Commun. Math. Phys. 150, 385–413 (1992)

Communications in
Mathematical
Physics

Coherent States and Geometric Quantization*

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Received September 11, 1990; in revised form June 16, 1992

Abstract. Based on the concept of generalized coherent states, a theory of mechanical systems is formulated in a way which naturally exhibits the mutual relation of classical and quantum aspects of physical phenomena.

1. Introduction

In this work we present a description of physical systems which, in a sense, unifies into one theory the formalisms of both quantum and classical mechanics. The construction is based on the conviction that all experimentally achievable states of any physical system are parameterized with help of an appropriate finite-dimensional manifold M. Simultaneously, as a basic object of both experiment and theory, a transition amplitude is chosen. This is to be interpreted as a probability amplitude for a system in the state parameterized by $q \in M$ to be in the state parameterized by $p \in M$. One assumes, in accordance with well established experience, that the transition amplitude possesses some natural properties. These are the properties which allow one to construct a map $K: M \rightarrow \mathbb{CP}(\mathcal{M})$ of the manifold M into the complex projective Hilbert space $\mathbb{CP}(\mathcal{M})$. The Hilbert space $\mathcal{M}$ and the map K are uniquely defined by the transition amplitude. And vice versa, once the map $K: M \rightarrow \mathbb{CP}(\mathcal{M})$ is given, the transition amplitude for a system can be recovered.

In the paper we limit ourselves to the case in which K is a symplectic embedding. This means that the pull-back $K^*\omega_{\mathrm{FS}}$ of the Fubini-Study form ω_{FS} of $\mathbb{CP}(\mathcal{M})$ is again a symplectic form. We define a mechanical system as a triple $(M, \mathcal{M}, K: M \rightarrow \mathbb{CP}(\mathcal{M}))$, where $(M, K^*\omega_{\mathrm{FS}})$ is interpreted as the phase space of classical states of the system, while $(\mathbb{CP}(\mathcal{M}), \omega_{\mathrm{FS}})$ is the phase space of its pure quantum states. M might be considered to represent a family of all admissible results of measurements with classical devices used to measure parameters

* This work was supported in part by The Swedish Institute

characterizing states of the system. Such a point of view entitles us to interpret the states $K(q)$ (with $q \in M$) as coherent states of the system. The analysis of many examples (see e.g. [14, 15, 20]) is the source of the conviction that the crucial feature which distinguishes that type of states is their property to follow the behavior of classical states. Therefore, it seems to be natural to call the map K – "a state quantization."

One also considers here the mixed states $P(\varrho)$ parameterized by $\varrho \in L^1(M, d\mu_L)$ with $d\mu_L$ being the Liouville measure determined by $K^*\omega_{FS}$ while P denotes a quantization procedure as proposed by Berezin (see [4, 5]). Among mixed states one can distinguish, in a natural way, a finitely parameterized family of states which can be interpreted as the equilibrium states of the physical system (see Sect. 3). In that chapter of the paper we show that for systems in equilibrium states $P(\varrho)$ their characteristics such as action, energy, transition amplitude, and interaction with an external field functionally depend on K and ϱ. In that way a mechanical system in an equilibrium state is completely described by a pair of maps $K: M \to \mathbb{CP}(\mathcal{M})$ and $\varrho: M \to \mathbb{R}_+$ related one to the other by certain natural conditions [see formulas (3.10) and (3.11)]. These maps are the primary ones for the approach exposed here. They are also primary from the point of view of experiments testing a given physical system. These are the transition amplitudes between states parameterized by M, which are measured directly and which lead to the experimental determination of K and ϱ (see Sect. 2).

At the same time in the Hamiltonian mechanics the fundamental quantity which characterizes the system is the action functional. A procedure proposed by Feynman, expresses the transition amplitude between states $q, p \in M$ as a formal integral of amplitudes $e^{iS[\gamma]}$ over all possible trajectories γ between the given states. In our approach, the calculation of the Feynman integral reduces to the construction of an embedding $K: M \to \mathbb{CP}(\mathcal{M})$ of the classical phase space into the quantum phase space. This means that "integration over trajectories" cannot be treated literally as a real integration procedure but should be viewed as a certain geometrical construction for the realization of which one can use various ways, including analytical methods. One can see this in numerous contexts providing examples of embeddings $K: M \to \mathbb{CP}(\mathcal{M})$ well known from the algebraic geometry and complex analysis (see e.g. [10, 22, 20]). As for the complex analysis, it is worth to point out that the problem of finding such an embedding $K: M \to \mathbb{CP}(\mathcal{M})$ is equivalent to the calculation of the Bergman type reproducing kernels (see [12, 20, 22]). Some of those already have a transparent physical interpretation (see [20, 22]). Another domain of mathematics which provides one with means of construction of $K: M \to \mathbb{CP}(\mathcal{M})$ is the Lie group representation theory. This is directly related to the theory of coherent states in the sense of Perelomov (see [21]).

Consequently, in Sect. 4 we discuss, within our model, a canonical relation between classical and quantum observables. One admits here as natural ones those observables that satisfy conditions directly originating from the Ehrenfest theorem (see [7]). We indicate a quantization procedure and discuss its relation with the Kostant-Souriau (see [16, 19]) and the Berezin (see [4, 5]) quantizations.

The last section consists of applications of our theory to three systems of special interest, i.e. to the finite-dimensional Schrödinger mechanics, to the isotropic harmonic oscillator and to the scalar massive relativistic particle. These examples confirm the efficiency of the theory proposed in this paper.

The other examples such as the Kepler system and the massive relativistic particle with spin are discussed in [11, 22, 14]. A specially interesting insight into the structure of space-time is obtained after including into our model the twistor theory constructions (see [19, 20]). In this way one can obtain what could be called a "quantization" of Minkowski space points.

2. The Main Categories for Microsystems

In this section we choose the language of categories in order to reveal the geometric, analytic, and quantum character of microevents and microprocesses. The language of categories enables one to express, in canonical way, the functorial equivalence of the three natural ways of viewing states of matter. We show the mutual equivalence between:
i) *the category $\mathscr{A}$ of complex line bundles with a distinguished non-negative Hermitian kernel (a transition amplitude point of view);*
ii) *the category $\mathscr{B}$ of complex line bundles with a distinguished Hilbert space of sections (an analytical point of view);*
iii) *the category $\mathscr{C}$ of maps from manifolds into complex projective Hilbert spaces (a geometrical point of view).*

From now on, we assume objects and morphisms of all categories considered here, to be smooth.

Let us start with the definition of the first category $\mathscr{A}$. The situation which one encounters in experiments with microscopic events could be shortly expressed in the following way:
i) one wants to know the transition amplitude between states which could be achieved by the considered system:
ii) the states taken into account are parameterized by a finite number of parameters.
In our approach we assume that these parameters form an n-dimensional manifold M. Then, after fixing an atlas $\{\Omega_\alpha, \phi_\alpha\}_{\alpha\in I}$, where Ω_α is the open domain of chart $\phi_\alpha : \Omega_\alpha \to \mathbb{R}^n$ for M, the transition amplitude between the two states parameterized by $q\in\Omega_\alpha$ and $p\in\Omega_\beta$, respectively, is given as a function $A_{\bar\alpha\beta}: \Omega_\alpha \times \Omega_\beta \to \mathbb{C}$. Let us take $p\in\Omega_\beta\cap\Omega_\gamma\cap\Omega_\delta$. From the independence of transition probability $|A_{\bar\alpha\beta}(q,p)|^2$ on the coordinate description, we have

$$A_{\bar\alpha\beta}(q,p) = g_{\beta\gamma} A_{\bar\alpha\gamma}(q,p), \tag{2.1}$$

where $g_{\beta\gamma}: \Omega_\beta\cap\Omega_\gamma \to \mathbb{S}^1$. From (2.1) we obtain the cocycle property

$$g_{\beta\gamma} = g_{\beta\delta} g_{\delta\gamma} \tag{2.2}$$

for $p\in\Omega_\beta\cap\Omega_\gamma\cap\Omega_\delta$. The analogous consideration results in $g_{\bar\beta\bar\gamma} = g_{\bar\beta\bar\delta} g_{\bar\delta\bar\gamma}$ for $p\in\Omega_\beta\cap\Omega_\gamma\cap\Omega_\delta$. Also, one assumes that

$$A_{\bar\alpha\alpha}(q,q) = 1, \tag{2.3}$$

$$\overline{A_{\bar\alpha\beta}(q,p)} = A_{\bar\beta\alpha}(p,q). \tag{2.4}$$

From (2.3) and (2.4) it follows that $g_{\alpha\alpha} = 1$ and $\bar g_{\alpha\beta} = g_{\bar\alpha\bar\beta}$. Additionally, we assume here the non-negativity of the Hermitian matrix

$$\begin{bmatrix} A_{\bar\alpha_1\alpha_1}(q_1,q_1) & \dots & A_{\bar\alpha_1\alpha_n}(q_1,q_n) \\ & \dots\dots\dots\dots\dots & \\ A_{\bar\alpha_n\alpha_1}(q_n,q_1) & \dots & A_{\bar\alpha_n\alpha_n}(q_n,q_n) \end{bmatrix} \geqq 0 \tag{2.5}$$

for all possible choices of $n\in\mathbb{N}$ and $q_1\in\Omega_{\alpha_1},\ldots,q_n\in\Omega_{\alpha_n}$. For example if $n=2$, condition (2.5) gives the physically obvious condition $|A_{\bar{\alpha}_1\alpha_2}(q_1,q_2)|^2\leqq 1$. Translating the above into the language of vector bundles theory we could say that due to the fixing of transition amplitudes we have obtained a complex line bundle $\mathbb{L}\to M$ whose transition cocycle is given by (2.2), with a distinguished section K of the bundle $\mathrm{pr}_1^*\bar{\mathbb{L}}^*\otimes\mathrm{pr}_2^*\mathbb{L}^*\to M\times M$. The functions $A_{\bar{\alpha}\beta}:\Omega_\alpha\times\Omega_\beta\to\mathbb{C}$ are coordinates of K in the unitary gauge connected with the cocycle (2.2). According to (2.3), (2.4), and (2.5) the section K is non-negative Hermitian kernel, i.e. K satisfies

$$\overline{K_{\bar{\alpha}\beta}(q,p)}=K_{\bar{\beta}\alpha}(p,q)\,, \tag{2.6}$$

$$K_{\bar{\alpha}\alpha}(q,q)>0\,, \tag{2.7}$$

$$\sum_{i,j=1}^{N} K_{\bar{\alpha}_i\alpha_j}(q_i,q_j)\bar{v}^i v^j\geqq 0 \tag{2.8}$$

for any $q\in\Omega_\alpha$, $p\in\Omega_\beta$, $v^1,\ldots,v^N\in\mathbb{C}$ and any set of indices $\alpha,\beta,\alpha_1,\ldots,\alpha_N$ resulting from covering of M by the family $\{\Omega_\alpha\}_{\alpha\in I}$ of open sets Ω_α. The covering is chosen so that the $\pi^{-1}(\Omega_\alpha)$ is trivial, which is equivalent to the existence of a smooth non-vanishing section $s_\alpha:\Omega_\alpha\to\mathbb{L}$. $K_{\bar{\alpha}\beta}\in C^\infty(\Omega_\alpha\times\Omega_\beta)$ are coordinates of $K=K_{\bar{\alpha}\beta}(q,p)\,\mathrm{pr}_1^*\bar{s}_\alpha^*(q)\otimes\mathrm{pr}_2^*s_\beta^*(p)$, where $s_\alpha^*:\Omega_\alpha\to\mathbb{L}^*$ is dual to $s_\alpha:\Omega_\alpha\to\mathbb{L}$, i.e. $s_\alpha^*(s_\alpha)=1$. The above definition of the non-negative Hermitian kernel K is trivialization-independent.

The transition amplitudes are obtained as coordinates of K taken in frame $[K_{\bar{\alpha}\alpha}(q,q)]^{-1/2}\,\mathrm{pr}_1^*\bar{s}_\alpha^*(q)\otimes[K_{\bar{\beta}\beta}(p,p)]^{-1/2}\,\mathrm{pr}_2^*s_\beta^*(p)$, i.e.

$$A_{\bar{\alpha}\beta}(q,p)=\frac{K_{\bar{\alpha}\beta}(q,p)}{K_{\bar{\alpha}\alpha}(q,q)^{1/2}K_{\bar{\beta}\beta}(p,p)^{1/2}}\,. \tag{2.9}$$

Now, let us define the objects of the category $\mathscr{A}$ as pairs $(\mathbb{L}\xrightarrow{\pi}M,K)$.

The set of morphisms $H((\mathbb{L}_1\to M_1,K_1),(\mathbb{L}_2\to M_2,K_2))$ between the objects $(\mathbb{L}_1\to M_1,K_1)$ and $(\mathbb{L}_2\to M_2,K_2)$ of category $\mathscr{A}$ consists of maps $f:M_2\to M_1$ such that $f^*\mathbb{L}_1=\mathbb{L}_2$ and $f^*K_1=K_2$ (f^*K_1 is a non-negative Hermitian kernel if K_1 is).

The objects of the second category $\mathscr{B}$ are complex line bundles $\mathbb{L}\to M$ with Hilbert space $\mathscr{H}_{\mathbb{L}}$ realized as a vector subspace of $\Gamma(M,\bar{\mathbb{L}}^*)$. Additionally, we assume that the objects $(\mathbb{L}\to M,\mathscr{H}_{\mathbb{L}})$ have the following property.

For each frame section $s_\alpha:\Omega_\alpha\to\mathbb{L}$ and $q\in\Omega_\alpha$ let us introduce the evaluation functionals $e_{\alpha,q}:\mathscr{H}_{\mathbb{L}}\to\mathbb{C}$,

$$e_{\alpha,q}(\psi)=\psi_\alpha(q)\,, \tag{2.10}$$

where $\psi=\psi_\alpha(q)\bar{s}_\alpha^*(q)\in\mathscr{H}_{\mathbb{L}}\subset\Gamma(M,\bar{\mathbb{L}}^*)$. These are linear functionals and we assume that they are continuous:

$$|\psi_\alpha(q)|\leqq M_{\alpha,q}\|\psi\|\,, \tag{2.11}$$

non-zero and smoothly dependent on $q\in\Omega_\alpha$.

The set of morphisms $H((\mathbb{L}_1\to M_1,\mathscr{H}_{\mathbb{L}_1}),(\mathbb{L}_2\to M_2,\mathscr{H}_{\mathbb{L}_2}))$ consists of the set of maps $f:M_2\to M_1$ which satisfy $f^*\mathbb{L}_1=\mathbb{L}_2$ and $f^*\mathscr{H}_{\mathbb{L}_1}=\mathscr{H}_{\mathbb{L}_2}$. In order to prove the correctness of this definition let us show that the vector space $f^*\mathscr{H}_{\mathbb{L}_1}=\{f^*\psi:\ \psi\in\mathscr{H}_{\mathbb{L}_1}\}$ of inverse image sections possesses a canonically defined Hilbert space structure with continuous evaluation functionals.

