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# $p$-Adic Heisenberg Group and Maslov Index

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**Abstract.** A "system of coordinates" on a set $\Lambda$ of selfdual lattices in a two-dimensional $p$-adic symplectic space $(\mathscr{V}, \mathscr{B})$ is suggested. A unitary irreducible representation of the Heisenberg group of the space $(\mathscr{V}, \mathscr{B})$ depending on a lattice $\mathscr{L} \in \Lambda$ (an analogue of the Cartier representation) is constructed and its properties are investigated. By the use of such representations for three different lattices $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \Lambda$ one defines the Maslov index $\mu = \mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ of a triple of lattices. Properties of the index $\mu$ are investigated and values of $\mu$ in coordinates for different triples of lattices are calculated.

## 1. Introduction

As it is known one of the profitable methods to study a quantization procedure is to construct and to investigate topological characteristics associated with this procedure. An example of such a characteristic is the Maslov index [Ma]. Let us discuss generally one way to obtain such characteristics. Let $G$ be a group and $(H_i, U_i)$, $i = 1, 2, 3$ be its unitary irreducible representations in the Hilbert spaces $H_i$, $i = 1, 2, 3$ respectively. Let us assume that these representations are unitary equivalent and $F_{21}$, $F_{32}$ and $F_{13}$ be unitary intertwining operators. That is, say for $F_{21}$, $F_{21} : H_1 \to H_2$ and for all $g \in G$ the relation

$$F_{21}^{-1} U_2(g) F_{21} = U_1(g)$$

holds (and similarly for operators $F_{32}$ and $F_{13}$). By the last formula the operator $F = F_{13} F_{32} F_{21} : H_1 \to H_1$ commutes with all operators $U_1(g)$, $g \in G$. In view of irreducibility of $(H_1, U_1)$ the operator $F$ is proportional to the identity operator, that is $F = \mu \,\mathrm{Id}$ for some $\mu \in \mathbb{T}$ ($\mathbb{T}$ denotes a unit circle in the field $\mathbb{C}$ of complex numbers). Hence we obtain a numerical characteristic $\mu$ of a group $G$ and a triple of its unitary irreducible representations.

Let us take an example, see [LV]. Let $(\mathcal{V}, \mathcal{B})$ be a two-dimensional symplectic vector space over the field $\mathbb{R}$ of real numbers and $\tilde{\mathcal{V}}$ be the Heisenberg group of the space $(\mathcal{V}, \mathcal{B})$ (that is $\tilde{\mathcal{V}}$ is the three-dimensional Heisenberg group). Let also $L$ be a lagrangian (that is one-dimensional for $\dim \mathcal{V} = 2$) subspace of $\mathcal{V}$ provided with the natural Haar measure $dm(L)$. As it is known there is a unitary irreducible representation $(H(L), U_L)$ of the group $\tilde{\mathcal{V}}$ in the Hilbert space $H(L) = L^2(L, dm(L))$. For two different lagrangian subspaces $L_1$ and $L_2$ these representations are unitary equivalent. Let now $L_1$, $L_2$ and $L_3$ be different lagrangian subspaces in $\mathcal{V}$. By applying the procedure discussed above for the group $\tilde{\mathcal{V}}$ and for the representations $U_{L_1}$, $U_{L_2}$ and $U_{L_3}$ we obtain a numerical characteristic $\mu(L_1, L_2, L_3)$ of these representations. It turns out that in this case $\mu = \exp(i\pi\tau/4)$, where $\tau = \tau(L_1, L_2, L_3) \in \mathbb{Z}$ is the Maslov index of lagrangian subspaces $L_1$, $L_2$ and $L_3$, see [LV].

As a different example we consider the Cartier representation [C] of the Heisenberg group $\tilde{\mathcal{V}}$. This representation is unitary, irreducible and depends on a selfdual $\mathbb{Z}$-lattice $\mathcal{L}$ in the space $\mathcal{V}$. By using the procedure discussed above for the Cartier representations associated with lattices $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ we obtain an index of a triple $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ of selfdual $\mathbb{Z}$-lattices, see [LV].

As $p$-adic numbers find expanding applications in mathematical physics (the active advancement began from the paper [V]) it is interesting to extend the construction discussed above for the field $\mathbb{Q}_p$ of $p$-adic numbers. Let now $(\mathcal{V}, \mathcal{B})$ be a two-dimensional symplectic vector space over $\mathbb{Q}_p$ and $\tilde{\mathcal{V}}$ be the Heisenberg group of this space (for the definition of the group $\tilde{\mathcal{V}}$ see Sect. 3 of this paper). As for the field $\mathbb{R}$ there is a unitary irreducible representation of $\tilde{\mathcal{V}}$ in the space $L^2(L, dm(L))$, where $L$ is a lagrangian subspace of the space $\mathcal{V}$ and $dm(L)$ is the Haar measure on $L$, as to the corresponding index see [LV] and bibliography there.

There exist also a unitary irreducible representation of the $p$-adic Heisenberg group depending on a selfdual $\mathbb{Z}_p$-lattice in the space $\mathcal{V}$. ($\mathbb{Z}_p$ denotes a ring of $p$-adic integers.) This representation is an analogue of the Cartier representation mentioned above. By applying the procedure discussed above for the $p$-adic Heisenberg group and a triple of its representations associated with lattices $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ we obtain a complex number $\mu = \mu(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \in \mathbb{T}$. This number $\mu$ we call the Maslov index of a triple $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ of selfdual $\mathbb{Z}_p$-lattices. This index is the subject of our investigation. It is not improbable that this index will be useful for $p$-adic quantum mechanics constructed in [VV] (see also [Me, R]).

The structure of this paper is the following. In Sect. 2 one considers $\mathbb{Z}_p$-lattices and their properties. In particular one constructs a "system of coordinates" on a set $\Lambda$ of selfdual $\mathbb{Z}_p$-lattices in a two-dimensional symplectic space $(\mathcal{V}, \mathcal{B})$ over $\mathbb{Q}_p$ (Proposition 1). In Sect. 3 we define the Heisenberg group $\tilde{\mathcal{V}}$ of the space $(\mathcal{V}, \mathcal{B})$ and construct a unitary irreducible representation $(H(\mathcal{L}), W_{\mathcal{L}})$ of this group depending on a lattice $\mathcal{L} \in \Lambda$. We prove also some properties of this representation (Proposition 2). In Sect. 4 an intertwining operator of two such representation is constructed and its properties are investigated (Proposition 3). In Sect. 5 we construct the Maslov index of a triple of selfdual $\mathbb{Z}_p$-lattices. We also obtain an explicit formula for this index (Proposition 4) and prove some natural properties of the index (Proposition 5). Section 6 is devoted to calculations of the Maslov index in coordinates defined in Sect. 2.

