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THE RE-EXAMINATION OF THE WEAKLY NONLINEAR THEORY OF HYDRODYNAMIC STABILITY*

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Abstract

The weakly nonlinear theory has been widely applied in the problem of hydrodynamic stability and also in other fields However although its application has been successful for some problems, yet, for other problems, the results obtained are not satisfactory, especially for problems like transition or the evolution of the vortex in the free shear flow, for which the goal of the theoretical investigation is not seeking for a steady state, but predicting an evolutional process In this paper we shall examine the reason for the unsuccessfulness and suggest ways for its amendment

Key words hydrodynamic stability, weakly nonlinear theory, resonance

I. The Landau-Stuart Amplitude Equation

For the convenience of discussion, let us take the 2-D,parallel flow as an example The Navier-Stokes equation and the continuity equation read

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\frac{1}{R}\nabla^2\mathbf{u},\quad \nabla\cdot\mathbf{u}=0 \tag{1.1}$$

where $\mathbf{u}=\{u,v\}^T$ is the velocity, u and v are its components in the direction of x and y , the Cartesian coordinates, t is the time, p is the pressure, and ∇ is the gradient operator All variables have been suitably non-dimensionized, and R is the Reynolds number

Assuming that the solution can be expanded in a power series of a certain small parameter, say, the small amplitude a of the fundamental, such that

$$\{\mathbf{u},p\}^T=\{\mathbf{u}_0,p_0\}^T+a\{\mathbf{u}_1,p_1\}^T+a^2\{\mathbf{u}_2,p_2\}^T+\cdots \tag{1.2}$$

where $\mathbf{u}_0=\{u_0(y),0\}^T$ and p_0 is the basic laminar flow, $\mathbf{u}_1$ is the solution of the linearized problem, suitably normalized

In most cases, we assume

$$\{\mathbf{u}_1,p_1\}^T=\{\hat{\mathbf{u}}_1(y),\hat{p}_1(y)\}^T\exp[i(\alpha x-\omega t)]+\text{C.C.} \tag{1.3}$$

and in what follows, we write $\theta=\alpha x-\omega t$, and depending on whether we use temporal mode or spatial mode, either α or ω is assumed to be real and given, and then the other one will be obtained

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together with $\mathbf{u}_1$ as the solution of an eigenvalue problem In this paper, we use the temporal mode as the prototype problem

According to the weakly nonlinear theory, a and θ satisfy the Landau-Stuart equation

$$\frac{da}{dt}=\omega_i a+A_3a^3+A_5a^5+\cdots \tag{1 4}$$

$$\frac{d\theta}{dt}=-\omega_r+C_2a^2+C_4a^4+ \tag{1 5}$$

where ω_r and ω_i are the real and imaginary parts of ω respectively, and A_i and C_i are constants to be determined during the course of solution by a certain condition, say, the solvability condition

By closely examining (1 2), (1 4) and (1 5) it is not difficult to see that once α is given, the solution depends only on one essential parameter, i e the amplitude of the fundamental On the other hand, assuming the flow field is periodic in x with a period $2\pi/\alpha$, , any velocity field satisfying the continuity equation and boundary conditions, and having the same period in x can be an initial condition for an actual flow realization Apparently, the flow field is in general not the same as those given by the solution of the weakly nonlinear theory

Even if we start with an initial condition given by

$$\{\mathbf{u},p\}^T=\{\mathbf{u}_0,p_0\}^T+a\{\mathbf{u}_1,p_1\}^T \tag{1.6}$$

which apparently is permissible because it satisfies the continuity equation and the boundary condition, then for (1 2), (1 4) and (1 5) to be valid, terms in (1 2), starting with order a^2, must be generated spontaneously which in a real situation, say, in an experimental realization, is impossible If we examine the data given by Kachanov[1], the amplitude of the second harmonics at different locations are always significantly smaller than those calculated by us by applying the weakly nonlinear, perturbation scheme, with a taken from the experiment For example, the calculated values were two or three times larger than those observed in the experiment, and the closer the location was to the vibrating ribbon, which generated the instability wave, the larger was the descrepency If we consider the problem from the physical point of view, then for the experiments of Nishioka et al [2] or Kachanov[1], at the initial stage of the development, there was no harmonics So the harmonics must be generated through nonlinear process, which can not be spontaneous Therefore, the second harmonics is not necessarily proportional to a^2, the third harmonics is not necessarily proportional to a^3, etc , as the weakly nonlinear theory predicted The way of the amendment of this discrepency is that we should assign an independent amplitude for each harmonics, and to find the evolution equation for each of them, so that the gradual evolutional process of them can be taken into consideration

