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A Quiver Quantum Group

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Abstract: We construct quantum groups at a root of unity and we describe their monoidal module category using techniques from the representation theory of finite dimensional associative algebras.

1. Introduction

In the representation theory of finite dimensional algebras, important techniques as quivers, almost-split sequences and irreducible morphisms have been developed these last two decades. Our purpose is to use these methods in order to study finite dimensional Hopf algebras which are neither commutative nor cocommutative together with their monoidal category of modules. Hopf algebras of this sort produce non-trivial braided categories through the representations of the quasi-triangular Drinfeld's double ([8]) or through the "center construction" ([10, 13, 18, 15]); according to Drinfeld [8] or Manin [19] these Hopf algebras are considered as quantum groups or rather better as functions on some hypothetic quantum group. Their representation theory through the double constructions has applications in various parts of physics and mathematics: one can use them to obtain solutions of the quantum Yang–Baxter equation or topological invariants (see for instance [24, 25, 17]).

Quivers with relations allow to present finite dimensional associative algebras in a useful way, see [11]. If the algebra is of finite representation type, its module category can be presented using the Auslander–Reiten quiver of the category of indecomposable modules and relations given by the almost-split sequences.

In this note we construct a family of infinite dimensional Hopf algebras over an arbitrary field k. To each integer $n \geqq 2$ and each n^{th} root of unity q in k we attach a Hopf algebra $H_n(q)$ which is the path algebra of the cyclic quiver Z_n. These Hopf

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algebras are neither commutative nor cocommutative, except for $q = 1$ when they are cocommutative.

Let d be the precise order of q; we will show that $H_n(q)$ has a finite dimensional quotient $\Lambda_n(q)$ of dimension nd with antipode of order $2d$. Those quantum groups are not quasitriangular unless n is even and $q = -1$. If the characteristic of k does not divide n and if in addition q is primitive then the Hopf algebras $\Lambda_n(q)$ are isomorphic to the Hopf algebras obtained by Taft [28]. If moreover n is odd, the Drinfeld double of $\Lambda_n(q)$ has been considered by several authors; it has a quotient isomorphic to a finite dimensional quotient of $U_q(sl_2)$ constructed by Reshetikhin and Turaev [25]. The presentation we obtain for these algebras allows to study their representation theory; the finitely generated $\Lambda_n(q)$ modules form a Krull–Schmidt category with only a finite number of isomorphism classes of indecomposable modules whose tensor product can be described. These algebras are not quasicommutative, now it turns out that the tensor product of modules is commutative up to canonical isomorphisms. The non-existence of an R-matrix insures that the category of modules is not braided, however left braidings can be considered an attempt to describe the module category of the Drinfeld double, see [10, 13, 18, 15]. The behaviour of almost-split sequences under tensor product is indicative in this respect: commutativity is only valid up to a non-zero scalar, and this is related to the R-matrix of Drinfeld's double. This aspect will be developed in a forthcoming publication.

Actually the universal cover of the algebra we consider admits a Hopf algebra structure over $k(q)$; most of the properties described for the finite dimensional quotients can already be stated in the setting of this algebra, the associative side of which is isomorphic to the algebra of infinite lower triangular matrices.

C. Kassel (see [14]) has made very useful comments at an early stage of this work, my special thanks to him.

2. Path Hopf Algebras

Let Z_n be the oriented graph with n vertices $\{s_0, s_1, \ldots, s_{n-1}\}$ and with n oriented edges $\{a_0, a_1, \ldots, a_{n-1}\}$, where a_i is an arrow from the vertex s_i to the vertex s_{i+1}. The indices belong to the cyclic group $\mathbf{Z}/n\mathbf{Z}$.

Any finite oriented graph enables to construct a tensor algebra called the path algebra, and in that case an oriented graph is called a quiver, see [11]. For Z_n the path algebra kZ_n over a commutative ring k has the following description: let γ_i^m be the path of length m starting at the vertex s_i. This simply means that γ_i^m is the sequence of arrows $a_{i+m-1} \ldots a_{i+1}a_i$. We agree that $\gamma_i^0 = s_i$ and notice that $\gamma_i^1 = a_i$. Let kZ_n be the free k-module with basis the set $\{\gamma_i^m\}_{i \in \mathbf{Z}/n\mathbf{Z}, m \in \mathbf{N}}$. The product is given by composition of paths whenever it can be performed and zero otherwise. Observe that vertices are orthogonal idempotents and that $a_i s_j = \delta_{i,j} a_i$ and $s_j a_i = \delta_{j,i+1} a_i$.

Consider the commutative k-algebra $A = \times_{i \in \mathbf{Z}/n\mathbf{Z}} ks_i$ and let M be the k-free A-bimodule with k-basis the set of arrows and bimodule structure provided by the formulas above. Clearly kZ_n is the tensor algebra $T_A(M)$ over A of the bimodule M, hence any morphism of algebras Δ: $kZ_n \to X$ is determined by an algebra map Δ_0: $A \to X$ and an A-bimodule map Δ_1: $M \to X$, where the A-bimodule structure on X is the one conferred by Δ_0.

Theorem 2.1. *For each n^{th} root of unity q there is a Hopf algebra structure $kZ_n(q)$ on kZ_n.*

For the proof of this theorem we shall need the following easy lemma which is well known in its dual version, see [1, pages 159–160].

Lemma 2.2. *Let A be a semisimple split commutative finite dimensional algebra over a field k. Let $A = \times_{e \in E} ke$ be its unique decomposition as a product of simple subalgebras, where E is the complete set of primitive orthogonal idempotents. There is a one-to-one correspondence between Hopf algebra structures on A and group structures on E, given by $\Delta(e) = \sum_{fg=e} f \otimes g$ in case E has a group structure.*

Remark 2.3. The A-module category is semisimple; the simple modules are one dimensional vector spaces $\{U_e\}_{e \in E}$ in bijection with the set E. We have $U_e \otimes U_f = U_{ef}$, where ef is the group product of e and f.

Proof of Theorem 2.1. We first give a Hopf algebra structure to the k-algebra A generated by the vertices. The cyclic group $\mathbf{Z}/n\mathbf{Z}$ provides such a structure, as in Lemma 2.2: $\Delta_0(s_i) = \sum_{j+k=i} s_j \otimes s_k$, where $j + k$ is sum in $\mathbf{Z}/n\mathbf{Z}$. The antipode is given by $S(s_i) = s_{-i}$ and the counit is $\varepsilon(s_i) = 0$ if $i \neq 0$ and $\varepsilon(s_0) = 1$. We now provide an A-bimodule map Δ_1: $M \to kZ_n \otimes kZ_n$; it will be completely determined by the values of $\Delta_1(a_i)$ which must belong to the free k-module $\Delta_0(s_{i+1})\,(kZ_n \otimes kZ_n)\,\Delta_0(s_i)$ with basis $\{\gamma_j^u \otimes \gamma_k^v\}_{j+k=i,\,u+v=1}$. We restrict ourselves to the case where

$$\Delta_1(a_i) = \sum_{j+k=i} \alpha_{j,k}\, s_j \otimes a_k + \sum_{j+k=i} \beta_{j,k}\, a_j \otimes s_k\,,$$

ignoring possible values for u and v different from 0 and 1; other Hopf algebra structures may exist when considering general values.

In order to insure coassociativity for the algebra morphism Δ we turn to the dual and look for associativity (we keep the same symbols for the dual basis). The cyclic group $\{s_0, s_1, \ldots, s_{n-1}\}$ transforms elements of the basis $\{a_0, a_1, \ldots, a_{n-1}\}$ through the formulas

$$s_j a_k = \alpha_{j,k} a_{j+k} \quad \text{and} \quad a_j s_k = \beta_{j,k} a_{j+k}\,.$$

The associativity is guaranteed by requiring that this gives a two-sided module structure over the cyclic group. Therefore the left and right transformations obtained with the generator s_1 determines the actions, provided that their n^{th} power acts trivially. This translates into $\prod \alpha_{1,k} = 1$, $\prod \beta_{1,k} = 1$, $\alpha_{j,k} = (\alpha_{1,k})^j$ and $\beta_{j,k} = (\beta_{1,k})^j$.

