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Jackson Integrals of Jordan–Pochhammer Type and Quantum Knizhnik–Zamolodchikov Equations

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Abstract. We show that the q-difference systems satisfied by Jackson integrals of Jordan–Pochhammer type give a class of the quantum Knizhnik–Zamolodchikov equation for $U_q(\widehat{\mathfrak{sl}}_2)$ in the sense of Frenkel and Reshetikhin.

1. Introduction

One of the most interesting features of the Knizhnik–Zamolodchikov equation originated in conformal field theory is the relation between its connection matrix and the trigonometric solutions of the quantum Yang–Baxter equation [TK, K, D]. It is related to the fact that certain hypergeometric type integrals give solutions to the Knizhnik–Zamolodchikov equation [DJMM, Ma, Ch, SV], etc. This fact is also looked at from the viewpoint of the free field realization, e.g. [Ku, ATY]. Besides them, the structure of the hypergeometric type integrals had been studied, e.g. [A1, A2]. Recently it attracts attention to construct a q-analogue of these theories.

The Jackson integrals of Jordan–Pochhammer type are the simplest multivariable generalizations of Heine's basic hypergeometric function which is a q-analogue of Gauss' hypergeometric function. They satisfy a system of first order q-difference equations, whose connection problem was solved by Mimachi [Mi]. Recently Aomoto and others [AKM] showed that the connection matrix determined by Mimachi is related to the ABF-solution of the quantum Yang–Baxter equation [ABF]. On the other hand, Frenkel and Reshetikhin [FR] studied a q-analogue of the chiral vertex operators of the WZNW model, along the line of Tsuchiya and Kanie [TK]. In particular, they introduced a q-difference system called the quantum Knizhnik–Zamolodchikov equation, and discussed the relation of the connection matrix with elliptic solutions of the quantum Yang–Baxter equation. Then it seems possible to understand the result of [AKM] in the framework of Frenkel and Reshetikhin.

In this article, we shall explicitly give solutions to a certain class of the quantum Knizhnik–Zamolodchikov equation for $U_q(\widehat{\mathfrak{sl}}_2)$ by Jackson integrals of Jordan–Pochhammer type. More precisely, we show that the q-difference system for the

Jackson integrals of Jordan–Pochhammer type is written in terms of trigonometric quantum R-matrix, and that this equation gives a class of the quantum Knizhnik–Zamolodchikov equation. When q goes to 1, our expressions of solutions go to the integral solutions of the Knizhnik–Zamolodchikov equation given by [Ch] in the trigonometric form.

The paper is organized as follows. In Sect. 2, we write the q-difference equation for Jackson integrals of Jordan–Pochhammer type, whose proof will be given in Sect. 4. In Sect. 3, we identify the equation with the quantum Knizhnik–Zamolodchikov equation. In Sect. 5, we give some comments on the connection problem according to current literatures.

2. q-Difference System for Jackson Integrals

Let p be a fixed complex number such as $0 < |p| < 1$. Let us denote

$$(a)_\infty = \prod_{n=0}^{\infty} (1 - ap^n) \tag{2.1}$$

as usual. For a value $s \in \mathbf{C}^*$ and for a function $\phi(t)$, we define

$$\int_0^{s\infty} \phi(t) d_p t = s(1-p) \sum_{n=-\infty}^{\infty} \phi(sp^n) p^n \tag{2.2}$$

whenever it is convergent. This is called the Jackson integral along a q-interval $[0, s\infty]$, which is a q-analogue of the ordinary integration. The q-shift operator T_k is defined by

$$(T_k F)(x_1, \ldots, x_n) = F(x_1, \ldots, px_k, \ldots, x_n) \tag{2.3}$$

for a function $F(x_1, \ldots, x_n)$.