If we follow Stuart[3] or Zhou[5], then in the frame work of the weakly nonlinear theory, we can write

$$\begin{aligned}\{\mathbf{u}_2,\ p_2\}^T&=\{\hat{\mathbf{u}}_{20}(y),\hat{p}_{20}(y)\}^T\exp[\ 2\theta_i]+\{\hat{\mathbf{u}}_{22}(y),\hat{p}_{22}(y)\}^T\exp[2i\theta]\\&\equiv\{\mathbf{u}_{20},p_{20}\}^T+\{\mathbf{u}_{22},p_{22}\}\end{aligned} \tag{1 7}$$

where the appearance of θ_i in the exponent is due to the fact that the eigenvalue problem for the linearized problem may correspond to a non-neutral case, otherwise $\theta_i=0$ Since the amplitude of $\mathbf{u}_{20}$, etc is not necessarily proportional to a^2, in place of the third term on the right-hand side of eq (1 2), we should write

$$b_{20}\{\hat{u}_{20},\hat{p}_{20}\}^T\exp[-2\theta_i]+b_{22}\{\hat{u}_{22},\hat{p}_{22}\}^T\exp[2i\theta] \tag{1.8}$$

According to the perturbation method of the weakly nonlinear theory, the equations satisfied by $\mathbf{u}_{20}$, etc are

$$\left.\begin{aligned}&\frac{\partial \mathbf{u}_{20}}{\partial t}+u_0\frac{\partial \mathbf{u}_{20}}{\partial x}+v_{20}\frac{d\mathbf{u}_0}{dy}+\nabla p_{20}-\frac{1}{R}\nabla^2\mathbf{u}_{20}=-2\mathrm{Re}[(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1^*]\\&\nabla\cdot\mathbf{u}_{20}=0\end{aligned}\right\} \tag{1.9}$$

$$\left.\begin{aligned}&\frac{\partial \mathbf{u}_{22}}{\partial t}+u_0\frac{\partial \mathbf{u}_{22}}{\partial x}+v_{22}\frac{d\mathbf{u}_0}{dy}+\nabla p_{22}-\frac{1}{R}\nabla^2\mathbf{u}_{22}=-2\mathrm{Re}[(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1]\\&\nabla\cdot\mathbf{u}_{22}=0\end{aligned}\right\} \tag{1.10}$$

where the * stands for the complex conjugate, Re stands for the real part of a complex variable

Following Zhou[5], the actual equations for $\hat{\mathbf{u}}_{20}$, etc are

$$\left.\begin{aligned}&2\omega_i\hat{u}_{20}-\frac{1}{R}\frac{d^2\hat{u}_{20}}{dy^2}=-2\mathrm{Re}\left(\hat{v}_1\frac{d\hat{v}_1^*}{dy}\right)\\&\hat{v}_{20}=0\end{aligned}\right\} \tag{1.11}$$

$$\left.\begin{aligned}&-2i\omega\hat{\mathbf{u}}_{22}+2i\alpha u_0\hat{\mathbf{u}}_{22}+v_{22}\frac{d\mathbf{u}_0}{dy}+\left\{2i\alpha,\frac{d}{dy}\right\}^T\hat{p}_{22}-\frac{1}{R}\left(\frac{d^2}{dy}-4\alpha^2\right)\hat{\mathbf{u}}_{22}\\&\quad=-2\mathrm{Re}\left[\left(i\alpha\hat{u}_1+\hat{v}_1\frac{d}{dy}\right)\hat{\mathbf{u}}_1\right]\\&\left\{2i\alpha,\frac{d}{dy}\right\}\mathbf{u}_{22}=0\end{aligned}\right\} \tag{1 12}$$

and if we let the amplitude factor for $\hat{\mathbf{u}}_{20}$, etc be a^2, then the result of the weakly nonlinear theory is restored

As stated above, independent amplitudes should be allowed for $\hat{\mathbf{u}}_{20}$, etc But we assume that the shape of their velocity distribution can be borrowed from the perturbation method In this case, the following expression can be the proper expression for the terms replacing the third term on the right-hand side of eq (1 2)

$$b_{20}(t)\{\hat{u}_{20}(y),\hat{p}_{20}(y)\}^T+b_{22}(t)\{\hat{u}_{22}(y),\hat{p}_{22}(y)\}^T \tag{1 13}$$

where the shape is given but the amplitudes b_{20} and b_{22} are allowed to vary in time with a slow time scale as for a