The left action can be normalized by replacing a_i by $\prod \alpha_{1,k} a_k$ (we keep the same symbol for the new a_i). This change does not affect the multiplication formulas since each arrow has been replaced by a scalar multiple of itself. The left action is now the regular one, the bimodule associativity condition immediately translates into $\beta_{1,0} = \beta_{1,1} = \ldots = \beta_{1,n-1}$ and taking into account that $\prod \beta_{1,k} = 1$ we obtain that the common value of the beta's is a n^{th} root of unity q. Finally we have

$$\Delta_1(a_i) = \sum_{j+k=i} s_j \otimes a_k + \sum_{j+k=i} q^j a_j \otimes s_k\,.$$

Actually the coassociativity property can be checked directly on this formula. The counit is extended by zero to the whole algebra. The antipode S has to be an anti-isomorphism of algebras, so $S(a_i) \in S(s_i)\ kZ_n\, S(s_{i+1}) = s_{-i} kZ_n s_{-i-1}$. Let $S(a_i) = x_i a_{i-1}$ with x_i in k. The condition $(S \otimes 1)\,(a_i) = 0$ translates into $x_i = -q^{i+1}$ and this already gives the antipode for kZ_n. Notice that $S^2(a_i) = qa_i$, hence the antipode is of order $2d$, where d is the precise order of q.

Remark 2.4. The above construction suggests that a tensor algebra $T_A(M)$ can be equipped with a Hopf algebra structure, provided that A is a Hopf algebra and M is an A-bimodule with some extra-structure. Indeed, if M is an A-bicomodule whose structure maps are A-bimodule maps, then $T_A(M)$ is a Hopf algebra. This fact has been established by Nichols [20], and Woronowicz in [29] has highlighted the role of these bicovariant bimodules (or Hopf bimodules) in relation to the braid equation. I am indebted to M. Rosso for pointing this out; in a forthcoming publication we are going to develop this aspect in order to obtain quantum groups based on the quiver of a group.

Notice that the left normalisation we have made in the proof of the theorem corresponds to the fact that a Hopf bimodule is always free on the left with cotrivial left coaction, see [27, 29].

3. Quotients and Covers

In order to get two-sided ideals of $kZ_n(q)$ preserved by Δ or to replace the root of unity q by an indeterminate, we first compute Δ of an arbitrary path γ_i^m of the quiver Z_n. At the end of this section we consider the universal cover of the Hopf algebras that we have obtained.

Recall that if x is a variable and $m \geqq u$ are positive integers, the Gauss polynomial $\binom{m}{u}_x$ is $\frac{m!_x}{(m-u)!_x u!_x}$, where $m!_x = m_x(m-1)_x \dots 2_x 1_x$, for $m_x = 1 + x + x^2 + \dots + x^{m-1}$. We agree that $0!_x = 1$ and notice that $\binom{m}{u}_{x=1} = \binom{m}{u}$.

The Pascal relation holds:

$$\binom{m+1}{u}_x = \binom{m}{u}_x + x^{m-u+1}\binom{m}{u-1}_x .$$

This is used to prove that $\binom{m}{u}_x$ is actually a polynomial with coefficients in $\mathbf{Z}$.

Lemma 3.1. *Let $kZ_n(q)$ be the Hopf algebra obtained from the quiver Z_n and an n^{th} root of unity q. Then*

$$\Delta(\gamma_i^m) = \sum_{j+k=i\ u+v=m} \binom{m}{u}_q q^{uk}\, \gamma_j^u \otimes \gamma_k^v .$$

Proof. For $m = 1$ the formula agrees with the comultiplication of an arrow obtained in the preceding section. We have

$$\Delta(\gamma_i^{m+1}) = \Delta(a_{i+m}\gamma_i^m) = \Delta(a_{i+m})\, \Delta(\gamma_i^m) .$$

The proof follows by induction on m, using the Pascal relation.

Lemma 3.2. *Let q be a root of unity of order d. Then $\binom{d}{u}_q = 0$ for $1 \leqq u \leqq d-1$.*

Proof. The polynomial m_x is the product of the cyclotomic polynomials Φ_l, where l divides m and $l \neq 1$. Writing the numerator and the denominator of $\binom{m}{u}_x$ as a product of cyclotomic polynomials leads to the fact that Φ_m divides $\binom{m}{u}_x$ if $u \neq 0$ and $u \neq m$.

Proposition 3.3. *Let q be an n^{th} root of unity of order d and let $kZ_n(q)$ be the corresponding Hopf algebra. Let I_d be the two-sided ideal generated by the set of paths of length d. Then*

$$\Delta(I_d) \subset kZ_n(q) \otimes I_d + I_d \otimes kZ_n(q)$$

and the antipode preserves I_d. Consequently $kZ_n(q)/I_d$ is a Hopf algebra which is k-free of dimension nd, and antipode of order $2d$.

Proof. By the preceding lemmas we have that

$$\Delta(\gamma_i^d) = \sum_{j+k=i} \gamma_j^0 \otimes \gamma_k^d + \sum_{j+k=i} q^{dk} \gamma_j^d \otimes \gamma_k^0 .$$

The value of the counit is zero on any path of positive length and the antipode is graded with respect to the length-grading. A basis for the quotient is provided by all the paths of length less than d; there are exactly nd such paths starting at each vertex.

Remark 3.4. If the characteristic of k is p, there is a cocommutative Hopf algebra $kZ_n(1)/I_{p^a}$ of dimension np^a for any positive integers n and a.

Remark 3.5. Let k be a field. An associative finite dimensional algebra Λ is selfinjective if the module Λ is isomorphic to the left Λ-module $\mathrm{Hom}_k(\Lambda, k)$, which is an injective module. The algebra is said to be symmetric if the resulting pairing is symmetric.

The underlying associative algebra of a finite dimensional Hopf algebra which is Morita reduced has to be a Frobenius algebra, see [16, 22]. This is a well known property of kZ_n/I_d: the linear form $\sum(\gamma_i^{d-1})^*$ is the appropriate trace, i.e. the free generator of the dual. The corresponding bilinear form is symmetric if and only if $d-1$ is a multiple of n. Since d divides n, we infer that the underlying finite dimensional algebra of $kZ_n(q)\ /I_d$ is never symmetric.

Remark 3.6. Recall that a left (resp. right) integral of a Hopf algebra is an element of the left (resp. right) socle which generates the left (resp. right) trivial simple module. By [27] the isotypic component of the left (resp. right) trivial module in the socle of the algebra is one-dimensional. Clearly γ_{-d+1}^{d-1} (resp. γ_0^{d-1}) and their scalar multiples are the left (resp. right) integrals of $kZ_n(q)/I_d$. They never coincide.

Remark 3.7. Let q and q' be n^{th} and n'^{th} non-trivial roots of unity of order d and d'. The tensor product $kZ_n(q)/I_d \otimes kZ_{n'}(q')/I_{d'}$ is again a Hopf algebra with quiver lying on a torus; this quiver is the evident "tensor product" of Z_n and $Z_{n'}$. The path algebra of this quiver presents the tensor product algebra by means of the two-sided ideal generated by the commutativity relations of each square and the monomial relations arising from I_d and $I_{d'}$. Despite this quite simple presentation, the module theory cannot be recovered from the representation theory of the tensorands; in fact the latter are of finite representation type while their tensor product is of wild representation type.

