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MARKETING-PRODUCTION DECISIONS UNDER INDEPENDENT AND INTEGRATED CHANNEL STRUCTURE

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Abstract

The topic of channel structure has recently attracted much attention among researchers in the marketing and economics area. However, in a majority of the existing literature the cost considerations are extremely simplified with the major focus being pricing policy. What happens when cost incurring decisions are strongly connected with pricing policies? This is the theme we wish to explore in the present paper. The non-trivial costs considered are production, inventory, and retailer effort rate, i.e. we seek to explore the marketing-production channel. We have used the methodology of differential games. The open-loop Stackelberg solution concept has been used to solve the manufacturer and retailer's problem. The Pareto solution concept has been used to solve the problem of the vertically integrated firm. The production, pricing, and effort rate policies thus derived have been compared to obtain insights into the impact of channel structure on these policies. Also, to examine the relation between channel structure and the retailing operation requiring effort, we derive the Stackelberg and Pareto solutions with and without effort rate as a decision variable. We show that once the production rate becomes positive, it does not become zero again. This implies production smoothing. However, none of the gains of production smoothing are passed on to the retailer. The optimal production rate and the inventory policy are a linear combination of the nominal demand rate, the peak demand factor, the salvage value, and the initial inventory. Also, as opposed to some of the existing literature, the optimal policies need not necessarily be concave in nature. In the scenario where the relating operation does not require effort, the pricing policies of the manufacturer and the retailer, and the production policy of the manufacturer have a synergistic effect. However, in the scenario where the retailing operation does benefit from effort, the retailer's pricing policy need not necessarily be synergistic with other policies. With regard to channel structures, it seems that production smoothing will be done more efficiently in the integrated setup. Also, we show that the price paid by the consumer need not necessarily be lower in the integrated setup. But despite higher prices, the channel profits are higher in the integrated setup. This implies a conflict between the interests of the consumers and the firm. Also, this contradicts the results of some of the earlier papers that have used simple static models.

1. Introduction

Starting as a raw material and ending with the final consumer, goods pass through many stages of transfer. One or more of these stages of transfer may be controlled by a single agent. How do these agents interact? Which stages should be

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grouped together under one agent and under what circumstances? The present paper attempts to answer some of these questions in the context of a marketing–production channel. In the case of a single manufacturer and a single retailer, Spengler [23] proved that the price paid by the consumer will be higher under a decentralized channel structure. He argued that this happens because of what is known as "double marginalization", i.e. the manufacturer as well as the retailer add their profit margins to the cost of the product before arriving at the price to be charged to the consumer. As a result of the higher prices under the decentralized structure, the quantity sold is less and the total channel profits might actually be lower. Thus, an integrated channel structure will be preferred by the consumers as well as the manufacturer. Under a very similar setting, Jeuland and Shugan [10] showed that the efficiency of an integrated channel structure can be achieved in a decentralized setting if the manufacturer adopts a nonlinear pricing policy, for example, offers quantity discounts.

Research in this area was further enriched by introducing competition. In the case of two manufacturers and two retailers, with each retailer carrying only one manufacturer's product, McGuire and Staelin [14] show that if the products of the two manufacturers are highly substitutable, both the manufacturers will prefer a decentralized structure to a vertically integrated structure. They suggest that this happens because the manufacturers want to shield themselves from the competition and hence they insert privately owned profit maximizers between themselves and the ultimate retail markets. More recently, Coughlan and Wernerfelt [4] have examined the robustness of channel decisions under competition by allowing nonlinear pricing within the channel. They show that if intra-channel contracts are observable to competitors, the existence of "more vertical middlemen levels always enhance profitability". However, if the intra-channel contracts are not observable, the external channel structure ceases to be of relevance, and they conclude that reasons other than strategic ones must be reponsible for the existence of decentralized channels.

While these and other papers along these lines make interesting contributions, their main focus is on pricing and profit sharing. With the exception of Moorthy [15], they completely ignore questions that might arise due to cost considerations. For example, McGuire and Staelin [14], Jeuland and Shugan [10], and Couglan [3] all assume constant marginal costs at the manufacturer as well as the retailer level. Under this assumption, the cost-incurring decision is highly simplified, and the pricing policy has no bearing upon the per unit cost of the product. Moorthy [15] considers an example in which the retailers have decreasing marginal costs. The rest of the model is the same as McGuire and Staelin [14]. Moorthy shows that decentralization is never a Nash equilibrium strategy, no matter how high the demand substitutability. This is exactly the opposite of the result obtained by McGuire and Staelin [14]. Hence, it seems reasonable to suspect that other situations in which price and cost decisions are interrelated might produce interesting results. In fact, as we show in this paper, when cost-incurring decisions such as production rate and retailer effort rate are interrelated with the pricing policy, the price charged by the vertically integrated firm need not be less than the price charged by the retailer throughout the season. Also, despite higher prices, the channel profits can be higher under the integrated setup. This represents a conflict between the interests of the consumer and the firm.

What happens when cost-incurring decisions are strongly connected with pricing policies? This is the theme we seek to explore in the current paper. We bring in cost-incurring decisions that might be influenced by prices and profit margins by introducing the time element into the channel problem. We assume that demand for the product is seasonal. The manufacturer has to decide how much to produce and at what price to sell the product to the retailer. The unsold amount is carried as inventory. Evidence of the linkage between price and production decisions in these types of settings is well documented in the production literature. For example, Feichtinger and Hartl [8] consider a simultaneous price–production decision model with general functional forms and static demand. They show that price as well as production rate have a "synergistic" effect (i.e. an increase in price or an increase in production rate both increase inventory; cf. Feichtinger [7] for more details on "synergy"). In the case of seasonal demand, they present a heuristic argument that the optimum inventory level will change in a dynamic way. Thus, if the channel structure has an influence on the prices and proft margins, one might expect the channel structure to affect the production and inventory decision also. Conversely, inventory decisions can also affect the pricing policy and hence the channel structure. Since inventories are an important feature in many markets, exploring a channel problem with inventories can provide insights for such markets. Another feature commonly observed in many markets is the active role played by the retail level in promoting and selling products. The marketing literature is abundant with articles that examine simultaneous price–advertising decisions. Hence once again, if the level of promotion effort (for example, advertising) is influenced by the profit margin at the retail level, the channel structure will have an affect on the level of promotional effort, and vice versa.

To answer the questions raised by the above discussion, we have formulated the problems as differential games. The Stackelberg solution concept has been used to solve the manufacturer and retailer's problem, with the manufacturer acting as a leader. The problem of the vertically integrated firm has been solved using the Pareto solution concept. The production, pricing, and effort policies thus derived have been compared to obtain insights into the impact of channel structure on these policies. Also, to examine the relation between channel structure and the retailing operations requiring effort, we derive the Stackelberg and Pareto solutions with and without effort rate as a decision variable.

There are two papers that we know of that look at the dynamic pricing–production problem under a channel setting. However, they do not compare pricing, production, or inventory policies under the different channel structures. We briefly describe their main results here so that comparisons can be made with our policies. Jorgensen [11] derives open-loop Nash equilibrium solutions for the case of a bilateral monopoly under static demand. The optimal price for the manufacturer is the maximum price it can quote. The optimal price for the retailer is either monotonically increasing as a function of time, or first increasing and then decreasing. The optimal ordering policy for the retailer is zero throughout or zero in the initial as well as terminal interval, with a positive purchase rate during an intermediate interval. The production policy either resembles the purchase policy, in which case the intermediate interval where production rate is positive, it simply equals the ordering rate. In the other case where the ordering rate exceeds the production rate in some intermediate interval, the production rate is a sequence of zero, singular, and maximal rate. As Jorgensen admits, some of these results arise because of the special structure of the model (i.e. because the objective function is linear in controls). He suggests that the derivation of Stackelberg as well as Pareto solutions would be an interesting extension [11, p. 76]. In contrast to these results, the optimal policies obtained by the paper we discuss next are more smooth. This is the case with the optimal policies derived in our paper also.

Eliashberg and Steinberg [6] were perhaps the first to consider a Stackelberg marketing–production game with time-dependent demand. The optimal processing policy for the retailer is to first process at a constantly increasing rate, and then precisely at the market demand rate. The price charged by the retailer is first increasing at a decreasing rate and then decreasing at an increasing rate. The inventory builds up for a while and then reaches zero; from then on, the retailer processes just enough to meet demand.

For the manufacturer, the optimal production policy is to first produce at a constantly increasing rate, and then precisely at the retailer's processing rate. Thus, the inventory first goes up and then down. Like Jorgensen, Eliashberg and Steinberg suggest that it might be interesting to look at cooperation and bargaining modes of behavior in the distribution channel [6, p. 996].