Let $\operatorname{Ker} f^* \subset \mathscr{H}_{\mathbb{L}_1}$ denote the vector subspace of those sections ψ from $\mathscr{H}_{\mathbb{L}_1}$ for which $f^*\psi=0$. It is easy to see that $\operatorname{Ker} f^*$ is a closed subspace of $\mathscr{H}_{\mathbb{L}_1}$. Now, taking into account the vector spaces isomorphisms $f^*\mathscr{H}_{\mathbb{L}_1} \simeq \mathscr{H}_{\mathbb{L}_1}/\operatorname{Ker} f^* \simeq \operatorname{Ker} f^{*\perp}$ we are able to define the Hilbert space structure on $f^*\mathscr{H}_{\mathbb{L}_1}$ as the structure of Hilbert space $\operatorname{Ker} f^{*\perp}$ (i.e. $\langle f^*\psi_1, f^*\psi_2\rangle_{f^*\mathbb{L}_1} = \langle \psi_1^\perp, \psi_2^\perp\rangle_{\mathbb{L}_1}$, where $\psi_i=\psi_i^0+\psi_i^\perp$, $i=1,2$, is decomposition given by $\mathscr{H}_{\mathbb{L}_1}=\operatorname{Ker} f^*\oplus \operatorname{Ker} f^{*\perp}$). Now, we may rewrite (2.11) as follows:

$$|(f^*\psi)_\alpha(p)| = |\psi_\alpha(f(p))| \leqq M_{\alpha, f(p)}(\|\psi^0\|_{\mathbb{L}_1} + \|\psi^\perp\|_{\mathbb{L}_1}), \tag{2.12}$$

where $p \in f^{-1}(\Omega_\alpha)$. Because of $\psi_\alpha^0(f(p))=0$ for $p\in f^{-1}(\Omega_\alpha)$, the left-hand side of inequality (2.12) does not depend on ψ^0. This results in

$$\begin{aligned}|(f^*\psi)_\alpha(p)| &\leqq M_{\alpha,f(p)} \min_{\psi_0 \in \operatorname{Ker} f^*} (\|\psi^0\|_{\mathbb{L}_1} + \|\psi^\perp\|_{\mathbb{L}_1}) = M_{\alpha, f(p)} \|\psi^\perp\|_{\mathbb{L}_1} \\ &= M_{\alpha, f(p)} \| f^*\psi\|_{f^*\mathbb{L}_1}.\end{aligned}$$

The above proves the continuity of the evaluation functionals $e_{\alpha, f(p)}$ for the Hilbert space $f^*\mathscr{H}_{\mathbb{L}_1}$. (Also smooth dependence of $e_{\alpha, f(p)}$ on p follows from the same statement for $e_{\alpha,q}$ due to the smoothness of the map f.)

Since $(f\circ g)^*\mathbb{L}_1 = g^*(f^*\mathbb{L}_1)$ and $(f\circ g)^*\mathscr{H}_{\mathbb{L}_1} = g^*(f^*\mathscr{H}_{\mathbb{L}_1})$, where $f: M_2\to M_1$ and $g: M_3 \to M_2$, the composition rules for the morphisms of the category $\mathscr{A}$ reduce to the composition rules for the category of smooth manifolds.

The third category under consideration – let us denote it by $\mathscr{C}$ – consists of triples $(M, \mathscr{M}, K: M\to \mathbb{CP}(\mathscr{M}))$, where M denotes a smooth manifold, $\mathscr{M}$ is a complex Hilbert space and K stands for a smooth map from M into complex projective Hilbert space $\mathbb{CP}(\mathscr{M})$, such that the image $K(M)$ is linearly dense in $\mathscr{M}$. The morphisms of this category are defined by the following commutative diagrams

$$\begin{array}{ccc} M_1 & \xrightarrow{K_1} & \mathbb{CP}(\mathscr{M}_1) \\ \uparrow f & & \uparrow [\varphi] \\ M_2 & \xrightarrow{K_2} & \mathbb{CP}(\mathscr{M}_2), \end{array} \tag{2.13}$$

where f is a smooth map of manifolds while the map $[\varphi]$ is induced by a Hilbert space monomorphism $\varphi: \mathscr{M}_2 \to \mathscr{M}_1$ which allows to identify $[\varphi]^*\mathbb{E}_1 = \mathbb{E}_2$, where $\mathbb{E} \to \mathbb{CP}(\mathscr{M}_i)$, $i=1,2$, is universal line bundle on $\mathbb{CP}(\mathscr{M}_i)$, i.e. $\mathbb{E}_i = \{(v, l) \in \mathscr{M}_i \times \mathbb{CP}(\mathscr{M}_i : v \in l\}$. The morphisms composition again reduces to the composition of maps f and φ, which determine these morphisms. Due to the diagram (2.13) the monomorphism φ is determined by f up to phase factor. This freedom in the choice of φ does not show up in $[\varphi]$. Therefore, thinking about the morphisms set $H[(M_2, \mathscr{M}_2, K_2: M_2 \to \mathbb{CP}(\mathscr{M}_2)), (M_1, \mathscr{M}_1, K_1: M_1 \to \mathbb{CP}(\mathscr{M}_1)]$ we will have in mind only the maps $f: M_2 \to M_1$.

Example 1. We shall consider now examples of objects for all three categories defined above.

i) Let $\mathbb{E} \xrightarrow{\pi} \mathbb{CP}(\mathscr{M})$ be the universal line bundle over a complex projective Hilbert space $\mathbb{CP}(\mathscr{M})$, i.e. $\mathbb{E} = \{(v,l) \in \mathscr{M} \times \mathbb{CP}(\mathscr{M}): v \in l\}$. With the use of projection $\imath: \mathbb{E} \to \mathscr{M}$ on the first factor of the product $\mathscr{M}\times \mathbb{CP}(\mathscr{M})$, we obtain the Hermitian kernel $K_{\mathbb{E}}(m,n): \pi^{-1}(m)\times \pi^{-1}(n) \to \mathbb{C}$ given by $K_{\mathbb{E}}(m,n)(\xi,\eta) := \langle \imath(\xi), \imath(\eta)\rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathscr{M}$. Coordinates of $K_{\mathbb{E}}$ satisfy the conditions (2.1), (2.2), and (2.3). Concluding, we see that $(\mathbb{E}\xrightarrow{\pi}\mathbb{CP}(\mathcal{M}), K_{\mathbb{E}})$ belongs to the category $\mathcal{A}$.

ii) The map $\mathcal{M}\ni v\xrightarrow{I}\langle\iota(\cdot),v\rangle\in\Gamma(\mathbb{CP}(\mathcal{M}),\bar{\mathbb{E}}^*)$ defines monomorphism of vector spaces. Its image $\mathcal{H}_{\mathbb{E}}:=I(\mathcal{M})\subset\Gamma(\mathbb{CP}(\mathcal{M}),\bar{\mathbb{E}}^*)$ is a Hilbert space with the scalar product defined by

$$\langle\langle\iota(\cdot),v\rangle,\langle\iota(\cdot),w\rangle\rangle_{\mathbb{E}}:=\langle v,w\rangle\,. \tag{2.14}$$

Then, after fixing the frame section $s_\alpha:\Omega_\alpha\to\mathbb{E}$ one finds $\langle\iota(\cdot),v\rangle(m)=\langle\iota(s_\alpha(m)),v\rangle\bar{s}_\alpha^*(m)=:v_\alpha(m)\bar{s}_\alpha^*(m)$. Here, $\{\Omega_\alpha\}_{\alpha\in I}$ stands for a covering of $\mathbb{CP}(\mathcal{M})$ by open subset Ω_α such that $\pi^{-1}(\Omega_\alpha)\simeq\Omega_\alpha\times\mathbb{C}$. Due to the Schwartz inequality one gets

$$|v_\alpha(m)|=|\langle\iota(s_\alpha(m),v\rangle|\leqq\langle\iota(s_\alpha(m)),\iota(s_\alpha(m))\rangle^{1/2}\|v\|\,, \tag{2.15}$$

which shows that the evaluation functional $e_{\alpha,m}:\mathcal{H}_{\mathbb{L}}\to\mathbb{C}$ is continuous. Smooth dependence of $e_{\alpha,m}$ on $m\in\Omega_\alpha$ follows from the smoothness of s_α, ι, and $\langle\cdot,\cdot\rangle$. In consequence $(\mathbb{E}\xrightarrow{\pi}\mathbb{CP}(\mathcal{M}),\mathcal{H}_{\mathbb{E}})$ belongs to the category $\mathcal{B}$.

iii) It is a trivial statement that $(\mathbb{CP}(\mathcal{M}),\mathcal{M},\mathrm{id}:\mathbb{CP}(\mathcal{M})\to\mathbb{CP}(\mathcal{M}))$ (for $\dim\mathcal{M}<+\infty$) is an object of the category $\mathcal{C}$.

The importance of Example 1 will become clear from the further considerations.

Now, we shall provide a construction of natural functors between the categories defined above. For that sake let us denote by $\mathcal{F}_{yx}$ the functor from the category $\mathcal{X}$ to the category $\mathcal{Y}$, where $\mathcal{X},\mathcal{Y}=\mathcal{A},\mathcal{B},\mathcal{C}$ and $x,y=a,b,c$.

We begin with the construction of the functor $\mathcal{F}_{ab}:\mathcal{A}\to\mathcal{B}$. Let $(\mathbb{L}\to M,K)$ be an object of $\mathcal{A}$. Let us take the unitary vector space U_K of finite linear combinations of sections

$$K_\beta(q)=K_{\bar{\alpha}\beta}(p,q)\bar{s}_\alpha^*(p)\in\Gamma(M,\bar{\mathbb{L}}^*), \tag{2.16}$$

where $q\in\Omega_\beta$, with the scalar product of the vectors $v=\sum_{i=1}^N v^iK_{\beta_i}(q_i)$ and $w=\sum_{j=1}^N w^jK_{\beta_j}(q_j)$ defined by

$$\langle v,w\rangle:=\sum_{i,j=1}^N\bar{v}^iw^jK_{\bar{\beta}_i\beta_j}(q_i,q_j)\,. \tag{2.17}$$

Because of (2.2) and $\left|\sum_{i=1}^N\bar{v}^iK_{\bar{\beta}_i\beta}(q_i,p)\right|^2=|\langle v,K_\beta(p)\rangle|^2\leqq\langle v,v\rangle K_{\bar{\beta}\beta}(p,p)$ we see that coordinates $\sum_{i=1}^N v^iK_{\bar{\alpha}\beta_i}(p,q_i)$ of the section $v\in U_K$ are equal to zero iff $\langle v,v\rangle=0$. This proves the positivity of the scalar product. The question is whether completion to Hilbert space of the unitary space U_K can be realized by sections of bundle $\bar{\mathbb{L}}^*$? The following statement gives the positive answer to this question.

Proposition 1. *The unitary space U_K extends in the canonical and unique way to Hilbert space $\mathcal{H}_K$ which is a vector subspace of $\Gamma(M,\bar{\mathbb{L}}^*)$.*

Proof. Let us denote by $\bar{U}_K$ the abstract completion of U_K. There exists a canonically defined morphism $I:\bar{U}_K\to\Gamma(M,\bar{\mathbb{L}}^*)$ of the vector spaces defined by

$$I([\{v_n\}])(p):=\left(\lim_{n\to\infty}v_{n\alpha}(p)\right)\bar{s}_\alpha^*(p), \tag{2.18}$$

where $v_n = v_{n\alpha}\bar{s}^*_\alpha \in U_K$ is a Cauchy sequence, i.e. $[\{v_n\}] \in \bar{U}_K$. Let us now define Hilbert space $\mathscr{H}_K$ as $I(\bar{U}_K)$ with the scalar product given by $\langle t, s\rangle := \langle I^{-1}(t), I^{-1}(s)\rangle_{\bar{U}_K}$ for $s, t \in \mathscr{H}_K$. $\mathscr{H}_K$ is realized by sections of $\mathbb{L}^*$ and extends uniquely the unitary space U_K. □

Let $v = \lim_{n\to\infty} v_n$, where $v_n \in U_K$. Then we find from

$$v_{n\alpha}(p) = \langle K_\alpha(p), v_n\rangle \tag{2.19}$$

that $\lim_{n\to\infty} v_{n\alpha}(p) = v_\alpha(p)$ and

$$v_\alpha(p) = \langle K_\alpha(p), v\rangle, \tag{2.20}$$

where $v_\alpha(p)$ is coordinate of $v \in \mathscr{H}_K$. Applying the Schwartz inequality to (2.19) we prove the continuity of evaluation functionals for $\mathscr{H}_K$. The smooth dependence of $e_{\alpha,p}$ on $p \in \Omega_\alpha$ is a consequence of formula (2.19) and smoothness of $K_\alpha(p)$. The condition $e_{\alpha,p} \neq 0$, for $p \in \Omega_\alpha$, follows from (2.7). Taking the above into account let us put

$$\mathscr{F}_{ba}[(\mathbb{L} \to M, K)] := (\mathbb{L} \to M, \mathscr{H}_K). \tag{2.21}$$

Now, let $f^* \in H((\mathbb{L}_1 \to M_1, K), (\mathbb{L}_2 \to M_2, K_2))$, be given by a map $f: M_2 \to M_1$ such that $f^*\mathbb{L}_1 = \mathbb{L}_2$ and $f^*K_1 = K_2$. The last means $K_{1\alpha}(f(p)) = K_{2\alpha}(p)$ for $p \in f^{-1}(\Omega_{1\alpha})$ and $s_{2\alpha} = f^*s_{1\alpha}: f^{-1}(\Omega_{1\alpha}) \to \mathbb{L}_2$. Because $K_{1\alpha}(f(p))$ are linearly dense in $f^*\mathscr{H}_{K_1}$ and $K_{2\alpha}(p)$ are linearly dense in $\mathscr{H}_{K_2}$, this shows that $f^*\mathscr{H}_{K_1} = \mathscr{H}_{K_2}$. The last proves the functorial property of $\mathscr{F}_{ba}$.

In order to define the functor $\mathscr{F}_{ab}: \mathscr{B} \to \mathscr{A}$ let us take the evaluation functional $e_{\alpha,p} \in \mathscr{H}^*_{\mathbb{L}}$, which is continuous and non-zero according to the definition of $(\mathbb{L} \to M, \mathscr{H}_{\mathbb{L}})$. Hence, by Riesz theorem there exists non-zero $K^{\mathbb{L}}_\alpha(p) \in \mathscr{H}_{\mathbb{L}}$ such that

$$\psi_\alpha(p) = e_{\alpha,p}(\psi) = \langle K^{\mathbb{L}}_\alpha(p), \psi\rangle. \tag{2.22}$$

Expressing $K^{\mathbb{L}}_\alpha(p)$ in coordinates $K^{\mathbb{L}}_\alpha(p) = K^{\mathbb{L}}_{\beta\alpha}(q,p)\bar{s}^*_\beta(q)$ we find that

$$\langle K_\beta(q), K_\alpha(p)\rangle = K_{\beta\alpha}(q,p). \tag{2.23}$$

Because of (2.23) and since $K^{\mathbb{L}}_\alpha(p) \neq 0$, $K^{\mathbb{L}} := K^{\mathbb{L}}_{\beta\alpha}\,\mathrm{pr}^*_1\bar{s}^*_\beta \otimes \mathrm{pr}^*_2 s^*_\alpha$ is a non-negative Hermitian kernel for the complex line bundle $\mathbb{L}$. We shall define $\mathscr{F}_{ab}$ by

$$\mathscr{F}_{ab}[(\mathbb{L} \to M, \mathscr{H}_{\mathbb{L}})] := (\mathbb{L} \to M, K^{\mathbb{L}}). \tag{2.24}$$

The following proposition proves the functoriality of $\mathscr{F}_{ab}$.