## 2. Lattices

Let $(\mathscr{V}, \mathscr{B})$ be a two dimensional symplectic space over $\mathbb{Q}_p$ and $\mathscr{L}$ be a *lattice* in $(\mathscr{V}, \mathscr{B})$ (that is $\mathscr{L}$ is a finitely generated $\mathbb{Z}_p$-submodule of the space $\mathscr{V}$ containing a basis of $\mathscr{V}$). A dual lattice $\mathscr{L}^*$ is defined as follows:

$$\mathscr{L}^* = \{x \in \mathscr{V} : \mathscr{B}(x, y) \in \mathbb{Z}_p \, \forall y \in \mathscr{L}\}.$$

If $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is a *selfdual* lattice. Let $\Lambda = \Lambda(\mathscr{V}, \mathscr{B})$ denote the set of all selfdual lattices in $(\mathscr{V}, \mathscr{B})$. Note that if $\mathscr{L} \in \Lambda(\mathscr{V}, \mathscr{B})$, then $(\mathscr{L}, \mathscr{B})$ is a space with symplectic inner product.

As $\mathbb{Z}_p$ is a local ring, then there exists a symplectic basis $\{e, f\}$ of the space $(\mathscr{V}, \mathscr{B})$ (symplectic means that $\mathscr{B}(e, f) = 1$) wherein (see [MH])

$$\mathscr{L} = \mathbb{Z}_p e \oplus \mathbb{Z}_p f.$$

Moreover for any $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$ there is a symplectic basis $\{e, f\}$ wherein these lattices have the form

$$\mathscr{L}_1 = \mathbb{Z}_p e \oplus \mathbb{Z}_p f, \qquad \mathscr{L}_2 = p^m \mathbb{Z}_p e \oplus p^{-m} \mathbb{Z}_p f$$

for some nonnegative integer $m$. For the proof of existence of such basis (but is not of necessity symplectic) see for example [W1], reduction to symplectic case is rather obvious.

Now we define a "system of coordinates" on the set $\Lambda$. Let $Sp(\mathscr{V})$ denote the group of all linear automorphisms of $\mathscr{V}$ preserving the form $\mathscr{B}$ (symplectic group) and $Sp(\mathscr{L})$ be a stabilizer of a selfdual lattice $\mathscr{L}$ in $Sp(\mathscr{V})$. $Sp(\mathscr{V})$ acts on $\Lambda$ in a standard manner, this action is transitive. Thus $\Lambda$ can be identified with the homogeneous space $Sp(\mathscr{V})/Sp(\mathscr{L})$.

**Proposition 1.** *Let $\{e, f\}$ be a symplectic basis in $(\mathscr{V}, \mathscr{B})$. Then the map $\varphi: \mathbb{Z} \times \mathbb{Q}_p/\mathbb{Z}_p \to \Lambda$,*

$$\mathbb{Z} \times \mathbb{Q}_p/\mathbb{Z}_p \ni (m, \bar{\mu}) \overset{\varphi}{\mapsto} \mathbb{Z}_p p^m e \oplus \mathbb{Z}_p(\mu p^m e + p^{-m} f) \in \Lambda$$

*defines a one-to-one correspondence between $\mathbb{Z} \times \mathbb{Q}_p/\mathbb{Z}_p$ and $\Lambda$. (In the right-hand part of the last formula $\mu$ denotes an arbitrary element of a coset $\bar{\mu}$.)*

*Proof.* Let $\mathscr{L}_0$ denote the following lattice:

$$\mathscr{L}_0 = \mathbb{Z}_p e + \mathbb{Z}_p f.$$

In the basis $\{e, f\}$ $Sp(\mathscr{V})$ and $Sp(\mathscr{L}_0)$ have the matrix realizations: $Sp(\mathscr{V}) \cong SL(2, \mathbb{Q}_p)$, $Sp(\mathscr{L}_0) \cong SL(2, \mathbb{Z}_p)$. Let $\mathscr{L}$ be an arbitrary lattice from $\Lambda$. Then there is an element $g \in SL(2, \mathbb{Q}_p)$ such that $\mathscr{L} = g\mathscr{L}_0$. By the Iwasawa decomposition (see [PR]) $g$ can be represented in the form:

$$g = \begin{pmatrix} p^m & 0 \\ 0 & p^{-m} \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} g_0$$

for some $m \in \mathbb{Z}$, $\mu \in \mathbb{Q}_p$ and $g_0 \in SL(2, \mathbb{Z}_p)$. Thus $\mathscr{L}$ has the form

$$\mathscr{L} = \mathbb{Z}_p p^m e \oplus \mathbb{Z}_p(\mu p^m e + p^{-m} f)$$

and the map $\varphi$ is surjective. As for $m, m' \in \mathbb{Z}$ and $\mu, \mu' \in \mathbb{Q}_p$ we have

$$\left[\begin{pmatrix} p^{m'} & 0 \\ 0 & p^{-m'} \end{pmatrix}\begin{pmatrix} 1 & \mu' \\ 0 & 1 \end{pmatrix}\right]^{-1}\begin{pmatrix} p^{m} & 0 \\ 0 & p^{-m} \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} p^{m-m'} & p^{m-m'}\mu - p^{m'-m}\mu' \\ 0 & p^{m'-m} \end{pmatrix} \in SL(2, \mathbb{Z}_p)$$

if and only if $m = m'$ and $\mu - \mu' \in \mathbb{Z}_p$, then the definition of the map $\varphi$ is correct (that is it doesn't depend on a choice of $\mu$ in a coset $\bar{\mu}$). This finishes the proof.

**Corollary.** *For any $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \Lambda$ there is a symplectic basis $\{e, f\}$ wherein*

$$\mathscr{L}_1 = \mathbb{Z}_p e \oplus \mathbb{Z}_p f\,,$$
$$\mathscr{L}_2 = p^m \mathbb{Z}_p e \oplus p^{-m} \mathbb{Z}_p f\,,$$
$$\mathscr{L}_3 = \mathbb{Z}_p p^n e \oplus \mathbb{Z}_p(\nu p^n e + p^{-n} f)$$

*for some $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$, $\nu \in \mathbb{Q}_p$.*