The latter paper is of special relevance to our work, and hence we discuss it in some detail. There are three very important differences. First, as Eliashberg and Steinberg (ES) point out, their "optimal control solution procedure involves the 'indirect adjoining' approach, which has not often been used in the literature" [6, p. 983]. The models in our paper are formulated as differential games, and the solution procedure used gives a standard open-loop Stackelberg solution (e.g. Simaan and Cruz, Jr. [21,22], Dockner and Jorgensen [5], and Karp [12]). Second, in ES, even though the demand is seasonal and all other variables are allowed to change with time (including the retailer's price), the manufacturer's price remains constant throughout the season. In our models, even the manufacturer is allowed to change its price with time. Thus, the manufacturer's problem is more complicated in our problem formulation. As we mentioned earlier, the interactions between channel members as well as channel structure and the production inventory policies occur through the price variable. Hence, we believe that this is an important difference. However, we find that these modifications complicate the solution procedure considerably. Hence, in order to keep the problem mathematically tractable, we simplify the retailer's problem by not allowing any inventory at the retail level. As evidenced by the results, the formulations still provide many interesting answers. There are two other differences. ES assume that the starting and closing inventories for the manufacturer as well as the retailer are zero. In our case, we start with a nonzero initial inventory and have a nonnegative salvage value for the ending inventory. ES specifically assume that the price charged by the manufacturer is such that the quantity demanded by the retailer is always positive. In our models, we discuss the circumstances in which this is optimal for the manufacturer. Hence, our models produce results that are quite different from ES. For example, all the policies derived by ES have a concave shape, i.e. the optimal production rate, or processing rate, or retail price, first increase and then decrease in a concave manner. Thus, they essentially have the same shape as the seasonal component of the demand function. In contrast to these results, the optimal policies we derive need not necessarily be concave or convex. Of course, we also discuss the cooperative solutions that ES suggest as fruitful future research.

2. Model I

As already mentioned, we have used the Stackelberg solution concept to solve the manufacturer and the retailer's problem. The manufacturer acts as the leader and the retailer acts as the follower. In real life, one would expect the retailer to wait for the manufacturer to announce its price before deciding its own price. This is exactly what happens in the Stackelberg solution. Thus, the Stackelberg solution concept seems appropriate to use.

We derive an open-loop Stackelberg and a Pareto optimal solution for each of the models studied. Other control structures such as closed loop and feedback loops are possible. Which structure is desirable, and when, has been widely discussed [2,24], but in general there is as yet no firm answer. Closed loop and feedback structures may look more attractive, but it has been shown that in certain cases, payoffs for each player are higher when they choose open-loop controls rather than feedback or closed-loop controls [16, 25]. This might be the case because the game is non-cooperative and the more information a player has about the opponent, the more harm it can inflict. Also, when players choose feedback controls, neither existence nor uniqueness of the solution to the state equation is guaranteed in general [1]. Lastly, open-loop solutions are easier to derive because one has to solve only a system of ordinary differential equations rather than a system of partial differential equations. Even when one uses open-loop controls, solutions are possible only for limited classes of problems. If the state equations are linear and the objective functions are quadratic, one obtains a system of linear differential equations which is possible to solve. In the present paper, we have used a linear demand function and quadratic cost functions, which gives us the desired linear–quadratic structure. The linear demand function is quite commonly found in the research literature since it leads to mathematical tractability. For example, McGuire and Staelin [14] and

Moorthy [15] use a linear demand function that is dependent upon the prices of both the retailers. Among dynamic models, Pekelman [18], Jorgensen [11], and Eliashberg and Steinberg [6] all use linear demand functions. Note that while we will be referring to it as a demand function, it is actually the rate at which quantity is demanded by consumers at time t. Pekelman [18] uses a general time varying demand function of the type $q(t) = a(t) - b(t)p(t)$, where $q(t)$ is the demand rate at price $p(t)$. Jorgensen [11] assumes both $a(t)$ and $b(t)$ are constant. Eliashberg and Steinberg [6] assume $b(t)$ is constant. In the present paper, we assume that $q(t) = \alpha_1 + \alpha_2 \sin \alpha_3 t - bp(t)$, with $\alpha_3 T = \pi$. Here, T is the duration of the season and $a(t)$ is the potential demand at time t. Quadratic total production and holding cost functions of the form $cx^2(t)$ and $hs^2(t)$, respectively, have been considered. Here, c and h are constant, $x(t)$ is the production rate, and $s(t)$ is the inventory at time t. The use of quadratic cost functions is common and has a long history, including one of the earliest models [9], where production and holding costs are simultaneously considered in a planning problem. A model of recent vintage is a stochastic production planning model by Parlar [17]. Jorgensen [11] uses a linear production cost function and a quadratic holding cost function to obtain bang-bang control solutions. Pekelman [18] and Eliashberg and Steinberg [6] use a linear holding cost function. This, together with a state constraint on the inventory, leads to better mathematical tractability for both. Pekelman uses a strictly convex increasing production cost function, whereas Eliashberg and Steinberg [6] use a quadratic production cost function.

2.1. STACKELBERG PROBLEM SPECIFICATION

The retailer (R) buys the goods from the manufacturer at price $p_m(t)$ and sells it to the consumer at price $p_r(t)$ at time t. The demand faced by the retailer is given by $\alpha_1 + \alpha_2 \sin \alpha_3 t - bp_r(t)$. Thus, as the season progresses, the potential demand first rises and then falls. The retailer does not carry inventory. This gives us the retailer's objective function:

$$\text{R:} \quad \underset{\text{w.r.t. } p_r}{\text{maximize}}\ J_R = \int_0^T \big(p_r(t) - p_m(t)\big)\big(a(t) - bp_r(t)\big)\, dt.$$

The manufacturer (M) produces $x_m(t)$ units at a cost of $c_m x_m^2(t)$. The difference between the amount sold and amount produced is carried as inventory $s_m(t)$. The inventory holding cost is $h_m s_m^2(t)$. At the end of the season, unsold inventory fetches a salvage value v_m per unit. This gives us the manufacturer's objective function:

$$\text{M:} \quad \underset{\text{w.r.t. } p_m, x_m}{\text{maximize}}\ J_M = \int_0^T \big(p_m(t)\big(a(t) - bp_r(t)\big) - c_m x_m^2(t) - h_m s_m^2(t)\big)\, dt + v_m s_m(T).$$

The state dynamics are as follows:

$$\dot{s}_m = \frac{\mathrm{d}s_m}{\mathrm{d}t} = x_m(t) - a(t) + bp_r(t).$$

The initial condition is $s_m(0) = s_{m0}$.

The nonnegativity conditions are $x_m(t) \geq 0$ (i.e. production rate cannot be negative) and $a(t)/b - p_r(t) \geq 0$ (i.e. quantity purchased cannot be negative).

Finally, we will assume that $\alpha_1/b \geq v_m \geq 0$ (to avoid a situation where the salvage value is so high that consumers will buy a negative quantity if charged a price equaling the salvage value).

We will omit the time argument in the rest of the discussion unless otherwise required to clarify a point. Also, the optimal policies will be indicated by an *.

STACKELBERG SOLUTION

The necessary conditions to be satisfied by a Stackelberg solution have been obtained by Simaan and Cruz, Jr. [21,22]. The solution procedure involves formulating Hamiltonians for each player as follows:

$$H^R = (p_r - p_m)(a - bp_r) + \lambda_r(x_m - a + bp_r). \tag{1}$$

$$H^M = (p_m(a - bp_r) - c_m x_m^2 - h_m s_m^2) + \lambda_m(x_m - a + bp_r). \tag{2}$$

To account for constraints on some of the control variables, one attaches Lagrange multipliers to form Lagrangians as follows:

$$L^R = H^R + \eta_r\left(\frac{a}{b} - p_r\right). \tag{3}$$

$$L^M = H^M + \eta_m x_m. \tag{4}$$

To obtain the necessary conditions, we first solve the retailer's (the follower's) problem, taking the manufacturer's control variables p_m and x_m as parameters. We set $L^R_{p_r} = 0$ to obtain

$$p_r^* = \frac{1}{2b}(a + bp_m + b\lambda_r - \eta_r). \tag{5}$$

Also, setting $\dot{\lambda}_r = \mathrm{d}\lambda_r/\mathrm{d}t = -H^R_{s_m}$, we obtain $\dot{\lambda}_r = 0$. Using the transversality condition, we obtain $\lambda_r^*(T) = 0$. Therefore,

$$\lambda_r^*(t) = 0. \tag{6}$$

Next, we solve the manufacturer's (the leader's) problem by substituting into the retailer's variables.

Substituting (5) and (6) into (2) gives

$$L^M = \left(p_m \tfrac{1}{2}(a - bp_m + \eta_r) - c_m x_m^2 - h_m s_m^2\right) + \lambda_m \left(x_m - \tfrac{1}{2}(a - bp_m + \eta_r)\right).$$

Setting $L^M_{x_m} = 0$, we obtain

$$x_m^* = \frac{\lambda_m + \eta_m}{2c_m}. \tag{7}$$

Also, setting $L^M_{p_m} = 0$, we obtain

$$p_m^* = \frac{1}{2b}(a + b\lambda_m^* + \eta_r) \tag{8}$$

and substituting (8) into (5), we obtain

$$p_r^* = \frac{1}{4b}(3a + b\lambda_m^* - \eta_r). \tag{9}$$

The complementary slackness conditions are

$$\eta_r \geq 0,\ \eta_r\left(\frac{a}{b} - p_r^*\right) = 0 \quad \text{and} \quad \eta_m \geq 0,\ \eta_m x_m^* = 0. \tag{10}$$

Also, using $\dot{\lambda}_m = \mathrm{d}\lambda_m/\mathrm{d}t = -H^M_{s_m}$, we obtain

$$\dot{\lambda}_m = 2h_m s_m, \tag{11}$$

and using the transversality condition $\lambda_m(T) = \partial(v_m s_m)/\partial s_m$, we obtain $\lambda_m^*(T) = v_m$.

The sufficiency condition requires the Lagrangians to be concave in the state and control variables [20, p. 46]. As shown below, this condition is satisfied, and hence the solutions satisfying the necessary conditions will indeed be optimal solutions to the Stackelberg problem.

$$\frac{\partial L^R}{\partial p_r^2} = -2b < 0,\quad \frac{\partial L^M}{\partial x_m^2} = -2c_m < 0,\quad \frac{\partial L^M}{\partial p_m^2} = -b < 0.$$

The necessary conditions derived above give rise to a system of two ordinary linear differential equations. We first discuss the significance of the equations derived using the adjoint variables. Next, we present results derived after solving the system of differential equations.

