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Primitive Ideals of $\mathbf{C}_q[SL(3)]$

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Abstract. The primitive ideals of the Hopf algebra $\mathbf{C}_q[SL(3)]$ are classified. In particular it is shown that the orbits in $\operatorname{Prim}\mathbf{C}_q[SL(3)]$ under the action of the representation group $H \cong \mathbf{C}^* \times \mathbf{C}^*$ are parameterized naturally by $W \times W$, where W is the associated Weyl group. It is shown that there is a natural one-to-one correspondence between primitive ideals of $\mathbf{C}_q[SL(3)]$ and symplectic leaves of the associated Poisson algebraic group $SL(3, \mathbf{C})$.

Introduction

The primitive spectrum of a noncommutative affine algebra is the natural generalization of the variety associated to a commutative affine algebra. When the noncommutative algebra A is a deformation of a commutative algebra B, one expects to find a close correspondence between the primitive ideals of A and the symplectic leaves of the associated Poisson structure on the variety $\operatorname{Max}(B)$. For instance if $\mathbf{g}$ is a solvable complex Lie algebra, then the primitive ideals of the enveloping algebra $U(\mathbf{g})$ correspond to the coadjoint orbits in $\mathbf{g}^*$, which are the symplectic leaves for the Kostant–Kirillov Poisson structure on $\mathbf{g}^*$.

A similar close correspondence seems likely to occur for quantum groups and related algebras. Let G be a semi-simple complex Lie group and let $\mathbf{C}_q[G]$ be the associated quantum group as defined in [16]. There is a standard Poisson Lie group structure on G associated to $\mathbf{C}_q[G]$. The primitive ideals of $\mathbf{C}_q[G]$ are expected to correspond bijectively to the symplectic leaves of G. This correspondence may be verified for $SL(2)$ by direct calculation. In this paper we study the primitive ideals of $\mathbf{C}_q[SL(n)]$ and prove that the primitive ideals of $\mathbf{C}_q[SL(3)]$ correspond exactly to the symplectic leaves of $SL(3)$.

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When q is real, $q \neq 1$, $\mathbf{C}_q[G]$ together with a natural involution * can be viewed as a deformation of $\mathbf{C}[K]$, the algebra of functions on a maximal compact subgroup K of G. In a series of articles in [18, 19, 20] Soibelman, and Vaksman showed that the unitary representations of $\mathbf{C}_q[K]$ correspond to the symplectic leaves of K.

Fix a maximal torus H in G. Then G has a natural H-invariant Poisson structure [4]. A description of the symplectic leaves of G may be deduced from the work of Semenov-Tian-Shansky and Lu and Weinstein [11, 17]; an outline of this description is given in Appendix A. Let W be the Weyl group of G. The symplectic leaves fall into H-orbits parameterized by $W \times W$. Let $D = G \times G$, identify G with the diagonal subgroup of D and let G_r be the dual group. Denote by p the natural projection $G \to D/G_r$. The symplectic leaves of G are precisely the connected components of the inverse images of the left G_r-orbits in D/G_r. Set $\Gamma = \ker p$ and $\bar{G} = p(G)$. Then Γ is a finite subgroup of H and $\bar{G} = G/\Gamma$ is an open subset of D/G_r. For each $w \in W \times W$, let $\mathscr{C}_w$ be the image of the corresponding Bruhat cell of D in D/G_r. Denote by $\mathscr{C}_{\dot{w}}$ a fixed G_r-orbit in $\mathscr{C}_w$. Then $\mathscr{C}_{\dot{w}} \cong \mathbf{C}^l \times (\mathbf{C}^*)^s$ and $\mathscr{C}_w$ is the union of the H-translates of $\mathscr{C}_{\dot{w}}$. Each symplectic leaf of G is then a finite cover of $h\mathscr{C}_{\dot{w}} \cap \bar{G}$ for some $w \in W \times W$ and some $h \in H$.

In section two we prove some preliminary results about the primitive spectrum of $\mathbf{C}_q[SL(n)]$. The group H occurs again in the quantum case as the character group and $\operatorname{Prim} \mathbf{C}_q[SL(n)]$ therefore decomposes into the union of the H-orbits. Following ideas of Soibelman [18, 19], we define for each $w \in W \times W$ a locally closed H-invariant subset Prim_w of $\operatorname{Prim} \mathbf{C}_q[SL(n)]$. It may be shown that Prim_w is nonempty for all w and that $\operatorname{Prim} \mathbf{C}_q[SL(n)] = \bigsqcup_w \operatorname{Prim}_w$. We conjecture that each Prim_w is a single H-orbit and that the elements of Prim_w are in bijection with the leaves of type w. This conjecture is proved in sections three and four for $\mathbf{C}_q[SL(3)]$. The truth of the conjecture for $\mathbf{C}_q[SL(2)]$ was proved earlier by S.P. Smith and the first author. This result is outlined in Appendix B.

In order to describe the symplectic leaves of G one passes first to $\bar{G}$. Similarly, in order to describe the primitive ideals of $\mathbf{C}_q[G]$, we first study the invariant subalgebra $\mathbf{C}_q[\bar{G}] = \mathbf{C}_q[G]^\Gamma$. The quantum analog of $\mathscr{C}_w \cap \bar{G}$ is a certain localization of a homomorphic image of $\mathbf{C}_q[G]$ denoted by B_w. The key result in section three is the decomposition of B_w as the tensor product $B_{\dot{w}} \otimes \mathbf{C}[H_w]$, where $B_{\dot{w}}$ is a quantum analog of $\mathscr{C}_{\dot{w}} \cap \bar{G}$ and $\mathbf{C}[H_w]$ is the algebra of functions on the torus $H_w = H/\operatorname{Stab}_H \mathscr{C}_{\dot{w}}$.

1. Preliminaries

1.1. In this section we introduce the basic definitions and notation that we shall be using. We denote by $\mathbf{g}$ the Lie algebra $sl(n, \mathbf{C})$ and by G the Lie group $SL(n, \mathbf{C})$. We follow the standard notation in Bourbaki for the roots, weights, Weyl group etc. associated to $\mathbf{g}$. Other notation is listed at the end of the paper.

1.2. Let $q \in \mathbf{C}^*$. We shall assume throughout this paper that *q is not a root of unity.* We denote by $\mathscr{Q}$ the set $\{q^n | n \in \mathbf{Z}\}$. Let $[a_{ij}]$ be the Cartan matrix associated to $\mathbf{g}$. Recall that the quantum universal enveloping algebra associated to $\mathbf{g}$ is defined to be the algebra $U_q(\mathbf{g})$ generated by $K_i^{\pm 1}, X_i^{\pm}, 1 \leqq i \leqq n-1$ with relations

$$K_i^{-1}K_i = K_iK_i^{-1} = 1\,, \quad K_iX_j^{\pm} = q^{\pm a_{ij}}X_j^{\pm}K_i\,,$$

$$[X_i^+, X_j^-] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}}\,, \quad K_iK_j = K_jK_i\,,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0, \quad \text{if } i \neq j\,,$$

$$\text{where} \begin{bmatrix} m \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{q^{2(m-j+1)} - q^{-2(m-j+1)}}{q^{2j} - q^{-2j}}$$

(see, for example, [12]). The algebra $U_q(\mathbf{g})$ is a Hopf algebra. The comultiplication Δ is defined by

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i^{-1} + K_i \otimes X_i^{\pm}\,, \quad \Delta(K_i) = K_i \otimes K_i\,,$$

and the counit and antipode by

$$\varepsilon(X_i^{\pm}) = 0, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i^{-1}, \quad S(X_i^{\pm}) = -q^{\mp 2}X_i^{\pm}\,.$$

There is also a **C**-linear antiautomorphism $a \mapsto a^*$ given by $(X_i^{\pm})^* = X_i^{\mp}$, $(K_i)^* = K_i$. It is easily verified that $\Delta(a^*) = \Delta(a)^*$ (where $(a \otimes b)^* = a^* \otimes b^*$) and $S(S(a)^*) = a^*$.

1.3. Set $U^o = \mathbf{C}[K_i^{\pm 1} \mid 1 \leqq i \leqq n-1]$. Let M be a U^o-module. If χ is a character of U^o define the χ-weight space of M by $M_\chi = \{x \in M \mid ux = \chi(u)x,\ \forall u \in U^o\}$. Set $\Omega(M) = \{\chi \mid M_\chi \neq 0\}$. Let P be the set of weights of $\mathbf{g}$ and let $\{\alpha_1, \ldots, \alpha_{n-1}\}$ be a fixed set of positive roots. Each weight $\lambda \in P$ induces a character of U^o via $\lambda(K_i) = q^{(\lambda, \alpha_i)}$, $1 \leqq i \leqq n-1$. We denote by M_λ the associated weight space.

Define $\mathscr{C}$ to be the category of finite dimensional $U_q(\mathbf{g})$ modules such that $M = \bigoplus_{\mu \in P} M_\mu$. Since $\mathscr{C}$ is closed under finite direct sums, tensor products and passage to the dual module, we may define the restricted dual of $U_q(\mathbf{g})$ with respect to $\mathscr{C}$. This is the associated quantum group $\mathscr{C}_q[G]$. Thus

$$\mathbf{C}_q[G] = \{f \in U_q(\mathbf{g})^* \mid \operatorname{Ker} f \supseteq \operatorname{Ann} M \text{ for some } M \in \mathscr{C}\}$$

The algebra $\mathbf{C}_q[G]$ then has a natural Hopf algebra structure induced in the usual way from that on $U_q(\mathbf{g})$. There is also an anti-automorphism on $\mathbf{C}_q[G]$ induced from that on $U_q(\mathbf{g})$ by $\ell^*(u) = \ell(S(u)^*)$ for all $\ell \in \mathbf{C}_q[G]$ and all $u \in U_q(\mathbf{g})$.

Let $\pi\colon U_q(\mathbf{g}) \to \operatorname{End}(M) \cong M_m(\mathbf{C})$, $\pi(a) = [\pi_{ij}(a)]$, be an m-dimensional representation of $U_q(\mathbf{g})$, where M is an object of $\mathscr{C}$. The elements $\pi_{ij} \in U_q(\mathbf{g})^*$ are called the matrix elements or matrix coefficients of the representation π. It is clear that these π_{ij} belong to $\mathbf{C}_q[G]$ and that the set of all such π_{ij} for all possible M in $\mathscr{C}$, spans $\mathbf{C}_q[G]$ as a vector space. Recall the following useful formulas:

$$\Delta\pi_{ij} = \sum_k \pi_{ik} \otimes \pi_{kj}\,, \quad \pi_{ij}\pi_{kl} = (\pi_{ij} \otimes \pi_{kl}) \circ \Delta, \quad S(\pi_{ij}) = \pi_{ij} \circ S, \quad \varepsilon(\pi_{ij}) = \pi_{ij}(1)\,.$$

1.4. The category $\mathscr{C}$ is in some sense a deformation of the category of finite dimensional modules over the Lie algebra $\mathbf{g}$ [12]. Denote by P_+ the set of dominant weights of $\mathbf{g}$. For each dominant weight $\Lambda \in P_+$ there is a simple module $L(\Lambda)$ in $\mathscr{C}$ and an element $v_\Lambda \in L(\Lambda)$ such that

1. $L(\Lambda) = U_q(\mathbf{g})v_\Lambda = \bigoplus_{\mu\in P, \mu\leqq\Lambda} L(\Lambda)_\mu$;
2. $L(\Lambda)_\Lambda = \mathbf{C}v_\Lambda$, $X_i^+ v_\Lambda = 0, 1 \leqq i \leqq n-1$; ($v_\Lambda$ is called the highest weight vector of $L(\Lambda)$)
3. the set of weights $\Omega(\Lambda) = \Omega(L(\Lambda))$ and the multiplicities are the same as for the corresponding simple $\mathbf{g}$-module.

Any $M \in \mathscr{C}$ decomposes as $M = \bigoplus_{\Lambda\in P_+} L(\Lambda)^{m_\Lambda}$. The representation ring of $\mathscr{C}$ is generated by the classes of the simple modules $L(\varpi_i)$ corresponding to the fundamental dominant weights ϖ_i, $1 \leqq i \leqq n-1$. Moreover each $L(\Lambda)$ occurs as a subquotient of a suitable power of the standard representation $L(\varpi_1)$. On the other hand the dual of $L(\varpi_1)$ is isomorphic to $L(\varpi_{n-1})$ which is isomorphic to the $(n-1)$-th quantum exterior power of $L(\varpi_1)$. Hence if the matrix coefficients with respect to the natural basis $e_1, \dots, e_n$ of $L(\varpi_1)$ are denoted by X_{ij} then the matrix coefficients corresponding to $L(\varpi_{n-1})$ are the quantum minors defined by:

$$D_{ij} = \sum_{\sigma\in S_{n-1}} (-q^2)^{\ell(\sigma)} X_{1,\sigma(1)} \dots X_{i-1,\sigma(i-1)} X_{i+1,\sigma(i+1)} \dots X_{n,\sigma(n)},$$

where S_{n-1} denotes the symmetric group acting in the usual way as bijections from $\{1, \dots, i-1, i+1, \dots, n\}$ to $\{1, \dots, j-1, j+1, \dots, n\}$.

From these and related facts one deduces the following well-known description of the Hopf algebra $\mathbf{C}_q[G]$.