Proposition 2. *Let the map $f: M_2 \to M_1$ be such that $f^*\mathscr{H}_{\mathbb{L}_1} = \mathscr{H}_{\mathbb{L}_2}$ and $f^*\mathbb{L}_1 = \mathbb{L}_2$. Then $f^*K^{\mathbb{L}_1} = K^{\mathbb{L}_2}$.*

Proof. The elements $K^{\mathbb{L}_1}_\alpha(q)$, where $q \in \Omega_\alpha$ and $\alpha \in I$, form linearly dense subset in $\mathscr{H}_{\mathbb{L}_1}$. It is so, because due to (2.22) the set of vectors orthogonal to $\{K^{\mathbb{L}_1}_\alpha(p)\}_{p\in\Omega_\alpha,\,\alpha\in I}$ is $\{0\}$. Thus, $\{K^{\mathbb{L}_1}_\alpha(f(q))\}$, $q \in f^{-1}(\Omega_\alpha)$ and $\alpha \in I$, is linearly dense in $f^*\mathscr{H}_{\mathbb{L}_1} = \mathscr{H}_{\mathbb{L}_2}$. In accordance with the decomposition $\mathscr{H}_{\mathbb{L}_1} = \mathrm{Ker} f^* \otimes \mathrm{Ker} f^{*\perp}$ we have

$$K^{\mathbb{L}_1}_\alpha(p) = K^{0\mathbb{L}_1}_\alpha(p) + K^{\perp\mathbb{L}_1}_\alpha(p). \tag{2.25}$$

Because $\mathscr{H}_{\mathbb{L}_2} = f^*\mathscr{H}_{\mathbb{L}_1} \cong \mathrm{Ker} f^{*\perp}$ and $K^{0\mathbb{L}_1}_\alpha(f(p)) = 0$ for $p \in \Omega_\alpha$ and $\alpha \in I$, vector $K^{\perp\mathbb{L}_1}_\alpha(f(p)) = K^{\mathbb{L}_1}_\alpha(f(p))$ represents evaluation functional $e_{2\alpha p}$ for the space $\mathscr{H}_{\mathbb{L}_2}$. Hence, applying (2.23) to $\mathscr{H}_{\mathbb{L}_1}$ and $\mathscr{H}_{\mathbb{L}_2}$, we get $K^{\mathbb{L}_2}_{\bar{\alpha}\beta}(p,r) = K^{\mathbb{L}_1}_{\bar{\alpha}\beta}(f(p), f(r))$ for $(p,r) \in f^{-1}(\Omega_\alpha) \times f^{-1}(\Omega_\beta)$. This ends the proof. □

Let us discuss now the relation between categories $\mathscr{B}$ and $\mathscr{C}$. Let $(\mathbb{L}\to M, \mathscr{H}_{\mathbb{L}})$ be an object of category $\mathscr{B}$. Taking vector $K_\alpha(q)\in\mathscr{H}_{\mathbb{L}}$, which represents the evaluation functional $e_{\alpha,q}$, we construct a smooth map $K^{\mathbb{L}}: M\to\mathbb{CP}(\mathscr{H}_{\mathbb{L}})$ given by

$$K^{\mathbb{L}}(q):=[K_\alpha(q)], \tag{2.26}$$

where $[K_\alpha(q)]$ denotes one-dimensional subspace spanned by $K_\alpha(q)\neq 0$. Because of $K_\alpha(q)=g_{\alpha\beta}K_\beta(q)$ the definition is independent of the choice of frame. The smoothness of $K^{\mathbb{L}}$ is ensured by smooth dependence of $e_{\alpha,q}$ on $q\in\Omega_\alpha$. From $\mathscr{H}_{\mathbb{L}_2}=f^*\mathscr{H}_{\mathbb{L}_1}\cong \operatorname{Ker} f^{*\perp}\to\mathscr{H}_{\mathbb{L}_1}$ one obtains the monomorphism of Hilbert spaces $\varphi_f:\mathscr{H}_{\mathbb{L}_2}\to\mathscr{H}_{\mathbb{L}_1}$ which satisfies

$$K_{1\alpha}(f(p))=K_{1\alpha}^{\perp}(f(p))=\varphi_f(K_{2\alpha}(p)) \tag{2.27}$$

for all $p\in f^{-1}(\Omega_\alpha)$ and $\alpha\in I$. Equation (2.27) implies the commutativity of the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{K^{\mathbb{L}_1}} & \mathbb{CP}(\mathscr{H}^{\mathbb{L}_1}) \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle [\varphi_f]} \\ M_2 & \xrightarrow{K^{\mathbb{L}_2}} & \mathbb{CP}(\mathscr{H}^{\mathbb{L}_2}). \end{array}$$

Thus, by definition we put

$$\begin{gathered}\mathscr{F}_{cb}[(\mathbb{L}\to M, \mathscr{H}_{\mathbb{L}})]=(M,\mathscr{H}_{\mathbb{L}}, K^{\mathbb{L}}: M\to\mathbb{CP}(\mathscr{H}_{\mathbb{L}})),\\ \mathscr{F}_{cb}(f^*):=(f,[\varphi_f]),\end{gathered} \tag{2.28}$$

where $(\mathbb{L}\to M,\ \mathscr{H}_{\mathbb{L}})\in\mathscr{B}$ and $f^*\in H((\mathbb{L}_1\to M_1,\ \mathscr{H}_{\mathbb{L}_1}),\ (\mathbb{L}_2\to M_2,\ \mathscr{H}_{\mathbb{L}_2}))$. A simple examination shows that $\mathscr{F}_{cb}(g^*\circ f^*)=\mathscr{F}_{cb}(f^*)\circ\mathscr{F}_{cb}(g^*)$, where $g^*\in H((\mathbb{L}_2\to M_2, \mathscr{H}_{\mathbb{L}_2}), (\mathbb{L}_3\to M_3, \mathscr{H}_{\mathbb{L}_3}))$.

Conclusion. *$\mathscr{F}_{cb}$ is a contravariant functor from that category $\mathscr{B}$ to the category $\mathscr{C}$.*

For the definition of $\mathscr{F}_{bc}:\mathscr{C}\to\mathscr{B}$ we employ the construction ii) of Example 1. Let $(M,\mathscr{M},K: M\to\mathbb{CP}(\mathscr{M}))$ be an object of the category $\mathscr{C}$. By definition we put

$$\begin{gathered}\mathscr{F}_{bc}[(M,\mathscr{M},K: M\to\mathbb{CP}(\mathscr{M}))]:=(K^*\mathbb{E}\to M, K^*\mathscr{H}_{\mathbb{E}}),\\ \mathscr{F}_{bc}[(f,[\varphi])]:=f^*,\end{gathered} \tag{2.29}$$

where $\mathbb{E}\to\mathbb{CP}(\mathscr{M})$ is the universal line bundle on $\mathbb{CP}(\mathscr{M})$ and $(f,[\varphi])$ is the morphism given by diagram (2.13). The commutativity of (2.13) and $\mathbb{E}_2=[\varphi]^*\mathbb{E}_1$ implies $f^*K_1^*\mathbb{E}_1=K_2^*\mathbb{E}_2$, therefore, $f^*\in H((K_1^*\mathbb{E}_1$“$M_1, K_1^*\mathscr{H}_{\mathbb{E}_1}), (K_2^*\mathbb{E}_2\to M_2, K_2^*\mathscr{H}_{\mathbb{E}_2}))$. This shows the correctness of the definition (2.29).

Defining $\mathscr{F}_{ac}:\mathscr{C}\to\mathscr{A}$ we proceed similarly as we did in the case of functor $\mathscr{F}_{bc}$. Namely, we put

$$\begin{gathered}\mathscr{F}_{ac}[(M,\mathscr{M},K: M\to\mathbb{CP}(\mathscr{M}))]:=(K^*\mathbb{E}\to M, K^*K_E),\\ \mathscr{F}_{ac}[(f,[\varphi])]:=f^*,\end{gathered} \tag{2.30}$$

where K_E is the canonical non-negative Hermitian kernel defined in point (i) of Example 1. The correctness of the definition results again from $\mathbb{E}_2=[\varphi]^*\mathbb{E}_1$ and (2.13).

At the end let us define $\mathscr{F}_{ca}:\mathscr{A}\to\mathscr{C}$ as a composition $\mathscr{F}_{ca}:=\mathscr{F}_{cb}\circ\mathscr{F}_{ba}$ of the functors $\mathscr{F}_{cb}$ and $\mathscr{F}_{ba}$. Also, we will use notation $\mathscr{F}_{xx}:=\mathrm{id}$, for $x=a,b,c$, where id denotes identity functor.

The following definition introduces an equivalence among the objects of the categories taken into consideration.

Definition 1. (i) The objects $(\mathbb{L}\to M, K_{\mathbb{L}}), (\mathbb{L}'\to M', K_{\mathbb{L}'})\in \mathrm{Ob}(\mathscr{A})$ are equivalent iff $M=M'$ and there exists a bundle isomorphism $\kappa:\mathbb{L}\to\mathbb{L}'$ such that $\kappa^* K_{\mathbb{L}'}=K_{\mathbb{L}}$.
(ii) The objects $(\mathbb{L}\to M, \mathscr{H}_{\mathbb{L}}), (\mathbb{L}'\to M', \mathscr{H}_{\mathbb{L}'})\in \mathrm{Ob}(\mathscr{B})$ are equivalent iff $M=M'$ and there exists a bundle isomorphism $\kappa:\mathbb{L}\to\mathbb{L}'$ such that $\kappa^*\mathscr{H}_{\mathbb{L}'}=\mathscr{H}_{\mathbb{L}}$.
(iii) The objects $(M,\mathscr{M},K:M\to\mathbb{CP}(\mathscr{M}))$, $(M',\mathscr{M}',K':M'\to\mathbb{CP}(\mathscr{M}))\in\mathrm{Ob}(\mathscr{C})$ are equivalent iff $M=M'$ and there exists a Hilbert space isomorphism $\lambda:\mathscr{M}\to\mathscr{M}'$ such that $K'=[\lambda]\circ K$.

These equivalences are preserved by morphisms for all three considered cases. This allows us to define the new category $\tilde{\mathscr{X}}=\tilde{\mathscr{A}},\tilde{\mathscr{B}},\tilde{\mathscr{C}}$ whose $\mathrm{Ob}(\tilde{\mathscr{X}})$ consists of the classes of equivalent objects and morphisms set $H(\tilde{\mathscr{X}})$ is canonically generated by morphisms of the category $\mathscr{X}$.

Applying the functor $\mathscr{F}_{yx}:\mathscr{X}\to\mathscr{Y}$ to two equivalent $X\simeq X'$ of the category $\mathscr{X}$ one obtains $\mathscr{F}_{yx}(X)\simeq\mathscr{F}_{yx}(X')$. Hence, one can define functors $\tilde{\mathscr{F}}_{yx}:\tilde{\mathscr{X}}\to\tilde{\mathscr{Y}}$.

The following theorem is fundamental for our considerations.

Theorem 3. *The diagram*

$$\begin{array}{ccc} & \tilde{\mathscr{Z}} & \\ {}^{\tilde{\mathscr{F}}_{xz}}\swarrow & & \searrow^{\tilde{\mathscr{F}}_{yz}} \\ \tilde{\mathscr{X}} & \xleftarrow{\tilde{\mathscr{F}}_{xy}} & \tilde{\mathscr{Y}} \end{array}$$

is commutative, i.e.

$$\tilde{\mathscr{F}}_{xz}=\tilde{\mathscr{F}}_{xy}\circ\tilde{\mathscr{F}}_{yz}, \tag{2.31}$$

for all possible substitutions $\tilde{\mathscr{X}},\tilde{\mathscr{Y}},\tilde{\mathscr{Z}}=\tilde{\mathscr{A}},\tilde{\mathscr{B}},\tilde{\mathscr{C}}$ *and* $x,y,z=a,b,c$.

Let us recall [6] that two categories $\mathscr{X}$ and $\mathscr{Y}$ are said to be isomorphic if there exists a functor $\mathscr{F}:\mathscr{X}\to\mathscr{Y}$ such that: 1) for any object Y of $\mathscr{Y}$ there exists a unique object X of $\mathscr{X}$ such that $\mathscr{F}(X)=Y$; 2) for any pair (X_1,X_2) of objects of $\mathscr{X}$, the map which associates to each morphism $f^*:X_1\to X_2$ the morphism $\mathscr{F}(f^*):\mathscr{F}(X_1)\to\mathscr{F}(X_2)$ is a bijection of the sets of morphisms.

Corollary 4. *The categories* $\tilde{\mathscr{A}}$, $\tilde{\mathscr{B}}$, *and* $\tilde{\mathscr{C}}$ *are mutually isomorphic.*

The proof of Theorem 3 follows by a straightforward verification. We shall omit it here.

For the proof of Corollary 4 it is enough to put $x=z$ in (2.31), which gives $\tilde{\mathscr{F}}_{xy}\circ\tilde{\mathscr{F}}_{yx}=\mathrm{id}$.

Now, let $\mathscr{L}$ denote the category of complex line bundles $\mathscr{L}\xrightarrow{\pi}M$ with fixed Hermitian metric $H\in C^\infty(M,\bar{\mathbb{L}}^*\otimes\mathbb{L}^*)$ and metrical connection $\nabla:C^\infty(\Omega,\mathbb{L})\to C^\infty(\Omega,\mathbb{L}\otimes T^*M)$, i.e.

(i) $$\nabla(fs)=df\otimes s+f\nabla s,$$

(ii) $$dH(s,t)=H(\nabla s,t)+H(s,\nabla t)$$

for any local smooth sections $s,t\in C^\infty(\Omega,\mathbb{L})$ and $f\in C^\infty(\Omega)$, where $\Omega\subset M$.

Now we shall discuss the relation between category $\mathscr{A}$ and $\mathscr{L}$. For an object $(\mathbb{L}\to M,K)$ of $\mathscr{A}$ let us define the differential 2-forms $\omega_{1,2}$ and $\omega_{2,1}$ on the product $M\times M$,

$$\omega_{1,2}:=\mathrm{id}_1 d_2\log K_{\bar{\alpha}_1\alpha_2}, \qquad \omega_{2,1}:=\mathrm{id}_2 d_1\log K_{\bar{\alpha}_2\alpha_1}, \tag{2.32}$$

where $K_{\bar{\alpha}_1\alpha_2}$ are coordinates of K in the local frame $\bar{s}^*_{\alpha_1}\otimes s^*_{\alpha_2}:\Omega_{\alpha_1}\times\Omega_{\alpha_2}\to \mathrm{pr}_1^*\bar{\mathbb{L}}^*\otimes \mathrm{pr}_2\mathbb{L}^*$. d_1 and d_2 denote the differentials with respect to the first and the second component of the product $M\times M$, respectively. (The complete differential on $M\times M$ is given as the sum $d=d_1+d_2$.)

From the transformation rule

$$K_{\bar{\alpha}_1\alpha_2}(m_1,m_2)=\overline{g_{\alpha_1\gamma_1}(m_1)}g_{\alpha_2\gamma_2}(m_2)K_{\bar{\gamma}_1\gamma_2}(m_1,m_2), \tag{2.33}$$

where $\{g_{\alpha\gamma}\}$ is transition cocycle for $\mathbb{L}$, and from the hermiticity of K we get the following properties of $\omega_{1,2}$.

Proposition 5. (i) $\omega_{1,2}$ *does not depend on the choice of frames;*
(ii) $\overline{\omega_{1,2}}=\omega_{2,1}$ *and* $\omega_{1,2}=-\omega_{2,1}$*;*
(iii) $d\omega_{1,2}=0$.