If now we put the first term of (1 13) into eq (1 9), scalarly multiply the first equation of (1 9) by $b_{20}\hat{\mathbf{u}}_{20}(y)$ and integrate, following the procedure for the derivation of the energy equation in the theory of hydrodynamic stability, we shall arrive at an evolution equation for b_{20} as

$$\frac{db_{20}}{dt}=B_{01}b_{20}+B_{02}a^2 \tag{1.14}$$

and in the same way, an equation for b_{22} can be derived as

$$\frac{db_{22}}{dt}=B_{21}b_{22}+B_{22}a^2 \tag{1.15}$$

and in account of eqs (1 11) and (1 12), it is readily shown that

$$B_{01}+B_{02}=B_{21}+B_{22}=2\omega_i \tag{1.16}$$

If we continue the perturbation calculation to the next step, then the Landau-Stuart amplitude equation would appear as

$$\frac{da}{dt}=\omega_i a+(A_{31}b_{20}+A_{32}b_{22})a \tag{1.17}$$

and a similar equation can be derived for θ Therefore, up to this order, we should have three amplitude equations (1 14), (1 15) and (1 17) in place of a single equation (1 4), and the evolution of the second harmonics and the mean flow distortion has been taken into consideration

Apparently, the above procedure can be continued and more and more amplitude equations will appear for more and more harmonics, etc

Eqs (1 14), (1 15) and (1 17) can also be used to determine the equilibrium state by putting the right-hand side of them equal to zero Here we notice that if in eq (1 17) we put

$$b_{20}=b_{22}=a^2$$

then the original Landau-Stuart equation is recovered However, in view of eq (1 16), when $b_{20}=b_{22}=a^2$, eqs (1 14) and (1 15) do not yield $db_{20}/dt\neq 0$ and $db_{22}/dt\neq 0$ Therefore, the equilibrium state determined by eqs. (1 14), (1 15) and (1 17) is different from those determined by the original Landau-Stuart equation, except when $\omega_i=0$ Since the original Landau-Stuart equation is a special case of the present formulation, and eqs (1 14) and (1 15) were obtained from the energy consideration with a shape assumption, which is apparently correct for the weakly nonlinear theory itself, so one would reach a conclusion that for the weakly nonlinear theory, only a formulation starting with a linear problem corresponding to a neutral case, either really being so as Stuart[3] and Watson[4], or artificially made to be so as Zhou[5], can yield a result consistent both from the analytical point of view and the energy trasfer point of view Of course, if we can really carry out the perturbation calculation to the infinite order and the resulting series converges, perhaps a consistent solution can be claimed to be found even for $\omega_i\neq 0$, but so far, no one can prove it does work in this way

Once the equilibrium state has been determined, we can readily determine its stability by the method of small perturbation from the equilibrium state We have calculated a number of cases for plane Poiseuille flow with parameters close to the neutral curve so that the weakly nonlinear calculations can start from a linearly neutral case So far, the stability analysis yield the same conclusions as those obtained from the original Landau-Stuart formulation, and the cases studied include both the supercritical and subcritical equilibrium states

We have also tried to calculate the evolution of the second harmonics of Kachanov's experiment, by using eq (1 15) with a taken from the measured value of the experiment.Kachanov's vibrating ribbon is at the location $x=250$mm, so we assume the amplitude of the second harmonics is nil there We calculated the coefficients of eq (1 15), and then integrated numerically to obtain b_{22}, the amplitude of the second harmonics We find the result is quite good up to $x=350$mm, compared with the experimental observations Notice that since it is an evolutional process, the shape of the second harmonics can not be simply the shape calculated at the local station either, so a certain kind of evolutional accumulation effect has been taken into consideration, otherwise the result will not be so good

We have not carried out the calculation further downstream, because the measured shape of the second harmonics has already been distorted there

II. The Resonant Idea in the Weakly Nonlinear Theory

Resonant idea has played an important role in the theory of hydrodynamic stability, probably in other fields as well In the context of the above section, two questions may be raised

Let us take the idea of resonant triad as an example[6-8] The following three waves are said to be in resonance,

$$\left.\begin{array}{l}\phi_1(y)\exp[i(\alpha_1 x-\omega_1 t)],\ \phi_2(y)\exp[i(\alpha_2 x+\beta z-\omega_2 t)]\\ \phi_3(y)\exp[i(\alpha_2 x-\beta y-\omega_2 t)]\end{array}\right\} \qquad (2\ 1)$$

if their wave number and frequency satisfy the so-called resonant condition,

$$\alpha_1=2\alpha_2,\quad \omega_{1r}=2\omega_{2r},\quad \omega_{1i}=\omega_{2i}=0 \qquad (2.2)$$

Some authors did not ask the third equation to hold to form a resonant triad, but the first two were thought to be necessary by every one The reason is, among other things, that only under this condition, all the three waves can travel with the same phase speed, so their relative phase remains unchanged, and thus guarantees the mutual excitation

However, such a stringent condition is only needed if we want to follow the evolution of the triad indefinitely, a situation only possible for strictly parallel flow, and as plane Poiseuille flow, besides, temporal mode must be used While in a real situation, the spatial mode is more suitable, and the time for evolution only lasts for a short period, say, several wavelengths or cycles, as in the problems of transition or the evolution of the vortex in a developing free shear flow In such cases, a slight difference of the wave speed will not cause much difference For example, a 3% difference of the wave speed will cause to the relative phase a change of 0 3 π within an interval of five wavelengths, which will not significantly change the energy transfer rate between different modes, while within that period, transition may have already taken place