Theorem 1.4.1. (a) *The algebra $\mathbf{C}_q[G]$ is generated by the X_{ij}, $1 \leqq i, j \leqq n$, with relations:*

$$X_{i\ell}X_{j\ell} = q^2 X_{j\ell}X_{i\ell}, \quad \forall \ell, \forall i<j, \quad X_{\ell i}X_{\ell j} = q^2 X_{\ell j}X_{\ell i}, \quad \forall \ell, \forall i<j,$$
$$X_{\ell i}X_{mj} = X_{mj}X_{\ell i}, \quad \forall \ell < m, \forall i > j,$$
$$X_{\ell i}X_{mj} - X_{mj}X_{\ell i} = (q^2 - q^{-2})X_{\ell j}X_{mi}, \quad \forall \ell < m, \forall i < j,$$
$$\mathrm{Det}_q = \sum_{\sigma\in S_n} (-q^2)^{\ell(\sigma)} X_{\sigma(1),1} \dots X_{\sigma(n),n} = 1.$$

(b) *The Hopf algebra structure is given by*

$$\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}, \quad S(X_{ij}) = (-q^2)^{i-j} D_{ji}, \quad \varepsilon(X_{ij}) = \delta_{ij}.$$

(c) *The involution * is given by* $(X_{ij})^* = (-q^2)^{j-i} D_{ij}$.
(d) *Furthermore*

$$\delta_{ij} = \textstyle\sum_k (-q^2)^{k-j} X_{ik} D_{jk} = \sum_k (-q^2)^{i-k} D_{ki} X_{kj} = \sum_k (-q^2)^{j-k} D_{jk} X_{ik}$$
$$= \textstyle\sum_k (-q^2)^{k-i} X_{kj} D_{ki}.$$

The reader is referred to [16] and [14] for further details concerning this algebra.

1.5. The generators described in the above section are not well suited to the study of the primitive ideals. A more natural set of generators is the following. This notation was first introduced by Soibelman in [18].

Recall that $L(\varpi_k) \cong \bigwedge^k L(\varpi_1)$ (the k^{th} quantum exterior power of $L(\varpi_1)$) and that $\Omega(\varpi_k) = W\varpi_k$, where W denotes the Weyl group. Recall that W may be naturally identified with the symmetric group S_n by letting the reflection with respect to the simple root α_i correspond to the transposition $(i, i+1)$. Let $i = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, n-1\}$ such that $i_1 < \dots < i_k$. Define

$e_{\underline{i}} = e_{i_1} \wedge \ldots \wedge e_{i_k}$. Then the weight spaces of $L(\varpi_k)$ are exactly the $\mathbf{C}e_{\underline{i}}$. For any $w \in W$ define $e_{w\varpi_k}$ to be the element of $e_{\underline{i}}$ of $L(\varpi_k)$, where $\underline{i}$ is the ordered multi-index associated to $\{w(1), \ldots, w(k)\}$. It is easily verified that $e_{w\varpi_k} \in L(\varpi_k)_{w\varpi_k}$. Define $e^*_{-w\varpi_k} \in L(\varpi_k)^*$ to be the dual basis element corresponding to $e_{w\varpi_k}$ and denote by $\langle -, - \rangle$ the natural pairing between a vector space and its dual.

Definition. *For each* $k = 1, \ldots, n-1$ *and each* $w \in W$ *we define elements* $c^{\pm}_{k,w} \in \mathbf{C}_q[G]$ *by:* $\forall u \in U_q(\mathfrak{g})$, $c^+_{k,w}(u) = \langle e^*_{-w\varpi_k}, ue_{\varpi_k} \rangle$, $c^-_{k,w}(u) = \langle e^*_{-ww_0\varpi_{n-k}}, ue_{w_0\varpi_{n-k}} \rangle$.

Thus $c^+_{k,w}$ (respectively $c^-_{k,w}$) is a matrix coefficient of $L(\varpi_k)$ (respectively $L(\varpi_{n-k})$). In particular we have that $c^+_{1,(1,i)} = X_{i1}$, $c^+_{n-1,(i,n)} = D_{in}$, $c^-_{1,(1,i)} = D_{i1}$, $c^-_{n-1,(i,n)} = X_{in}$. The general element $c^{\pm}_{k,w}$ can be interpreted as a general quantum minor as defined in [14]. In the notation of that article,

$$c^+_{j,w} = \xi^{w\{1,\ldots,j\}}_{\{1,\ldots,j\}}, \quad c^-_{j,w} = \xi^{w\{j+1,\ldots,n\}}_{\{j+1,\ldots,n\}},$$

where $w\{1, \ldots, k\} = \{w(1), \ldots, w(k)\}$ etc.

One of the key properties of these matrix elements is that they generate $\mathbf{C}_q[G]$. In fact a slightly stronger statement is true. Let A_+ be the subalgebra of $\mathbf{C}_q[G]$ generated by the elements of the form $c^+_{i,w}$ and let A_- be the algebra generated by the elements of the form $c^-_{i,w}$.

Theorem 1.5.1. *The linear map* $A_- \otimes A_+ \to \mathbf{C}_q[G]$ *given by* $a \otimes b \mapsto ab$ *is an epimorphism of* $\mathbf{C}$*-vector spaces.*

Proof. This result is Theorem 3.1 of [19]. It suffices to check that the definition of $A_{\pm}$ given there is in fact the same as the one given above. □

1.6. On occasion we will need a notation for a coordinate function coming from an arbitrary representation in $\mathscr{C}$. Our notation again follows Soibelman [19].

Let $\Lambda \in P_+$. Recall that $L(\Lambda) = \bigoplus_{\lambda \in \Omega(\Lambda)} L(\Lambda)_\lambda$, $L(\Lambda)^* \cong L(-w_0\Lambda)$ and $L(\Lambda)^*_{-\mu} = [L(\Lambda)_\mu]^*$. Each module $L(\Lambda)$ carries a non-degenerate bilinear contravariant form $(-|-)_\Lambda$ such that $(av|w)_\Lambda = (v|a^*w)_\Lambda$ for all $a \in U_q(\mathfrak{g})$ and $v, w \in L(\Lambda)$. Such a form is unique up to a scalar multiple [7]. Choose an orthonormal basis $\{v^{(j)}_\mu | \mu \in \Omega(\Lambda), 1 \leq j \leq \dim L(\Lambda)_\mu\}$ of $L(\Lambda)$ with respect to $(-|-)_\Lambda$. Let $\{\ell^{(i)}_{-\lambda}\}$ be the dual basis in $L(\Lambda)^*$. Then each $\ell^{(i)}_{-\lambda}$ identifies with $(v^{(i)}_\lambda | -)_\Lambda$ and $\ell^{(i)}_{-\lambda} \in L(\Lambda)^*_{-\lambda}$. Hence $\langle \ell^{(i)}_{-\lambda}, v^{(j)}_\mu \rangle = (v^{(i)}_\lambda | v^{(j)}_\mu)_\Lambda = \delta_{\lambda\mu}\delta_{ij}$. We define elements $c^\Lambda_{-\lambda,i,\mu,j}$ of $\mathbf{C}_q[G]$ by:

$$\forall u \in U_q(\mathfrak{g}), \quad c^\Lambda_{-\lambda,i,\mu,j}(u) = \langle \ell^{(i)}_{-\lambda}, uv^{(j)}_\mu \rangle = (v^{(i)}_\lambda | uv^{(j)}_\mu)_\Lambda .$$

For convenience we use the following abbreviations:

$$c^\Lambda_{-\lambda,i,\mu,j} = \begin{cases} c^\Lambda_{-\lambda,\mu,j} & \text{if } \dim L(\Lambda)_\lambda = 1 \\ c^\Lambda_{-\lambda,i,\mu} & \text{if } \dim L(\Lambda)_\mu = 1 \\ c^\Lambda_{-\lambda,\mu} & \text{if } \dim L(\Lambda)_\lambda = \dim L(\Lambda)_\mu = 1 \end{cases}$$

The first two parts of the following lemma are taken from [19]. The third part is a consequence of the general formula in Sect. 1.3.

Lemma 1.6.1. (a) $S(c^{\Lambda}_{-\lambda,i,\mu,j}) \in \mathbf{C}c^{-w_0\Lambda}_{\mu,j,-\lambda,i}$,

(b) $(c^{\Lambda}_{-\lambda,i,\mu,j})^* \in \mathbf{C}c^{-w_0\Lambda}_{\lambda,i,-\mu,j}$,

(c) $\Delta(c^{\Lambda}_{-\lambda,i,\mu,j}) = \sum_{\nu,k} c^{\Lambda}_{-\lambda,i,\nu,k} \otimes c^{\Lambda}_{-\nu,k,\mu,j}$.

Notice that $c^+_{k,w} \in \mathbf{C}c^{\varpi_k}_{-w\varpi_k,\varpi_k}$ and $c^-_{k,w} \in \mathbf{C}c^{\varpi_{n-k}}_{w\varpi_k,-\varpi_k} = \mathbf{C}(c^+_{k,w})^*$.

1.7. Let $R^{\pm}$ denote the set of positive and negative roots respectively. Denote by $\mathbf{b}^{\pm} = \mathbf{h} \oplus \mathbf{n}^{\pm}$ the Borel subalgebras associated to $R^{\pm}$. We denote by $U_q(\mathbf{b}^{\pm})$ the Hopf subalgebras of $U_q(\mathbf{g})$ generated by $\{K_i, X_i^{\pm} \mid 1 \leqq i \leqq n-1\}$ respectively (we call them the Borel subalgebras).

As in [19] we define the following ideals of $\mathbf{C}_q[G]$ which play a fundamental role in what follows:

$$I^+(w,\Lambda) = \langle c^{\Lambda}_{-\mu,i,\Lambda} \mid 1 \leqq i \leqq \dim L(\Lambda)_\mu,\ v^{(i)}_\mu \notin U_q(\mathbf{b}^+)v_{w\Lambda} \rangle\,,$$

$$I^-(w,\Lambda) = \langle c^{-w_0\Lambda}_{\mu,i,-\Lambda} \mid 1 \leqq i \leqq \dim L(-w_0\Lambda)_{-\mu},\ v^{(i)}_{-\mu} \notin U_q(\mathbf{b}^-)v_{-w\Lambda} \rangle\,.$$

Notice that in the definition of $I^-(w,\Lambda)$ the $v^{(i)}_{-\mu}$'s belong to $L(-w_0\Lambda)_{-\mu}$. Notice also that the condition $v^{(i)}_\mu \notin U_q(\mathbf{b}^+)v_{w\Lambda}$ can be expressed in the form $c^{\Lambda}_{-\mu,i,w\Lambda}(u) = 0$ for all $u \in U_q(\mathbf{b}^+)$.

Define τ to be the involutive automorphism $\tau = * \circ S$. For any $U_q(\mathbf{g})$-module M we denote by M^τ the twisted module where the action of an element $u \in U_q(\mathbf{g})$ on an element $v \in M$ is given by $u \cdot v = \tau(u)v$. Then it is easily verified that $L(-w_0\Lambda) \cong L(\Lambda)^\tau$. This isomorphism takes $v^{(i)}_{-\mu} \in L(-w_0\Lambda)$ onto $v^{(i)}_\mu \in L(\Lambda)_\mu$. Since $\tau(U_q(\mathbf{b}^+)) = U_q(\mathbf{b}^-)$ we obtain that

$$v^{(i)}_\mu \notin U_q(\mathbf{b}^+)v_{w\Lambda} \Leftrightarrow v^{(i)}_{-\mu} \notin U_q(\mathbf{b}^-)v_{-w\Lambda}\,.$$

Therefore Lemma 1.6.1 shows that $I^+(w,\Lambda)^* = I^-(w,\Lambda)$.

1.8. We shall need some elementary facts about the Bruhat ordering on W. We take the reverse of the usual Bruhat ordering introduced in [3]. Thus $e \leqq w \leqq w_0$ for all $w \in W$. For each fundamental weight ϖ_i we denote the stabiliser of ϖ_i in W by $W_i = \mathrm{Stab}(\varpi_i)$. Denote by $\hat{W}_i$ a fixed transversal of W_i in W.

Definition. *Fix $i \in \{1, \ldots, n-1\}$. Let $y, w \in W$. We say that $y \leqq_i w$ if and only if $y\varpi_i \geqq w\varpi_i$.*

It is clear that $\leqq_i$ is right W_i-invariant and that the induced ordering on W/W_i is a partial ordering. In order to keep the notation consistent, we shall sometimes use the notation $=_i$ for equivalence modulo W_i. The proof of the following proposition is similar to standard arguments concerning the Bruhat ordering (for instance [3, §7.7]).

Proposition 1.8.1. *Let $i \in \{1, \ldots, n-1\}$ and let $y, w \in W$.*

1. *The following are equivalent:* (a) $y \leqq_i w$; (b) $v_{y\varpi_i} \in U_q(\mathbf{b}^+)v_{w\varpi_i}$; (c) $v_{w\varpi_i} \in U_q(\mathbf{b}^-)v_{y\varpi_i}$.
2. $y \leqq_i w \Leftrightarrow yw_0 \geqq_{n-i} ww_0$.
3. $y \leqq w \Leftrightarrow y \leqq_i w$ *for all i.*

Example. If we identify W as above with the symmetric group S_n, then the subgroup $W_1 = \mathrm{Stab}_W(\varpi_1)$ identifies with the group $S_{n-1} = \mathrm{Sym}\{2, \ldots, n\}$ and we may take $\hat{W}_1$ to be $\{e = (1,1), (1,2), \ldots, (1,n)\}$. The ordering $\leqq_1$ is then given by

$$W_1 <_1 (1,2) W_1 <_1 \ldots <_1 (1,n) W_1 .$$

Similarly $W_{n-1} = \mathrm{Sym}\{1, \ldots, n-1\}$ and we may take the transversal $\hat{W}_{n-1}$ to be $\{(n,1), (n,2), \ldots, (n,n) = e\}$. The ordering $\leqq_{n-1}$ is then given by

$$W_{n-1} <_{n-1} (n, n-1) W_{n-1} <_{n-1} \ldots <_{n-1} (n,1) W_{n-1} .$$

In the case we shall be most interested in (when $n = 3$) these are of course the only two cases.

2. The Algebras A_w, B_w and C_w

2.1. In order to study $\mathbf{C}_q[G]$ in more detail we introduce algebras A_w, B_w and C_w defined for each $w \in W \times W$. The motivation for the definitions of these algebras comes from the structure of the symplectic leaves in G. Recall the notation of Appendix A. There are natural maps $G \to \bar{G} \hookrightarrow D/G_r$ and a symplectic leaf of G is a connected component of the inverse image of a left G_r-orbit of D/G_r. The Bruhat cells $\mathscr{C}_w$ of D/G_r are disjoint unions of isomorphic leaves of "type w." Just as in this geometric case it is natural to study the symplectic leaves by type, so in the study of $\mathbf{C}_q[G]$ it is natural to classify primitive ideals by type. The algebras C_w, B_w and A_w correspond to the cell $\mathscr{C}_w$ and its inverse image in $\bar{G}$ and G respectively.