Proposition 3.8. *Let n be a positive integer and let k be a field of characteristic not dividing n. If q is a primitive n^{th} root of unity, the Hopf algebras $kZ_n(q)$ and $kZ_n(q)/I_d$ are selfdual.*

Proof. The dual Hopf algebra has the following structure with respect to the dual basis $\{\check{\gamma}_i^u\}$:

$$\Delta(\check{s}_j) = \check{s}_j \otimes \check{s}_j, \quad \Delta(\check{\gamma}_j^u) = \check{\gamma}_j^u \otimes \check{s}_j + \check{s}_{j+u} \otimes \check{\gamma}_j^u ,$$

$$\check{\gamma}_j^u \check{\gamma}_k^v = q^{uk} \binom{u+v}{u}_q \check{\gamma}_{j+k}^{u+v} .$$

The elements $\sigma_i = \frac{1}{n}\sum_{l \in \mathbf{Z}/n\mathbf{Z}} q^{il}\check{s}_l$ and $\alpha_i = q^i \sum_{l \in \mathbf{Z}/n\mathbf{Z}} q^{(i+1)l}\check{a}_l$ can be considered as the Fourier transforms of the given ones. They satisfy the multiplication formulas of $kZ_n(q)$ and the map which assigns σ_i to s_i and α_i to a_i is a Hopf algebra isomorphism. The same map gives also an isomorphism at the level of the finite dimensional quotients.

Remark 3.9. If the characteristic of k divides n the Hopf algebra $kZ_n(q)/I_d$ is not selfdual. Indeed, $\check{\gamma}_1^0$ is an element of order exactly n in the dual algebra while the original one has no such elements.

For q a primitive n^{th} root of unity Taft has considered in [28] the Hopf algebra generated by X and Y subjected to the relations $X^n = 1$, $Y^n = 0$ and $YX = qXY$. The comultiplication is given by $\Delta(X) = X \otimes X$ and $\Delta(Y) = Y \otimes X + 1 \otimes X$. When q is a primitive n^{th} root of unity, this algebra is isomorphic to $(kZ_n(q)/I_n)^*$ by sending X to $\check{\gamma}_1^0 = \check{s}_1$ and Y to $\check{\gamma}_0^1 = a_0$. In case $n = 2$ and $q = -1$ the Hopf algebra is isomorphic to Sweedler's 4-dimensional neither commutative nor cocommutative example constructed in [27] and quoted in [8].

Recently these algebras have been considered by several authors. It has been shown that their Drinfeld double (see [8]) for n odd has a quotient isomorphic to a finite dimensional quotient of $U_q(sl_2)$ introduced by Reshetikhin and Turaev in [25]. Simple $U_q(sl_2)$-modules have been described in [26]; in the next section we consider the finitely generated indecomposable $\Lambda_n(q)$-modules and study their tensor product.

In [23] D. Radford has shown that Sweedler's 4-dimensional Hopf algebra is quasitriangular (see [8] for the definition). In fact there is a narrow subfamily of the Hopf algebras $kZ_n(q)/I_d$ sharing this property, as the next result shows.

Proposition 3.10. *The Hopf algebra $kZ_n(q)/I_d$ is quasitriangular if and only if n is even and $q = -1$. In all other cases the Hopf algebra is not even quasicocommutative.*

Proof. Let $R = \sum d_{i,j}^{u,v} \gamma_i^u \otimes \gamma_j^v$ be an invertible element such that $\Delta' R = R\Delta$, where Δ' stands for the opposite comultiplication. This quasicommutativity condition expressed on the vertices of the quiver translates into the fact that the coefficient of $\beta \otimes \alpha$ in R has to be zero if the product of the sources of the paths β and α differs from the product of their targets. In our case this means that $d_{i,j}^{u,v} = 0$ if $u + v$ is not divisible by n. Quasicommutativity expressed on the arrows gives the following condition on the coefficients

$$d_{j,k+1}^{0,0} = q^j d_{j,k}^{0,0} , \qquad q^k d_{j+1,k}^{0,0} = d_{j,k}^{0,0} ,$$

and for $u \neq 0$, $v \neq 0$, $u + v = 1(n)$,

$$d_{j,k+1}^{u,v-1} + q^k d_{j+1,k}^{u-1,v} = d_{j,k}^{u-1,v} + q^{j+u} d_{j,k}^{u,v-1} .$$

In particular we have $d_{0,1}^{0,0} = d_{0,0}^{0,0}$ and $d_{1,0}^{0,0} = d_{0,0}^{0,0}$. But $d_{1,1}^{0,0} = q d_{1,0}^{0,0}$ and $d_{1,1}^{0,0} = \frac{1}{q} d_{0,1}^{0,0}$, so either $q^2 = 1$ or $d_{j,k}^{0,0} = 0$ for all (j, k). Since R is invertible it does not lie in the Jacobson radical of $kZ_n(q)/I_d \otimes kZ_n(q)/I_d$, which means that at least one $d_{j,k}^{0,0}$ is non-zero. Therefore $q^2 = 1$, and notice that $q = 1$ is not allowed since $kZ_n(q)/I_d$ inherits a Hopf algebra quotient structure only for d equal to the precise order of q.

If $q = -1$, let us consider $kZ_n(-1)/I_2$ for n even, together with its canonical basis given by the set of vertices and arrows of the quiver. The condition $u + v = 0(n)$ implies now $u = v = 0$ for $n > 2$. This leads to $d_{j,k}^{0,0} = (-1)^{jk}x$, where x is an arbitrary element of k. The two remaining conditions for quasitriangularity (see [8]) enforces $x = \pm 1$, therefore $R = \pm \sum (-1)^{jk} s_j \otimes s_k$ are the two R-matrices of the considered $2n$-dimensional algebra.

In case $n = 2$, solutions with $(u, v) = (1, 1)$ are allowed. The set of R-matrices (obtained previously by Radford in [23]) is

$$\pm (s_0 \otimes s_0 - s_1 \otimes s_1 + s_0 \otimes s_1 + s_1 \otimes s_0)$$

$$+ x(a_0 \otimes a_0 + a_1 \otimes a_1 + a_0 \otimes a_1 - a_1 \otimes a_0) ,$$

where x is an arbitrary element of the field k.

Remark 3.11. In [9] Drinfeld pointed out that the square of the antipode of a quasicommutative Hopf algebra is an interior automorphism. It is interesting to note that this is true for each Hopf algebra $kZ_n(q)/I_d$ regardless of its quasicommutativity property. The conjugating element is $u = s_0 + q s_1 + \cdots + q^{n-1} s_{n-1}$.

Let us consider now the quiver A_∞^∞ with set of vertices $\{s_i\}_{i \in \mathbf{Z}}$ and arrows $\{a_i\}_{i \in \mathbf{Z}}$, where a_i goes from s_i to s_{i+1}. Let $K = k(q)$ or $K = \mathbf{Z}[q, q^{-1}]$, where q is an indeterminate and let $\mathscr{H}$ be the K-module of possibly infinite K-linear combinations of elements of the set C of finite length paths of this quiver. Alternatively $\mathscr{H}$ is the K-module K^C of all functions from K to C and we identify each element of C with the corresponding Dirac function having value 1 on it and 0-value otherwise; then for $f \in K^C$ we have $f = \sum_{\gamma \in C} f(\gamma)\, \gamma$.

The set C is the union of objects and morphisms of the cofinite category determined by A_∞^∞, where cofinite means that for any given morphism γ there is only a finite number of couples of morphisms having composition γ. In this situation $K^C = \mathscr{H}$ is an associative K-algebra: if f and g are in K^C their product fg is well defined by $fg(\gamma) = \sum_{\beta\alpha = \gamma} f(\beta) g(\alpha)$; the unit element is the sum of all the objects and two non-composable morphisms are orthogonal as elements of K^C. Notice that $\mathscr{H} = K^C$ is actually isomorphic to the algebra of infinite lower triangular matrices when C is deduced from A_∞^∞.

The proof of Theorem 2.1 suggests that $\mathscr{H}$ is a Hopf algebra provided that we consider the adequate tensor product as a receptacle for the comultiplication since $\Delta(a_i) = \sum_{j+k=i} s_j \otimes a_k + \sum_{j+k=i} q^j\, a_j \otimes s_k$ is not an element of $K^C \otimes K^C$. Nevertheless $C \times C$ is again obtained through a cofinite category and the expression above lies in the algebra $K^{C \times C}$ of possibly infinite sums of couples $(\beta, \alpha) = \beta \otimes \alpha$.