Adjoint variables give shadow prices of the corresponding state variables. More precisely, $\lambda_m(t) = \partial J_M^* / \partial s_m(t)$, i.e. $\lambda_m(t)$ represents an increase in the manufacturer's profit for a unit increase in the manufacturer's inventory at time t [19, p. 212]. Similarly, $\lambda_r(t)$ represents an increase in the retailer's profit for a unit increase in the manufacturer's inventory at time t.

According to (6), the shadow price of the manufacturer's inventory to the retailer is zero. As we show in lemma 1, the manufacturer carries the inventory to smooth production. However, none of the gain from production smoothing is passed on to the retailer.

According to (10) and (7), if $\lambda_m^*(t) \le 0$, $\eta_m(t) = -\lambda_m^*(t)$ and $x_m^*(t) = 0$, i.e. if the manufacturer's profit decreases when there is a unit increase in the manufacturer's inventory, the production rate will be zero.

According to (10) and (9), if $\lambda_m^*(t) \ge a/b$, $\eta_r(t) = b\lambda_m^*(t) - a$ and $a - bp_r^*(t) = 0$, i.e. if the manufacturer's profit increases beyond $a(t)/b$ when there is a unit increase in the manufacturer's inventory, the amount sold by the retailer and hence also the manufacturer will be zero.

Based on the above, one can say that there are three situations possible. Either the production is zero (i.e. when $\lambda_m^*(t) \le 0$), or the production rate and amount sold are both positive (i.e. $0 < \lambda_m^*(t) < a(t)/b$), or the production rate is positive but the amount sold is zero (i.e. when $a(t)/b \le \lambda_m^*(t)$). One can imagine any of these situations arising during the beginning, middle or the end of the season. However, as elaborated in lemma 1 below, once the production rate and amount sold become positive, they remain positive throughout the season (this implies production smoothing by the manufacturer). Thus, three cases are possible and are described in the following lemma.

LEMMA 1

Only three cases are possible. During a season, only one of the three cases will occur.

Case 1: Initially, the production rate is zero but the amount sold is positive (i.e. $\lambda_m^*(0) \le 0$). Eventually, even the production rate becomes positive. Once the production rate and the amount sold become positive, neither become zero again during the rest of the season.

Case 2: Initially, the production rate and the amount sold are both positive (i.e. $0 < \lambda_m^*(0) < \alpha_1/b$). The production rate and the amount sold will remain positive during the rest of the season.

Case 3: Initially, the production rate is positive but the amount sold is zero (i.e. $\alpha_1/b \le \lambda_m^*(0)$). Eventually, even the amount sold becomes positive. Once the production rate and the amount sold become positive, neither become zero during the rest of the season.

Proof

See appendix 1.

Thus, situations wherein the production rate is intermittently zero are not going to occur. Also eliminated are situations in which the amount sold will be zero in an intermittent manner.

In the first case, we have to solve two two-point boundary value problems (TPBVPs), one for the period when production rate is zero and one for the period when production rate becomes positive. In the second case, production rate and amount sold are positive throughout the season; hence, we have to solve only one TPBVP. In the third case, once again we have to solve two TPBVPs, one for the period when amount sold is zero and one for the period when amount sold becomes positive. The derivations of solutions of these TPBVPs are lengthy, and are available from the author upon request. The final solution to each of these cases has been given in appendix 1. Based on these solutions for the pricing policies, the production policy, and the inventory policy can be characterized as follows.

PROPOSITION 1

Characteristics of the production policy:

(1) Initially, the production rate is zero if the nominal demand (α_1) is low, and/or the initial inventory (s_{m0}) is high.

(2) Initially, the amount sold is zero if the peak demand factor (α_2) is high, and/or the nominal demand (α_1) is low, and/or the initial inventory (s_{m0}) is low.

(3) During the period when production rate is positive, the optimal production rate is a linear combination of the nominal demand rate (α_1), the peak demand factor (α_2), and the salvage value (v_m). If the production rate is positive throughout the season, the optimal policy is also a function of the initial inventory (s_{m0}).

The weights are functions of time such that:

(a) During the initial stage of the season, the manufacturer decides its production rate primarily on the basis of the initial inventory s_{m0}, the peak demand factor α_2, and the "nominal" demand α_1. As the season progresses, the importance of s_{m0} declines. However, the importance of α_1 may increase before becoming zero at the end of the season.

(b) Towards the end of the season, the manufacturer decides his production rate primarily on the basis of the salvage value v_m.

Proof

See appendix 1.

It is interesting to note that the optimal production policy need not necessarily be concave or convex, e.g. it can be a combination of both. This happens because of the four different factors driving the optimal policy and the change in weight on these factors as the season progresses. Also, the weight of nominal demand α_1 does not again start increasing towards the end of the season when the seasonal component of the demand rate function vanishes.

PROPOSITION 2

Characteristics of the inventory policy:

(1) If the production rate is initially zero, the inventory decreases until the production rate becomes positive. The higher the nominal demand, the faster is the decrease in inventory.

(2) If initially the amount sold is zero, the inventory will increase until the amount sold becomes positive. The higher the nominal demand rate, and the higher the peak demand factor, the faster will be the increase in inventory.

(3) During the period when the production rate and the amount sold are positive, the inventory is a linear combination of the nominal demand rate, the peak demand factor, and the salvage value. If the production rate and the amount sold are positive throughout the season, the inventory is also a function of the initial inventory.

The weights are functions of time such that:

(a) During the initial stage of the season, the manufacturer's inventory is determined primarily by its initial inventory and the peak demand factor. As the season progresses, the significance of the initial inventory in determining the manufacturer's current inventory declines.

(b) Towards the end of the season, the manufacturer's inventory is determined primarily by the nominal demand and the salvage value.

Proof

See appendix 1.

Once again, the shape of the inventory policy need not necessarily be concave or convex. It is interesting to note that the weight of the nominal demand increases with time throughout the season, and does not decrease and then again increase as would have been expected.

PROPOSITION 3

Characteristics of the manufacturer and retailer pricing policies:

(1) If the change in the production rate is in the same direction as the change in the potential demand, the pricing policies of the manufacturer and the retailer and the production policy of the manufacturer have a "synergistic" effect, i.e. an increase in the manufacturer's price or production rate or the retailer's price leads to an increase in the rate of change of inventory in all three cases.

(2) The manufacturer's pricing policy is based on equal weights on the coefficients of the demand rate function $a(t)/b$ and the value of inventory $\lambda_m(t)$, whereas the retailer's pricing policy is based on a 0.75 weight on the coefficients $a(t)/b$ and a 0.25 weight on the value of inventory $\lambda_m(t)$. Thus, the retailer is more sensitive to the consumer demand than the manufacturer.

Proof

See appendix 1.

Note that the results obtained here are different to those of Jorgensen [11] and Eliashberg and Steinberg [6]. One reason for this is the fact that the optimal production and inventory policies in our model are a linear combination of the nominal demand rate, the peak demand factor (in Jorgensen, the demand is static), the initial inventory and the salvage value (in ES, the initial and final inventory are assumed to be zero). The manufacturer places more weight upon one factor than the others at different times during the season, resulting in the different possible shapes of the optimal production and inventory policies. For example, if the nominal demand is low, the optimal policy might be to keep the production rate zero and create a backlog of inventory during the initial part of the season, and make the production rate positive once the demand has reached a significant level (in ES, the inventory is constrained to be nonnegative). If the nominal demand is low and the peak demand factor is high, the manufacturer will have a positive production rate, but the price charged to the retailer during the initial part of the season will be so high that the amount sold during the initial part of the season will be zero. Thus, the manufacturer will hoard the goods for sale during the peak of the season (this is not possible in ES because the manufacturer's price is constant throughout the season). In essence, the model we consider is less restrictive and hence leads to a richer set of optimal policies.

2.2. PARETO PROBLEM SPECIFICATION

To apply the necessary and sufficient conditions [13,24], one solves an associated optimal control problem where

$$J_P = \beta_1 J_M + \beta_2 J_R, \quad \beta_1, \beta_2 \geq 0, \ \beta_1 + \beta_2 = 1.$$

In the present paper, we have assumed that $\beta_1, \beta_2 = 0.5$, i.e. in the vertically integrated firm the bargaining powers of the manufacturing division and retailing division are identical. Thus, the integrated firm's (P) problem can be stated as follows.

$$\text{P:}\quad \underset{\text{w.r.t. } p_p, x_p}{\text{maximize}}\ J_P = \int_0^T (p_p(a - bp_p) - c_m x_p^2 - h_m s_p^2)\,dt + v_m s_p(T).$$

The state equation is as follows:

$$\dot{s}_p = x_p - a + bp_p\,.$$

The initial condition is $s_p(0) = s_{m0}$. The nonnegativity constraints are $x_p(t) \geq 0$ and $a/b \geq p_p$. Finally, we assume $\alpha_1/b \geq v_m \geq 0$.