## 3. $p$-Adic Heisenberg Group

Let $\chi_p$ be an additive character of $\mathbb{Q}_p$ of rank 0 (that is $\chi_p(x) = 1$ if and only if $x \in \mathbb{Z}_p$), $\mathbb{T}$ be a unit circle in the field $\mathbb{C}$ of complex numbers. *Heisenberg group* $\tilde{\mathscr{V}}$ of a space $(\mathscr{V}, \mathscr{B})$ is the set of pairs

$$\tilde{\mathscr{V}} = \{(\alpha, x), \alpha \in \mathbb{T}, x \in \mathscr{V}\}$$

with the composition law

$$(\alpha, x)(\beta, y) = (\alpha\beta\chi_p(1/2\mathscr{B}(x, y)), x + y)\,.$$

We assume that $p \neq 2$ below. Now we construct some representation of $\tilde{\mathscr{V}}$. This representation depends on a lattice $\mathscr{L} \in \Lambda$ and therefore we call it $\mathscr{L}$-representation. Let $\tilde{H}(\mathscr{L})$ denote the space of finite complex valued functions on $\mathscr{V}$ satisfying the relation

$$f(x + u) = \chi_p(1/2\mathscr{B}(x, u)) f(x)$$

for all $x \in \mathscr{V}$ and $u \in \mathscr{L}$. Note that if $f, g \in \tilde{H}(\mathscr{L})$ then $|f|$ and $f\bar{g}$ are constant on every coset in $\mathscr{V}/\mathscr{L}$ and nonzero only on a finite number of such cosets. For $f, g \in \tilde{H}(\mathscr{L})$ the formula

$$(f, g) = \sum_{\alpha \in \mathscr{V}/\mathscr{L}} f(\alpha)\bar{g}(\alpha)$$

defines a nonnegative hermitian form on $\tilde{H}(\mathscr{L})$ and thus $\tilde{H}(\mathscr{L})$ is provided by a prehilbertian structure. The *space $H(\mathscr{L})$ of $\mathscr{L}$-representation* is defined as the completion of $\tilde{H}(\mathscr{L})$ with respect to the norm $\|\cdot\|^2 = (\cdot\,, \cdot)$. As $\mathscr{V}/\mathscr{L}$ is a countable set, then $H(\mathscr{L})$ is a separable Hilbert space.

On the space $\tilde{H}(\mathscr{L})$ we define the following set of operators, $x, y \in \mathscr{V}$:

$$(W_{\mathscr{L}}(x) f)(y) = \chi_p(1/2\mathscr{B}(x, y)) f(y - x)\,.$$

These operators satisfy the co-called Weyl relation

$$W_{\mathscr{L}}(x)W_{\mathscr{L}}(y) = \chi_p(1/2\mathscr{B}(x,y))W_{\mathscr{L}}(x+y)\,.$$

It is easy to see that $W_{\mathscr{L}}(x)$, $x \in \mathscr{V}$ are isometric operators on $\tilde{H}(\mathscr{L})$ and therefore are uniquely extended to unitary operators on $H(\mathscr{L})$ (for these operators we retain the same notation $W_{\mathscr{L}}(x)$). $\mathscr{L}$-representation of $\tilde{\mathscr{V}}$ is defined as a pair $(H(\mathscr{L}), \tilde{W}_{\mathscr{L}})$, where $\tilde{W}_{\mathscr{L}}(\alpha, x) = \alpha W_{\mathscr{L}}(x)$. From the Weyl relation we see that this pair is in fact a unitary representation of $\tilde{\mathscr{V}}$. For the sake of convenience we use the term "$\mathscr{L}$-representation" for a pair $(H(\mathscr{L}), W_{\mathscr{L}}(x))$. A similar representation was considered in [W2]. Note that $\mathscr{L}$-representation is a $p$-adic analogue of the Cartier representation [C] of the Heisenberg group over real numbers.

Let $\phi_{\mathscr{L}}$ denote the following element of $H(\mathscr{L})$:

$$\phi_{\mathscr{L}}(u) = \begin{cases} 1\,, & u \in \mathscr{L}\,, \\ 0\,, & u \notin \mathscr{L}\,. \end{cases}$$

We call it a *vacuum vector* of $(H(\mathscr{L}), W_{\mathscr{L}}(x))$. It is easy to see that this vector satisfies the property

$$W_{\mathscr{L}}(x)\phi_{\mathscr{L}} = \phi_{\mathscr{L}} \tag{1}$$

for all $x \in \mathscr{L}$.

Let $\eta_{\mathscr{L}}\colon \mathscr{V} \to \mathbb{T}$ be a function satisfying the property

$$\eta_{\mathscr{L}}(x+u) = \chi_p(1/2\mathscr{B}(x,u))\eta_{\mathscr{L}}(x)$$

for all $x \in \mathscr{V}$ and $u \in \mathscr{L}$. It is quite easy to prove that the map $\mathscr{V} \to H(\mathscr{L})$:

$$\mathscr{V} \ni x \mapsto \eta_{\mathscr{L}}(x)W_{\mathscr{L}}(x)\phi_{\mathscr{L}}$$

is constant on every coset in $\mathscr{V}/\mathscr{L}$ and thus one defines a map $\psi\colon \mathscr{V}/\mathscr{L} \to H(\mathscr{L})$ by the same formula. The range of values of the map $\psi$ we call a set of *coherent states* of $\mathscr{L}$-representation.

**Proposition 2.** *The representation $(H(\mathscr{L}), W_{\mathscr{L}}(x))$ has the properties:*
(i) $(W_{\mathscr{L}}(x)\phi_{\mathscr{L}}, \phi_{\mathscr{L}}) = \phi_{\mathscr{L}}(x)$;
(ii) *the set of coherent states forms an orthonormal basis in $H(\mathscr{L})$;*
(iii) *the representation $(H(\mathscr{L}), W_{\mathscr{L}}(x))$ is irreducible.*

## 4. Intertwining Operator

Let for $\mathscr{L}_1, \mathscr{L}_2 \in \Lambda$ $\varrho^{-2}(\mathscr{L}_1, \mathscr{L}_2)$ denotes the number of elements of the group $\mathscr{L}_1/(\mathscr{L}_1 \cap \mathscr{L}_2)$.