Let us consider Δ: $K^C \to K^{C\times C}$ given by

$$\Delta(\gamma_i^m) = \sum_{j+k=i} \sum_{u+v=m} \binom{m}{u}_q q^{uk}\, \gamma_j^u \otimes \gamma_k^v ,$$

as suggested by Lemma 3.1. This gives a well defined map on the entire algebra K^C since for $f \in K^C$ the expression $\Delta f(\gamma_j^u, \gamma_k^v) = \binom{u+v}{u}_q q^{uk} f(\gamma_{j+k}^{u+v})$ does not involve infinite sums. Notice that Δ is an algebra map and that the coassociativity has to be understood in the following sense: there is a map that we still denote $\Delta \otimes 1$ from $K^{C\times C}$ to $K^{C\times C\times C}$ given by $(\Delta \otimes 1)(f)(\gamma_{j_1}^{u_1}, \gamma_{j_2}^{u_2}, \gamma_{j_3}^{u_3}) = \binom{u_1+u_2}{u_2}_q q^{u_1 j_2} f(\gamma_{j_1+j_2}^{u_1+u_2}, \gamma_{j_3}^{u_3})$ which is well defined. There is also an analogous map denoted $1 \otimes \Delta$ and it is easy to check that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. Finally there is an antipode S and a counit ε given by $[S(f)](\gamma_i^m) = (-1)^m f(\gamma_{-i-m}^m)\, q^{-m(i+m)+\frac{m(m+1)}{2}}$ and $\varepsilon(f) = f(\gamma_0^0) = f(s_0)$.

An important fact that we shall use is that the category of K-free finite rank left K^C-modules still forms a monoidal category even if the receptacle of Δ is $K^{C\times C}$ instead of $K^C \otimes K^C$. Indeed if M is a K-free finite rank K^C-module almost all elements of C has zero action on it since $M = \bigoplus_{i\in \mathbb{Z}} s_i M$. It follows that if M and N are two such modules the transformations of $M \otimes_K N$ associated to elements of $C \times C$ are almost all zero and this enables us to consider $M \otimes_K N$ as a $K^{C\times C}$-module. Using Δ: $K^C \to K^{C\times C}$ we obtain the K^C-module structure on $M \otimes_K N$.

The free group on one generator acts on A_∞^∞ by translations of amplitude n for each $n \geqq 2$; the orbits give the path algebra $k(q)Z_n$. Actually specializing q to an n^{th} root of unity allows to get a Hopf algebra structure on kZ_n which is of course the one we have considered before.

Remark 3.12. Let $\mathscr{H}$ be the $k(q)$-Hopf algebra attached to the quiver A_∞^∞ and let I_d be the two-sided ideal of $\mathscr{H}$ generated by the paths of length d. Specializing q to a primitive d^{th} root of unity enables us to consider the k-Hopf algebra $\mathscr{H}(q)/I_d$, as in Proposition 3.3; its quiver is still A_∞^∞. Using analogous considerations than in Proposition 3.10 we obtain that $\mathscr{H}$ is not quasicommutative. The only Hopf algebra $\mathscr{H}(q)/I_d$ which is quasicommutative (and is in fact quasitriangular) is $\mathscr{H}(-1)/I_2$.

Notice that the free abelian group with one generator acts on $\mathscr{H}(q)/I_d$ by translations of amplitude n provided d divides n. The quotient by this action is clearly the Hopf algebra $kZ_n(q)/I_d$ attached to the cyclic quiver Z_n and to an n^{th} root of unity q of order d.

4. Modules

Let k be a field, and let $\Lambda_n(q)$ be the nd-dimensional Hopf algebra $kZn(q)/I_d$, where as before Z_n is the cyclic quiver of length n and q is a n^{th} root of unity of order d. This algebra is not semisimple since its Jacobson radical is not zero; in fact its radical is of dimension $n(d-1)$ and is generated by the arrows of the quiver. Therefore there exist indecomposable modules which are not simple. Any finitely generated module M is a direct sum of indecomposable submodules, and by the Krull–Schmidt Theorem such decompositions only differ by an automorphism of M. Actually $\Lambda_n(q)$ is a uniserial algebra, meaning that each indecomposable projective module has a unique composition series; consequently this algebra is of finite representation type, i.e. there is only a finite number of iso-classes of indecomposable modules (each indecomposable module is a quotient of an indecomposable projective).

More precisely let P_i be the left ideal generated by the primitive idempotent s_i. This ideal is a projective module and any iso-class of a projective indecomposable module is obtained in this way. Let M_i^u be the quotient of P_i by the submodule generated by the path γ_i^u. In other words, M_i^u is the $u+1$ dimensional cyclic module with generator $s_i = \gamma_i^0$ subject to $\gamma_i^{u+1} = 0$. The vector space M_i^u has a canonical basis $\{\gamma_i^0, \gamma_i^1, \ldots, \gamma_i^u\}$ and the composition of paths provides the module structure. Notice that $M_i^{d-1} = P_i$ and that any simple module is isomorphic to $M_i^0 = S_i$ for some $i \in \mathbf{Z}/n\mathbf{Z}$.

From this discussion we infer the well known fact that $\{M_i^u\}_{i \in \mathbf{Z}/n\mathbf{Z},\, 1 \leqq u \leqq d}$ provides the complete list of iso-classes of finite dimensional indecomposable $\Lambda_n(q)$ left modules.

The module category of a k-Hopf algebra Λ is provided with an associative tensor product obtained via the comultiplication; notice that the tensor product is not necessarily commutative, the canonical switch map is not in general a map of Λ-modules. Actually the switch map from $\Lambda \otimes_k \Lambda$ to itself is a Λ-map if and only if the Hopf algebra is cocommutative.

However if there exists an invertible element R such that the comultiplication Δ is quasicocommutative (i.e. $\Delta' = R\Delta R^{-1}$) and if M and N are arbitrary Λ-modules, the modules $M \otimes_k N$ and $N \otimes_k M$ are isomorphic; indeed, the action of R followed by the switch map provides a Λ isomorphism which is natural in M and N.

The following Clebsch-Jordan-like result on tensoring indecomposable $\Lambda_n(q)$-modules shows that the tensor product is commutative despite the fact that the Hopf algebra is not quasicommutative in general.

Theorem 4.1. *Let M_i^u and M_j^v be indecomposable $\Lambda_n(q)$ -modules for $i, j \in \mathbf{Z}/n\mathbf{Z}$ and $0 \leqq u, v \leqq d-1$. Assume $u \geqq v$. There are isomorphisms*

a) *If $u + v \leqq d - 1$*

$$M_i^u \otimes M_j^v \cong M_{i+j}^{u+v} \oplus M_{i+j+1}^{u+v-2} \oplus \cdots \oplus M_{i+j+v}^{u-v} .$$

b) *If $u + v \geqq d - 1$, let the excess be $e = u + v - (d-1)$,*

$$M_i^u \otimes M_j^v \cong P_{i+j} \oplus P_{i+j+1} \oplus \cdots \oplus P_{i+j+e} \oplus M_{i+j+e+1}^{u+v-2(e+1)}$$

$$\oplus M_{i+j+e+2}^{u+v-2(e+2)} \oplus \cdots \oplus M_{i+j+v}^{u-v}$$

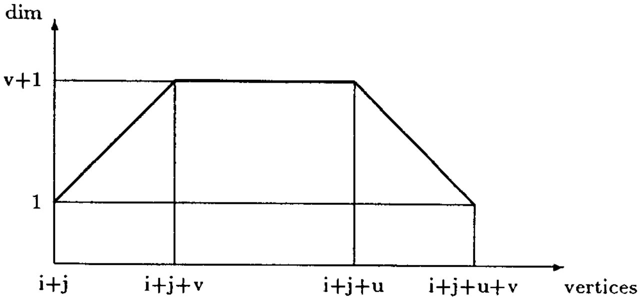
The same fact holds for $M_j^v \otimes M_i^u$.