On the other hand, the resonant idea gives people the impression that once the resonant condition is satisfied, the growth rate of the resonant waves would be enormously larger than those waves with other parameters However, calculations show that the energy transfer rate between different modes and the mean flow does not have a sharp peak around the resonant parameter, rather, it is quite flat Therefore, the first modification which should be applied to the resonant idea is that it does not provide a critereon which can single out one set of parameters as the mechanism of wave number selection Rather, we must test for a certain band, though narrow, of waves forming nearly resonant triads, and compare their growth rate

The second modification to apply the resonant idea is relevant to the calculation of its evolution process In [6,8], the weakly nonlinear theory was proposed However, as stated in the preceding section, for an evolutional problem, the evolution equation, i e the amplitude equation, should be reformulated, then perhaps we can provide a mechanism of wave number selection which might be different from the previous result

A correct formulation could be as follows First, we may start with three waves as in (2 1), but we may allow a certain degree of difference of the phase speed Then through nonlinear interaction of different modes, new waves may be generated having the following form,

$$\phi_4(y)\exp\{i[(\alpha_1-\alpha_2)x-(\omega_1-\omega_2^*)t+\beta z]\},\ \phi_5(y)\exp\{i[(\alpha_1-\alpha_2)x -(\omega_1-\omega_2^*)t-\beta z]\},\ \phi_6(y)\exp[2i(\alpha_2 x-\omega_2 t)] \qquad (2.3)$$

Then the newly excited waves may interact with the original waves to reinforce or weaken the energy transfer to or from other existing waves or the mean flow If now we apply the idea proposed in section I, we may get evolution equations for each mode Then through their integration, we can find the waves having the largest growth rate, which should provide a mechaism of wave number selection

III On the Mean Flow Distortion in Developing Free Shear Flows

In applying the weakly nonlinear theory to problems such as developing free shear flow, another difficulty may arise That is, if quasi-parallel assumption is made, then we will arrive at the following equations for the second order mean flow distortion $\hat{u}_{20}$,

$$\frac{1}{R}\frac{d^2\hat{u}_{20}}{dy^2}=\frac{d}{dy}\langle u_1v_1\rangle \tag{3.1}$$

where the right-hand side represents the force resulting from the Reynolds stress generated through nonlinear action by the fundamental If we integrate this equation, we will get a mean flow distortion which decays algebraically when $y\rightarrow\infty$, obviously incorrect compared with the experimental observations This difficulty arises from the fact that eq (3 1) is an equation only valid for steady state, while for a real free shear flow it is evolutional, the mean flow distortion at a certain location is actually the accumulated effect of the Reynolds stress generated by the disturbance upstream from that location within a finite time interval Thus, this is an evolutional process rather than a steady state situation The way to overcome this difficulty is to derive an equation for the unsteady, evolutional process In a frame moving with the convective speed of the disturbance, we would see an unsteady shear flow with its thickness growing But if we assume the growth rate is small, then after certain simplifications we will arrive at the following equation for the mean flow distortion,

$$\frac{\partial u_{20}(y,t)}{\partial t}=\frac{1}{R}\frac{\partial^2 u_{20}}{\partial y^2}-\frac{\partial}{\partial y}\langle u_1v_1\rangle \tag{3 2}$$

Since this is an equation of diffusion, its solution can be written as

$$u_{20}(y,t)=\frac{1}{2}\sqrt{\frac{R}{\pi}}\int_0^t\int_{-\infty}^{+\infty}\frac{1}{\sqrt{t-\tau}}\frac{\partial}{\partial y}\langle -u_1v_1\rangle \cdot\exp\left[-\frac{R(y-\eta)}{4(t-\tau)}\right]d\eta d\tau \tag{3 3}$$

but in actual integration, say, for a problem of mixing layer, we should start from a certain location where the mean flow distortion is supposed to be nil, and then calculate the fundamental disturbance, i e the T-S waves at each location, with its amplitude either taken from the experiment, or calculated by a certain kind of evolution equation Find the convective speed of the disturbance, convert the time variable to space variable according to the speed, and then integrate eq (3 3) step by step with a step length which can guaranttee the required accuracy In this way, we can get a mean flow distortion which decays exponentially when $y\rightarrow\infty$

One objection which people may raise to this scheme is that the effect of the growth of the thickness of the layer has not been taken into consideration in forming eq (3 2) The answer is that when we integrate the equation step by step, the thickness of the layer is assumed to be known, either from the experiment, or from some other equations which may be solved in advance or simultaneously So in fact, the effect of the growth of the thickness has already been included in the calculation

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