Now consider the Jackson integral of Jordan–Pochhammer type:

$$F_0(x) = \int_0^{s\infty} t^{\beta-1} \prod_{1 \leqq j \leqq n} \frac{(t/x_j)_\infty}{(p^{\beta_j} t/x_j)_\infty} d_p t\,, \tag{2.4}$$

where β_j are complex parameters and $x = (x_1, \ldots, x_n)$ is a variable in $(\mathbf{C}^\times)^n$. We are interested in the q-difference system associated with F_0. Take the set of functions $(F_1, \ldots, F_n)$ defined by

$$F_i(x) = \int_0^{s\infty} \Phi_i(t) d_p t \tag{2.5}$$

where, for each $i = 0, \ldots, n$, we have set

$$\Phi_i(t) = t^{\beta-1} \frac{\prod_{j=1}^{i} (pt/x_j)_\infty \prod_{j=i+1}^{n} (t/x_j)_\infty}{\prod_{j=1}^{i-1} (p^{\beta_j+1} t/x_j)_\infty \prod_{j=i}^{n} (p^{\beta_j} t/x_j)_\infty}\,. \tag{2.6}$$

Let us calculate the q-difference system satisfied by F_i. We set

$$x_{ij} = \begin{cases} x_i/x_j & \text{if } i < j\,, \\ 1 & \text{if } i = j\,, \\ px_i/x_j & \text{if } i > j\,. \end{cases} \tag{2.7}$$

Then the result is summarized as the following proposition.

Proposition 1. *We define the $n \times n$ matrix A_k with entries a_{ij}^k as follows. If $i = j + k$ then*

$$a_{ij}^k = \frac{x_{ki} - 1}{x_{ki} - p^{\beta_k}}\,. \tag{2.8}$$

If $i < j \leqq k$ or $k \leqq i < j$ then

$$a_{ij}^k = \frac{(1 - p^{\beta_i})x_{ki}}{x_{ki} - p^{\beta_k}} \frac{1 - p^{\beta_k}}{x_{kj} - p^{\beta_k}} \prod_{l=i+1}^{j-1} \frac{p^{\beta_l} x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}}\,. \tag{2.9}$$

If $j \leqq k \leqq i$ then

$$a_{ij}^k = p^{\beta} \frac{1 - p^{\beta_k}}{x_{kj} - p^{\beta_k}} \frac{(1 - p^{\beta_i})x_{ki}}{x_{ki} - p^{\beta_k}} \prod_{l=1}^{j-1} \frac{p^{\beta_l} x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}} \prod_{l=i+1}^{n} \frac{p^{\beta_l} x_{kl} - p^{\beta_k}}{x_{kl} - p^{\beta_k}}\,. \tag{2.10}$$

Otherwise $a_{ij}^k = 0\,.$ (2.11)

Then we have

$$(T_k F_1, \ldots, T_k F_n) = (F_1, \ldots, F_n) A_k\,. \tag{2.12}$$

Remark. For each i, j $(i \neq j)$, let $S_{i,j}$ denote the $n \times n$-matrix defined by

$$\begin{array}{cc}
\left(\begin{array}{ccccccccccc}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& \frac{p^{\beta_j} x_{ij} - p^{\beta_i}}{x_{ij} - p^{\beta_i}} &&&& \frac{1 - p^{\beta_i}}{x_{ij} - p^{\beta_i}} &&& \\
&&&& 1 &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& 1 &&&& \\
&&& \frac{(1 - p^{\beta_j}) x_{ij}}{x_{ij} - p^{\beta_i}} &&&& \frac{x_{ij} - 1}{x_{ij} - p^{\beta_i}} &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{array}\right)
&
\begin{array}{l}
\\ \\ \\ i^{\text{th}} \\ \\ \\ \\ j^{\text{th}} \\ \\ \\ \\
\end{array}
\\
\begin{array}{ccccccccccc}
&&& i^{\text{th}} &&&& j^{\text{th}} &&&
\end{array}
&
\end{array} \tag{2.13}$$

We also consider the $n \times n$-matrix P_k defined by

$$\left(\begin{array}{ccccccc}
1 &&&&&& \\
& \ddots &&&&& \\
&& 1 &&&& \\
&&& p^{\beta} &&& \\
&&&& 1 && \\
&&&&& \ddots & \\
&&&&&& 1
\end{array}\right)
\begin{array}{l}
\\ \\ \\ k^{\text{th}} \\ \\ \\ \\
\end{array} \tag{2.14}$$

Then, by an explicit calculation, we see

$$A_i = S_{k,k+1} \ldots S_{k,n} P_k S_{k,1} \ldots S_{k,k-1} . \tag{2.15}$$

Matrices $S_{i,j}$ form a set of unitary quantum R-matrices. Namely we have

$$S_{i,j}(T_i S_{j,i}) = \mathrm{id}, \quad \text{and} \quad S_{1,2}S_{2,3}S_{1,3} = S_{1,3}S_{2,3}S_{1,2} . \tag{2.16}$$

Finally, let us discuss the relation among $F_0, \ldots, F_n$.