2.2. Setting $\Lambda = \varpi_i$ in 1.7 we obtain the ideals $I^{\pm}(w, \varpi_i)$. From Lemma 1.6.2 and Proposition 1.8.1 it follows that

$$I^{\pm}(w, \varpi_i) = \langle c^{\pm}_{i,y} \mid y \nleqq_i w \rangle .$$

Henceforth, the principal objects of interest will be the ideals defined for each $w = (w_+, w_-) \in W \times W$ by:

$$I_w = \sum_{i=1}^{n-1} (I^+(w_+, \varpi_i) + I^-(w_-, \varpi_i)) = \langle c^{\varepsilon}_{i,y} \mid 1 \leqq i \leqq n-1, y \nleqq_i w_\varepsilon \rangle ,$$

and the sets, defined also for each $w = (w_+, w_-) \in W \times W$ by

$$\mathscr{E}_w = \{c^+_{i,w_+}, c^-_{i,w_-} \mid i = 1, \ldots, n-1\} .$$

We shall also occasionally use the following notation. For $y \in W$ we define $I^{\pm}(y) = \sum_{i=1}^{n-1} I^{\pm}(y, \varpi_i)$ and $\mathscr{E}^{\pm}(y) = \{c^{\pm}_{i,y} \mid i = 1, \ldots, n\}$. For $w = (w_+, w_-)$, we define $I^{\pm}_w = I^{\pm}(w_{\pm})$, and $\mathscr{E}^{\pm}_w = \mathscr{E}^{\pm}(w_{\pm})$.

Theorem 2.2.1. *Let $w \in W$. The image of $c^{\varepsilon}_{i,w}$ is normal in $\mathbf{C}_q[G]/I^{\varepsilon}(w, \varpi_i)$. In fact we have that*

$$c^{\Lambda}_{-\lambda,i,\mu,j} c^{\varepsilon}_{i,w} = \gamma c^{\varepsilon}_{i,w} c^{\Lambda}_{-\lambda,i,\mu,j} (\mathrm{mod}\, I^{\varepsilon}(w, \varpi_i)) \textit{ for some } \gamma \in \mathscr{Q} .$$

Proof. Recall that $\mathbf{C}_q[G] = \mathbf{C}[c^{\Lambda}_{-\lambda,i,\mu,j} \mid \Lambda \in P_+]$ and that $c^+_{i,w}$ is a scalar multiple of $c^{\varpi_i}_{-w\varpi_i,\varpi_i}$. The ideal $J_0(w\varpi_i, \varpi_i)$ defined in [19] is precisely the ideal $I^+(w, \varpi_i)$ defined above. The result for $c^+_{i,w}$ then follows from [19, Prop. 3.2]. Applying the involution $*$ yields the result for $c^-_{i,w}$. □

Corollary 2.2.2. *For any $w = (w_+, w_-) \in W \times W$, the elements of $\mathscr{E}_w$ (respectively $\mathscr{E}^\varepsilon_w$) are normal in $\mathbf{C}_q[G]/I_w$ (respectively $\mathbf{C}_q[G]/I^\varepsilon_w$).*

Now let $w = (w_+, w_-) \in W \times W$. Denote by E_w the multiplicatively closed set generated by the images of the elements of $\mathscr{E}_w$ in $\mathbf{C}_q[G]/I_w$. Since E_w consists of normal elements we may localize with respect to this set. Denote the localized algebra

$$A_w = (\mathbf{C}_q[G]/I_w)_{E_w}.$$

It is not immediately clear that $A_w \neq 0$ since it could happen that $E_w \cap I_w \neq \emptyset$. In the next few subsections we shall prove the following result:

Theorem 2.2.3. *For all $w \in W \times W$, $A_w \neq 0$.*

The idea of the proof is to construct a non-zero A_w-module by tensoring together certain "fundamental" $\mathbf{C}_q[G]$-modules. This technique was used by Soibelman in [19, §5]; the idea is apparently due to Drinfeld. It is a quantum analog of the proof that $p^{-1}(\mathscr{C}_w) \neq \emptyset$ given in Appendix A.

Definition. *A non-zero $\mathbf{C}_q[G]$-module is said to be of type $w \in W \times W$ if* (i) $I_w M = 0$ *and* (ii) $\forall c \in \mathscr{E}_w$, $M = cM$ (i.e., *M is $\mathscr{E}_w$-divisible*).

It is a standard fact that a module of type w has a natural structure as an A_w-module. Thus the theorem will be a consequence of the existence of a nontrivial module of type w for all $w \in W \times W$.

2.3. For each $i \in \{1, \dots, n-1\}$, denote by $U_q(\mathbf{sl}_i(2))$, the Hopf subalgebra generated by $\{X_i^+, X_i^-, K_i^{\pm 1}\}$; denote by $U_q(\mathbf{b}_i^\varepsilon)$ the subalgebra generated by $\{X_i^\varepsilon, K_i^{\pm 1}\}$. Consider the following commutative diagram of inclusions:

$$\begin{array}{ccc} U_q(\mathbf{b}_i^\varepsilon) & \rightarrow & U_q(\mathbf{sl}_i(2)) \\ \downarrow & \searrow^{\varphi_{\varepsilon,i}} & \downarrow^{\varphi_i} \\ U_q(\mathbf{b}^\varepsilon) & \underset{\varphi_\varepsilon}{\rightarrow} & U_q(\mathbf{g})\,. \end{array}$$

Since $U_q(\mathbf{b}^\varepsilon)$ is a Hopf subalgebra, the subspace $U_q(\mathbf{b}^\varepsilon)^\perp = \{f \in \mathbf{C}_q[G] \mid f(U_q(\mathbf{b}^\varepsilon)) = 0\}$ is an ideal of $\mathbf{C}[G]$. Define $\mathbf{C}_q[B^\varepsilon] = \mathbf{C}_q[G]/U_q(\mathbf{b}^\varepsilon)^\perp$ and define similarly $\mathbf{C}_q[B_i^\varepsilon]$ and $\mathbf{C}_q[SL_i(2)]$. Then we have a commutative diagram of surjections,

$$\begin{array}{ccc} \mathbf{C}_q[B_i^\varepsilon] & \leftarrow & \mathbf{C}_q[SL_i(2)] \\ \uparrow & \nwarrow^{\varphi^*_{\varepsilon,i}} & \uparrow^{\varphi_i^*} \\ \mathbf{C}_q(B^\varepsilon) & \underset{\varphi_\varepsilon^*}{\leftarrow} & \mathbf{C}_q(G)\,. \end{array}$$

It is easily verified that the canonical isomorphism, $U_q(\mathbf{sl}_i(2)) \cong U_q(\mathbf{sl}(2))$ induces an isomorphism $\mathbf{C}_q[SL(2)] \cong \mathbf{C}_q[SL_i(2)]$ such that the kernel of $\mathbf{C}_q[SL_i(2)] \to \mathbf{C}_q[B_i^+]$ is $I_{(e,s)}$ (and likewise $\mathrm{Ker}(\mathbf{C}_q[SL_i(2)] \to \mathbf{C}_q[B_i^-]) = I_{(s,e)}$). From the theorem in Appendix B, we know that there exist $\mathbf{C}_q[SL(2)]$ modules M^+ and M^- of type (s, e) and (e, s) respectively. Define $M_i^\pm$ to be the modules $M^\pm$ considered as $\mathbf{C}_q[G]$ modules via the map $\mathbf{C}_q[G] \to \mathbf{C}_q[SL_i(2)] \xrightarrow{\sim} \mathbf{C}_q[SL(2)]$. Then in particular we have that $\mathrm{Ann}\, M_i^\pm \supseteq \ker \varphi^*_{i,\mp}$.

Proposition 2.3.1. *The modules M_i^+ and M_i^- are of type (s_i, e) and (e, s_i) respectively.*

Proof. We give the proof for M_i^+. We first need to show that $I_{(s_i, e)} \subset \operatorname{Ann} M_i^+$; it is enough to show that $I^+(s_i, \varpi) + I^-(e, \varpi) \subset \operatorname{Ann} M_i^+$ for each fundamental weight ϖ. Notice that $I^-(e, \varpi) = \langle c_{\mu, -\varpi}^{-w_0\varpi} \mid v_{-\mu} \notin U_q(\mathbf{b}^-)v_{-\varpi} \rangle = \langle c_{\mu, -\varpi}^{-w_0\varpi} \mid \varphi_-^*(c_{\mu, -\varpi}^{-w_0\varpi}) = 0 \rangle \subset \operatorname{Ker} \varphi_-^* \subset \operatorname{Ann} M_i^+$. On the other hand, $I^+(s_i, \varpi) = \langle c_{-\lambda, \varpi}^{\varpi} \mid v_\lambda \notin U_q(\mathbf{b}^+)v_{s_i\varpi} \rangle$. Suppose that $I^+(s_i, \varpi) \not\subset \operatorname{Ker} \varphi_i^*$. Then there exists a λ such that $v_\lambda \notin U_q(\mathbf{b}^+)v_{s_i\varpi}$ and $v_\lambda \in U_q(\mathbf{sl}_i(2))v_\varpi$. Since $X_i^+ v_\varpi = 0$ and $X_i^+ v_{s_i\varpi} \in \mathbf{C}v_\varpi$, we obtain $v_\lambda \in U_q(\mathbf{sl}_i(2))v_\varpi = \mathbf{C}v_\varpi + \mathbf{C}v_{s_i\varpi} \subseteq U_q(\mathbf{b}^+)v_{s_i\varpi}$, a contradiction.

It remains to show that M_i^+ is $\mathscr{E}_{(s_i, e)}$-divisible. Recall that elements of $\mathscr{E}_{(s_i, e)}$ are of the form $c_j = c^{\varpi_j}_{-s_i\varpi_j, \varpi_j}$ or $c_j' = c^{-\varpi_0\varpi_j}_{\varpi_j, -\varpi_j}$. We first compute $\varphi_i^*(c_j)$ acting on M^+ via the identification $\mathbf{C}_q[SL_i(2)] = \mathbf{C}_q[SL(2)]$. The $U_q(\mathbf{sl}_i(2))$ module generated by v_{ϖ_j} is either trivial (when $(\varpi_j, \alpha_i) = 0$) or is the fundamental representation with highest weight vector v_{ϖ_i} (when $(\varpi_j, \alpha_i) = 1$). It follows that $\varphi_i^*(c_j) = (c^\rho_{-s\rho, \rho})^{(\varpi_j, \alpha_i)}$ for which M_i^+ is divisible by definition. A similar reasoning gives that $\varphi_i^*(c_j') = (c^\rho_{-s\rho, s\rho})^{(\varpi_j, \alpha_i)}$ which again acts divisibly by definition on M^+. □

2.4. We now show that modules of type $w = (w_+, w_-)$ can be constructed by forming the tensor product of modules of the form $M_i^\pm$ using the reduced decomposition of w_+ and w_-. The fundamental result is the following.

Theorem 2.4.1. *Let M be a $\mathbf{C}_q[G]$-module of type (w_+, w_-). If $s_i w_+ > w_+$ (respectively $s_i w_- > w_-$) then $M_i^+ \otimes M$ (respectively $M \otimes M_i^-$) is a $\mathbf{C}_q[G]$-module of type $(s_i w_+, w_-)$ (respectively of type $(w_+, s_i w_-)$).*

Proof. We prove the assertion in the case $s_i w_+ > w_+$.

(i) $I^-(w_-, \varpi) \subseteq \operatorname{Ann}(M_i^+ \otimes M)$ for all fundamental representations ϖ.

We denote $c_{\lambda, \mu}^{-w_0\varpi}$ by $c_{\lambda, \mu}$. A standard generator for $I^-(w_-, \varpi)$ is then of the form $c_{\lambda, -\varpi}$, where $v_{-\lambda} \notin U_q(\mathbf{b}^-)v_{-w_-\varpi}$. The action of $c_{\lambda, -\varpi}$ is given by the comultiplication $\Delta(c_{\lambda, -\varpi}) = \sum_{\mu \in \Omega(-w_0\varpi)} c_{\lambda, \mu} \otimes c_{-\mu, -\varpi}$. Suppose that the action is non-trivial. Then there exists a μ such that both factors $c_{\lambda, \mu}$ and $c_{-\mu, -\varpi}$ act non-trivially on M_i^+ and M respectively. Since M is of type (w_+, w_-) this implies that $v_\mu \in U_q(\mathbf{b}^-)v_{-w_-\varpi}$. Since $\operatorname{Ann}(M_i^+) \supseteq \operatorname{Ker}(\varphi_-^*)$, we must have that $\varphi_-^*(c_{\lambda, \mu}) \neq 0$; thus $v_{-\lambda} \in U_q(\mathbf{b}^-)v_\mu \subseteq U_q(\mathbf{b}^-)v_{-w_-\varpi}$, a contradiction.

(ii) $I^+(s_i w_+, \varpi) \subseteq \operatorname{Ann}(M_i^+ \otimes M)$ for all fundamental weights ϖ.

For these calculations we abbreviate $c^\varpi_{\lambda, \mu}$ by $c_{\lambda, \mu}$. Then a standard generator of $I^+(s_i w_+, \varpi)$ is $c_{-\lambda, \varpi}$, where $v_\lambda \notin U_q(\mathbf{b}^+)v_{s_i w_+ \varpi}$. The action on $M_i \otimes M$ is given by: $\Delta(c_{-\lambda, \varpi}) = \sum_{\mu \in \Omega(\varpi)} c_{-\lambda, \mu} \otimes c_{-\mu, \varpi}$. Suppose that there exists a μ such that both $c_{-\lambda, \mu}$ and $c_{-\mu, \varpi}$ act non-trivially on M_i^+ and M respectively. Then by definition and Proposition 1.8.1, $v_\mu \in U_q(\mathbf{b}^+)v_{w_+\varpi} \subseteq U_q(\mathbf{b}^+)v_{s_i w_+ \varpi}$. On the other hand, since $\operatorname{Ann}(M_i^+) \supseteq \operatorname{Ker}(\varphi_i^*)$ we must have $v_\lambda \in U_q(\mathbf{sl}_i(2))v_\mu$. Since $s_i w_+ > w_+$, $X_i^- v_{s_i w_+ \varpi} = 0$. Since moreover $[X_k^+, X_i^-] = \delta_{ik}(q^2 - q^{-2})^{-1}(K_i^2 - K_i^{-2})$, it follows easily that

$$U_q(\mathbf{sl}_i(2))U_q(\mathbf{b}^+)v_{s_i w_+ \varpi} \subseteq U_q(\mathbf{b}^+)v_{s_i w_+ \varpi}$$

which implies that $v_\lambda \in U_q(\mathbf{b}^+)v_{s_i w_+ \varpi}$, a contradiction.