Notice that the dimensions of the involved indecomposable modules starts at $u + v$ and decreases by 2 in case a) while in case b) the dimensions remains constant and equal to $d - 1$ until the vertex $i + j + e$ is reached.

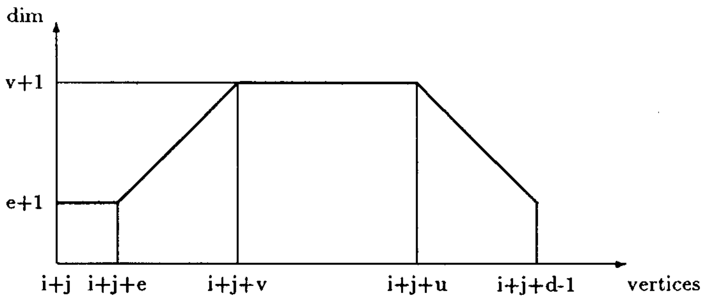
Proof. The dimension at the vertex s_k of a $\Lambda_n(q)$-module M is by definition the dimension of the vector space $s_k M$; the simple module S_k has dimension 1 at s_k and dimension 0 at any other vertex. Clearly the dimension at s_k of M is the multiplicity of S_k in a composition series of M.

Recall that $\{\gamma_i^x \otimes \gamma_j^y\}_{0 \leqq x \leqq u\, 0 \leqq y \leqq v}$ is a k-basis of the module $M_i^u \otimes M_j^v$ and that $\Delta(s_k) = \sum_{x+y=k} s_x \otimes s_y$; this provides the following diagrams of dimensions for $M_i^u \otimes M_j^v$

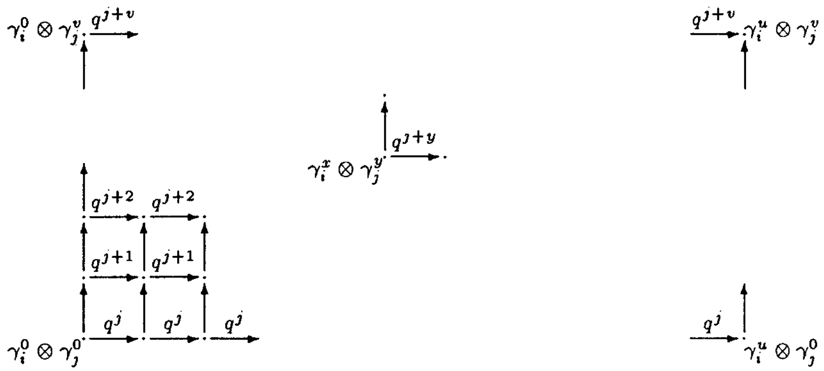
a) $u + v \leqq d - 1$



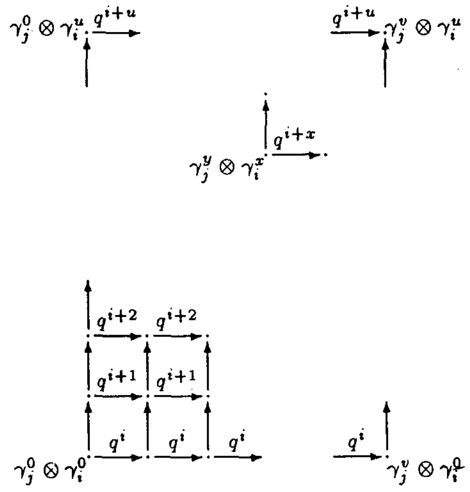
b) $u + v \geqq d - 1$ and $e = (u + v) - (d - 1)$



The dots of the next diagram represent the canonical basis elements of $M_i^u \otimes M_j^v$; the edges provides the action of the arrows of the quiver, the label indicates the scalar multiple obtained (no label means that the scalar value is 1). The dots located on a same antidiagonal give the basis at the corresponding vertex of the quiver. When $u + v$ exceeds $d - 1$ the appropriate identifications have to be performed in order to obtain the complete basis at some vertex.



The following is the diagram of the reverse tensor $M_j^v \otimes M_i^u$.



In order to determine the projective cover of $M_i^u \otimes M_j^v$, notice that the linear maps provided by the action of a_k from the vector space at s_k to the one at s_{k+1} are injective when the dimensions increase, bijective when it remains constant and surjective otherwise. The projective cover of $M_i^u \otimes M_j^v$ is determined by the simple modules at the top of the module, where the top is the quotient by the action of the Jacobson radical. The considerations on the action of the generators of the radical (i.e. the arrows of the quiver) shows that in both cases a) and b) $\text{top}(M_i^u \otimes M_j^v) = S_{i+j} \oplus S_{i+j+1} \oplus \cdots \oplus S_{i+j+v}$; the corresponding direct sum of indecomposable projectives supplies the projective cover. The comparison of dimensions at the vertices determines the result by uniseriality.

Remark 4.2. If $M_i^u = P_i$ is a projective module, then $e = v$ and the Theorem gives $P_i \otimes M_j^v = P_{i+j} \oplus P_{i+j+1} \oplus \cdots P_{i+j+v}$ which is a projective module. This was predicted since the tensor product of a projective module with any other module is projective; this fact is a consequence of the adjunction formula between Hom and $\otimes$, see [7].

Remark 4.3. Let $\mathscr{H}(q)/I_d$ be the infinite dimensional Hopf algebra associated to a primitive d^{th} root of unity (see Remark 3.12). The finite dimensional indecomposable modules are $\{M_i^u\}_{i \in \mathbf{Z}\, u \leq d-1}$, the projective ones are obtained with $u = d - 1$ while the simple ones are given by $u = 0$. The tensor product of this module is given by the formulas of Theorem 4.1.

Remark 4.4. Notice that the category of $\mathscr{H}(-1)/I_2$-modules can be identified with the category of complexes, and consequently this algebra is isomorphic to the dual of the algebra introduced by B. Pareigis in [21]. The tensor product of modules obtained through the comultiplication of this Hopf algebra corresponds of course to the usual tensor product of complexes. Recall that $\mathscr{H}(-1)/I_2$ is quasitriangular with R-matrix given by $\sum_{i,j \in \mathbf{Z}}(-1)^{ij} s_i \otimes s_j$; the braiding of the category of modules obtained via R is the usual isomorphism between $C. \otimes D.$ and $D. \otimes C.$ where $C.$ and $D.$ are complexes.

Our next aim is to show that the decomposition of Theorem 4.1 is unique; notice that in general there are various decompositions of a given module as a direct sum of indecomposable summands, the Krull–Schmidt Theorem "only" insures that such decompositions differ by an automorphism of the module.

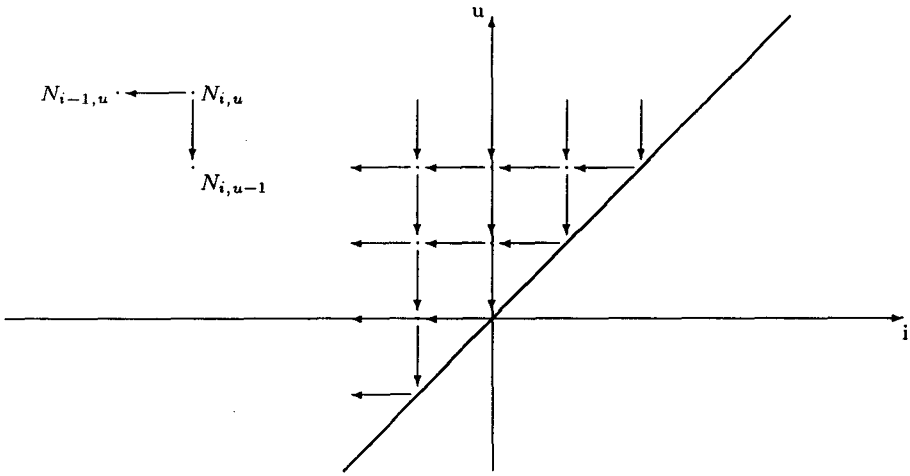
We consider from now on the category of finite dimensional modules over the K-Hopf algebra $\mathscr{H}$, where $K = k(q)$ and k is a field (see the end of the previous Section); the corresponding truncations can be easily performed in order to deal with the quotients of $\mathscr{H}$ that we have considered.