Let us also consider 1-forms

$$\theta_{2\alpha_2}:=d_2\log K_{\bar{\alpha}_1\alpha_2}, \tag{2.34a}$$

$$\theta_{1\bar{\alpha}_1}:=d_1\log K_{\bar{\alpha}_1\alpha_2}, \tag{2.34b}$$

which are independent of indices $\bar{\alpha}_1$ and α_2, respectively, and satisfy the transformation rules

$$\theta_{2\alpha_2}=\theta_{2\gamma_2}+d_2\log g_{\alpha_2\gamma_2}, \tag{2.35a}$$

$$\theta_{1\bar{\alpha}_1}=\theta_{1\bar{\gamma}_1}+d_1\log \overline{g_{\alpha_1\gamma_1}}. \tag{2.35b}$$

Let $\Delta:M\to M\times M$ be the diagonal embedding, i.e. $\Delta(m):=(m,m)$ for $m\in M$. We introduce the following notation:

$$\Delta^*K:=H,\quad \Delta^*\omega_{1,2}=:\omega,\quad \text{and}\quad \Delta^*\theta_{2\alpha_2}=\theta_{\alpha_2}. \tag{2.36}$$

Because of $\overline{\Delta^*\theta_{2\alpha}}=\Delta^*\theta_{2\bar{\alpha}}$, one has $\Delta^*\theta_{1\bar{\alpha}_1}=\overline{\theta_{\alpha_1}}$.

The following proposition helps to justify the above definitions.

Proposition 6. (i) *H is a Hermitian metric on* $\mathbb{L}$.
(ii) *The 1-forms $\theta_\beta\in C^\infty(\Omega_\beta,\mathbb{L}\otimes T^*M)$ ($\overline{\theta_\beta}\in C^\infty(\Omega_\beta,\overline{\mathbb{L}\otimes T^*M})$) define a connection ∇ ($\bar{\nabla}$) on the line bundle $\mathbb{L}$ ($\bar{\mathbb{L}}$) such that $\nabla s_\alpha:=\theta_\alpha\otimes s_\alpha$ ($\overline{\nabla s_\alpha}:=\bar{\theta}_\alpha\otimes\bar{s}_\alpha$).*
(iii) $i\cdot\mathrm{curv}\,\nabla=\omega$, *where* curv ∇ *denote the curvature 2-form for* ∇.
(iv) ∇ *is metric with respect to* H.

In such a way we obtain the map $\mathscr{F}_{la}:\mathrm{Ob}(\mathscr{A})\to\mathrm{Ob}(\mathscr{L})$, which maps the object $(\mathbb{L}\to M,K)$ of $\mathscr{A}$ on the object $(\mathbb{L}\to M,\nabla,H)$ of $\mathscr{L}$ given by Proposition 6. $\mathscr{F}_{la}$ satisfies functorial property, i.e. for $f:M_1\to M_2$ such that $f^*\mathbb{L}_2=\mathbb{L}_1$ and $f^*K_2=K_1$ the diagram

$$\begin{array}{ccc}(\mathbb{L}_1\to M_1,K_1) & \xrightarrow{f^*} & (\mathbb{L}_2\to M_2,K_2)\\ \downarrow{\scriptstyle \mathscr{F}_{la}} & & \downarrow{\scriptstyle \mathscr{F}_{la}}\\ (\mathbb{L}_1\to M_1,\nabla_1,H_1) & \xrightarrow{f^*} & (\mathbb{L}_2\to M_2,\nabla_2,H_2)\end{array}$$

is commutative if $H_1=f^*H_2$ and $\nabla_1=f^*\nabla_2$. Therefore, we have constructed a canonical covariant functor $\mathscr{F}_{la}$ from the category of complex line bundles with distinguished non-negative Hermitian kernels to the category of complex line bundles with distinguished metric and metric connection.

Example 2. An example of the object of category $\mathscr{L}$ is obtained by applying functor $\mathscr{F}_{la}$ to the object $(\mathbb{E} \to \mathbb{CP}(\mathscr{M}), K_{\mathbb{E}})$ of category $\mathscr{A}$ which has been defined in the point i) of Example 1. Applying the formulas (2.32), (2.33) to the kernel $K_{\mathbb{E}} = \langle \imath(s_k), \imath(s_l)\rangle \bar{s}_k^* \otimes s_l^*$ and using the notation given by (2.36) we find

$$H_{FS}(s_k, s_k) = \langle \imath(s_k), \imath(s_k)\rangle = 1 + \sum_l z_k^l \bar{z}_k^l, \tag{2.37}$$

$$\nabla_{FS} s_k = \partial \log \langle \imath(s_k), \imath(s_k)\rangle \otimes s_k, \tag{2.38}$$

$$\omega_{FS} = i \operatorname{curv} \nabla_{FS} = i\bar{\partial}\partial \log \langle \imath(s_k), \imath(s_k)\rangle, \tag{2.39}$$

where $z_k^i = \frac{\xi^i}{\xi^k}$ are affine coordinate on the set $\Omega_k := \{[\xi] \in \mathbb{CP}(\mathscr{M}): \xi^k \neq 0$ for $\xi = \xi^1 e_1 + \xi^2 e_2 + \ldots\}$ and the set of frames $s_k : \Omega_k \to \mathbb{E}$ is given by $s_k(z_k^1, z_k^2, \ldots) := (z_k^1, \ldots, z_k^{k-1}, 1, z_k^k, \ldots)$. This gives the object $(\mathbb{E} \to \mathbb{CP}(\mathscr{M}), H_{FS}, \nabla_{FS})$ of category $\mathscr{L}$. Because ω_{FS} is the Fubini-Study form we shall call ∇_{FS} and H_{FS} the Fubini-Study connection and the Fubini-Study metric, respectively.

Let $\mathbb{E}' = \mathbb{E} - s_0(\mathbb{CP}(\mathscr{M}))$, where $s_0 : \mathbb{CP}(\mathscr{M}) \to \mathbb{E}$ is zero section of bundle $\mathbb{E}$. $\mathbb{E}'$ is principal $\mathbb{C}^*$-bundle and $\imath : \mathbb{E}' \to \mathscr{M} - \{0\}$ gives a natural coordinate system for $\mathbb{E}'$. The Fubini-Study connection 1-form α_{FS} takes in this coordinate system the form

$$\alpha_{FS}(v \in l) = \frac{\langle v|dv\rangle}{\langle v|v\rangle}. \tag{2.40}$$

From (2.40) we see that $U(t) := \exp itH$, where $H^+ = H \in \mathscr{B}(\mathscr{M})$, is a one-parameter group of automorphisms of the bundle $(\mathbb{E} \to \mathbb{CP}(\mathscr{M}), H_{FS}, \nabla_{FS})$. The velocity vector field $x_U \in \Gamma(T(\mathbb{E}'))$ generated by $U(t)$ has the form

$$x_U = -i(Hv)^T \frac{\partial}{\partial v} + \overline{i(Hv)^T \frac{\partial}{\partial v}}. \tag{2.41}$$

One has the decomposition

$$x_U = i\frac{\langle v|H|v\rangle}{\langle v|v\rangle}\left(v^T \frac{\partial}{\partial v} - \overline{v^T \frac{\partial}{\partial v}}\right) + x_U^{\text{hor}}, \tag{2.42}$$

where the first and second components are the vertical and the horizontal parts of x_U. From $\mathscr{L}_x \alpha_{FS} = 0$ we obtain

$$d\left(-i\frac{\langle v|H|v\rangle}{\langle v|v\rangle}\right) + (d\alpha_{FS}) \lrcorner x_U = 0. \tag{2.43}$$

Projecting (2.43) by $\pi' : \mathbb{E}' \to \mathbb{CP}(\mathscr{M})$ on $\mathbb{CP}(\mathscr{M})$ we find that

$$\omega_{FS} \lrcorner \Pi_* x_U = d\frac{\langle v|H|v\rangle}{\langle v|v\rangle}, \tag{2.44}$$

which shows that the flow $[U(t)]$ obtained by projection π' is globally Hamiltonian with the average value of the Hermitian operator H as the generating function.

The example presented above is crucial for our considerations, especially for the next section. Here, we use it for the definition of a natural contravariant functor $\mathscr{F}_{lc} : \mathscr{C} \to \mathscr{L}$ given by

$$\mathscr{F}_{lc}[(M, \mathscr{M}, K : M \to \mathbb{CP}(\mathscr{M}))] := (K^*\mathbb{E} \to M, K^*\nabla_{FS}, K^*H_{FS}). \tag{2.45}$$

The functoriality of $\mathscr{F}_{lc}$ is obvious.

The following proposition expounds a relation between the functors $\mathcal{F}_{la}$ and $\mathcal{F}_{lc}$.

Proposition 6. i) *$\mathcal{F}_{la} \circ \mathcal{F}_{ac} = \mathcal{F}_{lc}$ and $\mathcal{F}_{lc} \circ \mathcal{F}_{ca} = \mathcal{F}_{la}$ (modulo isomorphisms).*
ii) *Each object of category $\mathcal{L}$ is obtained as $\mathcal{F}_{la}(A)$ and $\mathcal{F}_{lc}(C)$ for some $A \in \mathrm{Ob}(\mathcal{A})$ and $C \in \mathrm{Ob}(\mathcal{C})$.*

Proof. The point i) is proved by the direct check of two considered cases.

The point ii) is a corollary of the Narasimhan-Ramanan theorem (see [18]) and Theorem 3. □

Pursuing further these considerations let us define the symplectic subcategories of the categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{L}$. We choose to denote them by the same letters but with an index Sp added, i.e. $\mathcal{A}_{\mathrm{Sp}}$, $\mathcal{B}_{\mathrm{Sp}}$, $\mathcal{C}_{\mathrm{Sp}}$, and $\mathcal{L}_{\mathrm{Sp}}$.

Definition 2. i) $(\mathbb{L} \to M, K) \in \mathcal{O}(\mathcal{A}_{\mathrm{Sp}})$ iff K is non-degenerate kernel, i.e. 2-form $\omega = \Delta^* \omega_{1,2}$ [see formulas (2.32) and (2.36)] is non-degenerate.
ii) $(\mathbb{L} \to M, \mathcal{H}_{\mathbb{L}}) \in \mathcal{O}(\mathcal{B}_{\mathrm{Sp}})$ iff reproducing kernel of Hilbert space $\mathcal{H}_{\mathbb{L}}$, defined by evaluation functional, is non-degenerate [in the sense of point i) of this definition].
iii) $(M, \mathcal{M}, K : M \to \mathbb{CP}(\mathcal{M})) \in \mathcal{C}_{\mathrm{Sp}}$ iff $K^* \omega_{\mathrm{FS}}$ is non-degenerate.
iv) $(\mathbb{L} \to M, \nabla, H) \in \mathcal{O}(\mathcal{L}_{\mathrm{Sp}})$ iff $i \cdot \mathrm{curv}\, \nabla$ is non-degenerate.

Let us remark that these definitions of symplectic subcategories are consistent with the functional correspondence discussed above. Therefore, the symplecticity criterion is functorial invariant. As a consequence, Theorem 3 and Corollary 4 possess their symplectic counterparts. This allows us to formulate properties of the objects of one category in terms of the others. We provide an example of such a situation below.

Proposition 7. *Let $\mathcal{F}_{ca}[(\mathbb{L} \to M, K)] = (M, \mathcal{M}, K : M \to \mathbb{CP}(\mathcal{M}))$, $K : M \to \mathbb{CP}(\mathcal{M})$ is a symplectic embedding iff K is non-degenerate and satisfies the condition*

$$\frac{|K_{\bar{\alpha}\beta}(q,p)|^2}{K_{\bar{\alpha}\alpha}(q,q) K_{\bar{\beta}\beta}(p,p)} < 1\,. \tag{2.46}$$

Proof. Because i) and iii) are equivalent via functor $\mathcal{F}_{ca}$, we infer that $K : M \to \mathbb{CP}(\mathcal{M})$ is an immersion iff K is non-degenerate kernel. Next, because of $K_{\bar{\alpha}\beta}(q,p) = \langle K_\alpha(q), K_\beta(p) \rangle$ for $K_\alpha := \imath \circ s_\alpha \circ K : \Omega_\alpha \to \mathcal{M}$, where $s_\alpha : K_\alpha(\Omega_\alpha) \to \mathbb{E}$ are local frames of $\mathbb{E}$, condition (2.46) means that strict Schwartz inequality is satisfied which takes place iff K is an injection. □

The three alternative descriptions of a physical system, which have been proposed in this section, naturally correspond with well developed mathematical theories. In the case of categories $\mathcal{A}$ and $\mathcal{B}$ this is, first of all, the theory of reproducing kernels (see [1]) with its applications to the complex analysis (see i.e. [2, 3, 8]). We want to underline especially the aspect of applications, because, since the appearance of Bergman's work (see [2, 3]), the reproducing kernels became a well established and very efficient tool in the theory of holomorphic functions (see [8]) as well as an independent object of investigations (see [12]).

In order to distinguish the holomorphic subcategory $\mathcal{B}_{\mathrm{Hol}}$ within the category $\mathcal{B}$, we shall restrict ourselves to the investigation of holomorphic bundles $\mathbb{L} \to M$. The Hilbert spaces $\mathcal{H}_{\mathbb{L}}$ are then realized through the holomorphic sections of bundle $\mathbb{L}$ while the scalar product in $\mathcal{H}_{\mathbb{L}}$ is naturally given by the integration over

M that guarantees (see [25]) the continuity of the evaluation functional $e_{\alpha,q}$. The complex analysis provides many examples of objects $(\mathbb{L}\to M, \mathscr{H}_{\mathbb{L}})$ of category $\mathscr{B}_{\mathrm{Hol}}$ for which the reproducing kernels were found explicitly (see [12]). Some of those cases are also interesting from the physical point of view (see [20, 22]).

Another example of category $\mathscr{C}_G$ related with $\mathscr{C}$ by forgetting a functor can be obtained with $(G, M, \mathscr{M}, K: M\to \mathbb{CP}(\mathscr{M}), \sigma, T)$ as objects, where the Lie group G acts on M via $\sigma: G\to \mathrm{Diff}\, M$ and $T: G\to \mathrm{Aut}\,\mathscr{M}$ while K is a G-equivariant map. If G acts on M transitively then the construction of objects of $\mathscr{C}_G$ turns out to be related to the construction of an irreducible representation of G. In this way one arrives at the coherent states theory in the sense of Perelomov (see [21]). A quite interesting subcategory of $\mathscr{C}$, from the point of view of the theory of Kähler homogeneous spaces, is given by the intersection $\mathscr{C}_G\cap\mathscr{C}_{\mathrm{Hol}}\cap\mathscr{C}_{\mathrm{Sp}}$ (see [17]). Let now M be a compact complex algebraic manifold, $\dim\mathscr{M}<\infty$ and let K be an embedding of algebraic manifolds. The objects of that type form a subcategory $\mathscr{C}_{Alg}$ of category $\mathscr{C}$. In this case we may use the algebraic geometry where the necessary and sufficient conditions are formulated (Kodaira theorem) for the complex manifold to be algebraically embedded in $\mathbb{CP}(\mathscr{M})$, see [10, 23]. One can find there also explicit construction of such embeddings, as the Plucker, Segre, and Veronese embeddings, the last two ones closely related to the theory of spin.

As we see, the point of view presented here naturally encompasses such theories as complex algebraic geometry, complex analysis, representation theory and reproducing kernel theory, all appearing as efficient and natural tools for a description of physical phenomena in the microscale.