(iii) $M_i^+ \otimes M$ is $\mathscr{E}_{(s_i w_+, w_-)}$-divisible.

Let ϖ be a fundamental representation. We continue with the notation of part (ii). The action of $c_{s_i w_+ \varpi, \varpi}$ is given by:

$$\Delta(c_{-s_i w_+ \varpi, \varpi}) = \sum_{\mu \in \Omega(\varpi)} c_{-s_i w_+ \varpi, \mu} \otimes c_{-\mu, \varpi}\,.$$

Suppose μ is such that the corresponding summand is non-trivial. Then we have that (a) $v_\mu \in U_q(\mathbf{b}^+)v_{w_+\varpi}$ and (b) $v_{s_i w_+ \varpi} \in U_q(\mathbf{b}_i^-)v_\mu$. Consider the $U_q(\mathbf{sl}_i(2))$-submodule of $L(\varpi)$ containing $v_{w_+\varpi}$. Since $s_i w_+ > w_+$ it has highest weight $w_+\varpi$ and lowest weight $s_i w_+ \varpi$. If $(w_+\varpi, \alpha_i) = 0$, the representation is trivial; otherwise $(w_+\varpi, \alpha_i) = 1$ and the representation is the fundamental representation. From (b) we obtain $\mu = s_i w_+ \varpi + p\alpha_i = w_+\varpi + (p - (w_+\varpi, \alpha_i))\alpha_i$, where p is an integer between 0 and $(w_+\varpi, \alpha_i)$. From (a) we deduce that $p = (w_+\varpi, \alpha_i)$ and so $\mu = w_+\varpi$. Thus for any $m' \in M_i^+$ and $m \in M$,

$$c_{-s_i w_+ \varpi, \varpi} m' \otimes m = c_{-s_i w_+ \varpi, w_+ \varpi} m' \otimes c_{-w_+ \varpi, \varpi} m .$$

By hypothesis M is $c_{-w_+\varpi, \varpi}$-divisible. On the other hand, $\varphi_i^*(c_{-s_i w_+ \varpi, w_+ \varpi}) = (c^\rho_{-s\rho, \rho})^{(w_+\varpi, \alpha_i)}$ and M_i^+ is divisible with respect to this element. Hence $M_i^+ \otimes M$ is $c^\varpi_{-s_i w_+ \varpi, \varpi}$-divisible. The proof for elements of the form $c^{-w_0\varpi}_{w_-\varpi, -\varpi}$ is similar. □

Corollary 2.4.2. *Let $w_+ = s_{i_1} \ldots s_{i_k}$, $w_- = s_{j_1} \ldots s_{j_m}$ be reduced expressions for w_+ and w_- in w. Then*

$$M_{i_1}^+ \otimes \ldots \otimes M_{i_k}^+ \otimes M_{j_m}^- \otimes \ldots \otimes M_{j_1}^-$$

is a module of type (w_+, w_-).

This completes the proof of Theorem 2.2.3. These results generalize slightly [19, Propositions 5.1, 5.2].

2.5. Let $R(\mathbf{C}_q[G])$ denote the set of one-dimensional representations of $\mathbf{C}_q[G]$. Since $\mathbf{C}_q[G]$ is a Hopf algebra, $R(\mathbf{C}_q[G])$ has a natural group structure. Let $X = (X_{ij})$ be the matrix of coordinate functions as described in 1.4. Since the X_{ij} generate $\mathbf{C}_q[G]$, there is a natural map from $R(\mathbf{C}_q[G])$ to $M_n(\mathbf{C})$ given by $\chi \mapsto (\chi(X_{ij})) = \chi(X)$. It is easily verified that this is an isomorphism of $R(\mathbf{C}_q[G])$ onto the set of invertible diagonal matrices. Since $R(\mathbf{C}_q[G])$ is naturally isomorphic to this complex torus we shall denote it by H.

For any Hopf algebra A, there is a natural action of $R(A)$ as automorphisms of A given by $r_\chi(a) = \sum a_{(1)}\chi(a_{(2)})$ for all $\chi \in R(A)$ and $a \in A$. In the case $A = \mathbf{C}_q[G]$ the action of H on $\mathbf{C}_q[G]$ is therefore algebraic and given by $r_\chi(X) = X\chi(X)$.

Denote by Γ the subgroup of H corresponding to matrices with entries equal to ± 1. Denote by γ_i for $i = 1, \ldots, n-1$, the element with -1 in the (i, i) and $(i+1, i+1)$ position and 1's elsewhere. Obviously Γ is generated by the γ_i. Using the description of $c^\varepsilon_{i,w}$ as a quantum minor given in 1.5 it is easily verified that the action of γ_i on the elements $c^\varepsilon_{i,w}$ is given by

$$\gamma_i(c^\varepsilon_{j,w}) = \begin{cases} c^\varepsilon_{j,w} & \text{if } j \neq i \\ -c^\varepsilon_{j,w} & \text{if } j = i . \end{cases}$$

Definition. *We denote by $B = \mathbf{C}_q[\bar{G}] = \mathbf{C}_q[G]^\Gamma$ the algebra of elements of $\mathbf{C}_q[G]$ invariant under the action of Γ.*

Definition. *Let $w = (w_+, w_-) \in W \times W$. Recall that $A_w = (A/I_w)_{E_w}$. Since $\gamma(I_w) \subseteq I_w$ and $\gamma(E_w) \subseteq E_w$ for all $\gamma \in \Gamma$, there is a natural induced action of Γ on A_w. We define $B_w = A_w^\Gamma$.*

Notice that $B_w = (B/(I_w \cap B))_{(E_w \cap B)}$. In order to simplify the notation we continue to denote by $c^\varepsilon_{i,y}$ the image of $c^\varepsilon_{i,y}$ in A_w.

It is fairly easy to see that A_w has a natural structure as a crossed product of the dual group $\hat{\Gamma}$ over B_w. Denote by $\hat{\Gamma}$ the dual group of Γ and denote by $\hat{\gamma}_i$ the element of $\hat{\Gamma}$ such that $\hat{\gamma}_i(\gamma_j) = (-1)^{\delta_{ij}}$. Define a map $\phi: \hat{\Gamma} \to A_w$ by $\phi(\hat{\gamma}_{i_1} \dots \hat{\gamma}_{i_t}) = c^+_{i_1, w_+} \dots c^+_{i_t, w_+}$ if $i_1 < \dots < i_t$. Then A_w is a crossed product of $\hat{\Gamma}$ over B_w via ϕ in the sense of [13, 1.5.8].

2.6. Fix $w = (w_+, w_-) \in W \times W$.

Definition. *Let $y \in W$. In A_w set $z^\varepsilon_{i,y} = c^\varepsilon_{i,y}(c^\varepsilon_{i,w_\varepsilon})^{-1}$ and $t_i = c^-_{i,w_-}(c^+_{i,w_+})^{-1}$.*

Clearly these elements belong to B_w. We define C_w to be the subalgebra of B_w generated by the set

$$\{z^\varepsilon_{i,y} \mid \varepsilon = \pm, i = 1, \dots, n-1, y \in W\} \cup \{t_i^{\pm 1} \mid i = 1, \dots, n-1\}.$$

Clearly $z^\varepsilon_{i,y} = 0$ for $y >_i w_\varepsilon$ and $z^\varepsilon_{i,w_\varepsilon} = 1$. Thus

$$C_w = \mathbf{C}[t_i^{\pm 1}, z^\varepsilon_{i,y} \mid \varepsilon = \pm, y <_i w_\varepsilon, i = 1, \dots, n-1].$$

We now show that B_w is the localization of C_w with respect to an appropriate normal element. Recall [14, §2] that the relation $\mathrm{Det}_q = 1$ may be written, for each $i = 1, \dots, n-1$, as $1 = \sum_{y \in \hat{W}_i} \alpha_{i,y} c^+_{i,y} c^-_{i,y}$, where $\alpha_{i,y} \in \mathcal{Q}$ and $\hat{W}_i$ is a transversal of W_i in W. Using Theorem 2.2.1 and the description of the $c^\varepsilon_{i,w}$ as quantum minors given in 1.5, we obtain that C_w contains the elements

$$d_i = (c^+_{i,w_+})^{-1}(c^-_{i,w_-})^{-1} = \sum_{y \in \hat{W}_i} \beta_{i,y} z^+_{i,y} z^-_{i,y},$$

where $\beta_{i,y} \in \mathcal{Q}$. Define d to be $d_1 d_2 \dots d_{n-1}$.

Theorem 2.6.1. *The element d is a normal element of C_w and $B_w = C_w[d^{-1}]$.*

Proof. It follows easily from Theorem 2.2.1 that $dA_w = A_w d$. Since each $z^\varepsilon_{i,y}$ is an eigenvector for conjugation by d, it is clear that $dC_w d^{-1} = C_w$. Thus d is a normal element of C_w. It follows from Theorem 1.5.1 that A_w is spanned by elements of the form vd^t, where v is a word in the $c^\varepsilon_{i,y}$ and t is a non-negative integer. Such words are clearly eigenvectors for the action of Γ. Hence B_w is spanned by the words with eigenvalue 1; that is, words for which the number of occurrences in v of elements of the form $c^\varepsilon_{i,y}$, for a fixed i is even, say $2m_i$. For such words it follows from the normality of the elements $c^\varepsilon_{i,w_\varepsilon}$ (Theorem 2.2.1) that if $t > m_i$ for all i, then $vd^t \in C_w$. Hence for all $b \in B_w$, there exists a positive integer m such that $bd^m \in C_w$. □

2.7. We shall also be interested in the subalgebras of elements invariant under the action of the whole group H. There is a natural induced algebraic action of H on A_w and B_w. Let $\lambda \in \mathbf{C}^*$ and let $h = \lambda e_{ii} + \lambda^{-1} e_{i+1,i+1}$. Then it follows from the description of the $c^\varepsilon_{i,y}$ as quantum minors that

$$h(c^\pm_{j,y}) = \begin{cases} \lambda^{\pm 1} c^\pm_{j,y} & \text{if } j = i \\ c^\pm_{j,y} & \text{if } j \neq i. \end{cases}$$

It is thus clear that the elements $z^\varepsilon_{i,y}$ are H-invariant.

Theorem 2.7.1. (i) $C^H_w = \mathbf{C}[z^\varepsilon_{i,y} \mid \varepsilon = \pm, 1 \leq i \leq n-1, y \in W]$.

(ii) $A^H_w = B^H_w = C^H_w[d^{-1}]$.

(iii) *The monomials $t_1^{r_1} \dots t_{n-1}^{r_{n-1}}$ for $(r_1, \dots, r_{n-1}) \in \mathbf{Z}^{n-1}$ form a basis for C_w as a left or right C^H_w-module and a basis for B_w as a left or right B^H_w-module.*

Proof. Denote by S the subalgebra of C_w generated by the $z^{\varepsilon}_{i,y}$. Clearly $S \subseteq C^H_w$. On the other hand, since C_w is generated over S by the t_i which are invertible elements normalising S ($t_i S = S t_i$), it follows that the given monomials span C_w as a left or right S-module. It is also clear that if $h = \lambda e_{ii} + \lambda^{-1} e_{i+1,i+1}$, then $h(t_i) = \lambda^{-2} t_i$ and $h(t_j) = t_j$ for $j \neq i$. Thus each distinct monomial corresponds to a different character of H. Hence the monomials must be linearly independent over C^H_w. Thus $C_w = \bigoplus_{\mathbf{r} \in \mathbf{Z}^{n-1}} S t^{\mathbf{r}}$, where $t^{\mathbf{r}} = t_1^{r_1} \ldots t_{n-1}^{r_{n-1}}$ if $\mathbf{r} = (r_1, \ldots, r_{n-1})$. This proves (i) and the first part of (iii). The remaining assertions then follow easily. □

2.8. We are now in a position to formulate more precisely the conjectures made in the introduction concerning Prim $\mathbf{C}_q[G]$. Although we only consider here the case when $G = SL(n)$, similar conjectures may be made in the general case. The reader is referred to Appendix A for a description of the symplectic leaves of G. Denote by A the algebra $\mathbf{C}_q[G]$.

Definition. *For each* $w \in W \times W$, *define* $\operatorname{Spec}_w A = \{P \in \operatorname{Spec} A \mid P \supseteq I_w$ *and* $P \cap E_w = \emptyset\}$ *and* $\operatorname{Spec}_w B = \{P \in \operatorname{Spec} B \mid P \supseteq I_w \cap B_w$ *and* $P \cap E_w = \emptyset\}$. *Elements of* $\operatorname{Spec}_w A$ *and* $\operatorname{Spec}_w B$ *are said to be of type* w. *Set* $\operatorname{Prim}_w A = \operatorname{Spec}_w A \cap \operatorname{Prim} A$ *and* $\operatorname{Prim}_w B = \operatorname{Spec}_w B \cap \operatorname{Prim} B$.

The action of H on A described above induces an action of H on Prim A for which the locally closed subsets $\operatorname{Prim}_w A$ are invariant for all $w \in W \times W$. Since the action of H is algebraic, $\operatorname{Stab}_H P$ is a closed subgroup of H and $H/\operatorname{Stab}_H P$ is a torus for all $P \in \operatorname{Prim} A$.