PARETO SOLUTION

Hamiltonian:

$$H^P = (p_p(a - bp_p) - c_m x_p^2 - h_m s_p^2) + \lambda_p(x_p - a + bp_p). \tag{12}$$

Lagrangian:

$$L^P = H^P + \eta_1 x_p + \eta_2\left(\frac{a}{b} - p_p\right). \tag{13}$$

Necessary conditions for the control variables:

We set $L^P_{p_p} = 0$ to obtain

$$p_p^* = \frac{1}{2b}(a + \lambda_p^* b - \eta_2). \tag{14}$$

We set $L^P_{x_p} = 0$ to obtain

$$x_p^* = \frac{(\lambda_p^* + \eta_1)}{2c_m}. \tag{15}$$

Also, the complementary slackness conditions are

$$\eta_1 \geq 0,\quad \eta_1 x_p^* = 0,\quad \eta_2 \geq 0,\quad \eta_2\left(\frac{a}{b} - p_p^*\right) = 0. \tag{16}$$

Using $\dot{\lambda}_p = -H^P_{s_p}$,

$$\dot{\lambda}_p = 2h_m s_p. \tag{17}$$

The boundary conditions are $s_p(0) = s_{m0}$, $\lambda_p^*(T) = v_m$.

The complete solution is given in appendix 1. We find that lemma 1, as well as propositions 1 and 2 are still valid. Also, part 1 of proposition 3 is valid. Part 2 is different in that the integrated firm's pricing policy is based on equal weights on the coefficients of the demand function and the value of inventory. The main difference between the optimal policies under the two structures is the weights of the nominal demand rate, peak demand factor, initial inventory, and the salvage value. As we show in the numerical example, this leads to some interesting answers. As discussed earlier, the static case of vertical integration was first studied by Spengler [23]. He showed that the price charged by the vertically integrated firm will always be less than the price charged by the retailer (in the case where the manufacturing and retailing operations are conducted by independent firms). Similar results for bilateral monopolies have also been derived by Jeuland and Shugan [10], McGuire and Staelin [14] and references quoted therein. Spengler argued that this happens because of "double marginalization", i.e. the manufacturer as well as the retailer add their profit margins to their respective costs in arriving at the retail price.

In the present case, the price charged by the vertically integrated firm need not be less than the price charged by the retailer throughout the season. As claimed earlier in the paper, we show that this happens because of the interaction between the pricing and production–inventory policies. We show this using a numerical example.

2.3. A COUNTEREXAMPLE

The parameters used are $\alpha_1 = 4$, $\alpha_2 = 16$, $\alpha_3 = 0.5235$ (i.e. $6\alpha_3 = \pi$), $b = 0.4$, $c_m = 4$, $h_m = 1$, $v_m = 10$, $s_{m0} = 3$.

As can be seen from table 1, the retailer's price is less than the integrated firm's price for the first of the fifteen periods of the season. The inventory decreases for the first thirteen periods and then increases during the last two periods of the season. These results are also illustrated in fig. 1.

One possible explanation for this behavior is as follows. In the decentralized structure, the total channel profits are shared between the manufacturer and the retailer. Also, due to lack of cooperation, the manufacturer does not pass on the gain from production smoothing ($\lambda_r^*(t) = 0$). As opposed to this, in the centralized structure, profits do not have to be shared. Thus, the inventory carrying costs can be offset more easily under the centralized structure. This has an impact upon the tradeoff between selling the goods at a point in time and carrying it as inventory and selling it later during the season. In our example, during the beginning of the season this tradeoff is more in the favor of carrying inventory under the centralized structure. Thus, the value of inventory during the beginning of the season is much higher under the centralized structure, leading to higher prices during the beginning of the season. Also, in our model the peaks occurs exactly in the middle of the season. If the peak were to occur later in the season, the periods for which the integrated firm's price is higher will almost certainly be higher.

Table 1

s_m	s_p	λ_m	λ_p	p_r	p_p	p_m
3.0000	3.0000	4.3272	9.9865	8.5818	9.9932	7.1636
2.9295	2.7599	6.7069	12.5413	15.4141	15.4289	12.5117
2.7427	2.4799	8.9833	15.2344	21.9479	20.7519	17.6264
2.4476	2.1524	11.9662	17.8368	27.9001	25.6741	22.2888
2.0582	1.7782	12.8742	20.1556	33.0129	29.9407	26.3000
1.5930	1.3652	14.3389	22.0344	37.0655	33.3377	29.4900
1.0753	0.9270	15.4088	23.3541	39.8839	35.6982	31.7255
0.5316	0.4819	16.0522	24.0343	41.3487	36.9076	32.9165
-0.0086	0.0523	16.2599	24.0347	41.4006	36.9078	33.0204
-0.5143	-0.3367	16.0471	23.3553	40.0435	35.6988	32.0447
-0.9535	-0.6570	15.4541	22.0365	37.3443	33.3388	30.0476
-1.2939	-0.8778	14.5471	20.1589	33.4311	29.9424	27.1364
-1.5031	-0.9653	13.4178	17.8417	28.4880	25.6765	23.4646
-1.5496	-0.8815	12.1841	15.2413	22.7481	20.7554	19.2268
-1.4014	-0.5826	10.9886	12.5510	16.4845	15.4337	14.6525
-1.0261	-0.0154	10.0000	10.0000	10.0000	10.0000	10.0000

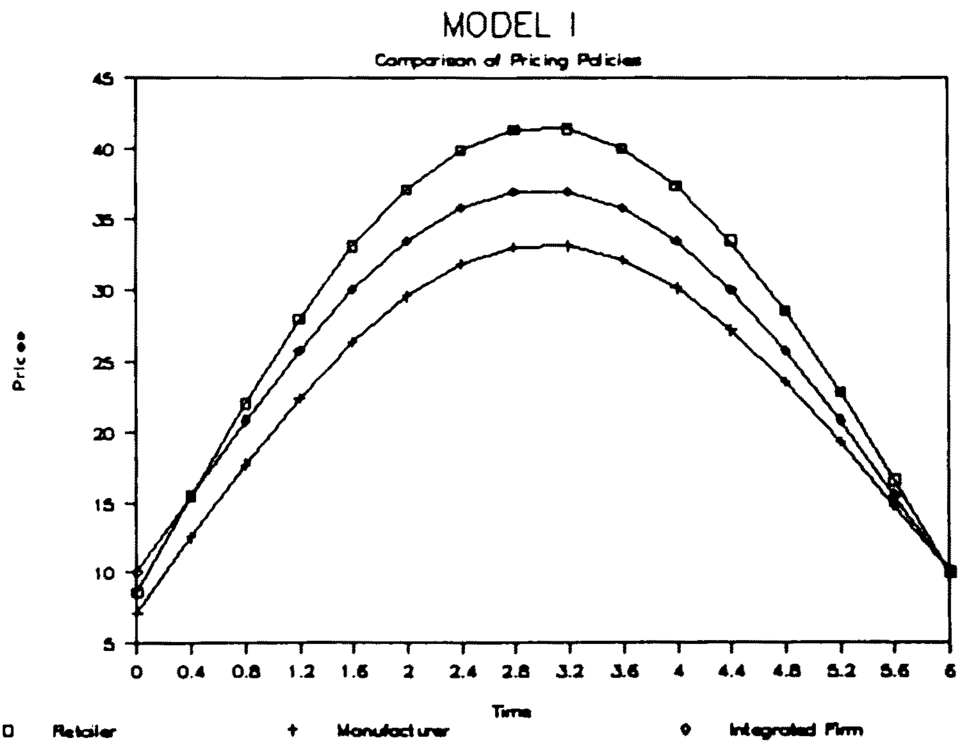

Fig. 1.

The above result implies that cost considerations should indeed be given importance while examining the channel problem; that cost considerations can change the generally accepted results in significant ways. We further reinforce this conclusion by adding one more cost consideration to the problem, namely retailer effort.

3. Model II

In this model, the retailer has an additional decision variable. It can exert effort to attract additional customers. Also, returns on this expense are diminishing.

3.1. STACKELBERG PROBLEM SPECIFICATION

The retailer's (R) problem is as follows:

$$\text{R:} \quad \underset{\text{w.r.t. } p_r, v}{\text{maximize}} \; J_R = \int_0^T \left(\left(p_r(t) - p_m(t)\right)\left(a(t) + v(t) - bp_r(t)\right) - c_a v^2(t) \right) \mathrm{d}t.$$

The manufacturer's (M) problem is as follows:

$$\text{M:} \quad \underset{\text{w.r.t. } p_m, x_m}{\text{maximize}} \; J_M$$
$$= \int_0^T \left\{ p_m(t)\left(a(t) + v(t) - bp_r(t)\right) - c_m x_m^2(t) - h_m s_m^2(t) \right\} \mathrm{d}t + v_m s_m(T).$$

The state dynamics are as follows:

$$\dot{s}_m = \frac{\mathrm{d}s_m}{\mathrm{d}t} = x_m(t) - \left(a(t) + v(t) - bp_r(t)\right).$$

The initial condition is $s_m(0) = s_{m0}$. The nonnegativity conditions are $x_m(t) \geq 0$ and $(a(t) + v(t))/b \geq p_r(t)$. Finally, we assume $\alpha_1/b \geq v_m \geq 0$.