Proposition 2. *We put $\beta_0 = -\beta - (\beta_1 + \cdots + \beta_n)$. Then the following relation holds:*

$$\sum_{i=0}^{n} p^{\beta_{i+1} + \cdots + \beta_n}(1 - p^{\beta_i})F_i = 0 . \tag{2.17}$$

Therefore F_0 is recovered from $F_1, \ldots, F_n$ if $p^{\beta_0} \neq 1$.

Remark. The identity (2.17) is a q-analogue of Aomoto's linear relation in the sense of [A2] and [DJMM].

3. Comparison with the Quantum Knizhnik–Zamolodchikov Equations

Let us briefly review the quantum enveloping algebra and the trigonometric R-matrix in the case of $\widehat{\mathfrak{sl}}_2$. The quantum enveloping algebra $\hat{U}_q = U_q(\widehat{\mathfrak{sl}}_2)$ is defined as an algebra with the generators:

$$X_0^{\pm}, X_1^{\pm}, K_0^{\pm 1}, K_1^{\pm 1} \tag{3.1}$$

and the relations:

$$\begin{aligned}
&K_0K_1 = K_1K_0, \quad K_0K_0^{-1} = K_1K_1^{-1} = 1 , \\
&K_iX_i^{\pm}K_i^{-1} = q^{\pm 2}X_i^{\pm}, \quad K_iX_j^{\pm}K_i^{-1} = q^{\mp 2}X_j^{\pm} \ (i \neq j) , \\
&[X_i^+, X_j^-] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}} , \\
&(X_i^{\pm})^3X_j^{\pm} - (q^2 + 1 + q^{-2})(X_i^{\pm})^2X_j^{\pm}X_i^{\pm} + (q^2 + 1 + q^{-2})X_i^{\pm}X_j^{\pm}(X_i^{\pm})^2 \\
&\qquad - X_j^{\pm}(X_i^{\pm})^3 = 0 \quad (i \neq j) .
\end{aligned} \tag{3.2}$$

Here, q denotes a general complex parameter. The comultiplication $\Delta\colon \hat{U}_q \to \hat{U}_q \otimes \hat{U}_q$ is defined by

$$\begin{aligned}
&\Delta(X_i^+) = X_i^+ \otimes 1 + K_i^{-1} \otimes X_i^+ , \\
&\Delta(X_i^-) = X_i^- \otimes K_i + 1 \otimes X_i^-, \quad \Delta(K_i) = K_i \otimes K_i .
\end{aligned} \tag{3.3}$$

We put $\Delta' = \sigma \circ \Delta$ where $\sigma(a \otimes b) = b \otimes a$ in $\hat{U}_q \otimes \hat{U}_q$. Next we consider the subalgebra $U_q = U_q(\mathfrak{sl}_2)$ generated by $X^{\pm} = X_1^{\pm}, K^{\pm} = K_1^{\pm}$. For each $x \in \mathbf{C}^*$, we define the algebra homomorphism $\varphi_x\colon \hat{U}_q \to U_q$ by

$$\begin{aligned}
&\varphi_x(X_0^{\pm}) = x^{\pm 1}X^{\mp}, \quad \varphi_x(X_1^{\pm}) = X^{\pm} , \\
&\varphi_x(K_0) = K^{-1}, \qquad \varphi_x(K_1) = K .
\end{aligned} \tag{3.4}$$