Conjecture 1. $\operatorname{Prim} A = \coprod_{w \in W \times W} \operatorname{Prim}_w A$ and $\operatorname{Prim}_w A$ is a non-empty H-orbit for all $w \in W \times W$. If $P_{\dot w}$ is a primitive ideal of type w, then $H/\operatorname{Stab}_H P_{\dot w}$ is a torus of rank $\operatorname{rk} G - s(w)$. Hence there is a bijection β: $\operatorname{Prim} A \to \operatorname{Symp} G$ such that $\beta(\operatorname{Prim}_w A) = \operatorname{Symp}_w G$.

Conjecture 2. One may define a bijection β: $\operatorname{Prim} A \to \operatorname{Symp} G$ as in Conjecture 1 such that β is order reversing and GKdim $A/P = \dim \beta(P)$ for all $P \in \operatorname{Prim} A$.

Both conjectures are known to be true in the case $G = SL(2, \mathbf{C})$ (see Appendix B). Conjecture 1 is proved in Sect. 4 in the case when $G = SL(3, \mathbf{C})$.

3. The Adjoint Action

3.1. Henceforth we restrict our attention to the case $G = SL(3)$. We shall denote the algebra $\mathbf{C}_q[SL(3)]$ by A. In order to study the ideals of A we look at the ideals of C_w and B_w invariant under the adjoint action. At the same time we study in detail the structure of the algebra C^H_w, showing that it is an iterated Ore extension in the sense of [13]. We shall therefore be interested in bases consisting of standard monomials as defined below.

Definition. *Let* $\mathscr{Y} = \{y_1, y_2, \ldots, y_t\}$ *be an indexed set of elements. The standard monomials in* $\mathscr{Y}$ *are defined to be the elements* $y^{\mathbf{r}} = y_1^{r_1} \ldots y_t^{r_t}$, *where* $\mathbf{r} = (r_1, \ldots, r_t) \in \mathbf{N}^t$.

3.2. We shall show that for each w, there exists a certain set of $z^{\varepsilon}_{i,y}$ such that for a suitably chosen ordering, the standard monomials in these z's form a basis for C^H_w. Clearly we should exclude from such a set all the $z^{\varepsilon}_{i,y}$ for which $y \nless_i w_\varepsilon$. The Plucker relations imply that certain other generators are redundant.

Definition. *Fix $w \in W \times W$. Define*

$$\mathscr{X} = \{z^{\varepsilon}_{i,y} | \varepsilon = \pm, y <_i w_\varepsilon, i = 1, 2\} - \{z^{\varepsilon}_{2, w_\varepsilon w_0} | \varepsilon = \pm\} .$$

Define **I** *to be the corresponding index set; that is,*

$$\mathbf{I} = \{(\varepsilon, y, i) | \varepsilon = \pm, y <_i w_\varepsilon, i = 1, 2\} - \{(\varepsilon, 2, w_\varepsilon w_0)\} .$$

Theorem 3.2.1. $C^H_w = \mathbf{C}[\mathscr{X}]$.

Proof. From Theorem 2.7.1 and the remarks at the beginning of 2.6, it suffices to show that if $w_\varepsilon w_0 < w_\varepsilon$, then $z^{\varepsilon}_{2, w_\varepsilon w_0} \in \mathbf{C}[\mathscr{X}]$. The Plucker relations given in Theorem 1.4.1 (d) imply that in A_w,

$$\sum_{y \in \hat{W}_1, y \leqq_1 w_+} \alpha_y c^+_{1,y} c^+_{2, y w_0} = 0, \quad \text{for some } \alpha_y \in \mathscr{Q} .$$

Multiplying by $(c^+_{1,w_+})^{-1}(c^+_{2,w_+})^{-1}$ and using the fact that $z^+_{1,w_+} = 1$, we obtain:

$$z^+_{2, w_+ w_0} = \sum_{y \in \hat{W}_1, y <_1 w_+} \gamma_y z^+_{1,y} z^+_{2, y w_0}, \text{ for some } \gamma_y \in \mathscr{Q} .$$

Now for $y <_1 w_+$, $z^+_{2, y w_0}$ is either 0 or an element of $\mathscr{X}$. Hence $z^+_{2, w_+ w_0} \in \mathbf{C}[\mathscr{X}]$, as required. A similar argument works for $z^-_{2, w_- w_0}$.

Remark. It is important to notice that if $w_+ <_2 w_+ w_0$, then the above relation collapses to $0 = 0$. Nontrivial relations for $z^+_{2, w_+ w_0}$ only occur when w_+ or w_- belongs to $\{(1, 3), (1, 3, 2), (1, 2, 3)\}$.

3.3. The ordered indexing on the set $\mathscr{X}$ will be induced from the following ordering on the set $\mathbf{S} = \{(\varepsilon, i, y) | \varepsilon = \pm, i = 1, 2 \text{ and } y \in \hat{W}_i\}$.

Definition. Define a total ordering on the set **S** by:

$$(\varepsilon', i', y') < (\varepsilon, i, y) \text{ iff } \begin{cases} i' < i; & \text{or} \\ i' = i \text{ and } y' >_i y; & \text{or} \\ i' = i \text{ and } y' =_i y \text{ and } \varepsilon' = +, \varepsilon = - . \end{cases}$$

Since $\hat{W}_i$ is totally ordered by $\leqq_i$, it is easy to see that this defines a total ordering on **S**.

The required commutation relations on the $z^{\varepsilon}_{i,y}$ follow from the following commuation relations in $\mathbf{C}_q[SL(3)]$.

Proposition 3.3.1. *Suppose that $(\varepsilon', i', y') < (\varepsilon, i, y)$. Then there exists an $\alpha \in \mathbf{C}^*$ such that*

$$c^{\varepsilon'}_{i',y'} c^{\varepsilon}_{i,y} = \alpha c^{\varepsilon}_{i,y} c^{\varepsilon'}_{i',y'} + \sum_j \beta_j a_j a'_j ,$$

where $\beta_j \in \mathbf{C}$, $a_j \in \{c^{\varepsilon}_{i,u} | (\varepsilon, i, u) < (\varepsilon, i, y)\}$ and $a'_j \in \{c^{\varepsilon'}_{i',u} | (\varepsilon', i', u) < (\varepsilon, i, y)\}$.

Proof. The result may be deduced from the commutation relations given in [8, 2.1, 2.2, 2.13–2.16] using the equations in Sect. 1.5. Alternatively, one may use the more general formula [19, §3.8] which follows from the form of the universal R-matrix for $U_q(\mathbf{sl}(3, \mathbf{C}))$. □

3.4. We define $R(\varepsilon, i, y) = \mathbf{C}[z^{\eta}_{j,u} | (\eta, j, u) < (\varepsilon, i, y)]$.

Proposition 3.4.1. *The algebra $R(\varepsilon, i, y)$ is spanned by the standard monomials in $\{z^{\varepsilon'}_{i',y'} \in \mathscr{Z} \mid (\varepsilon', i', y') < (\varepsilon, i, y)\}$. In particular, the algebra C^H_w is spanned by the standard monomials in the elements of $\mathscr{Z}$.*

Proof. It follows from the proof of Theorem 3.2.1 that $z^{\varepsilon}_{2, w_\varepsilon w_0} \in R(\varepsilon, 2, w_\varepsilon w_0)$. On the other hand Proposition 3.3.1 implies that $R(\varepsilon, i, y)[z^{\varepsilon}_{i,y}]$ is spanned as a left $R(\varepsilon, i, y)$ module by the powers of $z^{\varepsilon}_{i,y}$. The result then follows by induction. □

3.5. In order to show that the standard monomials from Proposition 3.4.1 form a basis for C^H_w, we consider the adjoint action of $\mathbf{C}_q[SL(3)]$. Let us recall the basic definitions and properties for the adjoint action of a Hopf algebra on a bimodule.

Let $(R, \Delta, \varepsilon, S)$ be a complex Hopf algebra and let M be an R-bimodule. The adjoint action of R on M is defined by: $(\operatorname{ad} h)x = h_{(1)} x S(h_{(2)})$ for all $h \in R$ and $x \in M$, where we are using the Sweedler notation together with the obvious summation convention. We set $M^{\mathrm{ad}} = \{x \in M \mid (\operatorname{ad} h)x = \varepsilon(h)x, \forall h \in R\}$. It is easily seen that $M^{\mathrm{ad}} = \{x \in M \mid hx = xh, \forall h \in R\}$.

The map ad: $R \to \operatorname{End}_{\mathbf{C}} M$ is a homomorphism of algebras and in this way M becomes a left R-module via ad. Suppose now that M also has the structure of a $\mathbf{C}$-algebra compatible with its bimodule structure; i.e.

$$\forall x, y \in M, \forall h \in R, \quad h(xy) = (hx)y \quad \text{and} \quad (xy)h = x(yh)\,.$$

Then under the adjoint action, M has the structure of a R-module algebra in the sense that $(\operatorname{ad} h)(xy) = (\operatorname{ad} h_{(1)})(x)(\operatorname{ad} h_{(2)})(y)$.

3.6. These generalities apply to the Hopf algebra $\mathbf{C}_q[SL(3)]$ and any bimodule M. Recall that $\mathbf{C}_q[SL(3)] = \mathbf{C}[X_{ij} \mid 1 \leqq i, j \leqq 3]$, where the X_{ij} are the coordinate functions for the standard 3-dimensional representation of $U_q(\mathbf{sl}(3, \mathbf{C}))$. Since $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$, the adjoint action of X_{ij} is given by $(\operatorname{ad} X_{ij})m = \sum_k X_{ik} m S(X_{kj})$ for all $m \in M$. Denote ad X_{ij} by ad_{ij}, and define the adjoint matrix of m to be $[\operatorname{ad} m] = [\operatorname{ad}_{ij} m]_{1 \leqq i,j \leqq n}$. Denote by X the matrix of coordinate functions $(X_{ij}) \in M_n(A)$ and by $S(X)$ the matrix $(S(X_{ij}))$. It follows easily from the coalgebra structure of A that $S(X) = X^{-1}$.

Proposition 3.6.1. *Let ϕ: $\mathbf{C}_q[SL(3)] \to B$ be a $\mathbf{C}$-algebra map. Then for any $b \in B$, $[\operatorname{ad} b] = \phi(X) b \phi(S(X))$. The map $[\operatorname{ad} -]$: $B \to M_n(B)$ is an algebra map. In particular, $[\operatorname{ad} bc] = [\operatorname{ad} b][\operatorname{ad} c]$ for all $b, c \in B$.*

Proof. The formula for $[\operatorname{ad} b]$ is clear. For simplicity, drop the ϕ and consider $M_n(A)$ as acting on $M_n(B)$ via ϕ. Then $[\operatorname{ad} bc] = XbcS(X) = XbIcS(X) = XbS(X)XcS(X) = [\operatorname{ad} b][\operatorname{ad} c]$. □

3.7. In this section we study the adjoint action of A on the subalgebra generated by the elements $t_1^{\pm 1}$, $t_2^{\pm 1}$ defined in Sect. 2.6. To simplify the notation a little, set

$$a = w_-(1), \quad b = w_+(1), \quad c = w_+(3), \quad d = w_-(3)\,.$$

In this notation, $t_1 = D_{a1} X_{b1}^{-1}$, $t_2 = X_{d3} D_{c3}^{-1}$ and $t_1 t_2 = q^{2(\delta_{a,c} - \delta_{b,d})} t_2 t_1$. Recall that by Theorem 2.7.1, the elements $t_1^n t_2^m$ for $n, m \in \mathbf{Z}$ form a basis for the subalgebra $\mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. Denote by $F_i(\alpha)$ the diagonal scalar matrix with the scalar α in the $(i, i)^{\text{th}}$ position and 1's elsewhere on the diagonal.

Lemma 3.7.1. *With the above notation we have that*

$$[\operatorname{ad} t_1] = F_b(q^2) F_a(q^{-2}) t_1 \quad \textit{and} \quad [\operatorname{ad} t_2] = F_c(q^{-2}) F_d(q^2) t_2\,.$$

Proof. It is easily verified that $X_{b1}X = F_b(q^2)XF_1(q^{-2})X_{b1} \pmod{I_w}$ and similarly that $D_{a1}S(X) = F_1(q^2)S(X)F_a(q^{-2})D_{a1} \pmod{I_w}$. Combining these two identities gives the formula for $[\operatorname{ad} t_1]$. The proof of the second equality is similar. □

Proposition 3.7.2. *The algebra* $\mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]^{ad}$ *is a subalgebra of the centre of* A_w *equal to*:

(i) $\mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ *if* $w_+ = w_-$;
(ii) $\mathbf{C}[t_i^{\pm 1}]$ *if* $w_+ = w_-(W_i)$ *and* $w_+ \neq w_-$;
(iii) $\mathbf{C}$ if $w_+ \neq w_-(W_i)$ *for* $i = 1$ *and* 2 *but* $w_- \neq w_+ w_0$;
(iv) $\mathbf{C}[(t_1 t_2^{-1})^{\pm 1}]$ *if* $w_- = w_+ w_0$.

Proof. It is clear that $\mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]^{ad}$ has as a basis the set of all monomials $t_1^n t_2^m$ which are ad A-invariant. Now

$$[\operatorname{ad} t_1^n t_2^m] = F_b(q^{2n})F_c(q^{-2m})F_a(q^{-2n})F_d(q^{2m})t_1^n t_2^m \,.$$

The result then follows easily. □

Notice that the dimension of $\mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]^{ad}$ is therefore $2 - s(w)$, where $s(w)$ is the length of a shortest expression for $w_+ w_-^{-1}$ as a product of reflections.

3.8. The adjoint action of A on C_w^H is a little more complicated. As usual let $w = (w_+, w_-) \in W \times W$. As before, set $a = w_-(1)$, $b = w_+(1)$, $c = w_+(3)$, $d = w_-(3)$ and set $p = q^2 - q^{-2}$.