**Proposition 3.** *Let $(H(\mathscr{L}_1), W_{\mathscr{L}_1})$ and $(H(\mathscr{L}_2), W_{\mathscr{L}_2})$ be $\mathscr{L}_1$- and $\mathscr{L}_2$-representations. Then the operator $F_{\mathscr{L}_2,\mathscr{L}_1}\colon H(\mathscr{L}_1) \to H(\mathscr{L}_2)$ defined by the formula*

$$F_{\mathscr{L}_2,\mathscr{L}_1} f(u) = \varrho(\mathscr{L}_1, \mathscr{L}_2) \sum_{\alpha \in \mathscr{L}_2/(\mathscr{L}_1 \cap \mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha, u)) f(u+\alpha) \tag{2}$$

*is a unitary operator. It satisfies the property*

$$F^{-1}_{\mathscr{L}_2,\mathscr{L}_1} = F_{\mathscr{L}_1,\mathscr{L}_2} \tag{3}$$

*and it is an intertwining operator for the $\mathscr{L}_1$- and $\mathscr{L}_2$-representations, that is for all $x \in \mathscr{V}$ the following relation holds:*

$$F^{-1}_{\mathscr{L}_2,\mathscr{L}_1} W_{\mathscr{L}_2}(x) F_{\mathscr{L}_2,\mathscr{L}_1} = W_{\mathscr{L}_1}(x)\,. \tag{4}$$

*Proof.* At first we check that the definition (1) is correct, that is the right-hand part of the formula (2) doesn't depend on a choice of an element in coset $\alpha \in \mathscr{L}_1/(\mathscr{L}_1 \cap \mathscr{L}_2)$. In fact taking into account that $f \in H(\mathscr{L}_1)$ for $\alpha' \in \mathscr{L}_1 \cap \mathscr{L}_2$ we have

$$\begin{aligned}
&\sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha+\alpha',u))\, f(u+\alpha+\alpha') \\
&= \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha+\alpha',u)+1/2\mathscr{B}(u+\alpha,\alpha'))\, f(u+\alpha) \\
&= \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha,\alpha'))\chi_p(1/2\mathscr{B}(\alpha,u))\, f(u+\alpha) \\
&= \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha,u))\, f(u+\alpha)\,.
\end{aligned}$$

It is easy to check that for $f \in H(\mathscr{L}_1)$ the condition $F_{\mathscr{L}_2,\mathscr{L}_1} f \in H(\mathscr{L}_2)$ holds.

Let us prove unitarity of $F_{\mathscr{L}_2,\mathscr{L}_1}$. From the definition of the operator $F_{\mathscr{L}_2,\mathscr{L}_1}$ we get

$$F_{\mathscr{L}_2,\mathscr{L}_1} f(u) = \varrho(\mathscr{L}_1,\mathscr{L}_2) \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(\mathscr{B}(\alpha,u))\, W_{\mathscr{L}_1}(-\alpha) f(u)\,. \tag{5}$$

From the definition of $\mathscr{L}$-representation, orthogonality of coherent states, Parseval-Stokes relation and the last formula we have

$$\|F_{\mathscr{L}_2,\mathscr{L}_1} f\|^2_{H(\mathscr{L}_2)} = \varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \|W_{\mathscr{L}_1}(-\alpha) f\|^2_{H(\mathscr{L}_1)} = \|f\|^2_{H(\mathscr{L}_1)}\,.$$

Now we prove the formula (3). Taking into account the condition $f \in H(\mathscr{L}_1)$ we get

$$\begin{aligned}
&F_{\mathscr{L}_1,\mathscr{L}_2} F_{\mathscr{L}_2,\mathscr{L}_1} f(u) \\
&= \varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\beta,u)) \\
&\quad\times \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha,u+\beta))\, f(u+\alpha+\beta) \\
&= \varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\substack{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)\\ \beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)}} \\
&\quad\times \chi_p(1/2\mathscr{B}(\beta,u)+1/2\mathscr{B}(\alpha,u+\beta)+1/2\mathscr{B}(u+\alpha,\beta))\, f(u+\alpha) \\
&= \varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\alpha\in\mathscr{L}_2/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(1/2\mathscr{B}(\alpha,u))\, f(u+\alpha) \sum_{\beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(\mathscr{B}(\alpha,\beta))
\end{aligned}$$

and (3) follows from the formula

$$\varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(\mathscr{B}(\alpha,\beta)) = \begin{cases} 1, & \alpha\in\mathscr{L}_1\cap\mathscr{L}_2, \\ 0, & \alpha\notin\mathscr{L}_2\cap\mathscr{L}_1. \end{cases} \tag{6}$$

For $\alpha\in\mathscr{L}_1\cap\mathscr{L}_2$ (6) obviously follows from the definition of $\varrho(\mathscr{L}_1,\mathscr{L}_2)$. For $\alpha\notin\mathscr{L}_2\cap\mathscr{L}_1$ let us choose $\beta'\in\mathscr{L}_1$ satisfying the condition $\chi_p(\mathscr{B}(\alpha,\beta'))\neq 1$ (by virtue of selfduality of $\mathscr{L}_1$ such $\beta'$ always exists). Then

$$\begin{aligned} &\varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(\mathscr{B}(\alpha,\beta)) \\ &= \varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(\mathscr{B}(\alpha,\beta+\beta')) \\ &= \chi_p(\mathscr{B}(\alpha,\beta'))\,\varrho^2(\mathscr{L}_1,\mathscr{L}_2) \sum_{\beta\in\mathscr{L}_1/(\mathscr{L}_1\cap\mathscr{L}_2)} \chi_p(\mathscr{B}(\alpha,\beta)) \end{aligned}$$

and therefore (6) is valid. The property (4) of the operator $F_{\mathscr{L}_2,\mathscr{L}_1}$ can be proved by analogy to that of (3).

The operator $F_{\mathscr{L}_2,\mathscr{L}_1}$ we call a *canonical intertwining operator.*

In particular from the last proposition it follows that $\mathscr{L}_1$- and $\mathscr{L}_2$-representations are unitary equivalent.

## 5. Maslov Index

Let $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in\Lambda$. Then the corresponding representations $(H(\mathscr{L}_1),W_{\mathscr{L}_1})$, $(H(\mathscr{L}_2), W_{\mathscr{L}_2})$ and $(H(\mathscr{L}_3),W_{\mathscr{L}_3})$ are unitary equivalent. Let us consider the unitary operator $\mathscr{F} = F_{\mathscr{L}_1,\mathscr{L}_3}F_{\mathscr{L}_3,\mathscr{L}_2}F_{\mathscr{L}_2,\mathscr{L}_1}$ on the space $H(\mathscr{L}_1)$. By using the formula (4) for intertwining operators $F_{\mathscr{L}_1,\mathscr{L}_3}$, $F_{\mathscr{L}_3,\mathscr{L}_2}$ and $F_{\mathscr{L}_2,\mathscr{L}_1}$ it is easy to see that the operator $\mathscr{F}$ commutes with all operators $W_{\mathscr{L}_1}(x)$, $x\in\mathscr{V}$ and by virtue of irreducibility of the $\mathscr{L}_1$-representation $(H(\mathscr{L}_1),W_{\mathscr{L}_1})$ it is proportional to the identity operator on $H(\mathscr{L}_1)$. Thus we have

$$\mathscr{F} = \mu(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\,\mathrm{Id}\,.$$

The number $\mu(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)\in\mathbb{T}$ we call the *Maslov index* of a triple of selfdual lattices.

Let us take an explicit formula for the Maslov index.