3. The Notion of a Mechanical System

In the foregoing section we have shown that the physical systems provided with the transition amplitude kernel could be identified with objects of the category $\mathscr{C}$ or of its isomorphic images $\mathscr{A}$ or $\mathscr{B}$. In this section we shall formulate a theory of quantum processes. In a sense a proposed formulation incorporates in a natural way both classical and quantum mechanical description of events.

We define *a mechanical system* to be an object $(M, \mathscr{M}, K: M\to\mathbb{CP}(\mathscr{M}))$ of the symplectic subcategory $\mathscr{C}_{\mathrm{Sp}}$. As a manifold of all attainable pure states of the system we take $\mathbb{CP}(\mathscr{M})$. States $K(m)$, where $m\in M$, will be called *coherent states.* The justification of this terminology will become clear after considerations and examples elaborated in what follows. We assume the family of all coherent states to be a submanifold of $\mathbb{CP}(\mathscr{M})$. This submanifold characterizes intrinsically the mechanical system under consideration. In addition, any chosen parameterization $K: M\to\mathbb{CP}(\mathscr{M})$ of the submanifold $K(M)$ corresponds to a way of measuring of transition amplitudes. The probability amplitude $A_{\bar\alpha\beta}(q,p)$ for the state $K_\alpha(q)$ to be found in the state $K_\beta(p)$, where $q\in\Omega_\alpha$ and $p\in\Omega_\beta$, $\Omega_\alpha, \Omega_\beta\subset M$, is given by

$$A_{\bar\alpha\beta}(q,p)=\frac{\langle K_\alpha(q), K_\beta(p)\rangle}{\langle K_\alpha(q), K_\alpha(q)\rangle^{1/2}\langle K_\beta(p), K_\beta(p)\rangle^{1/2}}, \tag{3.1}$$

where $K_\alpha:\Omega_\alpha\to\mathscr{M}$ are smooth maps such that, $K(q)=[K_\alpha(q)]$. Clearly, the probability density $|A_{\bar\alpha\beta}(q,p)|^2$ is independent of the choice of K_α.

It is convenient now to introduce a "transition operator" for coherent states $K(q)$ and $K(p)$ given by

$$\mathscr{A}_{\bar\alpha\beta}(q,p):=\frac{|K_\beta(p)\rangle\langle K_\alpha(q)|}{\langle K_\alpha(q)|K_\alpha(q)\rangle^{1/2}\langle K_\beta(p)|K_\beta(p)\rangle^{1/2}}. \tag{3.2}$$

So we have

$$A_{\bar{\alpha}\beta}(q,p)=\mathrm{Tr}\,\mathscr{A}_{\bar{\alpha}\beta}(q,p) \tag{3.3}$$

and

$$P(q):=\mathscr{A}_{\bar{\alpha}\alpha}(q,q)=\mathscr{A}_{\bar{\beta}\beta}(q,q) \quad \text{for} \quad q\in\Omega_\alpha\cap\Omega_\beta\,. \tag{3.4}$$

Hence, the projection operators $P(q)$, $q\in M$, do not depend on the choice of gauge. By our assumptions, manifold M used for parameterizing the coherent states is symplectic with symplectic form $\omega=K^*\omega_{\mathrm{FS}}$. Let $L^1(M,d\mu_L)$ denote the space of functions which are integrable with respect to the Liouville measure $d\mu_L:=\Lambda^n\omega$, where $2n=\dim M$. We may define a continuous linear map of $L^1(M,d\mu_L)$ into the algebra $\mathscr{B}(\mathscr{M})$ of bounded linear operators on $\mathscr{M}$ by

$$L^1(M,d\mu_L)\ni\varrho\to P(\varrho):=\int_M P(q)\varrho(q)d\mu_L(q)\,. \tag{3.5}$$

The integration in (3.5) is understood in the weak sense. One has the following inequality

$$\|P(\varrho)v\|\leqq\|\varrho\|_1\|v\|\,,\qquad \forall v\in\mathscr{M} \quad \text{and} \quad \forall\varrho\in L^1(M,d\mu_L)\,, \tag{3.6}$$

from which the continuity of the map $P:L^1(M,d\mu_L)\to\mathscr{B}(\mathscr{M})$ follows.

If the system is in a mixed state described by density operator $P(\varrho)$, given by (3.5) then we shall say that it is localized in $K(M)$ with the weight function $\varrho\in L^1(M,d\mu_L)$, $\int_M\varrho(q)d\mu_L(q)=1$. The probability $\langle P(\varrho)\rangle(p):=\mathrm{Tr}(P(\varrho)P(\varrho))$ of finding the system in a state $K(p)$ provided it is in state $P(\varrho)$ is equal to the ϱ-weighted "sum" of $|A_{\bar{\alpha}\beta}(q,p)|^2$

$$\langle P(\varrho)\rangle(p):=\mathrm{Tr}(P(\varrho)P(p))=\int_M|A_{\bar{\alpha}\beta}(q,p)|^2\varrho(q)d\mu_L(q)\,. \tag{3.7}$$

Assuming $\langle P(\varrho)\rangle:M\to\mathbb{R}_+$ as the weight function for the new mixed state $P(\langle P(\varrho)\rangle)$, we introduce the following definition.

Definition 1. We call the state $P(\varrho)$ an equilibrium state of the mechanical system if there exists a map $h:M\to\mathbb{R}_+$ and one-to-one maps $\varphi,\tilde{\varphi}:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\varrho=\varphi\circ h \quad \text{and} \quad \langle P(\varrho)\rangle=\tilde{\varphi}\circ h\,. \tag{3.8}$$

[We shall say then that the subsets of equally weighted coherent states for ϱ and $\langle P(\varrho)\rangle$ are identical.].

We shall consider a set $E(K,h)$ of equilibrium states which satisfies the following requirements:

a) The function h is fixed for all equilibrium states from $E(K,h)$.

b) $E(K,h)$ is closed with respect to transformation $\varrho\to\langle P(\varrho)\rangle$.

c) $E(K,h)$ is parameterized by a finite number of parameters $(\beta_1,\ldots,\beta_k)=\boldsymbol{\beta}\in\Delta_k\subset\mathbb{R}^k$, i.e. $\varrho(q)=\varphi(h(q);\boldsymbol{\beta})$.

d) There exists $\boldsymbol{\beta}_0\in\bar{\Delta}_k$ such that $\varphi(h(q);\boldsymbol{\beta}_0)=\varrho_0=\mathrm{const}$ and one has the following partition of the identity operator:

$$\mathbb{I}=P(\varrho_0)=\int_M P(q)\varrho_0 d\mu_L(q)\,. \tag{3.9}$$

Let us now present in brief the heuristic motivation for the above requirements. According to b) and c) the integral transform (3.7) results in the renormalization of parameters $\boldsymbol{\beta}$, i.e.

$$\langle P(\varrho)\rangle=\varphi(h(q);\tilde{\boldsymbol{\beta}}) \tag{3.10}$$

for

$$\varrho = \varphi(h(q); \boldsymbol{\beta}), \tag{3.11}$$

where $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\boldsymbol{\beta})$. Parameters $\beta_1, \ldots, \beta_k$ are to be interpreted in a thermodynamical sense, i.e. they characterize the equilibrium states of the system $K(\mathscr{M})$ within the "thermostate" $\mathbb{C}\mathbb{P}(\mathscr{M})$. Our assumptions guarantee that any "interference" into the system may transform it only from one state of equilibrium to another. By "interference" we mean experimental realization of the transform $\varrho \to \langle P(\varrho)\rangle$ as defined by (3.7).

The "interferences" that one admits here are such as not to destroy the system, i.e. such as do not change its intrinsic characteristics much, if at all. We consider the function $h: M \to \mathbb{R}_+$ to be one of such intrinsic characteristics of the system under observation. As functions h and φ, together with transition amplitudes, constitute the description of outcomes of observations, conditions a)–c) become natural. They guarantee a sufficient *stability* of the equilibrium states of the system under disturbances introduced by an experimental interference.

Requirement d) means that:

(i) the family of the coherent states forms a linearly dense subset in space of all pure states;

(ii) the composition law (with measure $\varrho_0 d\mu_L$) is satisfied for the transition amplitudes, i.e.

$$A_{\bar{\alpha}\beta}(q,p) = \int_M \sum_\delta h_\delta(r) A_{\bar{\alpha}\delta}(q,r) A_{\bar{\delta}\beta}(r,p) \varrho_0 d\mu_L(r), \tag{3.12}$$

where $\sum_\delta h_\delta = 1$ is a partition of unity subordinate to the covering $\bigcup_\alpha \Omega_\alpha = M$.

Formulas (3.11), (3.10), and (3.7) become the consistency equations for maps K and h. We shall not study in this paper the general problem of solving those relations. Instead, we describe at the end of the paper a number of physically important examples.

Now we pass to the calculation of the transition amplitude for a process along a given piecewise smooth trajectory.

Let the sequence $(q = q_1, q_2, \ldots, q_{N-1}, q_N = p)$ be such that $q_i \in \Omega_{\alpha_i}$, where we assume that $\Omega_{\alpha_1} = \Omega_\alpha$ and $\Omega_{\alpha_N} = \Omega_\beta$. According to the multiplicative property of the transition amplitude, the following expression gives the transition amplitude from the state $K(q)$ to the state $K(p)$,

$$A_{\bar{\alpha}\beta}(q, q_2, \ldots, q_{N-1}, p) := A_{\bar{\alpha}_1\alpha_2}(q_1, q_2) A_{\bar{\alpha}_2\alpha_3}(q_2, q_3) \cdot \ldots \cdot A_{\bar{\alpha}_{N-1}\alpha_N}(q_{N-1}, q_N), \tag{3.13}$$

under the condition that the system has gone through all the intermediate states $K(q_2), \ldots, K(q_{N-1})$. To express this the following terminology will be used. We shall call the sequence $(K(q_1), \ldots, K(q_N))$ a process starting at $K(q)$ and ending at $K(p)$. Consequently, $A_{\bar{\alpha}\beta}(q, q_2, \ldots, q_{N-1}, p)$ will be called the transition amplitude for that process. Let us investigate further a process of that kind. For that we shall consider a piecewise smooth curve $\gamma: [\tau_i, \tau_f] \to M$ and the partition $(\tau_1 = \tau_i, \tau_2, \ldots, \tau_{N-1}, \tau_N = \tau_f)$ of the interval $[\tau_i, \tau_f]$, where $\tau_{k+1} - \tau_k = \frac{1}{N-1}(\tau_f - \tau_i)$, such that $\gamma(\tau_i) = q_i$. Then in the limit $N \to \infty$, this curve γ may be viewed as a process approximately described by discrete processes $(q, q_2, \ldots, q_{N-1}, p)$. The transition amplitude for the process γ is then obtained from (3.13) by passing to the limit as $N \to \infty$,

$$A_{\bar{\alpha}\beta}(q; \gamma; p) = \lim_{N\to\infty} \prod_{k=1}^{N-1} A_{\bar{\alpha}_k\alpha_{k+1}}(\gamma(\tau_k), \gamma(\tau_{k+1})). \tag{3.14}$$

Taking into account the smoothness of $K_\alpha : \Omega_\alpha \to \mathcal{M}$ and piecewise smoothness of γ we could calculate (3.14) explicitly. In order to do this let us put $\Delta K(\gamma(\tau_k)) := K_{\alpha_{k+1}}(\gamma(\tau_{k]1})) - K_\alpha(\gamma(\tau_{k_k}))$ and use (3.14). Then

$$\begin{aligned}
A_{\bar{\alpha}\beta}(q;\gamma;p) &= \lim_{N\to\infty} \left(\frac{\langle K_\alpha(q)|K_\alpha(q)\rangle}{\langle K_\beta(p)|K_\beta(p)\rangle}\right)^{1/2} \\
&\quad \times \prod_{k=1}^{N-1} \left(1 - \frac{\langle K_{\alpha_k}(\gamma(\tau_k))|\Delta K(\gamma(\tau_k))\rangle}{\langle K_{\alpha_k}(\gamma(\tau_k))|K_{\alpha_k}((\tau_k))\rangle}\right) \\
&= \left(\frac{\langle K_\alpha(q)|K_\alpha(q)\rangle}{\langle K_\beta(p)|K_\beta(p)\rangle}\right)^{1/2} \\
&\quad \times \lim_{N\to\infty} \exp\left(\sum_{k=1}^{N-1} \frac{\langle K_{\alpha_k}(\gamma(\tau_k))|\Delta K(\gamma(\tau_k))\rangle}{\langle K_{\alpha_k}(\gamma(\tau_k))|K_{\alpha_k}(\gamma(\tau_k))\rangle}\right) \\
&= \left(\frac{\langle K_\alpha(q)|K_\alpha(q)\rangle}{\langle K_\beta(p)|K_\beta(p)\rangle}\right)^{1/2} \exp\int_\gamma \frac{\langle K|dK\rangle}{\langle K|K\rangle} \\
&= \exp i \int_\gamma \operatorname{Im} \frac{\langle K|dK\rangle}{\langle K|K\rangle}\,.
\end{aligned} \tag{3.15}$$

where the symbol $\frac{\langle K|dK\rangle}{\langle K|K\rangle}$ denotes a family of 1-forms which on Ω_α are given by $\frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle}$.

Now, we uncover a geometrical sense of the obtained expression. For that let us note that the metric structure $K^*H^{\mathrm{FS}}(s_\alpha, s_\alpha) =: \varrho_{\bar{\alpha}\alpha}$ and the connection one-form $K^*\nabla^{\mathrm{FS}} s_\alpha =: \theta_\alpha \otimes s_\alpha$ written in the frames s_α given by maps K_α take the following form:

$$\varrho_{\bar{\alpha}\alpha} = \langle K_\alpha|K_\alpha\rangle\,, \tag{3.16}$$

$$\theta_\alpha = \frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle}\,. \tag{3.17}$$

After expressing $K^*\nabla^{\mathrm{FS}}$ in the unitary gauge frame

$$u_\alpha := \frac{1}{H(s_\alpha, s_\alpha)^{1/2}} s_\alpha$$

one obtains

$$K^*\nabla^{\mathrm{FS}} u_\alpha = \operatorname{Im} \frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle} \otimes u_\alpha\,. \tag{3.18}$$

Formulas (3.15) and (3.18) exhibit the fact that the transition amplitude for the process γ is just a parallel transport factor for curve γ with respect to the connection $K^*\nabla^{\mathrm{FS}}$.