Lemma 3.8.1. *Let* y *be an arbitrary element of* W *and set* $r = y(1)$ *and* $s = y(3)$. *The adjoint action on* $z_{i,y}^{\varepsilon}$ *is given by*

$$[\operatorname{ad} z_{1,y}^+] = F_r(q^{-2})F_b(q^2)z_{1,y}^+ - \sum_{i=r+1}^{b} p z_{1,(1,i)}^+ e_{ir} \,,$$

$$[\operatorname{ad} z_{1,y}^-] = F_r(q^{-2})F_a(q^2)z_{1,y}^- + q^{2(r+1-a)} \sum_{i=r+1}^{a} (-1)^{r-i-1} p z_{1,(1,i)}^- e_{ri} \,,$$

$$[\operatorname{ad} z_{2,y}^+] = F_c(q^{-2})F_s(q^2)z_{2,y}^+ - q^{2(s-c-1)} \sum_{i=c}^{s-1} (-1)^{s-i-1} p z_{2,(i,3)}^+ e_{si} \,,$$

$$[\operatorname{ad} z_{2,y}^-] = F_d(q^{-2})F_s(q^2)z_{2,y}^- + \sum_{i=d}^{s-1} p z_{2,(i,3)}^- e_{is} \,.$$

Proof. Recall that $z_{1,y}^+ = X_{r1}X_{a1}^{-1}$. One verifies first that for $r \leqq j \leqq a$,

$$XX_{j1} = X_{j1}F_j(q^{-2})XF_1(q^2) - \left(\sum_{i=j+1}^{a} pX_{i1}e_{ij}\right)XF_1(q^2) \pmod{I_w} \,.$$

Hence $XX_{a1}^{-1} = X_{a1}^{-1}F_a(q^2)XF_1(q^{-2})$. Putting these two formulas together yields the desired result. A similar calculation proves the other three formulas. □

3.9. Thus for each $z_{i,y}^{\varepsilon}$ the matrix $[\operatorname{ad} z_{i,y}^{\varepsilon}]$ is of the form $D + N$, where D is diagonal and N is a strictly upper or lower triangular matrix with all its non-zero entries in a single row or column. Furthermore the nonzero entry in N that is furthest from the diagonal is a scalar. Since this entry is of particular importance we define ϕ to be the function that associates to $z_{i,y}^{\varepsilon}$ this position. That is, for a fixed w we define

$$\phi(z^+_{1,y}) = (w_+(1), y(1)), \quad \phi(z^+_{2,y}) = (y(3), w_+(3)),$$

$$\phi(z^-_{1,y}) = (y(1), w_-(1)), \quad \phi(z^-_{2,y}) = (w_-(3), y(3)).$$

This map is not injective on the set of all $z^\varepsilon_{i,y}$ since for instance when $w_+ = (13)$, $\phi(z^+_{2,e}) = (3,1) = \phi(z^+_{1,e})$. However when ϕ is restricted to $\mathscr{Z}$ we do have injectivity.

Lemma 3.9.1. *The map ϕ restricted to $\mathscr{Z}$ is injective.*

Proof. Clearly $\phi(z^+_{i,y}) \subset \{(k,l) | k > l\}$ and $\phi(z^-_{i,y}) \subset \{(k,l) | k < l\}$ so we may consider the two cases separately. Suppose that $\phi(z^+_{1,y}) = \phi(z^+_{2,y'})$, where $y <_1 w_+$ and $y' <_2 w_+$. This means that $(w_+(1), y(1)) = (y'(3), w_+(3))$. Hence $y = w_+ w_0(W_1)$ and $y' = w_+ w_0(W_2)$. Since $z^+_{2, w_+ w_0} \notin \mathscr{Z}$, the result follows. The other case is similar. □

Proposition 3.9.2. *Let $y \in W$ and suppose that $\phi(z^\varepsilon_{i,y}) = (k,l)$. Set $[\operatorname{ad} z^\varepsilon_{i,y}] = [a_{ij}]$. Then $[\operatorname{ad}(z^\varepsilon_{i,y})^n] = [\operatorname{ad} z^\varepsilon_{i,y}]^n = [a_{ij}(n)]$, where*
(i) $a_{ii}(n) = a^n_{ii} \in \mathbf{C}^*(z^\varepsilon_{i,y})^n$,
(ii) $a_{ij} = 0$ *implies* $a_{ij}(n) = 0$.
(iii) $a_{kl}(n) = \operatorname{ad}_{kl}(z^\varepsilon_{i,y})^n \in \mathbf{C}^*(z^\varepsilon_{i,y})^{n-1}$.

Proof. Write $[a_{ij}] = D + N$, where D is diagonal and N is strictly upper or lower triangular. Then because of the particular form of N, we have that $ND^iN = 0$ for any i. Hence $[a_{ij}(n)] = (D+N)^n = D^n + \sum_{s=0}^{n-1} D^s N D^{n-s-1}$. The first two assertions are then clear, as is the fact that

$$a_{kl}(n) = \sum_{s=0}^{n-1} (q^2)^{\pm(n-2s-1)} a_{kl}(z^\varepsilon_{i,y})^{n-1}.$$

Since q is not a root of unity, the coefficient on the right-hand side is non-zero. □

The lemma states that if $(k,l) = \phi(z^\varepsilon_{i,y})$, then ad_{kl} behaves rather like a partial differential operator with respect to $z^\varepsilon_{i,y}$. However, on an arbitrary standard monomial it is important to apply these operators in the correct order. This necessitates defining a new ordering on the standard monomials.

Let $\mathbf{I} = \{(\varepsilon, y, i) | \varepsilon = \pm, y <_i w_\varepsilon, \ i = 1,2\} - \{(\pm, 2, w_\varepsilon w_0)\}$ be the index set corresponding to the set $\mathscr{Z}$ and let $\mathbf{K} = \phi(\mathbf{I})$ (where ϕ is the obvious induced map on $\mathbf{I}$). For each w let $\prec$ be a total ordering on the set $\{(i,j) | i,j = 1,2,3, \ i \neq j\}$ satisfying

$$(1,i) \succ (1,i') \succ (2,3) \succ (3,j) \succ (3,j') \succ (2,1)$$

and i and i' are chosen so that if $(1,i)$ and $(1,i')$ are both in $\phi(\mathbf{I})$ then the ordering $\prec$ reverses the ordering induced by ϕ. We denote by $\prec$ the induced ordering on the subset $\mathbf{K}$. The ordering induced by $\prec$ on $\mathbf{I}$ via ϕ^{-1} will also be denoted by $\prec$. The ordering $\prec$ on $\mathbf{I}$ extends naturally to a lexicographic ordering on $\mathbf{N}^{\mathbf{I}}$ which will again be denoted by $\prec$.

Theorem 3.9.3. *Let $\mathbf{m} \in \mathbf{N}^{\mathbf{I}}$ and let $\phi(\mathbf{m})$ be its image in $\mathbf{N}^{\mathbf{K}}$. Let $M^{\mathbf{m}}$ be a standard monomial in the $z^\varepsilon_{i,y}$ with respect to the order defined in* 3.3 *and let $X^{\phi(\mathbf{m})}$ be the standard monomial in the X_{ij} with respect to the ordering on $\mathbf{K}$ defined above. Then* (i) $\operatorname{ad} X^{\phi(\mathbf{m})} M^{\mathbf{m}} \in \mathbf{C}^*$; (ii) $\operatorname{ad} X^{\phi(\mathbf{m})} M^{\mathbf{n}} = 0$ *for all* $\mathbf{n} \prec \mathbf{m}$.

Proof. Define $\mathrm{Supp}(m) = \{\eta \in \mathbf{I} \mid \mathbf{m}_\eta \neq 0\}$. For $(i,j) \in \mathbf{K}$, define f_{ij} to be the element of $\mathbf{N}^{\mathbf{I}}$ such that $(f_{ij})_\xi = \delta_{\xi, \varphi^{-1}(i,j)}$. It suffices to prove that for any monomial $M^{\mathbf{m}}$ and any $(i,j) \succeq \mathrm{Max}(\phi(\mathrm{Supp}(\mathbf{m}))$,

$$(\mathrm{ad}\, X_{ij}) M^{\mathbf{m}} = \begin{cases} cM^{\mathbf{m}-f_{ij}} & \text{if } (i,j) = \mathrm{Max}(\phi(\mathrm{Supp}(\mathbf{m}))\,; \\ 0 & \text{if } (i,j) \succ \mathrm{Max}(\phi(\mathrm{Supp}(\mathbf{m}))\,; \end{cases}$$

for some $c \in \mathbf{C}^*$. Suppose that $M^{\mathbf{m}} = Z_1^{m_1} \ldots Z_t^{m_t}$, where $Z_i \in \mathscr{X}$. Then $\mathrm{ad}_{ij} M^{\mathbf{m}}$ is the (i,j)-entry of $\mathrm{ad}(Z_1^{m_1}) \ldots \mathrm{ad}(Z_t^{m_t})$. The form of these matrices was computed in Proposition 3.9.1. A lengthy but routine calculation shows in all cases that if k is such that $(i,j) = \phi(Z_k) = \mathrm{Max}(\phi(\mathrm{Supp}(\mathbf{m}))$, then

$$(\mathrm{ad}\, X_{ij}) M^{\mathbf{m}} = \mathrm{ad}_{ii} Z_1^{m_1} \ldots \mathrm{ad}_{ii} Z_{k-1}^{m_{k-1}} \mathrm{ad}_{ij} Z_k^{m_k} \mathrm{ad}_{jj} Z_{k+1}^{m_{k+1}} \ldots \mathrm{ad}_{jj} Z_t^{m_t}\,,$$

and that if $(i,j) \succ \mathrm{Max}(\phi(\mathrm{Supp}(\mathbf{m}))$, then $(\mathrm{ad}\, X_{ij}) M^{\mathbf{m}} = 0$. Hence the result above follows from Proposition 3.9.1. □

3.10. We now come to the most important results of the section. For each character $\nu \in R(A)$ let us denote by C_w^ν the ν-isotypic part of C_w under the adjoint action. Denote by $\mathrm{Soc}\, C_w$ the socle of C_w under this action.

Theorem 3.10.1. 1. *The algebras C_w and C_w^H are iterated Ore extensions. Hence C_w, C_w^H and B_w are all domains.*

2. $\mathrm{Soc}\, C_w = \bigoplus_{\nu \in R(A)} C_w^\nu = \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. *Hence $C_w^{ad} = \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]^{ad}$ is as described in Sect.* 3.7.2.

3. If $\nu \in R(A)$ *is such that* $C_w^\nu \neq 0$, *then there exists a invertible element u_ν such that $C_w^\nu = u_\nu C_w^{ad}$.*

Proof. Theorem 3.9.3 implies that the standard monomials in the elements of $\mathscr{X}$ form a basis for C_w^H. The fact that C_w^H is an iterated Ore extension is an induction based on [2, 1.3] using Proposition 3.4.1. Theorem 2.7.1 implies that C_w is an Ore extension of C_w^H. Thus C_w and C_w^H are both domains. Since B_w is a localization of C_w (Theorem 2.6.1), it too is a domain.

Now let $f \in C_w$. We may write f in the form $f = \sum_{\mathbf{n} \leq \mathbf{m}} \alpha_{\mathbf{n}} M^{\mathbf{n}}$, where $M^{\mathbf{n}}$ is the monomial described in 3.9, $\alpha_{\mathbf{n}} \in \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ for all $\mathbf{n}$ and $\alpha_{\mathbf{m}} \neq 0$. By Theorem 3.9.3, there exists an $a \in A$ such that a is a product of elements of the form X_{ij} and such that

$$(\mathrm{ad}\, a) M_{\mathbf{n}} = \begin{cases} 0 & \text{if } \mathbf{n} < \mathbf{m} \\ 1 & \text{if } \mathbf{n} = \mathbf{m}\,. \end{cases}$$

Now $\alpha_{\mathbf{n}} = \sum_{\chi \in R(A)} \alpha_{\mathbf{n},\chi}$, where $\alpha_{\mathbf{n},\chi} \in \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]_\chi$. Moreover,

$$\begin{aligned}(\mathrm{ad}\, a)\alpha_{\mathbf{n},\chi} M_{\mathbf{n}} = (\mathrm{ad}\, a_{(1)})\alpha_{\mathbf{n},\chi}(\mathrm{ad}\, a_{(2)}) M_{\mathbf{n}} &= \chi(a_{(1)})\alpha_{\mathbf{n},\chi} \mathrm{ad}\, a_{(2)} M_{\mathbf{n}} \\ &= \alpha_{\mathbf{n},\chi} \mathrm{ad}(r_\chi(a)) M_{\mathbf{n}}\,.\end{aligned}$$

But $r_\chi(a) = \lambda_\chi a$ for some non-zero scalar λ_χ. Thus

$$(\mathrm{ad}\, a) f = \sum_\chi \lambda_\chi \alpha_{\mathbf{m},\chi} \in \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}] \setminus \{0\}\,.$$

Since $\mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ is a semi-simple ad-A module, this proves the second assertion. The third statement then follows easily from 3.7. □

Remark. We can identify C_w^{ad} with $\mathbf{C}[H_w]$, the algebra of functions on the torus $H_w = H/\mathrm{Stab}_H \mathscr{C}_{\dot{w}}$ (see Theorem A.3.1).

4. Primitive Spectrum of $\mathbf{C}_q[G]$

4.1. We begin with a result showing that the study of Spec A and Spec B may be reduced to the study of Spec A_w and Spec B_w, $w \in W \times W$ respectively.

Proposition 4.1.1. *Let* $P \in \mathrm{Spec}\, A$ (resp. Spec B). *Then there exists a unique* $w \in W \times W$ *such that* $P \supset I_w$ (resp. $P \supset I_w \cap B$) *and* $P \cap E_w = \emptyset$.