Let (V_i, π_i) be representations of U_q with the highest weights λ_i. Then $(V_i(x), \hat{\pi}_i) = (V_i, \pi_i \circ \varphi_x)$ gives a representation of $\hat{U}_q$ for each $x \in \mathbf{C}$. The operator

$$R_{V_iV_j}(x)\colon V_i(x) \otimes V_j(1) \to V_i(x) \otimes V_j(1) \tag{3.5}$$

such that

$$\Delta'(a)R_{V_iV_j}(x) = R_{V_iV_j}(x)\Delta(a), \quad a \in \hat{U}_q$$

gives a trigonometric R-matrix. Let v_i be the highest weight vector in V_i. We fix a choice of normalization such that

$$R_{V_iV_j}(x) v_i \otimes v_j = v_i \otimes v_j \,. \tag{3.6}$$

Then $R_{V_iV_j}(x)$ acts as

$$R_{V_iV_j}(x) X^- v_i \otimes v_j = \frac{xq^{m_j} - q^{m_i}}{x - q^{m_i+m_j}} X^- v_i \otimes v_j + \frac{1 - q^{2m_j}}{x - q^{m_i+m_j}} v_i \otimes X^- v_j \,,$$

$$R_{V_iV_j}(x) v_i \otimes X^- v_j = \frac{x(1 - q^{2m_i})}{x - q^{m_i+m_j}} X^- v_i \otimes v_j + \frac{xq^{m_i} - q^{m_j}}{x - q^{m_i+m_j}} v_i \otimes X^- v_j \,. \tag{3.7}$$

Here $m_i = (\lambda_i, \alpha)$ for the simple root α.

Let $\lambda_1, \dots, \lambda_n, \lambda$ be a set of weights. Let V_i be the irreducible representation of U_q with the highest weight λ_i and the highest weight vector v_i. Let ν be a complex parameter and put $p^\nu = q$. We set $\rho = \alpha/2$, the half sum of the positive roots. For a weight μ, we denote by $(q^\mu)_k$ the action of q^μ on the k^{th} component of the tensor product $V_1 \otimes \dots \otimes V_n$. For instance,

$$q^\mu(v_k) = q^{(\mu, \lambda_k)} v_k, \quad q^\mu(X^- v_k) = q^{(\mu, \lambda_k - \alpha)} X^- v_k \,. \tag{3.8}$$

The quantum Knizhnik–Zamolodchikov equation introduced by Frenkel and Reshetikhin [FR] is written as the following system of q-difference equations:

$$\begin{aligned} T_k \mathscr{F} &= R_{V_kV_{k-1}}(px_k/x_{k-1}) \dots R_{V_kV_1}(px_k/x_1)(q^{\lambda+2\rho})_k \\ &\quad \times q^{-(\lambda, \lambda_k)} R_{V_{k+1}V_k}(x_{k+1}/x_k)^{-1} \dots R_{V_nV_k}(x_n/x_k)^{-1} \mathscr{F} \,, \\ k &= 1, \dots, n \,, \end{aligned} \tag{3.9}$$

where $\mathscr{F} = \mathscr{F}(x_1, \dots, x_n)$ is a function valued in $V_1 \otimes \dots \otimes V_n$.

Let us compare Eqs. (2.12) and (3.9). Take the weights $\lambda_0, \lambda_\infty$ such that

$$\lambda_0 + \dots + \lambda_n - \lambda_\infty = \alpha, \quad \lambda_0 + \lambda_\infty = \lambda \,, \tag{3.10}$$

and put the parameters as:

$$\beta = -2(\lambda_\infty + \alpha, \alpha)\nu, \quad \beta_i = 2(\lambda_i, \alpha)\nu \,. \tag{3.11}$$

We set

$$\varphi_i(x_1, \dots, x_n) = p^{(\beta_{i+1} + \dots + \beta_n)/2} x_1^{\beta_1} \dots x_n^{\beta_n} F_i(p^{\beta_1/2} x_1, \dots, p^{\beta_n/2} x_n) \,, \tag{3.12}$$

for each $i = 1, \dots, n$, and define the $V_1 \otimes \dots \otimes V_n$-valued function $\mathscr{F}$ by

$$\mathscr{F} = \sum_{i=1}^{n} \varphi_i(x_1, \dots, x_n) v_1 \otimes \dots \otimes X^- v_i \otimes \dots \otimes v_n \,. \tag{3.13}$$

Then, by rewriting Eq. (2.12) in terms of $\mathscr{F}$, we have

Theorem 3. *The system* (2.12) *is equivalent to the restriction of the system* (3.9) *to the weight subspace with the weight* $\lambda_1 + \cdots + \lambda_n - \alpha$, *and the function* $\mathscr{F}$ *defined by* (3.13) *is a solution of* (3.9).