STACKELBERG SOLUTION

Hamiltonians:

$$H^R = \left((p_r - p_m)(a + v - bp_r) - c_a v^2\right) + \lambda_r\left(x_m - (a + v - bp_r)\right), \tag{18}$$

$$H^M = (p_m(a + v - bp_r) - c_m x_m^2 - h_m s_m^2) + \lambda_m\left(x_m - (a + v - bp_r)\right). \tag{19}$$

Lagrangians:

$$L^R = H^R + \eta_r\left(\frac{a+v}{b} - p_r\right), \tag{20}$$

$$L^M = H^M + \eta_m x_m. \tag{21}$$

Necessary conditions:

Setting $L^R_{p_r} = 0$, we obtain

$$p_r = \frac{a + v + bp_m - \eta_r + b\lambda_r}{2b}. \tag{22}$$

Setting $L^R_v = 0$, we obtain

$$v = \frac{p_r - p_m - \lambda_m + \eta_r/b}{2c_a}. \tag{23}$$

Setting $\dot{\lambda}_r = -H^R_{s_m}$, we obtain $\dot{\lambda}_r = 0$, and using the transversality condition $\lambda_r(T) = 0$, we obtain

$$\lambda_r^*(t) = 0. \tag{24}$$

Substituting (24) into (22) and (23), and solving to determine p_r and v independently,

$$p_r^* = \frac{2abc_a + b(2bc_a - 1)p_m - (2bc_a - 1)\eta_r}{b(4bc_a - 1)}, \tag{25}$$

$$v^* = \frac{a - bp_m + \eta_r}{(4bc_a - 1)}. \tag{26}$$

Substituting (25) and (26) into (19) gives

$$H^M = \left(\frac{p_m 2bc_a(a - bp_m + \eta_r)}{(4bc_a - 1)} - c_m x_m^2 - h_m s_m^2\right) + \lambda_m\left(x_m - \frac{2bc_a(a - bp_m + \eta_r)}{(4bc_a - 1)}\right).$$

Setting $L^M_{x_m} = 0$, we obtain

$$x_m^* = \frac{1}{2c_m}(\lambda_m + \eta_m). \tag{27}$$

Setting $H^M_{p_m} = 0$, we obtain

$$p^*_m = \frac{a + b\lambda_m + \eta_r}{2b}. \tag{28}$$

Also, the complementary slackness conditions are

$$\eta_m \geq 0, \quad \eta_m x^*_m = 0, \tag{29}$$

$$\eta_r \geq 0, \quad \eta_r \left(\frac{a + v^*}{b} - p^*_r \right) = 0. \tag{30}$$

Setting $\dot{\lambda}_m = -H^M_{s_m}$, we obtain

$$\dot{\lambda}_m = 2h_m s_m. \tag{31}$$

Using the transversality condition, $\lambda^*_m(T) = v_m$. Substituting (28) into (20) gives

$$v^* = \frac{1}{2(4bc_a - 1)} (a - \lambda^*_m b + \eta_r). \tag{32}$$

Substituting (28) into (25),

$$p^*_r = \frac{a(6bc_a - 1) + (2bc_a - 1)\lambda^*_m - (2bc_a - 1)\eta_r}{2b(4bc_a - 1)}. \tag{33}$$

Based on the above, like model I one can once again say that there are three situations possible. Either the production rate is zero (i.e. when $\lambda^*_m(t) \leq 0$) or the production rate, effort rate, and the amount sold are all positive (i.e. when $0 < \lambda^*_m(t) < a(t)/b$), or the production rate is positive but the amount sold and effort rate are both zero. Also, it can be easily seen that lemma 1 is still valid. The complete solution is given in appendix 2. From the solution, one can also see that propositions 1 and 2 are still valid. However, the pricing policy of the retailer will be different. Proposition 4 below describes the pricing policies of the manufacturer and the retailer, and the effort rate of the retailer.

PROPOSITION 4

Characteristics of the manufacturer and retailer pricing policies and the retailer effort rate policy:

(1) If the change in the production rate is in the same direction as the change in potential demand, the pricing and production policies of the manufacturer, and the effort rate of the retailer have a "synergistic" effect.

(2) If the change in the production rate is in the same direction as the change in the potential demand, and the proportionate cost of effort rate is high and/or the demand is price sensitive, the retailer's pricing policy will also be "synergistic" with the manufacturer's pricing and production policies, and the retailer's effort rate.

(3) If the proportionate cost of the effort rate is not high and/or the demand is not price sensitive, an increase in production rate or manufacturer's price can lead to a decrease in the retailer's price and may lead to decrease in inventory.

Proof

See appendix 1.

Proposition 4 is interesting because it says that it is possible to have a situation in which, if the manufacturer raises his price, the retailer will reduce his price. This is exactly the opposite of the result obtained in model I. A possible explanation is as follows. The effort rate of the retailer depends upon his profit margin. If the manufacturer increases his price, the retailer's profit margin reduces. Hence, he will reduce his effort rate. If the proportionate cost of effort rate is not high, this will not lead to much savings in cost. Thus, the retailer will have to maintain sales despite lower effort rate by reducing the price. Also, using (32) one can see that $dv(t)/db < 0$. Thus, if the demand is not price sensitive, once again the effort rate will be low.

3.2. PARETO PROBLEM SPECIFICATION

The integrated firm's (P) problem is as follows:

$$\text{P:} \quad \underset{\text{w.r.t. } p_p, x_p, v}{\text{maximize}} \; J_P = \int_0^T (p_p(a+v-bp_p) - c_m x_p^2 - c_a v^2 - h_m s_p^2)\,dt + v_m s_p(T).$$

The state dynamics are as follows:

$$\dot{s}_p = \frac{ds_p}{dt} = x_p - (a+v-bp_p).$$

The initial condition is $s_p(0) = s_{m0}$. The nonnegativity conditions are $x_p \geq 0$ and $(a+v)/b - p_p \geq 0$. Finally, we assume $a_1/b \geq v_p \geq 0$.

PARETO SOLUTION

Hamiltonian:

$$H^P = (p_p(a+v-bp_p) - c_m x_p^2 - c_a v^2 - h_m s_p^2) + \lambda_p(x_p - (a+v-bp_p)). \quad (34)$$

Lagrangian:

$$L^P = H^P + \eta_1 x_p + \eta_2\left(\frac{a+v}{b} - p_p\right). \tag{35}$$

Necessary conditions:

Setting $L^P_{p_p} = 0$, we obtain

$$p_p = \frac{a + v + b\lambda_p - \eta_2}{2b}. \tag{36}$$

Setting $L^P_{x_p} = 0$, we obtain

$$x_p = \frac{(\lambda_p + \eta_1)}{2c_m}. \tag{37}$$

Setting $L^P_v = 0$, we obtain

$$v = \frac{(p_p + \lambda_p + \eta_2/b)}{2c_a}. \tag{38}$$

The complementary slackness conditions are

$$\eta_1, \eta_2 \geq 0, \quad \eta_1 x_p^* = 0, \quad \eta_2\left(\frac{a+v^*}{b} - p_p^*\right) = 0. \tag{39}$$

Solving (36) and (38) to determine p_p and v independently,

$$p_p^* = \frac{2abc_a + b(2bc_a - 1)\lambda_p^* - (2bc_a - 1)\eta_2}{b(4bc_a - 1)}, \tag{40}$$

$$v^* = \frac{a - b\lambda_p^* + \eta_2}{(4bc_a - 1)}. \tag{41}$$

From the above-mentioned conditions, one can easily show that lemma 1 and propositions 1 and 2 are still valid. Once again, we compare the Pareto solution with the Stackelberg solution. In the previous model, we showed that the price charged by the vertically integrated firm may be more than that charges by the retailer for at least a small part of the season. For model II, we give an example in which the price charged by the vertically integrated firm is higher for almost the entire duration of the season.

Table 2

s_m	s_p	λ_m	λ_p	p_r	p_p	p_m	v_m	v_p
3.0000	3.0000	7.3411	9.8779	10.2099	10.0193	8.6705	0.6997	0.0643
3.4769	4.1571	11.0135	14.4858	18.8930	18.9213	14.6650	1.9218	2.0162
3.5907	4.5724	14.5871	19.1106	27.1918	27.3998	20.4283	3.0743	3.7678
3.4050	4.4268	17.8808	23.4436	34.7454	35.1011	25.6961	4.1133	5.2989
2.9801	3.8567	20.7342	27.2267	41.2251	41.6993	30.2300	4.9978	6.5784
2.3762	2.9729	23.0126	30.2512	46.3485	46.9131	33.8268	5.6917	7.5736
1.6557	1.8720	24.6123	32.3578	49.8920	50.5187	36.3273	6.1658	8.2550
0.8846	0.6435	25.4640	33.4392	51.7006	52.3611	37.6224	6.3992	8.6009
0.1339	-0.6261	25.5361	33.4413	51.6949	52.3608	37.6585	6.3802	8.5998
-0.5197	-1.8512	24.8369	32.3644	49.8743	50.5177	36.4396	6.1067	8.2515
-0.9921	-2.9449	23.4156	30.2636	46.3167	46.9111	34.0283	5.5856	7.5671
-1.1884	-3.8161	21.3619	27.2473	41.1756	41.6961	30.5438	4.8326	6.5676
-0.9993	-4.3658	18.8056	23.4762	34.6724	35.0959	26.1585	3.8699	5.2817
-0.2938	-4.4792	15.9156	19.1615	27.0869	27.3918	21.0925	2.7247	3.7410
1.0909	-4.0140	12.8986	14.5647	18.7442	18.9089	15.6075	1.4258	1.9746
3.3649	-2.7797	10.0000	10.0000	10.0000	10.0000	10.0000	0.0000	0.0000