Proof. Let $P \in \mathrm{Spec}\, A$. Define the elements $w_+^{(i)} \in \hat{W}_i$, $i = 1, 2$ to be the smallest elements of $\hat{W}_i$ such that $c_{i,y}^+ \in P$ for all $y > w_{(i)}^+$. We want to show that there exists a $w_+ \in W \times W$ such that $w_+ = w_+^{(i)}(W_i)$ for $i = 1, 2$. It is easily verified that this will occur if and only if $w_+^{(1)} w_0 \neq w_+^{(2)}(W_2)$. Suppose that $w_+^{(1)} w_0 = w_+^{(2)}(W_2)$. Recall the Plucker relation

$$\sum_{y \in \hat{W}_1} (-q^2)^{y(1)-1} c_{1,y}^+ c_{2,yw_0}^+ = 0 .$$

Now for $y >_1 w_+^{(1)}$, $c_{1,y}^+ \in P$ by definition. On the other hand, if $y <_1 w_+^{(1)}$, then $yw_0 >_2 w_+^{(1)} w_0 = w_+^{(2)}(W_2)$ by Proposition 1.8. Hence $c_{2,yw_0}^+ \in P$. The remaining term, which is a scalar multiple of $c_{1,w_+^{(1)}}^+ c_{2,w_+^{(2)}}^+$, must therefore lie in P also. However neither $c_{1,w_+^{(1)}}^+$ nor $c_{2,w_+^{(2)}}^+$ lie in P by hypothesis. Moreover $c_{1,w_+^{(1)}}^+$ is normal modulo P by Lemma 2.1. This contradicts the fact that P is prime.

A similar argument produces an analogous element w_-. Thus there exists an element $w = (w_+, w_-)$ such that $c_{i,y}^\pm \in P$ for all $y > w_\pm$ and $c_{i,w_\pm}^\pm \notin P$ for $i = 1, 2$. In other words, $P \supset I_w$ and $P \cap E_w = \emptyset$. It is clear that such an element must be unique.

Now let $P \in \mathrm{Spec}\, B$. By [13, 10.2.10], there exists a $Q \in \mathrm{Spec}\, A$ such that P is minimal over $Q \cap B$. By the first part of the proof there exists a w such that $Q \supset I_w$ and $Q \cap E_w = \emptyset$. Hence it is clear that $P \supset I_w \cap B$. Suppose that $c \in P \cap E_w$. From the minimality of P over $Q \cap B$ and the fact that c is normal modulo I_w it follows easily that $c \in Q$, a contradiction. □

Corollary 4.1.2. *Identify* Spec A *with* $\{P \in \mathrm{Spec}\, A \mid P \supset I_w,\ P \cap E_w = \emptyset\}$. *Then* $\mathrm{Spec}\, A = \bigsqcup_{w \in W \times W} \mathrm{Spec}\, A_w$, *where* $\bigsqcup$ *denotes the disjoint union. Similarly* $\mathrm{Spec}\, B = \bigsqcup_{w \in W \times W} \mathrm{Spec}\, B_w$.

The analogous result concerning the primitive spectrum is also true. However, this is a subtler question and the proof requires the characterization of the primitive ideals as the locally closed elements of Spec A.

4.2. We now return to the study of B_w and C_w. Define the algebra $C_{\dot{w}}$ by:

$$C_{\dot{w}} = \begin{cases} C_w^H, & \text{if } w_+ = w_-; \\ C_w^H[t_j^{\pm 1}], & \text{if } w_+ = w_-(W_i) \text{ and } w_+ \neq w_-(W_j); \\ C_w^H[t_1^{\pm 1}, t_2^{\pm 1}], & \text{if } w_+ \neq w_-(W_i) \text{ for } i = 1, 2 \text{ but } w_- \neq w_+ w_0; \\ C_w^H[t_1^{\pm 1}], & \text{if } w_- = w_+ w_0, \end{cases}$$

and define $B_{\dot{w}}$ to be $C_{\dot{w}}[d^{-1}]$. Then it is clear from 2.7.1 and 3.7.2 that $C_w \cong C_{\dot{w}} \otimes C_w^{ad}$ and $B_w \cong B_{\dot{w}} \otimes C_w^{ad}$. Moreover both $C_{\dot{w}}$ and $B_{\dot{w}}$ are integral domains by 3.10.1. We now show that $B_{\dot{w}}$ is simple. It will then follow that all prime ideals of B_w are induced from C_w^{ad}.

Lemma 4.2.1. *Let I be an ideal of B_w (respectively C_w). Then I is an* ad$A$*-submodule if and only if $I = (I \cap C_w^{ad})B_w$ (respectively $I = (I \cap C_w^{ad})C_w$).*

Proof. Since B_w is a localization of C_w and C_w is ad-invariant, it is enough to prove the result for C_w. Let I be an ideal of C_w and suppose that I strictly contains $(I \cap C_w^{ad})C_w$. Choose $f \in I \backslash (I \cap C_w^{ad})C_w$ and write f (as in the proof of 3.10.1) as $f = \sum_{\mathbf{n} \leq \mathbf{m}} \alpha_{\mathbf{n}} M^{\mathbf{n}}$, where $\alpha_{\mathbf{n}} \in \operatorname{Soc} C_w$ for all $\mathbf{n}$ and $\alpha_{\mathbf{m}} \neq 0$. Assume that $\mathbf{m}$ is minimal for such elements. The argument used in the proof of 3.10.1 implies that I contains $\sum_\chi \lambda_\chi \alpha_{\mathbf{m},\chi}$ for some non-zero scalars λ_χ. Since I is ad-invariant, it therefore contains each $\lambda_\chi \alpha_{\mathbf{m},\chi}$. But $\lambda_\chi \alpha_{\mathbf{m},\chi} \in (C_w)_\chi = u_\chi C_w^{ad}$ for some unit u_χ. Thus $\lambda_\chi \alpha_{\mathbf{m},\chi} \in (I \cap C_w^{ad})C_w$ and so $\alpha_{\mathbf{m}} M_{\mathbf{m}} \in (I \cap C_w^{ad})C_w$, contradicting the minimality of $\mathbf{m}$. □

Theorem 4.2.2. *$B_w \cong B_{\dot{w}} \otimes C_w^{ad}$, where $B_{\dot{w}}$ is a simple algebra. The center of B_w is C_w^{ad} and all ideals of B_w are generated by their intersection with the center. Thus* $\operatorname{Spec} B_w \cong \operatorname{Spec} C_w^{ad}$ *and* $\operatorname{Prim} B_w \cong \operatorname{Prim} C_w^{ad}$*. All primitive ideals of B_w are maximal and all prime ideals are completely prime. If $P \in \operatorname{Prim} B_w$ then* $\operatorname{GK dim} B_w/P = l(w) + s(w)$.

Proof. Let P_e be the ideal of B_w generated by elements of the form $t - 1$, where $t \in \{t_1^n t_2^m \mid n, m \in \mathbf{Z}\} \cap C_w^{ad}$. Then clearly $B_{\dot{w}} \cong B_w/P_e$. Hence P_e is a completely prime ideal of B_w. From the lemma we have that P_e is a maximal ad A-invariant ideal of B_w. Since A_w is a finite normalizing extension of B_w, it follows from "Lying over" and "Going up" [13, 10.2], that P_e is in fact a maximal ideal of B_w. Hence $B_{\dot{w}}$ is simple. Because $B_{\dot{w}}$ satisfies the nullstellensatz [13, 9.1], it follows that $B_{\dot{w}}$ is central simple and the assertion concerning the spectrum is a consequence of [3, 4.5.1]. By the nullstellensatz again, the primitive ideals are generated by the maximal ideals of C_w^{ad}. Since the quotient of B_w by such an ideal will always be isomorphic to $B_{\dot{w}}$, all the primitive ideals are completely prime. Since every prime ideal is an intersection of primitives it follows easily that all the prime ideals are completely prime. The assertion concerning the Gelfand–Kirillov dimension follows from the description of $B_{\dot{w}}$ as a localization of an Ore extension and a slight generalization of [13, 8.2.10]. □

4.3. We may now use Corollary 4.1.2 to deduce some global results about the primitive spectrum of B. We shall say that a Noetherian $\mathbf{C}$-algebra R satisfies the Dixmier–Moeglin condition if the following conditions are equivalent for a prime ideal P: (a) P is primitive; (b) P is rational (the center of the ring of fractions of R/P is $\mathbf{C}$); (c) P is locally closed in $\operatorname{Spec} R$. Recall that the action of H by right translation on B induces a natural action of H on $\operatorname{Prim} B$.

Theorem 4.3.1. *In the notation of Sect.* 2.8, *we have that*

$$\operatorname{Prim} B = \bigsqcup_{w \in W \times W} \operatorname{Prim}_w B .$$

Moreover $\operatorname{Prim}_w B$ *is a nonempty H-orbit for each $w \in W \times W$. If $Q_{\dot{w}}$ is a primitive ideal of type w, then $H/\operatorname{Stab}_H Q_{\dot{w}}$ is a torus of rank $2 - s(w)$. All primitive ideals of B are completely prime. B satisfies the Dixmier–Moeglin condition.*

Proof. Let P be a primitive ideal of B of type w. Then by the nullstellensatz [13, 9.1] and [3, 4.1.6] PB_w is maximal. On the other hand if P is a prime ideal of B of type w and PB_w is maximal, then any prime ideal strictly containing P intersects the set

$\mathscr{E}_w$ of regular elements nontrivially. Hence the set P is locally closed in Spec B and again by the nullstellensatz [13, 9.1.8], P must be primitive. The fact that all prime ideals of B are completely prime follows immediately from 4.1.2 and 4.2.2 by standard facts about localization. □

Remark. Notice that these results imply that for any primitive ideal P of B there exists an Ore set E_w and a normal element d such that $(B/P)_{E_w} \cong C_{\dot{w}}[d^{-1}]$ and $C_{\dot{w}}$ is an iterated Ore extension. This should be compared with the structure of primitive factors of the enveloping algebra of a solvable Lie algebra [13, §14.8].

4.4. We now deduce the main theorem. Recall that $A = \mathbf{C}_q[G]$.

Theorem 4.4.1. *In the notation of Sect. 2.8, we have that*

$$\operatorname{Prim} A = \bigsqcup_{w \in W \times W} \operatorname{Prim}_w A \ .$$

Moreover $\operatorname{Prim}_w A$ *is a nonempty H-orbit for each* $w \in W \times W$. *The map* $P \mapsto PA_w$ *is an isomorphism between* $\operatorname{Prim}_w A$ *and* $\operatorname{Prim} A_w$. *If* $P_{\dot{w}}$ *is a primitive ideal of type w, then* $H/\operatorname{Stab}_H P_{\dot{w}}$ *is a torus of rank* $\operatorname{rk} G - s(w)$. $\operatorname{GKdim} A/P_{\dot{w}} = l(w) + s(w)$. *A satisfies the Dixmier–Moeglin condition.*

Proof. Let $P_{\dot{w}}$ be a primitive ideal of A of type w. It follows from Sects. 4.2 and 4.3 that $P_{\dot{w}}A_w$ is a primitive ideal of A_w and that $P_{\dot{w}} \cap B$ is a primitive ideal of B of type w. Furthermore the prime ideals of A lying over a given primitive ideal of B form a Γ-orbit and are all primitive. The fact that the Dixmier–Moeglin condition passes from B to A follows from [9]. □

4.5. As noted in the proof of Theorem 4.4.1, it follows from the description of the primitive ideals of B_w that if $P \in \operatorname{Prim} A_w$, then $P \cap B_w$ is a primitive ideal of B_w and that the primitive ideals lying over a fixed primitive ideal of B_w form a nontrivial Γ-orbit. Using a detailed analysis of the structure of A_w as a crossed product of $\hat{\Gamma}$ over B_w, one can calculate the exact number of primitives of A_w lying over a given primitive of B_w.

Proposition 4.5.1. *Let* $P \in \operatorname{Prim} A_w$. *Then* $P \cap B_w$ *is a maximal ideal of* B_w. *Conversely for all maximal ideals* Q *of* B_w *the number of primitive ideals* P *of* A_w *such that* $P \cap B_w = Q$ *is:*

$$\begin{cases} 4 & \textit{if } w = (e, e); \\ 2 & \textit{if } w_+ = w_- = e(W_i) \textit{ and } w_+ \textit{ or } w_- \neq e(W_j) \\ 1 & \textit{otherwise}\ . \end{cases}$$

All primitive ideals of A_w *are maximal and completely prime.*

In particular this last result implies that all prime ideals of $\mathbf{C}_q[G]$ are completely prime. Goodearl and Letzter [6] have recently proved that all prime ideals of $\mathbf{C}_q[SL(n)]$ are completely prime.

Remark. The authors have recently generalized the results of this section, proving Conjecture 1 of 2.8 for $\mathbf{C}_q[SL(n)]$.

List of Notation:

1.2	$\mathcal{Q}, U_q(\mathbf{g})$	3.2	$\mathcal{X}, \mathbf{I}$
1.3	$\mathbf{C}_q[G]$	3.3	$\mathbf{S}$
1.4	X_{ij}	3.4	$R(\varepsilon, i, y)$
1.5	$c^{\varepsilon}_{i,w}$	3.5	R^{ad}
1.6	$c^{\Lambda}_{-\lambda,i,\mu,j}$	3.6	$[\operatorname{ad} m]$
1.7	$U_q(\mathbf{b}^{\pm}), I^{\pm}(w, \Lambda)$	3.7	$F_i(\alpha)$
1.9.	$W_i, \hat{W}_i, \leqq_i, =_i$	3.9	ϕ
2.2	$I_w, \mathscr{E}_w, E_w, A_w$	3.10	C^{ad}_w, H_w
2.5	H, Γ, B, B_w	4.2	$C_{\dot{w}}, B_{\dot{w}}$
2.6	$z^{\varepsilon}_{i,y}, t_i, C_w, d_i, d$		
2.7	C^H_w		
2.8	$\operatorname{Prim}_w, \operatorname{Symp}_w$		

A. Symplectic Leaves in a Semi-simple Poisson Lie Group

A.1. Let G be a connected complex semisimple Lie group with Lie algebra $\mathbf{g}$. Let $\mathbf{h}$ be a Cartan subalgebra of $\mathbf{g}$, let R be the associated root system and R^+ a choice of positive roots. Denote by $\kappa(-,-)$ the Killing form on $\mathbf{g}$. Let $\mathbf{n}^{\pm} = \bigoplus_{\alpha \in R} \pm \mathbf{g}_\alpha$, and let $\mathbf{b}^{\pm} = \mathbf{h} \oplus \mathbf{n}^{\pm}$. Let $\mathbf{d} = \mathbf{g} \times \mathbf{g}$. The Iwasawa decomposition of $\mathbf{d}$ (as defined in [3, 1.13.14]) is then $\mathbf{d} = \mathbf{g} \oplus \mathbf{a} \oplus \mathbf{u}^+$, where $\mathbf{g}$ is identified with the diagonal subalgebra of $\mathbf{d}$, $\mathbf{a} = \{(x, -x) | x \in \mathbf{h}\}$ and $\mathbf{u}^+ = \{(x, y) | x \in \mathbf{n}^+, y \in \mathbf{n}^-\}$. Define the bilinear form $\langle -, - \rangle$ on $\mathbf{d}$ by:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \frac{1}{2}(\kappa(x_1, x_2) - \kappa(y_1, y_2)) .$$

Denote $\mathbf{a} \oplus \mathbf{u}^+$ by $\mathbf{g}_r$. Then $(\mathbf{g}, \mathbf{g}_r, \mathbf{d})$ is a Manin triple in the sense of [4]. There is then a Poisson Lie group structure on G associated to this triple [4]. The corresponding Poisson tensor is the tensor π defined by $\pi(g) = l_{g*}R - r_{g*}R$, where $R = \frac{1}{2}\sum_{\alpha > 0} E_\alpha \wedge E_{-\alpha} \in \mathbf{g} \wedge \mathbf{g}$ and l_{g*} and r_{g*} are the differentials of left and right translation respectively. The associated local double Lie group is then (G, G_r, D), where $D = G \times G$; G is identified with the diagonal subgroup $\{(x, x) | x \in G\}$; $G_r = AU^+$, where $A = \{(x, x^{-1}) | x \in H\}$ and $U^+ = \{(x, y) \in N^+, y \in N^-\}$ and H, $N^{\pm}$ and $B^{\pm}$ are the closed connected subgroups of G associated to $\mathbf{h}$, $\mathbf{n}^{\pm}$ and $\mathbf{b}^{\pm}$ respectively.