The notion of transition amplitude for the process γ together with the composition law (3.12) constitutes a starting point for the expression of probability amplitude $A_{\bar{\alpha}\beta}(q,p)$ in terms of the path integral. We now proceed to obtain this expression. The multiple iteration of the composition law (3.12) ends up with the following expression for the transition amplitude:

$$\begin{aligned}
A_{\bar{\alpha}\beta}(q,p) &= \int_M \sum_{\delta_2} h_{\delta_2}(q_2)\varrho_0 d\mu_L(q_2) \cdot \ldots \cdot \int_M \sum_{\delta_{N-1}} h_{\delta_{N-1}}(q_{N-1})\varrho_0 d\mu_L(q_{N-1}) \\
&\quad \times A_{\bar{\alpha}\delta_2}(q,q_2) A_{\bar{\delta}_2\delta_3}(q_2,q_3) \cdot \ldots \cdot A_{\bar{\delta}_{N-1}\beta}(q_{N-1},p)\,.
\end{aligned} \tag{3.19}$$

Let $q=\gamma(\tau_i)$, $q_2=\gamma(\tau_2), \ldots, q_{N-1}=\gamma(\tau_{N-1})$, $p=\gamma(\tau_f)$. This and independence of the left-hand side of (3.19) on the number $N-1$ of iterations enable us to write down (3.19) in the form

$$\begin{aligned} A_{\bar{\alpha}\beta}(q,p) = \lim_{N\to\infty} \int_M \sum_{\delta_2} h_{\delta_2}(\gamma(\tau_2))\varrho_0 d\mu_L(\gamma(\tau_2)) \cdot \ldots \\ \times \int_M \sum_{\delta_{N-1}} h_{\delta_{N-1}}(\gamma(\tau_{N-1}))\varrho_0 d\mu_L(\gamma(\tau_{N-1})) \\ \times A_{\bar{\alpha}\delta_2}(q,\gamma(\tau_2)) A_{\bar{\delta}_2\delta_3}(\gamma(\tau_2),\gamma(\tau_3)) \cdot \ldots \cdot A_{\bar{\delta}_{N-1}\beta}(\gamma(\tau_{N-1}),p). \end{aligned} \tag{3.20}$$

Before letting N go to infinity in order to obtain the final expression we introduce a convenient notation,

$$\begin{aligned} \int \prod_{\tau\in[\tau_i,\tau_f]} d_K\gamma(\tau) := \lim_{N\to\infty} \int_M \sum_{\delta_2} h_{\delta_2}(\gamma(\tau_2))\varrho_0 d\mu_L(\gamma(\tau_2)) \cdot \ldots \\ \times \int_M \sum_{\delta_{N-1}} h_{\delta_{N-1}}(\gamma(\tau_{N-1}))\varrho_0 d\mu_L(\gamma(\tau_{N-1})). \end{aligned} \tag{3.21}$$

The expression being integrated in (3.20) tends to $A_{\bar{\alpha}\beta}(q;\gamma;p)$ as N goes to infinity – in accordance with (3.14). From that and (3.15) we get formally

$$A_{\bar{\alpha}\beta}(q,p) = \int \prod_{\tau\in[\tau_i,\tau_f]} d_K\gamma(\tau) \exp\left(i \int_{\tau_i}^{\tau_f} \operatorname{Im} \frac{\langle K|dK\rangle}{\langle K|K\rangle} \lrcorner \frac{d\gamma}{d\tau} d\tau \right) \tag{3.22}$$

which could be interpreted as a path integral expression for the transition amplitude, the integration in (3.22) being carried out over all piecewise smooth trajectories.

We now come to calculation of the transition amplitude between coherent states $K(q)\in h^{-1}(E)$ and $K(p)\in h^{-1}(E)$, however, under essential restriction. Namely, we admit in the process of integration (3.22) only those trajectories (representing the physical process) which are confined to the equiprobable hypersurface $h^{-1}(E)$, i.e. $h(\gamma(\tau))=E=\text{const}$ for $\tau\in[\tau_i,\tau_f]$. Let then $A_{\bar{\alpha}\beta}(q;p;h=E)$ denote the probability amplitude for the transition from $K(q)$ to $K(p)$ which is the result of the superposition of equiprobable γ processes. In order to express the amplitude $A_{\bar{\alpha}\beta}(q;p;h=E)$ as the path integral one should insert the appropriate δ-factor into the measure part in (3.21), i.e. we use

$$\begin{aligned} &\delta(h(\gamma(\tau_k))-E)\varrho_0 d\mu_L(\gamma(\tau_k)) \\ &= \int^{+\infty} e^{-i[h(\gamma(\tau_k))-E]\lambda(\tau_k)} d\lambda(\tau_k)\varrho_0 d\mu_L(\gamma(\tau_k)) \end{aligned} \tag{3.23}$$

instead of $\varrho_0 d\mu_L(\gamma(\tau_k))$. As a result we have

$$\begin{aligned} A_{\bar{\alpha}\beta}(q;p;h=E) = \int \prod_{\tau\in[\tau_i,\tau_f]} d_K\gamma(\tau) d\lambda(\tau) \\ \times \exp\left(i \int_{\tau_i}^{\tau_f} \left\{ \operatorname{Im} \frac{\langle K|dK\rangle}{\langle K|K\rangle} \lrcorner \frac{d\gamma}{d\tau} d\tau - [h(\gamma(\tau))-E]\lambda(\tau) \right\} d\tau \right). \end{aligned} \tag{3.24}$$

In integral (3.24), different parameterizations of the process γ give equal contributions. The reparameterization invariance may be fixed by introduction of $t=\int_{t_i}^{\tau} \lambda_0(s)ds$ as a time parameterizing the processes. This way, the integral becomes the integral over the equivalent choices of classical "clocks" and may be dropped out. The resulting amplitude is given by

$$A_{\bar{\alpha}\beta}(q;p;h=E)=e^{-iE(t_f-t_i)}\int \prod_{t\in[t_i,t_f]} d_K\gamma(t) \times \exp\left(i\int_{t_i}^{t_f}\left\{\operatorname{Im}\frac{\langle K|dK}{\langle K|K\rangle} \lrcorner \frac{d\gamma}{dt}-h(\gamma(t))\right\}dt\right). \tag{3.25}$$

According to Feynman's path integral interpretation of the above transition amplitude, the action functional $S_{K,h}$ for the mechanical system $(M,\mathcal{M},K:M\to\mathbb{CP}(\mathcal{M}))$ reads as follows:

$$S_{K,h}[\gamma]=\int_{t_i}^{t_f}\left\{\operatorname{Im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{dt}-h(\gamma(t))\right\}dt. \tag{3.26}$$

The extremals of the action $S_{K,h}$ are to be found from the Hamilton equation

$$\omega \lrcorner \frac{d\gamma}{dt}=dh. \tag{3.27}$$

It should be clear now that function h has to be identified with the Hamiltonian of the mechanical system. Evidently, the summand $\operatorname{Im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}\lrcorner\frac{d\gamma}{dt}$ is responsible for the interaction of the system with the effective external field resulting from the way the embedding $K:M\to\mathbb{CP}(\mathcal{M})$ has been realized. Consequently, $\operatorname{Im}\frac{\langle K|dK\rangle}{\langle K|K\rangle}$ is to be identified with generalized canonical momentum.

In any given situation it is convenient to fix a map $K_0:M\to\mathbb{CP}(\mathcal{M})$ in a homotopy equivalence class of maps from M into $\mathbb{CP}(\mathcal{M})$ and then describe the other maps of this equivalence class as deformations of K_0. The fixing of K_0 may be realized via imposing on K_0 some symmetry condition which are inherent to the physical system by assumption. The deformations K of K_0, breaking the symmetry of the system are at the same time responsible for interaction of the system with an external field. An example of such a situation is discussed in [20].

4. The Relation Between Classical and Quantum Observables

The model of the mechanical system developed in the two preceding sections is deeply found on the notion of coherent states. The coherent states are known to be close to the states of classical phase space. Due to this, the map $K:M\to\mathbb{CP}(\mathcal{M})$ could be treated as a quantization of states from (M,ω) describing the relation between classical and quantum states.

In this section we shall concentrate on relations between classical and quantum observables. We define the classical observable in a standard way as a function $h\in C^\infty(M)$, while the quantum observable is represented by a Hermitian operator $H\in B(\mathcal{M})$. For the sake of simplicity we restrict our consideration only to bounded operators.

The structure of our model of mechanics provides one with a natural (also from the physical point of view) way of prescribing to any given quantum observable its classical counterpart. This procedure consists of the calculation of the mean value on coherent states $K(q)$, $q \in M$, for $H \in B(\mathcal{M})$, i.e.

$$\langle H \rangle_K(q) := \frac{\langle K_\alpha(q) | H K_\alpha(q) \rangle}{\langle K_\alpha(q) | K_\alpha(q) \rangle}, \tag{4.1}$$

for $q \in \Omega_\alpha$. In this way one obtains functions $\langle H \rangle_K \in C^\infty(M)$.

The transition from classical to quantum quantities, known generally as the quantization procedure, is given by a homomorphism Q of an appropriate Lie subalgebra of the Poisson algebra $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ into the operator Lie algebra $(B(M), [\cdot, \cdot])$. Our goal is to introduce such a quantization procedure Q which is natural and consistent with the structures that we have introduced.

We shall start with two remarks originating from the very beginning of quantum theory (see [7, 24]).

(i) The motion of the wave packet corresponds to the motion of the classical particle under the condition that the potential varies slowly in the space occupied by the wave packet. Taking mean values of the position and the momentum operators we may exhibit this correspondence for the position and the momentum of the wave packet (see [7]). This is what is known in quantum mechanics as the "Ehrenfest theorem."

(ii) A year before Ehrenfest's work [7], Schrödinger discovered (see [24]) a family of states of the quantum harmonic oscillator, which is parameterized by the classical phase space and is invariant under time evolution which thus defines the evolution of the classical states. The classical dynamics obtained in this way is identical with the dynamics defined via the Hamiltonian of the classical harmonic oscillator.

In view of the above remarks we shall introduce the following notion. Let a function $h \in C^\infty(M, \mathbb{R})$ define a global Hamiltonian flow σ_h on (M, ω) and let $H^+ = H \in B(\mathcal{M})$.

Definition 1. We shall say that a pair $(h, H) \in C^\infty(M, \mathbb{R}) \times B(\mathcal{M})$ satisfies the Ehrenfest condition if for each $t \in \mathbb{R}$ one has

$$\begin{array}{ccc} M & \xrightarrow{K} & \mathbb{CP}(\mathcal{M}) \\ \sigma_h(t) \downarrow & & \downarrow [U(t)] \\ M & \xrightarrow{K} & \mathbb{CP}(\mathcal{M}), \end{array} \tag{4.2}$$

where $U(t) = \exp(itH)$.

Naturally, the function $h + \text{const}$ generates, the same flow as h, so the pair $(h + \text{const}, H)$ also satisfies the Ehrenfest condition. In order to avoid this non-uniqueness let us introduce a slightly stronger version of that condition. To this end let us introduce a one-parameter group of automorphisms

$$U'(t) = \iota^{-1} \circ U(t) \circ \iota, \quad \text{for} \quad t \in \mathbb{R},$$

of the principal bundle $\mathbb{E}' = \mathbb{E} \setminus \iota^{-1}(0) \overset{\iota}{\cong} \mathcal{M} \setminus \{0\}$. Note that the Fubini-Study connection 1-form $\alpha_{FS} \in \Gamma^\infty(\mathbb{E}', T^*\mathbb{E}')$ is invariant under $U'(t)$. Let $\mathbb{L}' := K^*\mathbb{E}'$ be $\mathbb{C}^*$-principal bundle over M defined as the pull back of $\mathbb{E}'$. The function h generates a one-parameter group σ'_h of automorphisms of $\mathbb{L}'$ which preserves the connection one-form $K^*\alpha_{FS}$ and such that $\pi' \circ \sigma'_h(t) = \sigma_h(t)$, where $\pi' : \mathbb{L}' \to M$ (see [16]). It is easy to see that (see [16]) each such one-parameter group of automorphisms of $(\mathbb{L}', K^*\alpha_{FS})$ is of the form σ'_h, where $h \in C^\infty(M, \mathbb{R})$. Because K is an embedding and $[U(t)]$ preserves $K(M)$ one has

$$\begin{array}{ccc} \mathbb{L}' & \xrightarrow{K'} & \mathbb{E}' \\ {\scriptstyle \sigma_h}\downarrow & & \downarrow{\scriptstyle U'(t)} \\ \mathbb{L}' & \xrightarrow{K'} & \mathbb{E}'. \end{array} \tag{4.3}$$

Here, $K':\mathbb{L}'\to\mathbb{E}'$ is a principal bundle morphism generated by K. Since from now on, by the Ehrenfest condition, we shall mean (4.3). Requirement (4.3) removes the non-uniqueness in the correspondence between functions h and operators H.

Lemma 1. *If (h_1,H_1) and (h_2,H_2) satisfy the Ehrenfest condition and $h_1=h_2$ or $H_1=H_2$ then $(h_1,H_1)=(h_2,H_2)$.*

Proof. If $h_1=h_2$ then $\sigma'_{h_1}\equiv\sigma'_{h_2}$, which gives

$$U_1(t)\circ\imath\circ K'=K'\circ\sigma'_{h_1}(t)=K'\circ\sigma'_{h_2}(t)=U_2(t)\circ\imath\circ K'.$$

Because $(\imath\circ K')(M)$ is linearly dense in $\mathcal{M}$, we obtain $U_1\equiv U_2$, which gives $H_1=H_2$. If $U_1=U_2$ then from the Ehrenfest condition we have

$$K'\circ\sigma'_{h_1}(t)=K'\circ\sigma'_{h_2}(t).$$

As K' is an injection, $\sigma'_{h_1}\equiv\sigma'_{h_2}$ and $h_1=h_2$. □

Let E denote the set of pairs $(h,H)\in C^\infty(M,\mathbb{R})\times B(\mathcal{M})$ satisfying the Ehrenfest condition. Let pr_i denote the projection of E on i^{th} component of the Cartesian product $C^\infty(M,\mathbb{R})\times B(\mathcal{M})$. We introduce the following notation

$$\mathrm{pr}_1(E)=:C^\infty_E(M,\mathbb{R}) \quad\text{and}\quad \mathrm{pr}_2(E)=:B_E(\mathcal{M}).$$

Because of Lemma 1 we get the following mutually inverse bijections

$$\mathrm{pr}_2\circ\mathrm{pr}_1^{-1}:C^\infty_E(M,\mathbb{R})\to B_E(\mathcal{M}),$$
$$\mathrm{pr}_1\circ\mathrm{pr}_2^{-1}:B_E(\mathcal{M})\to C^\infty_E(M,\mathbb{R}).$$

Proposition 2.

$$\mathrm{pr}_1\circ\mathrm{pr}_2^{-1}(H)=\langle H\rangle_K, \tag{4.4}$$

$$\mathrm{pr}_2\circ\mathrm{pr}_1^{-1}(h)=\varrho_0^{-1}P(h)-i\int_M P(q)X_hP(q)d\mu_L(q) \tag{4.5}$$

for $H^+=H\in B_E(\mathcal{M})$ and $h\in C^\infty_E(M,\mathbb{R})\cap L^1(M,d\mu_L)$, where X_h is given by $\omega\lrcorner X_h=dh$.

Proof. (i) Let h be a function satisfying the Ehrenfest condition with the operator $H^+=H$. Let us denote by X_H and X_h the vector fields generated by flows $[U(\cdot)]:=[\exp i(\cdot)H]$ and σ_h correspondingly. From this and from (2.44) one has

$$\omega\lrcorner X_h=dh, \tag{4.6}$$

$$\omega_{\mathrm{FS}}\lrcorner X_H=d\langle H\rangle. \tag{4.7}$$

Carrying over the expression (4.7) on M with help of K we get

$$K^*(\omega_{\mathrm{FS}}\lrcorner X_H)=d\langle H\rangle_K. \tag{4.8}$$

This, together with $K^*\omega_{\mathrm{FS}}=\omega$ and $K_*X_h=X_H$ results in

$$\omega\lrcorner X_h=d\langle H\rangle_K. \tag{4.9}$$

Subtracting (4.6) from (4.9) one gets $\langle H\rangle_K - h = \text{const}$ and the constant is zero because of (4.3).
(ii) Due to $\imath:\mathbb{E}'\to\mathcal{M}\setminus\{0\}$ and (4.3) one has

$$U(t)P(q)U^+(t)=P(\sigma(t)q), \tag{4.10}$$

which after differentiation at $t=0$ gives

$$i[H,P(q)]=(X_hP)(q). \tag{4.11}$$

In accordance with (i) of this proof, $h=\langle H\rangle_K$, hence

$$\begin{aligned} P(h) &= \int_M P(q)h(q)d\mu_L(q) \\ &= \int_M \frac{|K_\alpha(q)\rangle\langle K_\alpha(q)|}{\langle K_\alpha(q)|K_\alpha(q)\rangle}\frac{\langle K_\alpha(q)|H|K_\alpha(q)\rangle}{\langle K_\alpha(q)|K_\alpha(q)\rangle}d\mu_L(q) \\ &= \int_M P(q)HP(q)d\mu_L(q) \\ &= \left(\int_M P(q)d\mu_L(q)\right)H - i\int_M P(q)(X_hP)(q)d\mu_L(q) \\ &= \varrho_0^{-1}H - i\int_M P(q)(X_hP)(q)d\mu_L(q). \end{aligned} \tag{4.12}$$

We have used relation (3.9) for the proof of the last equality. □

Let us also note the following identities:

$$\int_M (X_hP)(q)d\mu_L(q)=0, \tag{4.13}$$

$$\int_M (X_hP)(q)P(q)d\mu_L(q)+\int_M P(q)(X_hP)(q)d\mu_L(q)=0 \tag{4.14}$$

for $h\in C^\infty(M,\mathbb{R})$, which are a consequence of invariance of the Liouville measure under the Hamiltonian flow σ_h.