Remark. When q goes to 1, $\mathscr{F}$ defined by (3.13) goes to a special case of the integral solutions to the Knizhnik–Zamolodchikov equation obtained by Cherednik [Ch] in the trigonometric form.

We shall give another description of the equation. Let $\lambda_0, \ldots, \lambda_n, \lambda_\infty$ be a set of weights such that

$$\lambda_0 + \cdots + \lambda_n - \lambda_\infty = \alpha . \tag{3.14}$$

Let V_i be the irreducible representation of U_q with the highest weight λ_i and the highest weight vector v_i. The quantum Knizhnik–Zamolodchikov equation for a $\mathrm{Hom}_{U_q}(V_\infty, V_0 \otimes \ldots \otimes V_n)$-valued function $\mathscr{F}$ is written as:

$$\begin{aligned} T_k\mathscr{F} &= R_{V_kV_{k-1}}(px_k/x_{k-1}) \ldots R_{V_kV_1}(px_k/x_1)R_{V_kV_0}(0)(q^{2\rho})_k \\ &\quad \times R_{V_\infty^* V_k}(0)^{-1}R_{V_{k+1}V_k}(x_{k+1}/x_k)^{-1} \ldots R_{V_nV_k}(x_n/x_k)^{-1}\mathscr{F} . \end{aligned} \tag{3.15}$$

Here we understand $\mathscr{F}$ as an element of $V_0 \otimes \ldots \otimes V_n \otimes V_\infty^*$. Next we consider the set $\mathscr{H}_{\lambda_\infty}(V_0 \otimes \ldots \otimes V_n)$ of highest weight vectors in $V_0 \otimes \ldots \otimes V_n$ with the weight λ_∞. We have an injection

$$\mathrm{Hom}_{U_q}(V_\infty, V_0 \otimes \ldots \otimes V_n) \to \mathscr{H}_{\lambda_\infty}(V_0 \otimes \ldots \otimes V_n) \tag{3.16}$$

by evaluating the highest weight vector v_∞. Then Eq. (3.15) is regarded as a restriction of the following system:

$$\begin{aligned} T_k\mathscr{F} &= R_{V_kV_{k-1}}(px_k/x_{k-1}) \ldots R_{V_kV_1}(px_k/x_1)R_{V_kV_0}(0)(q^{\lambda_\infty + 2\rho})_k q^{-(\lambda_\infty, \lambda_k)} \\ &\quad \times R_{V_{k+1}V_k}(x_{k+1}/x_k)^{-1} \ldots R_{V_nV_k}(x_n/x_k)^{-1}\mathscr{F} , \end{aligned} \tag{3.17}$$

where $\mathscr{F}$ is a $\mathscr{H}_{\lambda_\infty}(V_0 \otimes \ldots \otimes V_n)$-valued function.

Remarks. (1) If all V_i are the Verma modules or are the finite dimensional modules, then the linear map (3.16) is surjective, and the system (3.15) is the same as (3.17).

(2) If $q^{2(\lambda_0, \alpha)} \neq 1$, then the system (3.17) is same as the restriction of the system (3.9) to the weight subspace with the weight $\lambda_1 + \cdots + \lambda_n - \alpha$, hence is equivalent to the system (2.12).

We define the $V_0 \otimes \ldots \otimes V_n$-valued function $\mathscr{F}$ by

$$\mathscr{F} = \sum_{i=0}^{n} \varphi_i(x_1, \ldots, x_n)v_0 \otimes \ldots \otimes X^- v_i \otimes \ldots \otimes v_n , \tag{3.18}$$

where φ_i is defined by (3.12) for each $i = 0, \ldots, n$. Then, by interpreting the identity (2.17), we have

$$X^+\mathscr{F} = 0 . \tag{3.19}$$

Therefore $\mathscr{F}$ is one of the highest weight vectors in $V_0 \otimes \ldots \otimes V_n$ with the weight λ_∞. Thus we finally obtain:

Theorem 4. *The* $\mathscr{H}_{\lambda_\infty}(V_0 \otimes \ldots \otimes V_n)$*-valued function* $\mathscr{F}$ *defined by* (3.18) *is a solution of the quantum Knizhnik–Zamolodchikov equation* (3.17).