**Proposition 4.** *Let $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3\in\Lambda$. Then the following formula holds:*

$$\mu(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3) = \frac{\varrho(\mathscr{L}_1,\mathscr{L}_2)\,\varrho(\mathscr{L}_2,\mathscr{L}_3)}{\varrho(\mathscr{L}_3,\mathscr{L}_1)} \sum_{\substack{\alpha\in\mathscr{L}_2/(\mathscr{L}_2\cap\mathscr{L}_3)\\ \beta\in\mathscr{L}_3/(\mathscr{L}_3\cap\mathscr{L}_1)\\ \alpha+\beta\in\mathscr{L}_1}} \chi_p(1/2\mathscr{B}(\alpha,\beta))\,.$$

*Proof* leans upon the formula (2) for a canonical intertwining operator. Let $f \in H(\mathscr{L}_1)$, then we have

$$\mathscr{F} f(u) = \varrho(\mathscr{L}_1, \mathscr{L}_2)\, \varrho(\mathscr{L}_2, \mathscr{L}_3)\, \varrho(\mathscr{L}_3, \mathscr{L}_1) \sum_{\substack{\gamma \in \mathscr{L}_1/(\mathscr{L}_3 \cap \mathscr{L}_1) \\ \beta \in \mathscr{L}_3/(\mathscr{L}_2 \cap \mathscr{L}_3) \\ \alpha \in \mathscr{L}_2/(\mathscr{L}_1 \cap \mathscr{L}_2)}} \chi_p(1/2\mathscr{B}(\gamma, u)$$

$$+ 1/2\mathscr{B}(\beta, u+\gamma) + 1/2\mathscr{B}(\alpha, u+\beta+\gamma))\, f(u+\alpha+\beta+\gamma)$$

$$= \varrho(\mathscr{L}_1, \mathscr{L}_2)\, \varrho(\mathscr{L}_2, \mathscr{L}_3)\, \varrho(\mathscr{L}_3, \mathscr{L}_1) \sum_{\substack{\gamma \in \mathscr{L}_1/(\mathscr{L}_3 \cap \mathscr{L}_1) \\ \beta \in \mathscr{L}_3/(\mathscr{L}_2 \cap \mathscr{L}_3) \\ \alpha \in \mathscr{L}_2/(\mathscr{L}_1 \cap \mathscr{L}_2)}}$$

$$\times \chi_p(1/2\mathscr{B}(\alpha, \beta) + \mathscr{B}(\alpha+\beta, \gamma))\, f(u+\alpha+\beta)\,.$$

By using the last formula for $f = \phi_{\mathscr{L}_1}$ we get the needed formula:

$$\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = (\mathscr{F}\phi_{\mathscr{L}_1}, \phi_{\mathscr{L}_1})_{H(\mathscr{L}_1)}$$

$$= \frac{\varrho(\mathscr{L}_1, \mathscr{L}_2)\, \varrho(\mathscr{L}_2, \mathscr{L}_3)}{\varrho(\mathscr{L}_3, \mathscr{L}_1)} \sum_{\substack{\alpha \in \mathscr{L}_2/(\mathscr{L}_2 \cap \mathscr{L}_3) \\ \beta \in \mathscr{L}_3/(\mathscr{L}_3 \cap \mathscr{L}_1) \\ \alpha+\beta \in \mathscr{L}_1}} \chi_p(1/2\mathscr{B}(\alpha, \beta)) \qquad \square$$

Proposition 4 shows that the Maslov index of a triple of selfdual lattices does depend on only the "relative positions" of lattices, although in its definition one uses a representation of the Heisenberg group.

**Proposition 5.** *Let* $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4 \in \Lambda$. *The following statements are valid.*
(i) $\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = \mu(g\mathscr{L}_1, g\mathscr{L}_2, g\mathscr{L}_3)$ *for all* $g \in Sp(\mathscr{V})$;
(ii) $\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = 1$ *if at least two lattices in the triple coincide;*
(iii) $\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ *remains the same under an even permutation of lattices in the triple and transfers to a conjugate expression under an odd one;*
(iv) *the following cocycle relation holds:*

$$\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\, \mu(\mathscr{L}_1, \mathscr{L}_3, \mathscr{L}_4) = \mu(\mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4)\, \mu(\mathscr{L}_2, \mathscr{L}_4, \mathscr{L}_1)\,.$$

*Proof.* (i) follows directly from the explicit formula for $\mu$ (Proposition 4). The statement (ii)–(iv) one proves in a similar manner immediately from the definition of $\mu$. Let us prove the statement (iv). From the definition of the Maslow index we have:

$$\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)\, \mu(\mathscr{L}_1, \mathscr{L}_3, \mathscr{L}_4)\ \mathrm{Id}$$

$$= F_{\mathscr{L}_1,\mathscr{L}_4} F_{\mathscr{L}_4,\mathscr{L}_3} F_{\mathscr{L}_3,\mathscr{L}_2} F_{\mathscr{L}_2,\mathscr{L}_1} = F^{-1}_{\mathscr{L}_2,\mathscr{L}_1} (F_{\mathscr{L}_2,\mathscr{L}_1} F_{\mathscr{L}_1,\mathscr{L}_4} F_{\mathscr{L}_4,\mathscr{L}_2})$$

$$\times (F_{\mathscr{L}_2,\mathscr{L}_4} F_{\mathscr{L}_4,\mathscr{L}_3} F_{\mathscr{L}_3,\mathscr{L}_2}) F_{\mathscr{L}_2,\mathscr{L}_1} = \mu(\mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4)\, \mu(\mathscr{L}_2, \mathscr{L}_4, \mathscr{L}_1)\ \mathrm{Id}\,.$$

## 6. Calculations of the Maslov Index

Let us remind that any $x \in \mathbb{Q}_p^*$ can be uniquely represented in the following form:

$$x = p^{\mathrm{ord}_p(x)} \varepsilon(x)\,,$$

where $\text{ord}_p : \mathbb{Q}_p^* \to \mathbb{Z}$ and $|x|_p = p^{-\text{ord}_p(x)}$; $\varepsilon : \mathbb{Q}_p^* \to \mathbb{Z}_p^*$ and $\varepsilon(x) = x_0 + x_1 p + \ldots$, $x_j = 0, 1, \ldots, p-1$, $x_0 \neq 0$. Fractional part $\{x\}_p$ equals 0 if $x \in \mathbb{Z}_p$ and for $x \notin \mathbb{Z}_p$ is defined by the formula

$$\{x\}_p = p^{\text{ord}_p(x)}(x_0 + x_1 p + \ldots + x_{-\text{ord}_p(x)-1} p^{-\text{ord}_p(x)-1}).$$

Let $\lambda_p : \mathbb{Q}_p \to \mathbb{T}$ be a function defined by the formula (see [VV]):

$$\lambda_p(0) = 1.$$

$$\lambda_p(x) = \begin{cases} 1, & \text{ord}_p(x) = 2k, k \in \mathbb{Z}, \\ \left(\dfrac{\varepsilon(x)}{p}\right), & \text{ord}_p(x) = 2k+1, k \in \mathbb{Z}, p \equiv 1 \,(\text{mod}\, 4), \\ i\left(\dfrac{\varepsilon(x)}{p}\right), & \text{ord}_p(x) = 2k+1, k \in \mathbb{Z}, p \equiv 3 \,(\text{mod}\, 4), \end{cases}$$

where $\left(\dfrac{\varepsilon(x)}{p}\right)$ is the Legendre symbol of a $p$-adic unit $\varepsilon(x) \in \mathbb{Z}_p^*$. This function has the following properties.