It is convenient to denote $N_{i,u}$ the indecomposable module starting at the vertex s_i and ending at s_u; thus $u \geqq i$ and $N_{i,u}$ corresponds to M_i^{u-i} in the previous notation.

Recall that a morphism $f: M \to N$ is called irreducible if it has no factorisations but the trivial ones, that is any factorisation of f is isomorphic to $M \xrightarrow{\binom{1}{0}} M \oplus X \xrightarrow{(f,0)} N$ or $M \xrightarrow{\binom{f}{0}} N \oplus X \xrightarrow{(1,0)} N$.

The irreducible morphisms of the considered finite length category are of two sorts (see [12]): the canonical surjections $N_{i,u} \to N_{i,u-1}$ and the natural monomorphisms $N_{i,u} \to N_{i-1,u}$.

The following is the Auslander–Reiten quiver of the category of indecomposable modules: it has one vertex for each isomorphism class of indecomposable modules and one arrow from $[N]$ to $[N']$ if there is an irreducible morphism $N \to N'$.



Notice that on a fixed diagonal the dimension of the modules is constant; simples are located on the principal diagonal. If q is specialized to a primitive d^{th} root of unity, the Auslander–Reiten quiver of $\mathscr{H}(q)/I_d$ is obtained by deleting the vertices located above the $d-1$ diagonal. For $kZ_n(q)/I_d$, the corresponding identifications shows that the Auslander–Reiten quiver lies on a cylinder.

An almost-split sequence (see [3, 4]) ending at an indecomposable module N is a non-split exact sequence $0 \to M \to X \to N \to 0$ such that any morphism f: $Y \to N$ which is not a split surjection can be lifted to a morphism $Y \to X$. Such a sequence is unique and always exists in case N is a non-projective indecomposable object of a category of finitely generated modules over a finite dimensional algebra. In that case M is indecomposable and uniquely determined by N, it is called the Auslander–Reiten translate of N.

The almost-split sequences in our situation are

$$0 \to N_{i+1,u+1} \to N_{i+1,u} \oplus N_{i,u+1} \to N_{i,u} \to 0$$

for a non-simple indecomposable module $N_{i,u}$, and

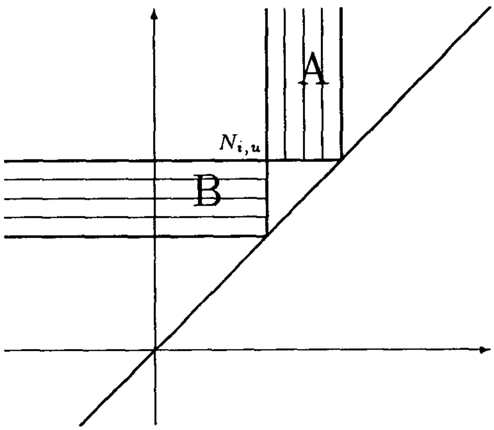
$$0 \to S_{i+1} \to N_{i,i+1} \to S_i \to 0$$

for a simple module $S_i = N_{i,i}$.

We recall some properties of this category (see [12]). The mesh relations arising from the Auslander–Reiten translate indicates that the squares of the quiver commute and that the composition of two arrows of any triangle over the principal diagonal is zero. The important fact is that these relations are sufficient in order to present the category of indecomposable modules, see for instance [12].

The following is an immediate consequence of the preceding discussion:

Proposition 4.5 ([12]). *Let $N_{i,u}$ be a fixed indecomposable module. The indecomposable modules X such that $\mathrm{Hom}_{\mathscr{H}}(X, N_{i,u}) \neq 0$ are located in the A-region of the following diagram; the indecomposable modules Y such that $\mathrm{Hom}_{\mathscr{H}}(N_{i,u}, Y) \neq 0$ are located in the B-region.*

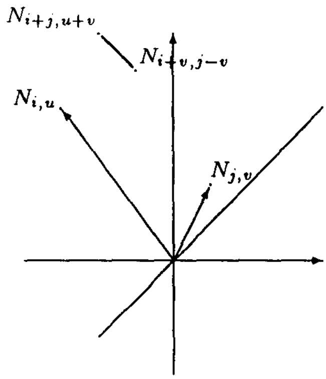


Theorem 4.6. *Let $N_{i,u}$ and $N_{j,v}$ be finite dimensional indecomposable $\mathscr{H}$- modules. The decomposition of $N_{i,u} \otimes N_{j,v}$ as a direct sum of indecomposable submodules is unique.*

Proof. By Proposition 4.1 we have

$$N_{i,u} \otimes N_{j,v} = N_{i+j,u+v} \oplus N_{i+j+1,u+v-1} \oplus \cdots \oplus N_{i+v,u+j}$$

when $u - i \geqq v - j$. The indecomposable direct summands of this decomposition lie on the antidiagonal thick segment of the following diagram.



We consider the A and B regions of an indecomposable summand of $N_{i,u} \otimes N_{j,v}$; there is no other indecomposable summand contained in these regions. Consequently any endomorphism of $N_{i,u} \otimes N_{j,v}$ preserves the considered decomposition; the Krull–Schmidt Theorem asserts that two decompositions of a given module into indecomposable summands differs by an automorphism, hence the given decomposition is unique. The result also holds for the reverse tensor product since this module is isomorphic to the original one.

Remark 4.7. Clearly the endomorphism ring of any finite dimensional $\mathscr{H}$-module is isomorphic to the ground field $K = k(q)$. The proof of the preceding theorem shows that $\mathrm{End}_{\mathscr{H}}(N_{i,u} \otimes N_{j,v})$ is a split basic semisimple K-algebra of K-dimension $\min\{\dim_K N_{i,u}, \dim_K N_{j,v}\}$.

5. Almost-Split Sequences

Almost-split sequences are an important tool in understanding finite length module categories. In case of modular group algebras, their behaviour under tensoring by an indecomposable module has been described in [2, 6]; this has been useful for decomposing a tensor product into a direct sum of indecomposables and for describing elements of the Green ring of representations.

In this section, we tensor on the left and on the right the almost-split sequence of the trivial module over $\mathscr{H}$ by a finite dimensional indecomposable module N. The result is always an almost-split sequence for N, so the two sequences can only differ by a scalar in the corresponding Ext-vector space. We compute this scalar which is non-trivial; let us point out that this is related to the left (or right) braided isomorphisms of functors between $N \otimes -$ and $- \otimes N$, which in turn are the objects of the double category of modules, see [18, 10].

Notice that the results for $\mathscr{H}$-modules lead easily to the corresponding results over the various truncations of $\mathscr{H}$, the only special case occurs when N is projective: tensoring the trivial almost-split sequence by N is then a split sequence, of course. None of the Hopf-algebras considered has antipode of order 2, so the results in [5, pages 48–50] are not in force. Moreover the results concerning the tensor product of a module with its dual highlights the splitting between the Hopf algebras with antipode of order 2 and those with bigger orders: the characteristic of the ground field does not intervene in the second case.