Notes. (1) In the situation of [FR], V_0 and V_∞ are integrable $\hat{U}_q$-modules and $V_1, \ldots, V_n$ are finite dimensional $\hat{U}_q$-modules, and ν corresponds to $-\dfrac{1}{2(k+g)}$, where k is the fixed level and g is the dual coxeter number. Moreover the quantum Knizhnik–Zamolodchikov equation for the correlation function is written in terms of the image of the universal R-matrix, which differs from our $R_{V_iV_j}$ by a certain scalar factor.

(2) For $n = 2$, our expressions of solutions to (3.9) coincide with those given in [FR, Sect. 7].

4. Proof of Propositions

We write $\phi_1(t) \sim \phi_2(t)$ if

$$\int_0^{s\infty} \phi_1(t)d_pt = \int_0^{s\infty} \phi_2(t)d_pt \tag{4.1}$$

holds for any $s \in \mathbf{C}^*$. For example, we have

$$\Phi_i(t) \sim p\Phi_i(pt)\,. \tag{4.2}$$

Proof of Proposition 1. The following is obvious from the definition:

$$T_kF_i = \int_0^{s\infty} T_k\Phi_i(t)d_pt\,. \tag{4.3}$$

Therefore the q-difference system (2.12) is equivalent to

$$T_k\Phi_j(t) \sim \sum_{i=1}^{n} a_{ij}^k\Phi_i(t)\,. \tag{4.4}$$

Now, because of (4.2), the following lemma is enough to prove the proposition.

Lemma 5.

(a) *For $j < k$, we have*

$$pT_k\Phi_j(pt) = p\sum_{i=1}^{j} a_{ij}^k\Phi_i(pt) + \sum_{i=k}^{n} a_{ij}^k\Phi_i(t)\,.$$

(b) *For $j = k$, we have*

$$pT_k\Phi_j(pt) = p\sum_{i=1}^{j-1} a_{ij}^k\Phi_i(pt) + \sum_{i=j}^{n} a_{ij}^k\Phi_i(t)\,.$$

(c) *For $k < j$, we have*

$$T_k\Phi_j(t) = \sum_{i=k}^{j} a_{ij}^k\Phi_i(t)\,.$$

Proof. Since all the cases are treated in a similar way, we will exhibit detailed calculations only for the most difficult case (b). We put $a_{ij} = a_{ij}^k$ for simplicity.

Multiplied by appropriate factors, (b) is equivalent to

$$
\begin{aligned}
p^{\beta} x_j \prod_{l=1}^{j-1} (p^{\beta_l} pt - x_l) \prod_{l=j+1}^{n} (p^{\beta_l} t - x_l) \\
= p^{\beta} \sum_{i=1}^{j-1} a_{ij} x_i \prod_{l=1}^{i-1} (p^{\beta_l} pt - x_l) \prod_{l=i+1}^{j-1} (pt - x_l) \prod_{l=j}^{n} (p^{\beta_l} t - x_l) \\
+ \sum_{i=j}^{n} a_{ij} x_i \prod_{l=1}^{j-1} (pt - x_l) \prod_{l=j}^{i-1} (p^{\beta_l} t - x_l) \prod_{l=i+1}^{n} (t - x_l) .
\end{aligned}
\tag{4.5}
$$

Since both sides are polynomials of degree $n - 1$ with respect to t, it suffices to check the equality at n different values of t. Putting $t = x_m/p$ for $m \leqq j - 1$ in (4.5), we have

$$
p x_j \prod_{l=m}^{j-1} (p^{\beta_l} x_m - x_l) - \sum_{i=m}^{j-1} a_{ij} x_i (p^{\beta_j} x_m - p x_j) \prod_{l=m}^{i-1} (p^{\beta_l} x_m - x_l) \prod_{l=i+1}^{j-1} (x_m - x_l) = 0 .
\tag{4.6}
$$