**Lemma 1.** *Function $\lambda_p$ has the properties:*

(i) $\lambda_p(-x) = \overline{\lambda_p(x)}$;

(ii) $\lambda_p(a^2 x) = \lambda_p(x)$, $a \in \mathbb{Q}_p^*$;

(iii) $\lambda_p(x)\lambda_p(y) = \lambda_p\left(\dfrac{x+y}{xy}\right)\lambda_p(x+y)$;

(iv) $\lambda_p(x)\lambda_p(y) = (x,y)\lambda_p(xy)$, *where $(x,y)$ is the Hilbert symbol.*

*Proof.* For the proof of the properties (i)–(iii) see [VV]. Taking into account that $\lambda_p(x) = 1$ for $x \in \mathbb{Z}_p^*$, statement (ii) and the symmetry of (iv) it is sufficient to check (iv) for the cases $x = y = p$, $x = y = \eta p$, $x = p$, $y = \eta p$, where $\eta \in \mathbb{Z}_p^*$, $\left(\dfrac{\eta}{p}\right) = -1$ that can be done by direct calculations.

From the definition of $\lambda_p$ it is easy to make out the connection of this function with the Gauss sum

$$\sum_{k=0}^{p^n-1} \exp\left(2\pi i a \frac{k^2}{p^n}\right) = p^{n/2}\lambda_p(ap^n), \tag{7}$$

where $a \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$ and $a$ is not divisible by $p$.

Let $m, n \in \mathbb{Z}$, $\mu, \nu \in \mathbb{Q}_p$ and $\{e, f\}$ be a symplectic basis of $(\mathscr{V}, \mathscr{B})$. We consider now the following triple of selfdual lattices in $(\mathscr{V}, \mathscr{B})$:

$$\mathscr{L}_1 = \mathbb{Z}_p e \oplus \mathbb{Z}_p f,$$
$$\mathscr{L}_2 = \mathbb{Z}_p p^m e \oplus \mathbb{Z}_p(\mu p^m e + p^{-m} f),$$
$$\mathscr{L}_3 = \mathbb{Z}_p p^n e \oplus \mathbb{Z}_p(\nu p^n e + p^{-n} f).$$

As it is evident from the foregoing the Maslov index of these triples can be represented as function of $m, n, \mu$ and $\nu$, that is $\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) = M(m, \mu; n, \nu)$ for some function $M : (\mathbb{Z} \times \mathbb{Q}_p) \times (\mathbb{Z} \times \mathbb{Q}_p) \to \mathbb{T}$. The explicit formulas for the function $M$ in simplest cases is given by the following theorem.

**Theorem.** *The following formulas are valid:*

$$\text{(i)} \qquad M(m,0;n,0) = 1 \quad \textit{for all} \quad m,n \in \mathbb{Z};$$

$$\text{(ii)} \qquad M(m,0;0,\nu) = \begin{cases} 1, & m \geq 0 \quad \textit{or} \quad \nu \in \mathbb{Z}_p, \\ \lambda_p(-\nu), & m < 0, 1 < |\nu|_p < p^{-2m}, \\ 1, & m < 0, p^{-2m} \leq |\nu|_p; \end{cases}$$

$$\text{(iii)} \qquad M(0,\mu;0,\nu) = \begin{cases} 1, & \mu \in \mathbb{Z}_p \quad \textit{or} \quad \nu \in \mathbb{Z}_p \quad \textit{or} \quad \mu-\nu \in \mathbb{Z}_p, \\ \lambda_p(\mu\nu(\mu-\nu)) & \textit{in other cases.} \end{cases}$$

*Proof.* Since $|\mu(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)| = 1$, then all calculations can be carried out up to some real positive factor and instead of the equality sign we shall write the sign $\sim$. By virtue of Proposition 4 and the last remark we have

$$M(m,\mu,n,\nu) \sim \sum_{\substack{\alpha \in \mathscr{L}_1/(\mathscr{L}_2 \cap \mathscr{L}_3) \\ \beta \in \mathscr{L}_3/(\mathscr{L}_3 \cap \mathscr{L}_1) \\ \alpha+\beta \in \mathscr{L}_1}} \chi_p(1/2\mathscr{B}(\alpha,\beta)).$$

(i) Taking into account Proposition 5 (ii) it is sufficient to consider the case $m \neq 0$, $m \neq n$, $n \neq 0$. Besides that we can reduce the general case to the case of $m > n$, $m > 0$ by means of changes of order of lattices in the triple and transformation of basis $e \to f$, $f \to -e$ if it is necessary. Since $\alpha \in \mathscr{L}_2$ and $\beta \in \mathscr{L}_3$ they can be represented in the following form:

$$\alpha = \alpha_1 p^m e + \alpha_2 p^{-m} f,$$
$$\beta = \beta_1 p^n e + \beta_2 p^{-n} f,$$

where $\alpha_1, a_2, \beta_1, \beta_2 \in \mathbb{Z}_p$. As $p^m\alpha_1 \in \mathbb{Z}_p$ if $m > 0$ and $\alpha_1 \in \mathbb{Z}_p$ then the condition $\alpha + \beta \in \mathscr{L}_1$ has the form:

$$p^n\beta_1 \in \mathbb{Z}_p, \qquad p^{-m}\alpha_2 + p^{-n}\beta_2 \in \mathbb{Z}_p. \tag{8}$$

Since $\chi_p$ is of rank 0 and taking into account the condition $m-n > 0$ and the formula (8) we get:

$$\begin{aligned} \chi_p(\mathscr{B}(\alpha,\beta)) &= \chi_p(p^{m-n}\alpha_1\beta_2 - p^{n-m}\alpha_2\beta_1) = \chi_p(-p^{n-m}\alpha_2\beta_1) \\ &= \chi_p(-p^n\beta_1(p^{-n}\beta_2 + p^{-m}\alpha_2 - p^{-n}\beta_2)) \\ &= \chi_p(-p^n\beta_1(p^{-n}\beta_2 + p^{-m}\alpha_2) + \beta_1\beta_2) = 1 \end{aligned}$$

and therefore $M(m,0;n,0) = 1$ for all $m,n \in \mathbb{Z}$.