Recall that a finite dimensional indecomposable $\mathscr{H}$-module $N_{i,u}$ has dimension $u - i + 1$ and a canonical basis $\{\gamma_{i,i}, \gamma_{i,i+1}, \ldots, \gamma_{i,u}\}$, where $\gamma_{i,j}$ lies in $s_j N_{i,u}$; the element $\gamma_{i,i}$ is a generator at s_i which is uniquely defined by this property, up to a non-zero scalar. The choice of this generator determines uniquely the generator $\gamma_{i,i+1}$ of the unique maximal submodule of $N_{i,u}$, by means of $\gamma_{i,i+1} = a_i \gamma_{i,i}$.

The irreducible morphisms are π: $N_{i,u} \to N_{i,u-1}$ and I: $N_{i,u} \to N_{i-1,u}$, where $I(\gamma_{i,i}) = \gamma_{i-1,i}$. The canonical almost-split sequence for $N_{i,u}$ with $u > i$ is

$$\alpha_{i,u}\colon 0 \to N_{i+1,u+1} \xrightarrow{\binom{\pi}{-I}} N_{i,u+1} \oplus N_{i,u-1} \xrightarrow{(I,\pi)} N_{i,u} \to 0\,,$$

and for $u = i$,

$$\alpha_{i,i}\colon 0 \to N_{i+1,i+1} \xrightarrow{I} N_{i,i+1} \xrightarrow{\pi} N_{i,i} \to 0\,.$$

Recall that $N_{0,0}$ is the trivial $\mathscr{H}$-module determined by the counit; in order to study $N_{i,u} \otimes \alpha_{0,0}$ and $\alpha_{0,0} \otimes N_{i,u}$ we need to identify the generators of the indecomposable direct summands of $N_{i,u} \otimes N_{0,1}$ and $N_{0,1} \otimes N_{i,u}$.

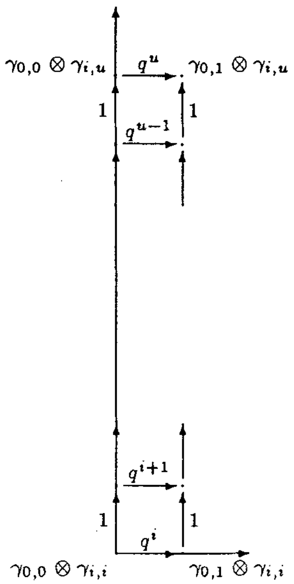
Lemma 5.1. *Let $i > u$ and let $N_{i,u}$ be the corresponding non-simple indecomposable module. In the decomposition $N_{i,u} \otimes N_{0,1} = N_{i,u+1} \oplus N_{i+1,u}$ the element $\gamma_{i,i} \otimes \gamma_{0,0}$ is a generator of $N_{i,u+1}$ at s_i and the element $\gamma_{i,i+1} \otimes \gamma_{0,0} + x_{i,u}\gamma_{i,i} \otimes \gamma_{0,1}$ where $x_{i,u} = -\dfrac{1 + q + \cdots + q^{u-i-1}}{q^{u-i}}$ is a generator of $N_{i+1,u}$ at s_{i+1}.*

Proof. The diagram representing $N_{i,u} \otimes N_{0,1}$ is

From Theorem 4.1 we have that $\gamma_{i,i} \otimes \gamma_{0,0}$ generates $N_{i,u+1}$ and that a linear combination of $\gamma_{i,i+1} \otimes \gamma_{0,0}$ and $\gamma_{i,i} \otimes \gamma_{0,1}$ generates $N_{i+1,u}$; this linear combination has image zero under the action of the path $\gamma_{i+1,u+i}$. The image of $\gamma_{i,i+1} \otimes \gamma_{0,1}$ under this action is $q^{u-1}\gamma_{i,u} \otimes \gamma_{0,1}$ while the image of $\gamma_{i,i+1} \otimes \gamma_{0,0}$ is $(1 + q + q^2 + \cdots + q^{u-i-1})\,\gamma_{i,u} \otimes \gamma_{0,1}$.

Lemma 5.2. *In the decomposition $N_{0,1} \otimes N_{i,u} = N_{i,u+1} \oplus N_{i+1,u}$ the element $z_{i,u}\gamma_{0,0} \otimes \gamma_{i,i} + \gamma_{0,0} \otimes \gamma_{i,i+1}$ is a generator of $N_{1+i,u}$ at s_{i+1}, where $z_{i,u} = -q^{i+1}(1 + q + \cdots + q^{u-i-1})$.*

Proof. The diagram representing $N_{0,1} \otimes N_{i,u}$ is



As in the previous lemma, we compute the action of the path $\gamma_{i+1,u+1}$ on $\gamma_{0,0} \otimes \gamma_{i,i+1}$ and on $\gamma_{0,1} \otimes \gamma_{i,i}$. We obtain respectively $(q^{i+1} + \cdots + q^u)\gamma_{0,1} \otimes \gamma_{i,u}$ and $\gamma_{0,1} \otimes \gamma_{i,u}$.

Remark 5.3. The element $x_{i,u}$ only depends on the dimension of the module while $z_{i,u}$ depends both on the dimension and on the starting vertex.

Proposition 5.4. *Let $N_{i,u}$ be the canonical non-simple indecomposable $\mathscr{H}$-module and let $\alpha_{i,u}$ be its canonical almost-split sequence. Let $\alpha_{0,0}$ be the canonical almost-split sequence of the trivial simple $\mathscr{H}$-module $N_{0,0}$. Then $N_{i,u} \otimes \alpha_{0,0}$ and $\alpha_{0,0} \otimes N_{i,u}$ are almost-split sequences ending at $N_{i,u}$ and*

$$N_{i,u} \otimes \alpha_{0,0} = \frac{1}{x_{i,u} - 1}\, \alpha_{i,u} \quad \alpha_{0,0} \otimes N_{i,u} = \frac{1}{z_{i,u} - q^i}\, \alpha_{i,u} .$$

Proof. We compare the tensored morphisms of $\alpha_{0,0}$ with the morphisms of $\alpha_{i,u}$ through the identifications obtained via the generators. First we consider π: $N_{0,1} \to N_{0,0}$ and $1 \otimes \pi$: $N_{i,u} \otimes N_{0,1} \to N_{i,u} \otimes N_{0,0} = N_{i,u}$. We have that $(1 \otimes \pi)\,(\gamma_{i,i} \otimes \gamma_{0,0}) = \gamma_{i,i} \otimes \gamma_{0,0} =$ the generator of $N_{i,u}$ and $(1 \otimes \pi)\,(\gamma_{i,i+1} \otimes \gamma_{0,0} + x_{i,u}\gamma_{i,0} \otimes \gamma 0, 1) = \gamma_{i,i+1} \otimes \gamma_{0,0} =$ the generator of the radical of $N_{i,u}$. Consequently $1 \otimes \pi$ coincides with the standard surjection of $\alpha_{i,u}$.

Next, consider I: $N_{1,1} \to N_{0,1}$ and $1 \otimes I$: $N_{i+1,u+1} = N_{i,u} \otimes N_{1,1} \to N_{i,u} \otimes N_{0,1}$. We have $(1 \otimes I)\,(\gamma_{i,i} \otimes \gamma_{1,1}) = \gamma_{i,i} \otimes \gamma_{0,1}$, while the standard monomorphism of the canonical almost-split sequence translated to this context sends the generator $\gamma_{i,i} \otimes \gamma_{1,1}$ to the generator of $N_{i+1,u}$ minus the generator of $N_{i,u-1}$, i.e. to

$$\gamma_{i,i+1} \otimes \gamma_{0,0} + x_{i,u}\gamma_{i,i} \otimes \gamma_{0,1} - (\gamma_{i,i+1} \otimes \gamma_{0,1} - \gamma_{i,i} \otimes \gamma_{0,1})$$
$$= (x_{i,u} - 1)\, \gamma_{i,i} \otimes \gamma_{0,1} .$$

Hence $1 \otimes I$ is the canonical injection multiplied by $\dfrac{1}{x_{i,u} - 1}$.