Consider the map $p: G \to D/G_r$. Define Γ to be $G \cap G_r = \ker p$. It is easily seen that $\Gamma = \{(h, h) \in H | h^2 = 1\}$. Hence Γ is a finite subgroup of D isomorphic to $\mathbf{Z}_2^{\operatorname{rk} G}$. Define $\bar{G}$ to be $G/\Gamma \cong GG_r/G_r$. Since GG_r is open in D, it follows that $\bar{G}$ is an open subset of D/G_r. Since π is H-invariant (and therefore Γ-invariant), it induces a Poisson tensor on $\bar{G}$.

Recall that a symplectic leaf of a Poisson variety is defined to be a maximal connected symplectic subvariety. We denote by Symp G the set of symplectic leaves of G. There is a natural partial order on Symp G by inclusions of closures.

Theorem A.1.1. 1) *The symplectic leaves of $\bar{G}$ are of the form $\bar{G} \cap G_r x G_r/G_r$ for some $x \in G$.*

2) *The symplectic leaves of G are the connected components of the inverse images of the symplectic leaves of $\bar{G}$.*

Proof. Since $p: G \to \bar{G}$ is étale, we have that for all $x \in G$, $T_x G \cong T_{p(x)}\bar{G} \cong T_{p(x)} D/G_r$. We recall some results from [11]. The left action of G_r on D/G_r induces a map σ from the Lie algebra $\mathbf{g}_r$ to the Lie algebra of vector fields on D/G_r. For $\alpha \in \mathbf{g}_r$ we denote by $\sigma_x(\alpha)$ the corresponding element of $T_x G$. The bilinear form $\langle -, - \rangle$ identifies $\mathbf{g}_r$ with $\mathbf{g}^*$. Therefore, each $\alpha \in \mathbf{g}_r$ induces a right invariant 1-form α_r on G. Define the right dressing vector field on G by $\langle \rho_x(\alpha), \xi \rangle = \pi_x(\alpha_r(x), \xi)$ for all $\xi \in T_x^* G$. By [11, 3.13], $\rho_x(\alpha) = -\sigma_x(\alpha)$ for all $\alpha \in \mathbf{g}_r$ and $x \in G$. Hence

$$\operatorname{rk} \pi_x = \dim \sigma_x(\mathbf{g}_r) = \dim G_r x G_r / G_r, \quad \forall x \in G .$$

It is easily seen that $G_r x G_r / G_r \cap \bar{G}$ is a Poisson subvariety of $\bar{G}$; hence it is a symplectic subvariety by the above equality. The theorem then follows easily. □

A.2. Denote by $Q = TU^+ = HG_r$ the positive Borel subgroup of D. Recall the Bruhat decomposition $D = \bigsqcup_{w \in W \times W} QwQ = \bigsqcup_{w \in W \times W} QwG_r$. For each $w \in W \times W$ we fix a representative $\dot{w}$ of w in the normaliser of T and we set: $\mathscr{C}_{\dot{w}} = G_r \dot{w} G_r / G_r$, $\mathscr{C}_w = QwG_r/G_r = \bigcup_{h \in H} h\mathscr{C}_{\dot{w}}$. Hence $D/G_r = \bigsqcup_{w \in W \times W} \mathscr{C}_w$. Set $\mathscr{B}_{\dot{w}} = \mathscr{C}_{\dot{w}} \cap \bar{G}$, $\mathscr{B}_w = \mathscr{C}_w \cap \bar{G}$, $\mathscr{A}_w = p^{-1}(\mathscr{B}_w)$. Fix a connected component $\mathscr{A}_{\dot{w}}$ of $p^{-1}(\mathscr{B}_{\dot{w}})$. Notice that $QwG_r \cap G \neq \emptyset$ for all $w \in W \times W$. This can be proved as follows by induction on $l(w)$ (the length of w). Assume that s is a simple reflection; so $s = (s_\alpha, e)$ or (e, s_α) for some $\alpha \in R^+$. If $s = (s_\alpha, e)$ we have that $QsQ \cap G = (B^+ s_\alpha B^+, B^-) \cap G \neq \emptyset$ since $B^+ s_\alpha B^+ \cap B^- \neq \emptyset$; similarly for $s = (e, s_\alpha)$. In the general case, set $w = sw'$, where s is a simple reflection and $l(w) = l(w') + 1$. Then by induction $QwQ \cap G \supset (QsQ \cap G)(Qw'Q \cap G) \neq \emptyset$. Therefore $\mathscr{B}_w = \mathscr{C}_w \cap \bar{G} \neq \emptyset$ and since $\mathscr{C}_w = \bigcup_{h \in H} h\mathscr{C}_{\dot{w}}$, we have that $h\mathscr{C}_{\dot{w}} \cap \bar{G} \neq \emptyset$ for all $h \in H$. These observations together with the theorem of section one give the following description of the symplectic leaves.

Theorem A.2.1. 1) *Each symplectic leaf of $\bar{G}$ is of the form $h\mathscr{B}_{\dot{w}}$ for some $h \in H$ and $w \in W \times W$.*

2) *Each symplectic leaf of G is of the form $h\mathscr{A}_{\dot{w}}$ for some $h \in H$ and some $w \in W \times W$.*

Let $w = (w_+, w_-) \in W \times W$. Define $A'_w = w(A) \cap A = \{a \in A \mid a\dot{w}G_r = \dot{w}G_r\}$. Set $A_w = A/A'_w$. Then A_w is a torus of rank $s(w) = \dim A - \dim A'_w = \operatorname{codim}_{\mathfrak{h}} \ker(w_+ w_-^{-1} - I)$. When $G = SL(n, \mathbf{C})$ we have that $s(w) = \min\{m \mid w_+ w_-^{-1} = r_1 \ldots r_m$, where r_i is a transposition for all $i\}$.

Define $U_w^\pm = w(U^\pm) \cap U^+$ and recall that we have an isomorphism of varieties $U^+ \cong U_w^- \times U_w^+$, and that $U_w^- \cong \mathbf{C}^{l(w)}$. Thus we have that $\mathscr{C}_{\dot{w}} = AU^+ \dot{w} G_r/G_r = AU_w^- \dot{w} G_r / G_r$. Using a standard argument one verifies that the multiplication $A_w \times U_w^- \to \mathscr{C}_{\dot{w}}$ is an isomorphism. Thus we have proved the following proposition.

Proposition A.2.2. *$\mathscr{C}_{\dot{w}} \cong A_w \times U_w^-$, where A_w is a torus of rank $s(w)$ and $U_w^- \cong \mathbf{C}^{l(w)}$. Hence* $\dim \mathscr{C}_{\dot{w}} = l(w) + s(w)$.

A.3. Let $w \in W \times W$. Set $H'_w = \{h \in H \mid hG_r \dot{w} G_r = G_r \dot{w} G_r\}$. Then H'_w is a closed subgroup of H and $H_w = H/H'_w$ is a torus of rank $\operatorname{rk} G - s(w)$. We have that $\mathscr{C}_w = H\mathscr{C}_{\dot{w}}$ and the same argument as in the previous subsection shows that the multiplication map $H_w \times \mathscr{C}_{\dot{w}} \to \mathscr{C}_w$ is an isomorphism.

The group G_r acts by left translation on $\mathscr{C}_w$ and therefore on the product $H_w \times \mathscr{C}_{\dot{w}}$. It is easily seen that the algebra of G_r-invariant functions on $\mathscr{C}_w$ is $\mathbf{C}[H_w]$.

This proves the first part of the theorem below. The second part is a consequence of the description given above.

Theorem A.3.1. 1) *The G_r-orbits in $\mathscr{C}_w$ are the fibres of the natural projection $\mathscr{C}_w \to G_r \backslash\backslash \mathscr{C}_w \cong H_w$.*

2) *The symplectic leaves of type w in $\bar{G}$ are the fibres of the induced projection $\mathscr{B}_w \to H_w$.*

We now summarize the results about the set Symp G of symplectic leaves in G. Denote by $\mathrm{Symp}_w G$ the set of symplectic leaves of type $w \in W \times W$.

Theorem A.3.2. 1) $\mathrm{Symp}\, G = \bigsqcup_{w \in W \times W} \mathrm{Symp}_w G$.

2) *For each $w \in W \times W$, $\mathrm{Symp}_w G$ is a nonempty H-orbit. If $\mathscr{A}_{\dot{w}}$ is a fixed symplectic leaf of type w, then $H/\mathrm{Stab}_H \mathscr{A}_{\dot{w}}$ is a torus of rank* $\mathrm{rk} G - s(w)$.

3) *The dimension of a leaf of type w is $l(w) + s(w)$.*

B. The Case $G = SL(2, \mathbf{C})$

B.1. In this appendix we outline the classification of primitive ideals of $\mathbf{C}_q[SL(2)]$ and of symplectic leaves of $SL(2, \mathbf{C})$. The proofs of the two theorems below are straightforward calculations. In the notation of Sect. 1.4, $\mathbf{C}_q[SL(2)]$ is generated by the elements $a = X_{11}$, $b = X_{12}$, $c = X_{21}$, and $d = X_{22}$ subject to the relations $ab = q^2 ba$, $ac = q^2 ca$, $bd = q^2 db$, $bc = cb$, $ad - da = (q^2 - q^{-2})bc$, and $ad - q^2 bc = 1$. The Weyl group in this case is just $W = \{e, s\}$, where $s^2 = e$. The ideals I_w for $w \in W \times W$ are given by $I_{(e,e)} = (b, c)$, $I_{(s,e)} = (b)$, $I_{(e,s)} = (c)$ and $I_{(s,s)} = (0)$.

Theorem B.1.1. *The following is a complete list by type of the primitive ideals of* $\mathbf{C}_q[SL(2)]$:

$$(e, e)\colon P_{(e,e),\lambda} = (b, c, a - \lambda, d - \lambda^{-1}), \quad \lambda \in \mathbf{C}^* ,$$

$$(s, e)\colon P_{(s,e)} = I_{(s,e)} = (b) ,$$

$$(e, s)\colon P_{(e,s)} = I_{(e,s)} = (c) ,$$

$$(s, s)\colon P_{(s,s),\lambda} = (b - \lambda c), \quad \lambda \in \mathbf{C}^* .$$

All prime ideals of $\mathbf{C}_q[SL(2)]$ are completely prime.

Remark. Let M^+ and M^- be modules with annihilators $P_{(s,e)}$ and $P_{(e,s)}$ respectively. Then M^+ and M_- are modules of type (s, e) and (e, s) respectively. The existence of such modules is used in Sect. 2.3.

B.2. We now describe explicitly the symplectic leaves of $SL(2, \mathbf{C})$. We continue to denote the coordinate functions of the standard representation of $SL(2, \mathbf{C})$ by a, b, c and d as above. The standard Poisson bracket is then given by: $\{a, b\} = -ab$, $\{a, c\} = -ac$, $\{b, d\} = -bd$, $\{b, c\} = 0$ and $\{a, d\} = -2bc$.

Theorem B.2.1. *The following is a complete list by type of the symplectic leaves of* $SL(2, \mathbf{C})$:

$$(e, e)\colon \left\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}\right\}, \quad \lambda \in \mathbf{C}^*$$

$$(s, e)\colon \left\{\begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix} \middle| \alpha, \gamma \in \mathbf{C}^* \right\}$$

$$(e, s)\colon \left\{\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \middle| \alpha, \beta \in \mathbf{C}^* \right\}$$

$$(s, s)\colon \left\{\begin{bmatrix} \alpha & \lambda\gamma \\ \gamma & \delta \end{bmatrix} \middle| \gamma \in \mathbf{C}^*, \alpha\delta - \lambda\gamma^2 = 1 \right\}, \quad \lambda \in \mathbf{C}^* .$$

Combining these two theorems yields a positive answer to all the conjectures given in Sect. 4.

Corollary B.2.2. *There is an order preserving bijection* β: $\operatorname{Prim} \mathbf{C}_q[SL(2)] \to \operatorname{Symp} SL(2, \mathbf{C})$. *Furthermore, if* $L = \beta(P)$, *then* $\dim L = GK \dim \mathbf{C}_q[SL(2)]/P$.

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Note added in proof. Conjecture 1 of Sect. 2.8 has recently been proved by A. Joseph for G a simply connected semi-simple complex Lie group.

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