(ii) Taking into account Proposition 1 and 5 (ii) it is sufficient to consider the case $m \neq 0$, $\nu \notin \mathbb{Z}_p$. Let $\alpha \in \mathscr{L}_2$ and $\beta \in \mathscr{L}_3$. Then we have

$$\alpha = \alpha_1 p^m e + \alpha_2 p^{-m} f,$$
$$\beta = \beta_1 e + \beta_2(\nu e + f),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_p$. The condition $\alpha + \beta \in \mathscr{L}_1$ has the form:

$$p^m\alpha_1 + \nu\beta_2 \in \mathbb{Z}_p, \qquad p^{-m}\alpha_2 \in \mathbb{Z}_p. \tag{9}$$

Since of $\chi_p$ is a character of rank 0 and taking into account the formula (9) we get:

$$\begin{aligned}\chi_p(\mathscr{B}(\alpha,\beta)) &= \chi_p(p^m\alpha_1\beta_2 - p^{-m}\nu\alpha_2\beta_2)\\ &= \chi_p(p^m\alpha_1\beta_2 - p^{-m}\alpha_2(p^m\alpha_1 + \nu\beta_2 - p^m\alpha_1))\\ &= \chi_p(p^m\alpha_1\beta_2 - p^{-m}\alpha_2(p^m\alpha_1 + \nu\beta_2) + \alpha_1\alpha_2) = \chi_p(p^m\alpha_1\beta_2). \quad (10)\end{aligned}$$

If $m \geq 0$ then as it follows from (10) $\chi_p(\mathscr{B}(\alpha,\beta)) = 1$ and $M(m,0;0,\nu) = 1$. Let now $m < 0$ and $|\nu|_p \geq p^{-2m}$, that is $\mathrm{ord}_p(\nu) \leq 2m$. By virtue of (9) and (10) we have

$$\begin{aligned}\chi_p(\mathscr{B}(\alpha,\beta)) &= \chi_p(p^{m-\mathrm{ord}_p(\nu)}\varepsilon^{-1}(\nu)\alpha_1(p^m\alpha_1 + \nu\beta_2 - p^m\alpha_1))\\ &= \chi_p(p^{m-\mathrm{ord}_p(\nu)}\varepsilon^{-1}(\nu)\alpha_1(p^m\alpha_1 + \nu\varphi_2) - p^{2m-\mathrm{ord}_p(\nu)}\varepsilon^{-1}(\nu)\alpha_1^2)\\ &= \chi_p(-p^{2m-\mathrm{ord}_p(\nu)}\varepsilon^{-1}(\nu)\alpha_1^2) = 1\end{aligned}$$

and $M(m,0;0,\nu) = 1$. In the last case $m < 0$ and $1 < |\nu|_p < p^{-2m}$ the proof is given below for the case $1 < |\nu|_p \leq p^{-m}$ (the case $p^{-m} < |\nu|_p < p^{-2m}$ one considers analogously). Let $a$ and $b$ denote $\alpha_1$ and $\beta_2$ respectively, $n$ denotes $\mathrm{ord}_p(\nu)$ and $\varepsilon$ denotes $\varepsilon(\nu)$. As any $x \in \mathbb{Z}_p$ can be represented in the form

$$x = x_0 + x_1p + x_2p^2 + \dots, \qquad x_j = 0,1,\dots,p-1,$$

then the condition (9) takes the form

$$p^m(a_0 + a_1p + \dots) + p^n(b_0 + b_1p + \dots)\varepsilon \in \mathbb{Z}_p.$$

From the last formula we get that the formula (9) is equivalent to the set of equations:

$$\begin{gathered}a_0 = a_1 = \dots = a_{n-m-1} = 0,\\ a_{n-m} + (b\varepsilon)_0 = 0,\\ \vdots\\ a_{-m-1} + (b\varepsilon)_{-n-1} = 0,\end{gathered}$$

thus from (10) we have

$$\begin{aligned}\chi_p(\mathscr{B}(\alpha,\beta)) &= \chi_p(p^n(a_{n-m} + a_{n-m+1}p + \dots)(b_0 + b_1p + \dots))\\ &= \chi_p(-p^n((b\varepsilon)_0 + (b\varepsilon)_1p + \dots + (b\varepsilon)_{-n-1}p^{-n-1})\\ &\quad\times(b_0 + b_1p + \dots + b_{-n-1}p^{-n-1}))\\ &= \chi_p(-p^n(b_0 + b_1p + \dots + b_{-n-1}p^{-n-1})^2\eta), \quad (11)\end{aligned}$$

where $\eta = \varepsilon_0 + \varepsilon_1p + \dots + \varepsilon_{-n-1}p^{-n-1}$. It is easy to see that the set $\mathscr{L}_3 \cap \mathscr{L}_1$ has the form:

$$\mathscr{L}_3 \cap \mathscr{L}_1 = \{\beta_1e + \beta_2(\nu e + f), \beta_1 \in \mathbb{Z}_p, \nu\beta_2 \in \mathbb{Z}_p\},$$

and from the last formula and (11) we have

$$M(m,0;0,\nu) \sim \sum_{b_0,b_1,\dots,b_{-n-1}=0}^{p-1} \chi_p(-p^n\eta(b_0 + \dots + b_{-n-1}p^{-n-1})^2),$$

whence it follows that

$$M(m,0;0,\nu) \sim \sum_{k=0}^{p^{-n}-1} \exp\left( -2\pi i \eta \frac{k^2}{p^{-n}} \right).$$

(Here we use the explicit form for the character $\chi_p(\xi) = \exp(2\pi i\{\xi\}_p)$. Taking into account the formula (7) we get the needed formula $M(m,0;0,\nu) = \lambda_p(-p^{-n}\eta) = \lambda_p(-\nu)$.