We put $t = x_j/p^{\beta_j}$, then we have

$$
\begin{aligned}
p^{\beta} \prod_{l=1}^{j-1} (p^{\beta_l} p x_j - p^{\beta_j} x_l) \prod_{l=j+1}^{n} (p^{\beta_l} x_j - p^{\beta_j} x_l) \\
= a_{jj} \prod_{l=1}^{j-1} (p x_j - p^{\beta_j} x_l) \prod_{l=j+1}^{n} (x_j - p^{\beta_j} x_l) .
\end{aligned}
\tag{4.7}
$$

We finally put $t = x_m/p^{\beta_m}$ for $j + 1 \leqq m$, then we have

$$
\sum_{i=j}^{m} a_{ij} x_i \prod_{l=j}^{i-1} (p^{\beta_l} x_m - p^{\beta_m} x_l) \prod_{l=i+1}^{n} (x_m - p^{\beta_m} x_l) = 0 .
\tag{4.8}
$$

Now let us consider the explicit values of a_{ij} defined by (2.8)–(2.10). Substitute them in the left of (4.6) inductively as $i = j - 1, j - 2, \ldots, N$. Then we have

$$
\begin{aligned}
p x_j \prod_{l=N}^{j-1} \frac{p^{\beta_l} p x_j - p^{\beta_j} x_l}{p x_j - p^{\beta_j} x_l} \prod_{l=m}^{N-1} (p^{\beta_l} x_m - x_l) \prod_{l=N}^{j-1} (x_m - x_l) \\
- \sum_{i=m}^{N} a_{ij} x_i (p^{\beta_j} x_m - p x_j) \prod_{l=m}^{i-1} (p^{\beta_l} x_m - x_l) \prod_{l=i+1}^{j-1} (x_m - x_l) .
\end{aligned}
$$

When $N = m$, this is zero and (4.6) is verified. Equation (4.7) follows easily from (2.10). To verify (4.8), it suffices to substitute the values of $a_{ij}, i = j, j + 1, \ldots, N$ inductively. Hence (4.5) is shown and the proof of (b) is completed. Q.E.D.

Proof of Proposition 2. By the relation (4.2), it suffices to show the following lemma.

Lemma 6. *We have the following relation:*

$$
p^{\beta_1 + \cdots + \beta_n} \Phi_0(t) - p^{-\beta + 1} \Phi_0(pt) = \sum_{i=1}^{n} p^{\beta_{i+1} + \cdots + \beta_n} (p^{\beta_i} - 1) \Phi_i(t) .
\tag{4.9}
$$

Proof. Multiplied by an appropriate factor, (4.9) is equivalent to

$$p^{\beta_1+\cdots+\beta_n}\prod_{j=1}^{n}(1-t/x_j)-\prod_{j=1}^{n}(1-p^{\beta_j}t/x_j)$$
$$=\sum_{i=1}^{n}p^{\beta_{i+1}+\cdots+\beta_n}(p^{\beta_i}-1)\prod_{j=1}^{i-1}(1-p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1-t/x_j)\,. \tag{4.10}$$

The right becomes

$$\sum_{i=1}^{n}p^{\beta_i+\cdots+\beta_n}\prod_{j=1}^{i-1}(1-p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1-t/x_j)$$
$$\times\sum_{i=1}^{n}p^{\beta_{i+1}+\cdots+\beta_n}\prod_{j=1}^{i-1}(1-p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1-t/x_j)$$
$$=\sum_{i=1}^{n}p^{\beta_i+\cdots+\beta_n}\prod_{j=1}^{i-1}(1-p^{\beta_j}t/x_j)\prod_{j=i}^{n}(1-t/x_j)$$
$$\times\sum_{i=1}^{n}p^{\beta_{i+1}+\cdots+\beta_n}\prod_{j=1}^{i}(1-p^{\beta_j}t/x_j)\prod_{j=i+1}^{n}(1-t/x_j)\,,$$

which yields the left of (4.10). Q.E.D.

5. Discussions

In this paper, we have constructed a Jackson integral representation of solutions to the quantum Knizhnik–Zamolodchikov equation in the simplest case for $U_q(\widehat{\mathfrak{sl}}_2)$. Let us briefly review the results of [AKM] and [FR], and discuss the relation of our result and the connection problem of q-difference equations.