Lemma 3. *The mean value operation defines the morphism of Lie algebras*

$$i\langle\cdot\rangle:(B(\mathcal{M}),[\cdot,\cdot])\to(C^\infty\mathbb{CP}(\mathcal{M}),\mathbb{C}),\{\cdot,\cdot\}_{\mathrm{FS}}).$$

Proof. The linearity of $i\langle\cdot\rangle$ is obvious and $\langle A\rangle\equiv 0$ iff $A=0$. This proves that $i\langle\cdot\rangle$ is a monomorphism. The equality

$$i\langle[A,B]\rangle=\{\langle A\rangle,\langle B\rangle\}_{\mathrm{FS}}, \tag{4.15}$$

which shows that $i\langle\cdot\rangle$ is a homomorphism of Lie algebras, may be checked through a straightforward calculation. □

Proposition 4. *The following conditions on $H=H^+\in B(\mathcal{M})$:*

(i) $H\in B_E(\mathcal{M})$,

(ii) $\forall A\in B(\mathcal{M}),\quad \{\langle H\rangle_K,\langle A\rangle_K\}=i\langle[H,A]\rangle_K$,

(iii) $\forall A\in B(\mathcal{M}),\quad \{\langle H\rangle_K,\langle A\rangle_K\}=\{\langle H\rangle,\langle A\rangle\}_{\mathrm{FS}}\circ K$

are equivalent.

Proof. (i) ⇒ (ii). As $H \in B_E(\mathcal{M})$, there exists a function $h \in C^\infty(M, \mathbb{R})$ which generates the global Hamiltonian flow σ_h fulfilling (4.2). Hence for every $A \in B(\mathcal{M})$,

$$\frac{\langle K_\alpha \circ \sigma_h(t)|AK_\alpha \circ \sigma_h(t)\rangle}{\langle K_\alpha \circ \sigma_h(t)|K_\alpha \circ \sigma_h(t)\rangle} = \frac{\langle e^{-itH}K_\alpha|Ae^{-itH}K_\alpha\rangle}{\langle e^{-itH}K_\alpha|e^{-itH}K_\alpha\rangle} = \frac{\langle K_\alpha|e^{itH}Ae^{-itH}K_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle}. \tag{4.16}$$

Differentiating (4.16) with respect to the parameter t at $t=0$ gives

$$\frac{d}{dt}\langle A\rangle_K|_{t=0} = \{h, \langle A\rangle_K\} = i\langle [H, A]\rangle_K. \tag{4.17}$$

In accordance with (4.4) $h = \langle H\rangle_K$, which together with (4.17) results in (ii).
(ii) ⇒ (iii). This follows from (4.15).
(iii) ⇒ (i). The expression (iii) can be rewritten in the form

$$\langle K_* X_{\langle H\rangle_K}, d\langle A\rangle\rangle \circ K = \langle X_{\langle H\rangle}, d\langle A\rangle\rangle \circ K, \quad \forall A \in B(\mathcal{M}), \tag{4.18}$$

where $X_{\langle H\rangle_K}$ and $X_{\langle H\rangle}$ stand for the vector fields tangent to the Hamiltonian flows $\sigma_{\langle H\rangle_K}$ and $[U(\cdot)]$, respectively. Because forms $d\langle A\rangle$, with $A \in B(\mathcal{M})$, span the cotangent spaces $T^*_{[v]}(\mathbb{CP}(\mathcal{M}))$ for any $[v] \in \mathbb{CP}(\mathcal{M})$, one obtains from (4.18)

$$K_* H_{\langle H\rangle_K}(q) = X_{\langle H\rangle}(K(q)) \tag{4.19}$$

for $q \in M$. From (4.19) one derives directly that the velocity field $X_{\langle H\rangle}$ for the flow $[U(\cdot)]$ is tangent to $K(M)$, i.e. the flow $[U(t)]$ preserves $K(M)$. As $K: M \to \mathbb{CP}(\mathcal{M})$ is an embedding of manifolds then $K \circ [U(\cdot)] \circ K^{-1}$ is a globally defined flow on M which due to (4.19) fulfills the condition $\sigma_{\langle H\rangle_K}(t) = K \circ [U(t)] \circ K^{-1}$, that is, we get a weaker version of the Ehrenfest condition. The stronger version follows from the equality $\langle H\rangle_K = \langle H\rangle \circ K$ (see [16]). □

The time evolution of a quantum observable $A \in B(\mathcal{M})$ is governed by the Heisenberg equation

$$\frac{d}{dt}A = i[A, H]. \tag{4.20}$$

For $H \in B_E(\mathcal{M})$, in accordance with point (ii) of Proposition 4, after performing the mean value operation $\langle \cdot \rangle_K$ on (4.20) one gets the evolution for the classical observable $\langle A\rangle_K$, which is a counterpart of the quantum observable A, i.e.

$$\frac{d}{dt}\langle A\rangle_K = \{\langle A\rangle_K, \langle H\rangle_K\}. \tag{4.21}$$

This additionally justifies naming (4.2) the "Ehrenfest condition."

Theorem 5. (i) *$(iB_E(\mathcal{M}), [\cdot, \cdot])$ is a Lie subalgebra of $(B(\mathcal{M}), [\cdot, \cdot])$;*
(ii) *$(C^\infty_E(M, \mathbb{R}), \{\cdot, \cdot\})$ is a Lie subalgebra of $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$;*
(iii) *$i\langle \cdot \rangle_K : iB_E(\mathcal{M}) \to C^\infty_E(M, \mathbb{R})$ is an isomorphism of Lie algebras.*

Proof. From point (ii) of Proposition 4 it follows that $iB_E(\mathcal{M})$ is an real-vector subspace of $B(\mathcal{M})$ and that

$$\begin{aligned} i\langle[[H_1,H_2],A]\rangle_K &= i\langle[[H_1,A],H_2]\rangle_K + i\langle[H_1,[H_2,A]]\rangle_K \\ &= \{\langle[H_1,A]\rangle_K,\langle H_2\rangle_K\} + \{\langle H_1\rangle_K,\langle[H_2,A]\rangle_K\} \\ &= -i\{\{\langle H_1\rangle_K,\langle A\rangle_K\},\langle H_2\rangle_K\} - i\{\langle H_1\rangle_K,\{\langle H_2\rangle_K,\langle A\rangle_K\}\} \\ &= -i\{\{\langle H_1\rangle_K,\langle H_2\rangle_K\},\langle A\rangle_K\} = \{\langle[H_1,H_2]\rangle_K,\langle A\rangle_K\} \end{aligned}$$

for $H_1,H_2\in B_E(\mathcal{M})$ and for any $A\in B(\mathcal{M})$. From the above it follows that $iB_E(\mathcal{M})$ is closed with respect to the commutation. This completes the proof of point (i).

Because of Lemma 1 and formula (4.4) the map $i\langle\cdot\rangle_K$ is an isomorphism of vector spaces $iB_E(\mathcal{M})$ and $C_E^\infty(M,\mathbb{R})$. Putting $H=H_1$, $A=H_2\in B_E(\mathcal{M})$ in (ii) of Proposition 4 one notices that $i\langle\cdot\rangle_K$ is also an isomorphism of the Lie algebras, which proves (ii) and (iii). □

Let $Q: C_E^\infty(M,\mathbb{R})\to B_E(\mathcal{M})$ be the inverse of $i\langle\cdot\rangle_K$. Hence, Q could be treated as a quantization procedure applied to classical observables belonging to $C_E^\infty(M,\mathbb{R})$.

Let us discuss the relation of the Q-quantization to the Kostant-Souriau quantization procedure. Let $L^2\Gamma(M,\bar{\mathbb{L}}^*)$, where $\mathbb{L}:=K^*\mathbb{E}$, denote the vector space of sections of $\bar{\mathbb{L}}^*$ square-integrable with respect to the Liouville measure. This means that $\psi\in L^2\Gamma(M,\bar{\mathbb{L}}^*)$ iff

$$\langle\psi|\psi\rangle_{\mathbb{L}} := \int_M \sum_\alpha h_\alpha(q)\frac{\overline{\psi_\alpha(q)}\psi_\alpha(q)}{K_{\bar\alpha\alpha}(q,q)}\varrho_0 d\mu_L(q) < \infty\,, \tag{4.22}$$

where

$$\psi=\psi_\alpha(q)\bar{s}_\alpha^*(q) \quad \text{for} \quad q\in\Omega_\alpha\,.$$

According to Sect. 2, we have the following natural monomorphism: $I:\mathcal{M}\to L^2\Gamma(M,\bar{\mathbb{L}}^*)$ of vector spaces with I defined as follows:

$$I(v) := \langle K_\alpha(q)|v\rangle\bar{s}_\alpha^*(q)\,. \tag{4.23}$$

Because of condition (3.9) one has $\langle I(v)|I(v)\rangle_{\mathbb{L}}=\langle v|v\rangle$, i.e. I is also a monomorphism of Hilbert spaces.

Proposition 6. *The Hilbert space* $I(\mathcal{M})$ *is preserved by the Kostant-Souriau prequantization operator* $\nabla_{X_h}+ih$ *and*

$$Q(h)=I^{-1}\circ(\nabla_{X_h}+ih)\circ I \tag{4.24}$$

for $h\in C_E^\infty(M,\mathbb{R})$.

Proof. Here we adopt the notation of (4.3). We have

$$\begin{aligned} \langle(\imath\circ K')(\xi)|U(t)v\rangle &= \langle U(-t)\circ\imath\circ K'(\xi)|v\rangle \\ &= \langle\imath\circ U'(-t)\circ K'(\xi)|v\rangle = \langle(\imath\circ K'\circ\sigma'(-t))(\xi)|v\rangle\,, \end{aligned} \tag{4.25}$$

where $\xi\in\mathbb{L}'$ and $v\in\mathcal{M}$. Differentiation with respect to t at $t=0$ results in

$$\mathcal{L}_{X_{h'}}(\langle\imath\circ K'(\cdot)|v\rangle) = \langle\imath\circ K'(\cdot)|iQ(h)v\rangle\,, \tag{4.26}$$

where $X_{h'}$ stands for the velocity field of the flow σ_h'. Simultaneously, one has the following natural isomorphism

$$\Phi(\langle\imath\circ K'(\cdot)|v\rangle) := \langle K_\alpha(\cdot)|v\rangle\bar{s}_\alpha^*$$

between the vector spaces $\{\langle \imath \circ K'(\cdot)|v\rangle : v \in \mathcal{M}\}$ and $I(\mathcal{M})$. Due to (4.26), $\{\langle \imath \circ K'(\cdot)|v\rangle : v \in \mathcal{M}\}$ is a module over the Lie algebra $(C_E^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$. Standard considerations (see [9, 16]) lead to the conclusion

$$\Phi \circ \mathcal{L}_{X_h} \circ \Phi^{-1} = \nabla_{X_h} + ih . \tag{4.27}$$

Applying the isomorphism Φ to the equality (4.26) and taking into account (4.27) one arrives at (4.24). The invariance of $I(\mathcal{M})$ under $\nabla_{X_h} + ih$ follows from the corresponding invariance of $\Phi^{-1}(I(\mathcal{M}))$ under $\mathcal{L}_{X_h}$. □

We have shown that quantization $Q: C_E^\infty(M,\mathbb{R}) \to B_E(\mathcal{M})$ is equivalent to the Kostant-Souriau quantization procedure provided that we realize the Hilbert space $\mathcal{M}$ as the subspace $I(\mathcal{M})$ in the Hilbert space $L^2\Gamma(M,\mathbb{L}^*)$.

Any bounded operator $B \in B(\mathcal{M})$ can be expressed in the $I(\mathcal{M})$ representation as the integral operator

$$(I \circ B \circ I^{-1}\psi)_{\bar{\alpha}}(q) = \int_M \sum_\alpha h_\alpha(p) B_{\bar{\alpha}\gamma}(q,p) \frac{\psi_{\bar{\gamma}}(p)}{\langle K_\gamma(p)|K_\gamma(p)\rangle} d\mu_L(p) \tag{4.28}$$

with the kernel

$$B_{\bar{\alpha}\gamma}(q,p) := \langle K_\alpha(q)|BK_\gamma(p)\rangle , \tag{4.29}$$

where $q \in \Omega_\alpha$ and $p \in \Omega_\gamma$, while $\psi_\alpha : \Omega_\alpha \to \mathbb{C}$ denotes the coordinate of the section

$$\psi = \psi_\alpha(q)\bar{s}_\alpha^*(q) \in I(\mathcal{M}) .$$

Functions $B_{\bar{\alpha}\gamma} : \Omega_\alpha \times \Omega_\gamma \to \mathbb{C}$ as well as $K_{\bar{\alpha}\gamma}$ might be considered as coordinates of a certain smooth section of the bundle $\mathrm{pr}_1^*\bar{\mathbb{L}}^* \otimes \mathrm{pr}_2^*\mathbb{L}^*$ taken in the frame $\mathrm{pr}_1^*\bar{s}_\alpha^* \otimes \mathrm{pr}_2^*\bar{s}_\gamma^*$. Hence

$$\check{B}(p,q) := \frac{B_{\bar{\alpha}\gamma}(q,p)}{K_{\bar{\alpha}\gamma}(q,p)} \tag{4.30}$$

is a function on $M \times M$ which, when restricted to diagonal Δ in $M \times M$ gives the mean value function of B, i.e. $\check{B}(q,q) = \langle B\rangle_K(q)$.

Formulas (4.29) and (4.30) provide a one-to-one linear map of $B(\mathcal{M})$ into $C^\infty(M \times M, \mathbb{C})$. This allows to describe the quantum observables via smooth functions defined on the product $M \times M$ of the classical phase spaces. If it happens that $B \in B_E(\mathcal{M})$, then due to Proposition 6 one infers that in order to recover the function $\check{B}: M \times M \to \mathbb{C}$ it is enough to know its restriction $B|_\Delta = \langle B\rangle_K$ to the diagonal Δ. Besides the properties listed in Proposition 2, Theorem 5, and Proposition 6, this is one more characteristic of $B_E(\mathcal{M})$.