(iii) Taking into account Propositions 1 and 5 (ii) it is sufficient to consider the case $\mu \notin \mathbb{Z}_p$, $\mu - \nu \notin \mathbb{Z}_p$, $\nu \notin \mathbb{Z}_p$. We present here the proof only for the case of $|\mu|_p \neq |\nu|_p$, otherwise (iii) can be proved analogously. By the symmetry we can suppose that $|\nu|_p < |\mu|_p$. Let $\alpha \in \mathscr{L}_2$, $\beta \in \mathscr{L}_3$, then

$$\alpha = \alpha_1 e + \alpha_2(\mu e + f),$$
$$\beta = \beta_1 e + \beta_2(\nu e + f),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_p$. The condition $\alpha + \beta \in \mathscr{L}_1$ takes the form:

$$\mu\alpha_2 + \nu\beta_2 \in \mathbb{Z}_p.$$

Since the rank of $\chi_p$ equals 0 we have:

$$\chi_p(\mathscr{B}(\alpha,\beta)) = \chi_p(\mu\alpha_2\beta_2 - \nu\alpha_2\beta_2) = \chi_p((\mu-\nu)\alpha_2\beta_2). \tag{12}$$

Let $\text{ord}_p(\mu) - m$, $\text{ord}_p(\nu) = -n$, $\alpha_2 = a$, $\beta_2 = b$. As for the proof of the statement (ii) from the formula (12) we get:

$$p^{-m}\varepsilon(\mu)(a_0 + a_1 p + \ldots) + p^{-n}\varepsilon(\nu)(b_0 + b_1 p + \ldots) \in \mathbb{Z}_p.$$

In the case of $m > n \geq 1$ from the last formula we have:

$$\begin{aligned} (\varepsilon(\mu(a)_0 = (\varepsilon(\mu)a)_1 &= (\varepsilon(\mu)a)_{m-n-1} = 0, \\ (\varepsilon(\mu)a)_{m-n} &+ (\varepsilon(\nu)b)_0 = 0, \\ &\vdots \\ (\varepsilon(\mu)a)_{m-1} &+ (\varepsilon(\nu)b)_{n-1} = 0. \end{aligned} \tag{13}$$

As for the proof of (ii) from (12) and (13) we have:

$$\chi_p(\mathscr{B}(\alpha,\beta)) = \chi_p\left( -(\mu-\nu)\frac{\varepsilon(\nu)}{\varepsilon(\mu)} p^{m-n}(b_0 + b_1 p + \ldots + b_{n-1}p^{n-1})^2 \right).$$

Since $\text{ord}_p(\mu - \nu) = \text{ord}_p(\mu)$ from the last formula we obtain:

$$\chi_p(\mathscr{B}(\alpha,\beta)) = \chi_p(p^{-n}\eta(b_0 + b_1 p + \ldots + b_{n-1}p^{n-1})^2),$$

where

$$\eta = \left(\varepsilon(\nu-\mu)\frac{\varepsilon(\nu)}{\varepsilon(\mu)}\right)_0 + \left(\varepsilon(\nu-\mu)\frac{\varepsilon(\nu)}{\varepsilon(\mu)}\right)_1 p + \ldots + \left(\varepsilon(\nu-\mu)\frac{\varepsilon(\nu)}{\varepsilon(\mu)}\right)_n p^{n-1}.$$

The set $\mathscr{L}_3 \cap \mathscr{L}_1$ has the form

$$\mathscr{L}_3 \cap \mathscr{L}_1 = \{\beta_1 e + \beta_2(\nu e + f), \beta_1 \in \mathbb{Z}_p, \nu\beta_2 \in \mathbb{Z}_p\},$$

and as for the proof of (ii) we have:

$$M(0,\mu;0,\nu) = \lambda_p(p^n\eta).$$

Taking into account the properties of the function $\lambda_p$ and the relation $\mathrm{ord}_p(\nu-\mu) = \mathrm{ord}_p(\mu)$ we derive from the last formula:

$$\begin{aligned} M(0,\mu;0,\nu) &= \lambda_p\left(p^n\varepsilon(\nu-\mu)\frac{\varepsilon(\nu)}{\varepsilon(\mu)}\right) \\ &= \lambda_p(p^n\varepsilon(\nu)p^m\varepsilon(\mu)p^m\varepsilon(\nu-\mu)) = \lambda_p(\nu(\nu-\mu)). \end{aligned}$$

The proved theorem makes possible to calculate the Maslov index in the general case. By Proposition 1 for an arbitrary triple $(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)$ of selfdual lattices there is a symplectic basis $\{e,f\}$ wherein

$$\begin{aligned} \mathscr{L}_1 &= \mathbb{Z}_p e \oplus \mathbb{Z}_p f, \\ \mathscr{L}_2 &= \mathbb{Z}_p p^m e \oplus \mathbb{Z}_p p^{-m} f, \\ \mathscr{L}_3 &= \mathbb{Z}_p p^n e \oplus \mathbb{Z}_p(\nu p^n e + p^{-n} f), \end{aligned} \tag{14}$$

where $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$, $\nu \in \mathbb{Q}_p$. Therefore the Maslov index of this triple is given by the relation

$$\mu(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3) = M(m,0;n,\nu).$$

Let $\mathscr{L}_4 = \mathbb{Z}_p p^n e \oplus \mathbb{Z}_p p^{-n} f$. In the symplectic basis $\{\tilde{e} = p^n e, \tilde{f} = p^{-n} f\}$ we have

$$\begin{aligned} \mathscr{L}_1 &= \mathbb{Z}_p p^{-n}\tilde{e} \oplus \mathbb{Z}_p p^n \tilde{f}, \\ \mathscr{L}_2 &= \mathbb{Z}_p p^{m-n}\tilde{e} \oplus \mathbb{Z}_p p^{n-m}\tilde{f}, \\ \mathscr{L}_3 &= \mathbb{Z}_p\tilde{e} \oplus \mathbb{Z}_p(\nu\tilde{e} + \tilde{f}), \\ \mathscr{L}_4 &= \mathbb{Z}_p\tilde{e} \oplus \mathbb{Z}_p\tilde{f}. \end{aligned}$$

Taking into account Proposition 5(i), (iii), (iv) we have

$$\begin{aligned} \mu(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3) &= \bar{\mu}(\mathscr{L}_1,\mathscr{L}_3,\mathscr{L}_4)\mu(\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_4)\mu(\mathscr{L}_2,\mathscr{L}_4,\mathscr{L}_1) \\ &= \bar{M}(-m,0;0,\nu)M(m-n,0;0,\nu)M(-n,0;m-n,0). \end{aligned}$$

By virtue of the theorem and the last formula the following corollary is valid.

**Corollary.** *For the lattices $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3$ of the form* (14) *we have*

$$\mu(\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3) = \begin{cases} 1, & m = 0 \quad or \quad \nu \in \mathbb{Z}_p \quad of \quad n \leq 0, \\ \lambda_p(\nu), & 0 < n \leq m, 1 < |\nu|_p < p^{2n}, \\ 1, & 0 < n \leq m, p^{2n} \leq |\nu|_p, \\ 1, & m < n, 1 < |\nu|_p < p^{2(n-m)}, \\ \lambda_p(\nu), & m < n, p^{2(n-m)} \leq |\nu|_p < p^{2n}, \\ 1, & m < n, p^{2n} \leq |\nu|_p. \end{cases}$$

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