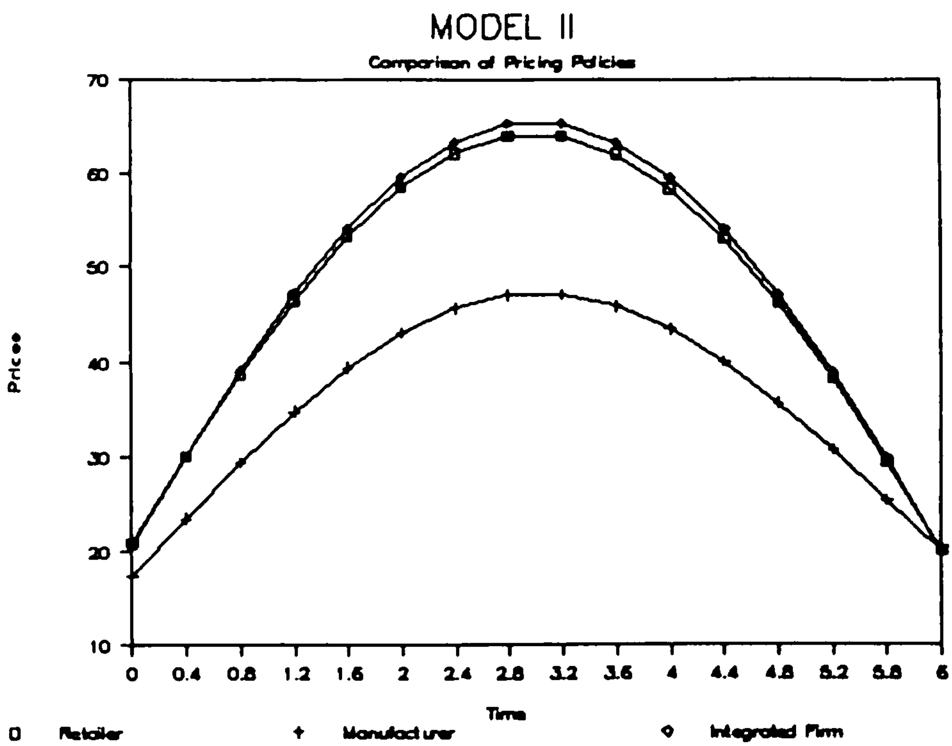

Fig. 2.

3.3. A COUNTEREXAMPLE

The additional parameter used in this model is $c_a = 1.1$. All the other parameter values are the same as the previous example. As can be seen from table 2 (note that v_m refers to the effort rate in the independent setup and v_p refers to the effort rate in the integrated setup) as well as fig. 2, the retailer's price is higher than the integrated firm's price only during the initial period. A possible explanation for this behavior is the fact that the pricing policies of the manufacturer and the retailer do not have a "synergistic" effect. Note that $2bc_a - 1 = -0.12 < 0$ in our example. Thus, an increase in the manufacturer's price, or production rate, leads to a decrease in the retailer's price.

Lastly, we compare the profit and the costs under the two structures for both the models using the numerical example. As can be seen from table 3, the total channel profits are higher under the centralized channel structure. Also, the inventory carrying costs are lower under the centralized channel structure. This implies more efficient production smoothing under the centralized structure. Since the prices are higher under the centralized structure for a part of the season in model I, and during almost the entire duration of the season in model II, this implies a conflict between the interests of the consumer and the firm. This is an interesting result because in the existing literature the interests of the two parties always seem to be aligned in the sense that integration leads to higher profits and lower prices.

Table 3

Profit		Costs		
Model I				
Manufacturer	Retailer	Inventory	Production	Effort
911.91	372.15	16.47	77.67	–
Integrated firm				
1782.75		12.43	171.11	–
Model II				
Manufacturer	Retailer			
1563.17	579.2	133.76	352.4	762.12
Integrated firm				
2818.32		398.39	1240.29	1983.7

4. Conclusions and future research

In this paper, we have attempted to examine the relationship between channel structure and optimal pricing and production–inventory policies. Towards this end, we first derived and then compared the policies for independent and integrated channel structures. Also, we tried to examine the impact of effort rate on the other policies under different structures. We showed that if the nominal demand is low, or the initial inventory is high, initially the production rate will be zero. Alternatively, if the initial inventory is low, or the peak demand is high, initially the amount sold will be zero even if the production rate is positive. Once the production rate becomes positive, it does not become zero again. This implies production smoothing. However, none of the gains of production smoothing are passed on to the retailer. This seems to provide a rationale for the retailer to carry his own inventory. While the retailer does not carry inventory in the model studied in the present paper, we are currently investigating that problem. Also, we show that the optimal production rate and inventory are a linear combination of the nominal demand, peak demand factor, salvage value, and the initial inventory. In the scenario where the retailing operation does not require any effort, the pricing policies of the manufacturer and the retailer and the production policy of the manufacturer have a synergistic effect, i.e. an increase in the manufacturer's price or production rate or the retailer's price leads to an increase in the rate of change of inventory. However, in the scenario where the retailing operation does benefit from effort, the retailer's pricing policy need not necessarily be synergistic with the other policies. If the proportionate cost of effort rate is not high, and/or the demand is not price sensitive, an increase in the production rate or the manufacturer's price leads to a decrease in the retailer's price. With regard to channel structure, the numerical example seems to suggest that production smoothing will be done more efficiently in the integrated setup. Also, we showed that the price paid by the consumer need not necessarily be lower in the integrated setup. If the proportionate cost of effort rate is not high, and/or the demand is not price sensitive, the price paid by the consumer is lower in the independent setup. However, despite higher prices, the channel profits are higher in the integrated setup.

Appendix 1: Model I

STACKELBERG SOLUTION

The two-point boundary value problem (TPBVP) can be stated as follows:

$$\dot{s}_m = \frac{1}{2c_m}(\eta_m + \lambda_m) - \frac{1}{2}\left(a - b\,\frac{1}{2b}(a + b\lambda_m + \eta_r) + \eta_r\right)$$

$$= \frac{1}{2c_m}(\eta_m + \lambda_m) - \frac{1}{4}(a - \lambda_m b + \eta_r), \tag{42}$$

$$\dot{\lambda}_m = 2h_m s_m. \tag{11}$$

The boundary conditions are $s_m(0) = s_{m0}$, $\lambda_m(T) = v_m$.

For the time intervals when $\lambda_m(t) \le 0$ using (7) and (10), we obtain $\eta_m(t) = -\lambda_m(t) \ge 0$. Also, using (9) and (10), $\eta_r(t) = 0$. Therefore,

$$\dot{s}_m = \frac{b}{4}\lambda_m - \frac{a}{4} \quad \text{and} \quad x_m(t) = 0,\ a(t) - bp_r(t) > 0. \tag{43}$$

For the time intervals when $a(t)/b > \lambda_m(t) > 0$, using (7), (9) and (10) we obtain $\eta_m(t) = 0$, $\eta_r(t) = 0$. Therefore,

$$\dot{s}_m = \left(\frac{1}{2c_m} + \frac{b}{4}\right)\lambda_m - \frac{a}{4} \quad \text{and} \quad x_m(t) > 0,\ a(t) - bp_r(t) > 0. \tag{44}$$

For the time intervals when $\lambda_m(t) \ge a(t)/b$, using (7) and (10), $\eta_m(t) = 0$. Also, using (9) and (10), $\eta_r(t) = b\lambda_m - a$. Therefore,

$$\dot{s}_m = \frac{1}{2c_m}\lambda_m \quad \text{and} \quad x_m(t) > 0,\ a(t) - bp_r(t) = 0. \tag{45}$$

Proof of lemma 1

The proof involves showing three things.

First, we must show that if the production rate is initially zero, it eventually becomes positive. From (11) and (43),

$$\ddot{\lambda}_m = 2h_m\left(\frac{b}{4}\lambda_m - \frac{a}{4}\right) < 0, \tag{46}$$

i.e. $\lambda_m(t)$ is concave for $\lambda_m(t) \le 0$ and hence it is possible for $\lambda_m(t)$ to become positive. Since $\lambda_m(T) = v_m \ge 0$ is required, $\lambda_m(t)$ must become positive before the end of the season.

Second, we must show that if the amount sold is initially zero, it eventually becomes positive. From (11) and (45),

$$\ddot{\lambda}_m = 2h_m\lambda_m > 0, \tag{47}$$

i.e. $\lambda_m(t)$ is convex for $\lambda_m(t) \ge a(t)/b$. Since $a(t)/b$ is concave for the complete season, it is possible for $\lambda_m(t)$ to become less than $a(t)/b$. Since $\lambda_m(T) = v_m \le \alpha_1/b$ is required, $\lambda_m(t)$ must become less than $a(t)/b$ before the end of the season.

Third, we must show that once the production rate and the amount sold become positive (i.e. $0 < \lambda_m(t) < a(t)/b$) they remain positive. The method we use involves showing that if either become zero, the end point condition $\lambda_m(T) = v_m$ ($\alpha_1/b \ge v_m \ge 0$) will not be satisfied. Suppose at some point during the season the amount sold becomes zero, i.e. $\lambda_m(t) \ge a(t)/b$. After this point, we know from (47) that $\lambda_m(t)$ is convex. Also, for $\lambda_m(t)$ to become greater than $a(t)/b$, the slope must be positive. However, if this is the case, $\lambda_m(t)$ will never decrease to become equal to v_m at the end of the season. Similarly, suppose at some point during the season the production rate becomes zero, i.e. $\lambda_m(t) \le 0$. After this point, we know from (46) that $\lambda_m(t)$ is concave. Also, for $\lambda_m(t)$ to become less than zero, the slope must be negative. However, if this is the case, $\lambda_m(t)$ will never increase to become equal to v_m at the end of the season. □

Given below are the solutions of the TPBVPs that arise for each of the cases:

Case 1 ($\lambda_m(0) \le 0$)

