $w_o \in \{0, c_d\}$.

In this section we prove this statement. First, we note from Lemma 2.2 that if $w_o > c_d$, then $(v(x), w(x))$ must blow-up at a finite x. Hence, we may assume $w_o < c_d$. If $0 < w_o < c_d$, then $v(x)$ has at least two positive zeros and by Proposition 4.1, below, and Proposition 2.1, $(v(x), w(x))$ must blow-up at a finite x. Finally, if $w_o < 0$, then $v(x)$ has at least one positive zero and by Proposition 4.2, below, and Proposition 2.1 $(v(x), w(x))$ must blow-up at a finite x.

Proposition 4.1. *Let (v, w) be the solution of (4) with $(\sigma, w_o) \in (1,3] \times \left(0, \frac{\sigma-1}{2\sigma}\right)$, $\bar{x} > 0$ be the second positive zero of v, then*

$$C_i(\sigma, \bar{x}, w(\bar{x})) > 0 \quad (i = 1,2)\,. \tag{26}$$

Proof. Let $w_{\min}$ and λ be defined as in Definition 3.2, and $\bar{x} > 0$ be the second positive zero of v. We note that if $\bar{x}^2 \geqq 3$ or $w(\bar{x}) \geqq \frac{\sigma-1}{2\sigma}$, then $C_i(\sigma, \bar{x}, w(\bar{x})) > 0$ $(i = 1,2)$. However, if $w_{\min} \in \left(-\infty, -\frac{2}{3}\right]$ then by Proposition 3.2, we have $w(\bar{x}) > \frac{\sigma-1}{2\sigma}$, and if $w_{\min} \in \left(-\frac{2}{3}, 0\right)$, then (15) $\lambda \in (1,7)$ and by Proposition 3.3, Item 4, $\bar{x}^2 \geqq 3$. □

Proposition 4.2. *Let (v, w) be a solution of (4) with $(\sigma, w_o) \in (1,3] \times (-\infty, 0)$ and $\bar{x} > 0$ be the first positive zero of v, then*

$$C_i(\sigma, \bar{x}, w(\bar{x})) > 0 \quad (i = 1,2)\,. \tag{27}$$

Before proving Proposition 4.2, we state Lemma 4.1, whose proof we postponed until the end of this section.

Lemma 4.1. *Let (v, w) be a solution of (4) with $(\sigma, w_o) \in (1,3] \times \left(-\frac{3}{10}, 0\right)$ and $\bar{x} > 0$ be the first positive zero of v, then*

$$C_i(\sigma, \bar{x}, w(\bar{x})) > 0 \quad (i = 1,2)\,. \tag{28}$$

Proof (*Proposition 4.2*). Let $\bar{x} > 0$ be the first positive zero of $v, w_{\min}$ and λ be as in Definition 3.2. Then $w_{\min} = w_o$ and $\lambda = (1 + 2|w_o|)\sigma$.

We first show that $C_1(\sigma, \bar{x}, w(\bar{x})) > 0$. We note that if one of the following conditions (i) $w(\bar{x}) > \frac{\sigma-1}{2\sigma}$ or (ii) $w(\bar{x}) > \frac{\sigma-2}{2\sigma}$ and $\bar{x} > 1$ or (iii) $w(\bar{x}) > \frac{5\sigma-14}{10\sigma}$ and $\bar{x}^2 > \frac{9}{5}$, holds, then $C_1(\sigma, \bar{x}, w(\bar{x})) > 0$.

Let $w_o < 0$ be given. If $w_o \in \left(-\infty, -\frac{2}{3}\right)$ then by Proposition 3.2, $w(\bar{x}) > \frac{\sigma-1}{2\sigma}$ and (i) holds. If $w_o \in \left(-\frac{2}{3}, -\frac{1}{3}\right]$ then by Proposition 3.2, $w(\bar{x}) > \frac{\sigma-2}{2\sigma}$ and since $\lambda \in (1,7)$, then by Proposition 3.3, Item 4, $\bar{x}^2 > 1$ and (ii) holds. If $w_o \in \left(-\frac{1}{3}, -\frac{3}{10}\right]$, then by Proposition 3.2, $w(\bar{x}) > \frac{5\sigma-14}{10\sigma}$ and since $\lambda \in (1,5]$, then by Proposition 3.3, Items 1 and 2, $\bar{x}^2 > \frac{9}{5}$ and (iii) holds. Next we note that if $w_o \in (-\frac{3}{10}, 0)$, then by Lemma 4.1, $C_1(\sigma, \bar{x}, w(\bar{x})) > 0$.

Next we show that $C_2(\sigma,\bar{x},w(\bar{x})) > 0$. By Lemma 4.1, if $w_o \in \left(-\frac{3}{10},0\right)$, then $C_2(\sigma,\bar{x},w(\bar{x})) > 0$, and by Proposition 3.2, if $w_o \in \left(-\infty,-\frac{3}{10}\right]$, then $w(\bar{x}) > \frac{\sigma-2}{6\sigma}$ and since $C_1(\sigma,\bar{x},w(\bar{x})) > 0$, then from (7), we are done. □

Next, we prove Lemma 4.1 and for that we need the following definition.

Definition 4.1. *Let* $\omega\colon \left(0,\frac{3}{10}\right] \to R_+$ *and* $\tau\colon (1,3]\times\left(0,\frac{3}{10}\right] \to R_+$ *be defined by*

$$\omega(t) = \left(\frac{1-2t}{1+2t}\right)t \tag{29}$$

and

$$\tau(\sigma,t) = \sqrt{\frac{6}{(1+2t)\sigma-1}}\,. \tag{30}$$

The following result is straightforward:

$$\frac{\partial C_i(\sigma,x,\omega)}{\partial x} > 0, \qquad \frac{\partial C_i(\sigma,x,\omega)}{\partial \omega} > 0\,, \tag{31}$$

$$\frac{\partial}{\partial\sigma}C_i(\sigma,\tau(\sigma,t),\omega(t)) < 0\,, \tag{32}$$

$$C_i(3,\tau(3,t),\omega(t)) > 0\,. \tag{33}$$

Proof (*Lemma 4.1*). Assume $(\sigma,w_o)\in(1,3]\times(-\frac{3}{10},0)$ and let $\bar{x}>0$ be the first positive zero of $v, w_{\min}$ and λ be defined by (14) and (15), respectively. Then $w_{\min} = w_o$ and $\lambda = (1+2|w_o|)\sigma$, which implies $\lambda\in(1,5]$. If $\lambda\in(1,3]$, then by Proposition 3.3, Item 1,$\bar{x}^2 \geqq 3$, which implies $C_i(\sigma,\bar{x},w(\bar{x})) > 0\,(i=1,2)$. Therefore, we may assume $\lambda\in(3,5]$. From Proposition 3.3, Item 3, it follows that

$$\bar{x} > \tau(\sigma,|w_o|)\,, \tag{34}$$

and from Proposition 3.1

$$w(\bar{x}) > \omega(|w_o|)\,. \tag{35}$$

From (31)–(35), we have

$$C_i(\sigma,\bar{x},w(\bar{x})) > C_i(\sigma,\tau(\sigma,|w_o|),\omega(|w_o|)) > C_i(3,\tau(3,|w_o|),\omega(|w_o|)) > 0\,,$$

which implies Lemma 4.1. □

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