We now apply the reverse functor $- \otimes N_{i,u}$ to the trivial almost-split sequence: $\pi \otimes 1\colon N_{0,1} \otimes N_{i,u} \to N_{0,0} \otimes N_{i,u}$ verifies $(\pi \otimes 1)\,(\gamma_{0,0} \otimes \gamma_{i,i}) = \gamma_{0,0} \otimes \gamma_{i,i} =$ the generator of $N_{i,u}$ and $(\pi \otimes 1)(z_{i,u}\gamma_{0,1} \otimes \gamma_{i,i} + \gamma_{0,0} \otimes \gamma_{i,i+1}) = \gamma_{0,0} \otimes \gamma_{i,i+1} =$ the generator of the radical of $N_{i,u}$. As before, $\pi \otimes 1$ coincides with the standard surjection of $\alpha_{i,u}$.

Now, $(I \otimes 1)\,(\gamma_{1,1} \otimes \gamma_{i,i}) = \gamma_{0,1} \otimes \gamma_{i,i}$, while the canonical monomorphism of the almost-split sequence $\alpha_{i,u}$ translated to this context sends $\gamma_{1,1} \otimes \gamma_{i,i}$ to $z_{i,u}\gamma_{0,1} \otimes \gamma_{i,i} + \gamma_{0,0} \otimes \gamma_{i,i+1} - (q^i\gamma_{0,1} \otimes \gamma_{i,i} + \gamma_{0,0} \otimes \gamma_{i,i+1}) = (z_{i,u} - q^i)\gamma_{0,1} \otimes \gamma_{i,i}$.

Notice that the canonical generator of the radical of $N_{i,u+1}$ is $q^i\gamma_{0,1} \otimes \gamma_{i,i} + \gamma_{0,0} \otimes \gamma_{i,i+1}$ because $\gamma_{0,0} \otimes \gamma_{i,i}$ is the chosen generator of $N_{i,u+1}$. Consequently the monomorphism of $\alpha_{0,0} \otimes N_{i,u}$ is the standard injection multiplied by $\dfrac{1}{z_{i,u} - q^i}$.

Theorem 5.5. *Let $N_{i,u}$ be the finite dimensional $\mathscr{H}$-module from the vertex s_i to the vertex s_u of A_∞^∞. Let $\alpha_{0,0}$ be the trivial almost split sequence of the module $N_{0,0}$. Then $N_{i,u} \otimes \alpha_{0,0} = q^u\, \alpha_{0,0} \otimes N_{i,u}$.*

Proof. For $u > i$, recall that $x_{i,u} = \dfrac{-(u-i-1)_q}{q^{u-i}}$ and $z_{i,u} = -q^{i+1}(u-i-1)_q$. Consequently $\dfrac{1}{x_{i,u}-1} = \dfrac{-q^{u-i}}{(u-i)_q}$, $\dfrac{1}{z_{i,u}-q^i} = -\dfrac{1}{q^i(u-i)_q}$ and their quotient value is q^u. The case $i = u$ (simple modules) follows from the fact that $S_i \otimes \alpha_{0,0} = \alpha_{i,i}$ and $\alpha_{0,0} \otimes S_i = \dfrac{1}{q^i}\alpha_{i,i}$.

The following is a consequence of Proposition 5.4, compare with [2].

Proposition 5.6. a) *Let N and M be indecomposable finite dimensional $\mathscr{H}$-modules, and let α_M be the almost-split sequence ending at M. Then $N \otimes \alpha_M$ and $\alpha_M \otimes N$ are direct sums of almost-split sequences.*
b) *If q is an n^{th} root of unity of order d, let N and M be $\mathscr{H}(q)/I_d$ or $kZ_n(q)/I_d$ indecomposable modules. Then $N \otimes \alpha_M$ and $\alpha_M \otimes N$ are direct sums of almost-split sequences and projective-injectives factors.*

Proof. Up to a non-zero scalar we have $\alpha_M = M \otimes \alpha_{0,0}$, where $\alpha_{0,0}$ is the almost-split sequence of the trivial module. Hence $N \otimes \alpha_M = N \otimes M \otimes \alpha_{0,0}$; the result follows from the decomposition of Theorem 4.1 and the fact that $X \otimes \alpha_{0,0}$ is almost-split. In case b) the eventual projective summands of $N \otimes M$ provide the split factors.

In [6, 2, 5] it is proved that for a modular k-group algebra (or a finite dimensional k-Hopf algebra with antipode of order 2), the trivial module is a direct summand of $M \otimes N$ if and only if $M \cong N^*$ and $(p, \dim_k N) = 1$, where M and N are indecomposable modules and p is the characteristic of the field k. This result allowed Benson and Carlson [6] to give a proof of the semisimplicity of the Green algebra of representations when the Hopf algebra is of finite representation type: they prove that there are no nilpotent elements in this commutative algebra.

The next result shows that for the $k(q)$-Hopf algebra $\mathscr{H}$ the condition concerning the characteristic of the field is dropped; for q an n^{th} root of unity of order d and for indecomposable $\mathscr{H}(q)$ or $kZ_n(q)/I_d$-modules this condition is replaced by requiring N not to be projective. Notice that both algebras have antipode of order $2d$, with $d > 1$. Moreover the analog of the Green algebra of representations of Sweedler's 4-dimensional example has nilpotent elements.

Proposition 5.7. a) *Let M and N be indecomposable finite dimensional $\mathscr{H}$-modules. The trivial module $N_{0,0}$ is a direct summand of $M \otimes N$ if and only if $M \cong N^*$.*

b) *For q an n^{th} root of unity of order d and for indecomposable $\mathscr{H}/I_d$ or $kZ_n(q)/I_d$-modules, the trivial module is a direct summand of $M \otimes N$ if and only if $M \cong N^*$ and N is not projective.*

Proof. The dual vector space of a left $\mathscr{H}$-module is a right module which becomes a left module through the antipode. Note that the double dual is isomorphic to the original module, since the square of the antipode is inner, see Remark 3.11. Hence the dual $N^*_{i,u}$ of the indecomposable module $N_{i,u}$ is indecomposable; it begins at s_{-u} and ends at s_{-i} since $N_{i,u}$ begins at s_i and ends at s_u, and $S(s_j) = s_{-j}$. From the proof of Theorem 4.6 we have that the indecomposable summands of $N_{i,u} \otimes N_{j,v}$ are the vertices lying on the antidiagonal segment from $N_{i+j,u+v}$ to $N_{i+v,u+j}$ which is entirely contained in the half plane up the principal diagonal. This segment contains $N_{0,0}$ if and only if $v = -i$ and $j = -u$.

In case b) the eventual end of the segment up to the $d-1$-diagonal has to be projected on this diagonal according to the vertical direction. The obtained broken segment records the indecomposable summands of $N_{i,u} \otimes N_{j,v}$. The result follows.

Remark 5.8. It would be interesting to know if part b) of the preceding Proposition holds for an arbitrary finite dimensional Hopf algebra with antipode of order bigger than 2.

The representation ring $a(H)$ of a finite dimensional Hopf algebra H is the Grothendieck group of the category of finite dimensional modules (iso-classes of indecomposable modules compose a basis) equipped with the multiplication given by the tensor product. The projective indecomposable modules form a basis of an ideal $\mathscr{P}$; the structure of the rings $a(H)$ and $a(H)/\mathscr{P}$ is certainly interesting from the point of view of quantum groups. The Sweedler's 4-dimensional Hopf algebra $kZ_2(-1)/I_2$ has 4 indecomposable modules: 2 simple ones $k = S_0$ and S_1, and 2 projective ones P_0 and P_1. The representation $\mathbf{C}$-algebra has a basis composed by these modules and we have $P_0^2 = P_1^2 = P_0P_1 = P_1P_0 = P_0 + P_1$ together with the cyclic group $\{S_0, S_1\}$ acting on $\{P_0, P_1\}$. This algebra is isomorphic to $\mathbf{C} \times \mathbf{C} \times \mathbf{C}[\varepsilon]/\varepsilon^2$, where $\varepsilon = P_0 - P_1$.

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