For the period $[0, t_1]$, using (43) and (11) with boundary conditions $s_m(0) = s_{m0}$, $\lambda_m(t_1) = 0$, we obtain:

$$s_m(t) = \left(s_{m0} - \frac{\alpha_2\alpha_3}{4\varepsilon}\right)\frac{\cosh\zeta(t_1 - t)}{\cosh\zeta t_1} - \frac{\sinh\zeta t}{\cosh\zeta t_1}\,\theta\left(\frac{\alpha_1}{b} + \frac{h_m\alpha_2\sin\alpha_3 t_1}{2\varepsilon}\right) + \frac{\alpha_2\alpha_3\cos\alpha_3 t}{4\varepsilon}, \tag{48}$$

$$\lambda_m(t) = \frac{1}{\theta}\left(\frac{\alpha_2\alpha_3}{4\varepsilon} - s_{m0}\right)\frac{\sinh\zeta(t_1 - t)}{\cosh\zeta t_1} + \frac{\alpha_1}{b}\left(1 - \frac{\cosh\zeta t}{\cosh\zeta t_1}\right) + \frac{h_m\alpha_2}{2\varepsilon}\left(\sin\alpha_3 t - \sin\alpha_3 t\frac{\cosh\zeta t}{\cosh\zeta t_1}\right). \tag{49}$$

For the period $[t_1, T]$ when $a(t)/b > \lambda_m(t) > 0$, using (44) and (11) with boundary conditions $\lambda_m(t_1) = 0$, $\lambda_m(T) = v_m$, we obtain:

$$s_m(t) = v_m\, \iota\,\frac{\cosh\kappa(t - t_1)}{\sinh\kappa(T - t_1)} + \xi\iota\left(e^{\kappa(T-t)} - e^{\kappa(T-t_1)} + 1\right) + \iota\,\frac{\cosh\kappa(T - t)}{\sinh\kappa(T - t_1)}\,\frac{h_m\alpha_2\sin\alpha_3 t}{2\rho} + \frac{\alpha_2\alpha_3\cos\alpha_3 t}{4\rho}, \tag{50}$$

$$\lambda_m(t) = v_m\frac{\sinh\kappa(t - t_1)}{\sinh\kappa(T - t_1)} + \xi\left\{\frac{\sinh\kappa(T - t_1) - \sinh\kappa(T - t) - \sinh\kappa(t - t_1)}{\sinh\kappa(T - t_1)}\right\} + \frac{h_m\alpha_2\sin\alpha_3 t}{2\rho}\left\{1 - \frac{\sinh\kappa(T - t)}{\sinh\kappa(T - t_1)}\right\}. \tag{51}$$

Using (49), $\lambda_m(0) \le 0$ if and only if

$$\frac{\alpha_1}{b}(\cosh \zeta t_1 - 1) < \frac{h_m \alpha_2 \sin \alpha_3 t_1}{2\varepsilon} + \left(s_{m0} - \frac{\alpha_2 \alpha_3}{4\varepsilon}\right)\frac{1}{\theta}\sinh \zeta t_1 . \tag{52}$$

Case 2 $(\alpha_1/b > \lambda_m(0) > 0)$

Using (44) and (11) with boundary conditions $s_m(0) = s_{m0}$, $\lambda_m(T) = v_m$, we obtain:

$$s_m(t) = (v_m - \xi)\iota \frac{\sinh \kappa t}{\cosh \kappa T} + \left(s_{m0} - \frac{\alpha_2 \alpha_3}{4\rho}\right)\frac{\cosh \kappa(t-T)}{\cosh \kappa T} + \frac{\alpha_2 \alpha_3 \cos \alpha_3 t}{4\rho}, \tag{53}$$

$$\lambda_m(t) = \xi\left(1 - \frac{\cosh \kappa t}{\cosh \kappa T}\right) + \left(\frac{\alpha_2 \alpha_3}{4\rho} - s_{m0}\right)\frac{1}{\iota}\left(e^{-\kappa t} - e^{-\kappa T}\right) + \frac{h_m \alpha_2 \sin \alpha_3 t}{2\rho} + \frac{\cosh \kappa t}{\cosh \kappa T} v_m . \tag{54}$$

Case 3 $(\lambda_p(0) \geq \alpha_1/b)$

For the period $[0, t_1]$, using (45) and (11) with boundary conditions $s_m(0) = s_{m0}$, $\lambda_m(t_1) = a(t_1)/b$, we obtain:

$$s_m(t) = \frac{s_{m0} \cosh \sigma(t-t_1)}{\cosh \sigma t_1} + \frac{\sinh \sigma t}{\cosh \sigma t_1}\left(\frac{\alpha_1 + \alpha_2 \sin \alpha_3 t_1}{2bv}\right), \tag{55}$$

$$\lambda_m(t) = \frac{\alpha_1 + \alpha_2 \sin \alpha_3 t_1}{b}\,\frac{\cosh \sigma t}{\cosh \sigma t_1} - 2v\, s_{m0}\frac{\sinh \sigma(t_1 - t)}{\cosh \sigma t_1}. \tag{56}$$

For the period $[t_1, T]$ when $a(t)/b > \lambda_m(t) > 0$, using (44) and (11) with boundary conditions $\lambda_m(t_1) = a(t_1)/b$ and $\lambda_m(T) = v_m$, we obtain:

$$s_m(t) = \xi\left\{\frac{\cosh \kappa(T-t) - \cosh \kappa(t-t_1)}{\sinh \kappa(T-t_1)}\right\}\iota + v_m\left\{\frac{\cosh \kappa(t-t_1)}{\sinh \kappa(T-t_1)}\right\}\iota - \iota\frac{\cosh \kappa(T-t)}{\sinh \kappa(T-t_1)}\left\{\frac{\alpha_1 + \alpha_2 \sin \alpha_3 t_1}{b} - \frac{h_m \alpha_2 \sin \alpha_3 t}{2\rho}\right\} + \frac{\alpha_2 \alpha_3 \cos \alpha_3 t}{4\rho}, \tag{57}$$

$$\lambda_m(t) = v_m \left\{ \frac{\sinh \kappa(t - t_1)}{\sinh \kappa(T - t_1)} \right\} + \frac{h_m \alpha_2 \sin \alpha_3 t}{2\rho} \left\{ 1 - \frac{\sinh \kappa(T - t)}{\sinh \kappa(T - t_1)} \right\}$$

$$+ \left\{ \frac{\alpha_1 + \alpha_2 \sin \alpha_3 t_1}{b} \right\} \frac{\sinh \kappa(T - t)}{\sinh \kappa(T - t_1)}. \tag{58}$$

Using (56), $\lambda_m(0) \geq a(0)/b$ if and only if

$$\frac{\alpha_2}{b} \sin \alpha_3 t_1 > \frac{\alpha_1}{b} (\cosh \sigma t_1 - 1) + 2v\, s_{m0} \sinh \sigma t_1 , \tag{59}$$

where

$$\varepsilon = \left(\frac{bh_m}{2} + \alpha_3^2 \right), \quad \zeta = \sqrt{\frac{bh_m}{2}}, \quad \theta = \sqrt{\frac{b}{8h_m}}, \quad \iota = \sqrt{\frac{2 + c_m b}{8h_m c_m}}, \quad \kappa = \sqrt{\frac{h_m}{c_m} + \frac{bh_m}{2}},$$

$$\xi = \frac{\alpha_1 c_m}{2 + c_m b}, \quad \rho = \left(\frac{h_m}{c_m} + \frac{bh_m}{2} + \alpha_3^2 \right), \quad \sigma = \sqrt{\frac{h_m}{c_m}}, \quad v = \sqrt{h_m c_m}.$$

Proof of proposition 1

(1) Using (49),

$$\frac{\partial \lambda_m(0)}{\partial \alpha_1} = \frac{1}{b} \left(1 - \frac{1}{\cosh \zeta t_1} \right) > 0$$

because $\cosh(v) \geq 1$ for $-\infty < v < \infty$. Thus, keeping everything else constant, $\lambda_m(0) \leq 0$ is more likely to be satisfied if a_1 is low.

Again, using (49),

$$\frac{\partial \lambda_m(0)}{\partial s_{m0}} = -\frac{1}{\theta} \frac{\sinh \zeta t_1}{\cosh \zeta t_1} < 0$$

because $\sinh(v) \geq 0$ for $0 \leq v < \infty$. Thus, keeping everything else constant, $\lambda_m(0) \leq 0$ is more likely to be satisfied if s_{m0} is high. This can be verified from (52).

(2) Using (56),

$$\frac{\partial \lambda_m(0)}{\partial s_{m0}} = -2v\, s_{m0} \frac{\sinh \sigma t_1}{\cosh \sigma t_1} < 0.$$

Thus, $\lambda_m(0) \geq \alpha_1/b$ is more likely to be satisfied if s_{m0} is low.

Again, using (56),

$$\frac{\partial \lambda_m(0)}{\partial \alpha_2} = \frac{\sin \alpha_3 t_1}{b} \frac{1}{\cosh \sigma t_1} > 0.$$

Thus, $\lambda_m(0) \geq \alpha_1/b$ is more likely to be satisfied if α_2 is high.

Lastly, using (56),

$$\frac{\partial(\lambda_m(0) - \alpha_1/b)}{\partial \alpha_1} = \frac{1}{b}\left(\frac{1}{\cosh \sigma t_1} - 1\right) < 0.$$

Thus, $\lambda_m(0) \geq \alpha_1/b$ is more likely to be satisfied if α_1 is low. This can also be verified from (59).

(3) If the production rate and amount sold are nonnegative throughout the season, the production rate is given by (54). In the case where the initial production rate is zero, the production rate, once it becomes positive, is given by (51). In the case where the initial amount sold is zero, the production rate, once it becomes positive, is given by (56) and (58). As can be readily seen in each of the cases, the production rate, which equals $\lambda_m(t)$, can be written as

$$\lambda_m(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t) + s_{m0} f_3(t) + v_m f_4(t). \tag{60}$$

Here, $f_1(t)$, $f_2(t)$, $f_3(t)$, and $f_4(t)$ are functions of time with some or all of parameters α_3, b, h_m and c_m.