Let $F'_i = F'_i(x_1, \ldots, x_n)$ be the function defined by

$$F'_i=\int_0^{s\infty}\frac{t^{\beta-1}}{1-t/x_i}\frac{\prod_{j=1}^{n}(t/x_j)_\infty}{\prod_{j=1}^{n}(p^{\beta_j}t/x_j)_\infty}\,d_qt\,.$$

Consider the system satisfied by F'_i:

$$(T_kF'_1,\ldots,T_kF'_n)=(F'_1,\ldots,F'_n)A'_k\,. \tag{5.1}$$

The asymptotic behavior in

$$\{(x_1,\ldots,x_n);|x_{\sigma(1)}|\gg\cdots\gg|x_{\sigma(n)}|\gg 1\}$$

characterizes the fundamental solution $\Xi_\sigma = \Xi_\sigma(x_1, \ldots, x_n)$ for a permutation $\sigma \in \mathfrak{S}_n$. Let e be the identity in $\mathfrak{S}_n$. In the sense of [Mi], the elementary connection matrix P_i is defined by $\Xi_{\sigma_i} = P_i\Xi_e$ for a transposition $\sigma_i = (i, i+1) \in \mathfrak{S}_n$. Then it is shown in [AKM], for $\beta_1 = \cdots = \beta_n$, that P_i depends only on the ratio x_i/x_{i+1} and satisfies the Yang–Baxter equation:

$$P_i(u)P_{i+1}(uv)P_i(v)=P_{i+1}(v)P_i(uv)P_{i+1}(u)\,.$$

This is equivalent to the Boltzmann weights of the eight vertex SOS model, i.e., the ABF-solution of the star-triangle relation (cf. [ABF, JMO]).

On the other hand, Frenkel and Reshetikhin [FR] studied a q-deformed chiral vertex operator along the line of [TK], for a quantum affine algebra $U_q(\hat{\mathfrak{g}})$. They showed that the correlation function satisfies the quantum Knizhnik–Zamolodchikov equation, which is written in terms of the universal R-matrix, and considered the connection matrix as a q-analogue of the braiding matrix in conformal field theory. In some situations, they proved that the connection matrix of the quantum Knizhnik–Zamolodchikov equation for a simple transposition depends only on the ratio of two arguments and it satisfies the quantum Yang–Baxter equation. The most remarkable point of their theory is the factorization property, from which it is possible to determine the connection matrix by computing it for $n = 2$, namely by considering the 4-point function as in the discussion of [TK]. Using this argument and considering Jackson integral solutions for $n = 2$, they calculated the connection matrix in the simplest case for $U_q(\widehat{\mathfrak{sl}}_2)$ which includes the ABF-solution [FR, Sect. 7]. Therefore the connection matrix of the quantum Knizhnik–Zamolodchikov equation for a special case coincides with that of [AKM].

Now our Eq. (2.12) for the function F_i defined by (2.5) is obviously equivalent to Eq. (5.1). In fact, F_i and F'_i are related to each other by a triangular matrix:

$$F_i = \sum_{j=1}^{i} b_{ij} F'_j .$$

The explicit form is given by

$$b_{ij} = \prod_{k=1}^{i} b_{ij}^k, \quad b_{ij}^k = \begin{cases} \dfrac{p^{\beta_j} x_j - x_k}{x_j - x_k} & (\text{if } k < i) \\ \dfrac{(p^{\beta_i} - 1) x_i}{x_j - x_i} & (\text{if } k = i) . \end{cases}$$

Since Theorem 3 says that Eq. (2.12) is equivalent to the quantum Knizhnik–Zamolodchikov equation (3.9), we have seen the coincidence above explicitly at the level of the q-difference equation before going to the connection matrix. Finally, combined with the discussions in [FR], the results in the present paper enable us to observe the surprising phenomenon revealed by [AKM], that a very rich structure is contained in such a simple expression:

$$\int_0^{s\infty} t^{\beta-1} \prod_{1 \leqq j \leqq n} \frac{(t/x_j)_\infty}{(p^{\beta_j} t/x_j)_\infty} d_q t ,$$

from the viewpoint of the representation theory of quantum enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

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