We shall end this section by calculating the transition amplitude $\mathcal{A}_{\bar{\alpha}\beta}(q;p;h=E)$ in the case $h = \langle H\rangle$. Reversing the consideration which led to (3.22), we obtain

$$\begin{aligned}
\mathcal{A}_{\bar{\alpha}\beta}(q;p;h=E) = e^{-iE(t_f-t_i)} \lim_{N\to\infty} & \left(\frac{\langle K_\alpha^{1/2}(q)|K_\alpha(q)\rangle}{\langle K_\beta(p)|K_\beta(p)\rangle} \right) \\
& \times \int_M \sum_{\delta_2} h_{\delta_2}(\gamma(\tau_2))\varrho_0 d\mu_L(\gamma(\tau_2)) \ldots \\
& \ldots \int_M \sum_{\delta_{N-1}} h_{\delta_{N-1}}(\gamma(\tau_{N-1}))\varrho_0 d\mu_L(\gamma(\tau_{N-1})) \\
& \times \exp\left(\sum_{k=1}^{N-1} \left\{ -i \frac{\langle K_{\alpha_k}(\gamma(t_k))|\Delta K(\gamma(t_k))\rangle}{\langle K_{\alpha_k}(\gamma(t_k))|K_{\alpha_k}(\gamma(t_k))\rangle} - \langle H\rangle(\gamma(t_k))\Delta t \right\} \right) \\
= e^{-iE(t_f-t_i)} & \frac{\langle K_\alpha(q)|e^{-i(t_f-t_i)H}K_\beta(p)\rangle}{\langle K_\alpha(q)|K_\alpha(q)^{1/2}\langle K_\beta(p)|K_\beta(p)\rangle^{1/2}} .
\end{aligned} \tag{4.31}$$

This is the Schrödinger equation propagator in the coherent state representation (see [15]).

5. Examples of Mechanical Systems

Now we shall illustrate the proposed model of the mechanical system with several examples which are important from the physical point of view. These are describing the Schrödinger finite-dimensional mechanics, n-dimensional isotropic harmonic oscillator and scalar massive relativistic particle. The last case was also presented in [20].

The efficiency of our model in the case of the Kepler system has been verified in [11].

In Example 2 and Example 3 we will apply the Ehrenfest condition to unbounded operators, which shows that it can be understood in a broader sense than we assumed in Sect. 4.

Example 1. Schrödinger Finite-Dimensional Mechanics. The mechanical system is defined by $M=\mathbb{C}\mathbb{P}(N)$, $\mathcal{M}=\mathbb{C}^{N+1}$ while $K=\mathrm{id}$. We shall express the physical quantities of interest in a chart (Ω_1, ϕ_1) and in a frame $s_1(z^1, \ldots, z^N)=(1, z^1, \ldots, z^N)$, where we have put $z_1^k=z^k$ (see Example 2 of Sect. 2). Thus, in accordance with (2.39) the symplectic form $\omega=\omega_{\mathrm{FS}}$ is given by

$$\omega=i\bar{\partial}\partial\log(1+z^+z), \tag{5.1}$$

where $z^+z=\bar{z}^1z^1+\ldots+\bar{z}^Nz^N$ and the Liouville measure is

$$d\mu_L(z)=\frac{|dz|}{(1+z^+z)^{N+1}}. \tag{5.2}$$

The transition amplitude between coherent states $z, w\in\mathbb{C}\mathbb{P}(N)$ in the frame $s_1:\Omega_1\to\mathbb{E}$ is of the form

$$A(\bar{z}, w)=\frac{1+z^+w}{(1+z^+z)^{1/2}(1+w^+w)^{1/2}}. \tag{5.3}$$

The projecting operator $P(z)$ on the coherent state $z\in\mathbb{C}\mathbb{P}(N)$ takes the form

$$P(z)=\frac{\begin{bmatrix}1\\ z\end{bmatrix}[1\ \ z^+]}{1+z^+z}. \tag{5.4}$$

Let now $A\in\mathrm{Mat}_{(N+1)\times(N+1)}(\mathbb{C})$ and $\mathbb{I}$ be the unit matrix. Then one has the following useful formula:

$$\int\limits_{\mathbb{C}\mathbb{P}(N)} P(z)\langle A\rangle(z)d\mu_L(z)=\pi^N\frac{1}{(N+1)!}(A+\mathrm{Tr}A\mathbb{I}). \tag{5.5}$$

Equation (5.5) could be checked by a straightforward calculation. As a corollary of (5.5) we get the unit operator decomposition

$$\int\limits_{\mathbb{C}\mathbb{P}(N)} P(z)\pi^{-N}\frac{1}{(N+1)!}d\mu_L(z)=\mathbb{I} \tag{5.6}$$

equivalent to the composition rule for transition amplitudes (5.3) with respect to the measure $\pi^{-N}\frac{1}{(N+1)!}d\mu_L$.

Let us consider now the following function:

$$\varrho=\varrho(\langle H\rangle;\beta_1,\beta_2):=\beta_1\langle H\rangle+\beta_2\,\mathrm{Tr}H\,, \tag{5.7}$$

where $H=H^+\in\mathrm{Mat}_{(N+1)\times(N+1)}(\mathbb{C})$ and $\beta_1,\beta_2\in\mathbb{R}$. The family of this function with H fixed, satisfies the conditions a)–d) of Definition 1 of Sect. 3, i.e. it consists of the equilibrium states of the system. In particular, the transform (3.7) maps $\varrho(\langle H\rangle;\beta_1,\beta_2)$ on $\varrho(\langle H\rangle;\tilde{\beta}_1,\tilde{\beta}_2)$, where $\tilde{\beta}_1=\frac{1}{N+2}\beta_1$ and $\tilde{\beta}_2=\frac{1}{N+2}\beta_1+\beta_2$. Equation (3.10) is also satisfied if one puts $\beta_1=0$ and $\beta_2=\frac{1}{\mathrm{Tr}H}\pi^{-N}(N+1)!$ (we assume that $\mathrm{Tr}H\neq 0$).

Assuming that the system is in the equilibrium state as described by (5.7) and applying the procedure discussed in Sect. 3 we get, with the help of (3.26) the expression for the action of the system:

$$S_{\mathrm{id},\langle H\rangle}[z(\cdot)]=\int_t^{t_f}\left[\mathrm{Im}\frac{1+z^+(t)\frac{dz}{dt}(t)}{1+z^+(t)z(t)}-\langle H\rangle(z(t))\right]dt\,. \tag{5.8}$$

The equation of motion obtained from $\delta S_{\mathrm{id},\langle H\rangle}=0$ is of the form (2.44), which after being solved results in the flow $[U(t)]=[\exp(itH)]$.

As the result, the family $P(\varrho(\langle H\rangle;\beta_1,\beta_2))$ of equilibrium states is assigned to every Hermitian operator H. After applying the procedure of Sect. 3 it provides us with the Schrödinger time-evolution of the system. Since $K=\mathrm{id}$, space $C_E^\infty(M,\mathbb{R})$ consists of functions $\langle H\rangle$, where $H^+=H\in B(\mathcal{M})$, while $B_E(\mathcal{M})$ is given by Hermitian operators from $B(\mathcal{M})$. The classical and the quantum description of the system are essentially identical.

Example 2. The Isotropic Harmonic Oscillator. Take $M=\mathbb{C}^n$. The Hilbert space $\mathcal{M}_\kappa$ will consist of those holomorphic functions $\psi\in\mathcal{O}(\mathbb{C}^n)$ which are square-integrable with respect to the Gauss measure $d\mu_{G,\kappa}=\exp\left(-\frac{1}{2\kappa}z^+z\right)d^{2n}z$, with the L^2 scalar product.

It is easy to check that the holomorphic functions of argument $w\in\mathbb{C}^n$

$$K_\kappa(\bar{z},w):=\frac{1}{(2\pi\kappa)^n}e^{\frac{1}{2\kappa}z^+w} \tag{5.9}$$

parameterized by $z\in\mathbb{C}^n$ belong to $\mathcal{M}$. Thus we get the map

$$\mathbb{C}^n\ni z\to K_\kappa(z):=K_\kappa(\bar{z},\cdot)\in\mathcal{M}_\kappa \tag{5.10}$$

which defines an embedding $K_\kappa=[K_\kappa]$ of $\mathbb{C}^n$ in $\mathbb{CP}(\mathcal{M}_\kappa)$. As the reproducing property $\langle K_\kappa(z),K_\kappa(w)\rangle=K_\kappa(\bar{w},z)$ is satisfied, we find that

$$\omega=K^*\omega_{\mathrm{FS}}=i\bar{\partial}\partial\log K(z,z)=\frac{1}{2\kappa i}\sum_{k=1}^n dz^k\wedge d\bar{z}^k\,, \tag{5.11}$$

and hence

$$d\mu_L=\Lambda^n\omega=\frac{n!}{\kappa^n}d^{2n}z\,.$$

The transition amplitude between the states $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$ takes the form

$$A(\bar{z}, w) = e^{\frac{1}{2\kappa}(z^+ w - \frac{1}{2} z^+ z - \frac{1}{2} w^+ w)}. \tag{5.12}$$

The family of equilibrium states $P(\varrho)$ is defined by

$$\varrho = \varphi(h(z); \beta) = \frac{\beta^n}{(2\pi)^n n!} e^{-\frac{\beta}{2\kappa} z^+ z}, \tag{5.13}$$

where $h(z) = \frac{1}{2\kappa} z^+ z$ and $\beta \in \mathbb{R}_+$. The simple calculation shows that

$$\langle P(\varphi(h(z); \beta)) \rangle = \varphi\left(h(z); \frac{\beta}{1+\beta}\right) \tag{5.14}$$

and

$$P(\varphi(h(z); 1)) = \mathbb{I}. \tag{5.15}$$

Now, interpreting $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ as position and momentum of the system in the equilibrium state $P(\varphi(h; \beta))$, we see that $(\mathbb{C}^n, \mathcal{M}_\kappa, K_\kappa : \mathbb{C}^n \to \mathbb{CP}(\mathcal{M}_\kappa))$ describe the n-dimensional isotropic harmonic oscillator. Map (5.10) defines the principal bundles morphism $K'_\kappa : \mathbb{L}' = \mathbb{C}^n \times \mathbb{C}^* \to \mathbb{E}'$ given by

$$K'(z, \xi) = \iota^{-1}(\xi K_\kappa(z)). \tag{5.16}$$

The composition rule

$$(e^{is_1}, \alpha_1) \circ (e^{is_2}, \alpha_2) := \left(e^{i\left[s_1 + s_2 - \frac{i}{2}(\alpha_1^+ \alpha_2 - \alpha_2^+ \alpha_1)\right]}, \alpha_1 + \alpha_2\right)$$

for $(e^{is_1}, \alpha_1), (e^{is_2}, \alpha_2) \in \mathbb{S}^1 \times \mathbb{C}^n$, defines the Heisenberg-Weyl group W_n structure on $\mathbb{S}^1 \times \mathbb{C}^n$. One has the natural representations $\sigma : W_n \to \mathrm{Aut}\,\mathbb{L}'$ and $U : W_n \to \mathrm{Aut}\,\mathcal{M}_\kappa$ given by

$$\sigma(\alpha, s)(z, \xi) = (z + (2\kappa)^{1/2}\alpha, \xi e^{is - \frac{1}{2}\alpha^+ \alpha - (2\kappa)^{-1/2} z^+ \alpha}) \tag{5.17}$$

and

$$(U(\alpha, s)\psi)(z) = e^{is - \frac{1}{2}\alpha^+ \alpha + (2\kappa)^{-1/2}\alpha^+ w} \psi(z - (2\kappa)^{1/2}\alpha), \tag{5.18}$$

respectively. The bundles morphism K' is W_n-equivariant with respect to the above actions, therefore, the Ehrenfest condition is satisfied for the elements of the Heisenberg-Weyl algebra $\mathfrak{W}_n$. Thus $\mathfrak{W}_n \subset C_E^\infty(\mathbb{C}^n, \mathbb{R})$ and $Q : \mathfrak{W}_n \to \mathrm{End}\,\mathcal{M}_\kappa$ gives the Bargmann-Fock representation of the canonical commutation relations. The energy function $h(z) = \frac{1}{2\kappa} z^+ z$ also belongs to $C_E^\infty(\mathbb{C}^n, \mathbb{R})$ and

$$Q(h) = \sum_{k=1}^{n} z^k \frac{d}{dz^k} + \mathrm{const}.$$

This equivalence of the classical and the quantum dynamics for the harmonic oscillator was first mentioned by Schrödinger in [24].

Example 3. *The Scalar Massive Relativistic Particle.* In this example we treat the future tube $\mathbb{M}^{++} := \{x + iy : x \in \mathbb{R}^4 \text{ and } y \in C_+\}$ as a classical phase space M, where

$C_+=\{y\in\mathbb{R}^4\colon\ y^2=y^{0^2}-y^{1^2}-y^{2^2}-y^{3^2}>0$ and $y^0>0\}$. The Hilbert space is $\mathcal{M}_\lambda:=\{\varphi\in\mathcal{O}(\mathbb{M}^{++})\colon \langle\psi|\psi\rangle_\lambda<+\infty\}$ with

$$\langle\psi|\psi\rangle_\lambda:=2^{-2(\lambda+4)}\int_{\mathbb{M}^{++}}\bar{\psi}(z)\psi(z)\left[\left(\frac{z-z}{2i}\right)^2\right]^\lambda d^4y\,d^4x\,, \tag{5.19}$$

where $\lambda>-3$, as the scalar product. Let us define the embedding $K_\lambda\colon\mathbb{M}^{++}\to\mathcal{M}_\lambda$ by the formula

$$K_\lambda(z)=K_\lambda(\bar{z},\cdot):=\left[\left(\frac{\bar{z}-z}{2i}\right)^2\right]^{-\lambda-4}. \tag{5.20}$$

For $-3<\lambda\in\mathbb{Z}$ the map $K_\lambda:=[K_\lambda]$ is equivariant with respect to the conformal group $SU(2,2)/\mathbb{Z}_4$ which acts on $\mathbb{M}^{++}$ as the group of biholomorphisms and in $\mathcal{M}_\lambda$ by the representations from the discrete series (see [13]).

As in the two previous examples the reproducing property $\langle K_\lambda(z)|K_\lambda(w)\rangle = K_\lambda(\bar{w},z)$ is satisfied. Hence, after introducing the coordinates $\left(x^\mu, p^\mu=(\lambda+4)\frac{y^\mu}{y^2}\right)$, we obtain

$$\omega=K_\lambda^*\omega_{\mathrm{FS}}=-i(\lambda+4)\bar{\partial}\partial\log\left(\frac{\bar{z}-z}{2i}\right)^2=dx_\mu\wedge dy^\mu. \tag{5.21}$$

Let us now parameterize the family of equilibrium states with the following system of weight functions:

$$\varrho(z)=\varphi(\log y^2;\beta)\simeq e^{\beta\log y^2}. \tag{5.22}$$

Applying the transform (3.7) to (5.22) we get

$$\langle P(\varrho)\rangle_{K_\lambda}\simeq e^{\beta\log y^2}.$$

In the equilibrium state the conformal symmetry is broken down to the Poincaré one.

Applying the machinery described in this paper to the above data we uncover the mechanics of a scalar massive relativistic particle with p_μ as the 4-momentum and x^μ as the space-like position. For an exhaustive discussion of this model see [19, 20].

Acknowledgements. I want to express my thanks to Dr. A. K. Kwaśniewski for friendly discussions and constant support during my work. The attention of Drs. M. Horowski and W. Lisiecki is also appreciated. Last but not least let me express my deep gratitude to Professor K. Maurin for having been my teacher for so many years. I would also like to thank the referee for suggesting changes which made the paper more readable.

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Communicated by K. Gawedzki