(a) In the first two cases, $\partial^2 |f_1(t)| / \partial t^2 < 0$. In the third case, $\partial^2 |f_1(t)| / \partial t^2 > 0$, but $\partial |f_1(t)| / \partial t > 0$ for $[0, t_1]$ and $\partial |f_1(t)| / \partial t < 0$ for $[t_1, T]$. Thus, the weight of $|f_1(t)|$ increases and then decreases over time, and becomes zero at the end of the season.

$\partial |f_3(t)| / \partial t < 0$ in all three cases. Thus, $|f_3(t)|$ decreases over time, and becomes zero at the end of the season.

(b) $\partial |f_4(t)| / \partial t > 0$ in all three cases. Thus, $|f_4(t)|$ increases with time, and becomes one at the end of the season. □

Proof of proposition 2

(1) If the initial production rate is zero, the inventory policy is given by (49). As can be readily seen from (44), the inventory decreases with time.

From (43) and (49),

$$\frac{\partial \dot{s}_m(t)}{\partial \alpha_1} = -\frac{\cosh \zeta t}{\cosh \zeta t_1} < 0.$$

Thus, one can see that the higher the value of α_1, the faster will be the decrease in the inventory.

(2) If the initial amount sold is zero, the inventory policy is given by (55). From (45), one can see that the inventory increases with time.

From (45) and (56),

$$\frac{\partial \dot{s}_m(t)}{\partial \alpha_1} = \frac{1}{b2c_m} \frac{\cosh \zeta t}{\cosh \zeta t_1} > 0,$$

$$\frac{\partial \dot{s}_m(t)}{\partial \alpha_2} = \frac{\sin \alpha_3 t_1}{b2c_m} \frac{\cosh \zeta t}{\cosh \zeta t_1} > 0.$$

Thus, the higher the value of α_1 and α_2, the faster will be the increase in inventory.

(3) If the initial production rate is zero, the inventory policy is given by (48) and (50). If the production rate and the amount sold are nonnegative throughout the season, the inventory policy is given by (53). If the initial amount sold is zero, the inventory policy is given by (55) and (57). As can be readily seen in each of these cases, the inventory policy can be written as

$$s_m(t) = \alpha_1 g_1(t) + \alpha_2 g_2(t) + s_{m0} g_3(t) + v_m g_4(t), \tag{61}$$

where $g_1(t)$, $g_2(t)$, $g_3(t)$, and $g_4(t)$ are functions of time with some or all of parameters α_3, b, h_m, and c_m.

(a) In all three cases, $\partial |g_3(t)| / \partial t < 0$. Thus, $|g_3(t)|$ decreases over the entire season, and becomes zero at the end of the season.

(b) In all three cases, $\partial |g_1(t)| / \partial t$, $\partial |g_4(t)| / \partial t > 0$. Thus, $|g_1(t)|$ and $|g_4(t)|$ increases over the entire season.

(c) One cannot say anything in general about $g_2(t)$.

Proof of proposition 3

(1) The state equation is $\dot{s}_m = x_m^*(t) - a(t) + bp_r^*(t)$. Also, from (8) and (9), we have $p_r^*(t) = (1/4b)(3a + b\lambda_m^* - \eta_r)$ and $p_m^*(t) = (1/2b)(a + b\lambda_m^* - \eta_r)$.

Since $\eta_r = \eta_m = 0$ if the production rate and amount sold are positive, $x_m(t) = (1/2c_m)\lambda_m(t)$, and

$$\frac{\mathrm{d}p_r^*(t)}{\mathrm{d}x_m^*(t)} = \frac{1}{4b}\left(3\frac{\mathrm{d}a}{\mathrm{d}x_m} + b2c_m\right) > 0,$$

$$\frac{\mathrm{d}p_m^*(t)}{\mathrm{d}x_m^*(t)} = \frac{1}{2b}\left(\frac{\mathrm{d}a}{\mathrm{d}x_m} + b2c_m\right) > 0$$

if $\mathrm{d}a/\mathrm{d}x_m > 0$.

(2) This is obvious from (8) and (9). □

PARETO SOLUTION

As can be seen from the necessary conditions, the system of equations generated for the Pareto solution is very similar to the one generated for the Stackelberg solution. It can be easily shown that lemma 1 is valid for the Pareto solution also. The only difference is in ε, ζ, θ, ι, κ, ξ, ρ, σ and v. Hence, the various policies are the same with

$$\varepsilon = (bh_m + \alpha_3^2)/2, \quad \zeta = \sqrt{bh_m}, \quad \theta = \sqrt{\frac{b}{4h_m}}, \quad \iota = \sqrt{\frac{1 + c_m b}{4h_m c_m}}, \quad \kappa = \sqrt{\frac{h_m}{c_m} + bh_m},$$

$$\xi = \frac{\alpha_1 c_m}{1 + c_m b}, \quad \rho = \left(\frac{h_m}{c_m} + bh_m + \alpha_3^2\right)/2, \quad \sigma = \sqrt{\frac{h_m}{c_m}}, \quad v = \sqrt{h_m c_m}.$$

Appendix 2: Model II

STACKELBERG SOLUTION

Once again, it can be easily shown that lemma 1 is valid for the Stackelberg solution of model II also. The only difference is in ε, ζ, θ, κ, ξ, ρ, σ, and v. Hence, the various policies are the same with

$$\varepsilon = \frac{(\alpha_3^2(4bc_a - 1) + b^2 c_a 2h_m)}{4bc_a}, \quad \zeta = \sqrt{\frac{b^2 c_a 2h_m}{(4bc_a - 1)}}, \quad \theta = \sqrt{\frac{b^2 c_a}{2h_m(4bc_a - 1)}},$$

$$\iota = \sqrt{\frac{(4bc_a - 1) + 2b^2 c_a c_m}{4h_m c_m (4bc_a - 1)}}, \quad \kappa = \sqrt{\frac{h_m}{c_m} + \frac{b^2 c_a 2h_m}{(4bc_a - 1)}}, \quad \xi = \frac{2c_m bc_a \alpha_1}{4bc_a - 1 + 2b^2 c_a c_m},$$

$$\rho = \frac{\left((\alpha_3^2 2c_m + 2h_m)(4bc_a - 1) + 4b^2 c_a h_m c_m\right)}{8bc_a c_m}, \quad \sigma = \sqrt{\frac{h_m}{c_m}}, \quad v = \sqrt{h_m c_m}.$$

Proof of proposition 4

The state equation is $\dot{s}_m = x_m^*(t) - a(t) - v^*(t) + bp_r^*(t)$. Also from (28), (32) and (33), we have

$$v^*(t) = \frac{a - b\lambda_m^* + \eta_r}{2(4bc_a - 1)},$$

$$p_r^*(t) = \frac{a(6bc_a - 1) + (2bc_a - 1)\lambda_m^* - (2bc_a - 1)\eta_r}{2b(4bc_a - 1)},$$

and

$$p_m^*(t) = \frac{a + \lambda_m^*(t) + \eta_r}{2b}.$$

Since η_r, $\eta_m = 0$ if the production rate and amount sold are positive,

$$x_m^*(t) = \frac{1}{2c_m}\lambda_m^*(t) \quad \text{and} \quad \frac{\mathrm{d}p_m^*(t)}{\mathrm{d}x_m^*(t)} > 0, \quad \frac{\mathrm{d}p_r^*(t)}{\mathrm{d}x_m^*(t)} > 0$$

if

$$2bc_a - 1 > 0, \quad \frac{\mathrm{d}a}{\mathrm{d}x_m} > 0 \quad \text{and} \quad \frac{\mathrm{d}v(t)}{\mathrm{d}x_m(t)} < 0.$$

(1) Thus, an increase in production rate of manufacturer's price will lead to an increase in inventory. If the production rate increases, effort rate will decrease and once again lead to an increase in inventory.

(2) If $2bc_a - 1 > 0$, an increase in production or retailer's price will lead to an increase in inventory.

(3) If $(2bc_a - 1) < 0$, $\mathrm{d}p_r^*(t)/\mathrm{d}x_m^*(t) < 0$ is possible. Thus, an increase in production rate will lead to a decrease in price and thus may lead to a decrease in inventory.

PARETO SOLUTION

Again, it can be easily shown that lemma 1 is valid for the Pareto solution also. The only difference is in ε, ζ, θ, ι, κ, ξ, ρ, σ, and v. Hence, the various policies are the same with

$$\varepsilon = \frac{(\alpha_3^2(4bc_a - 1) + b^2c_a4h_m)}{8bc_a}, \quad \zeta = \sqrt{\frac{b^2c_a4h_m}{(4bc_a - 1)}}, \quad \theta = \sqrt{\frac{b^2c_a}{h_m(4bc_a - 1)}},$$

$$\iota = \sqrt{\frac{(4bc_a - 1) + 4b^2c_ac_m}{4h_mc_m(4bc_a - 1)}}, \quad \kappa = \sqrt{\frac{h_m}{c_m} + \frac{b^2c_a4h_m}{(4bc_a - 1)}}, \quad \xi = \frac{4c_mbc_a\alpha_1}{4bc_a - 1 + 4b^2c_ac_m},$$

$$\rho = \frac{\left((\alpha_3^2 2c_m + 2h_m)(4bc_a - 1) + 8b^2c_ah_mc_m\right)}{16bc_ac_m}, \quad \sigma = \sqrt{\frac{h_m}{c_m}}, \quad v = \sqrt{h_mc_m}.$$

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