# Degeneracy in interior point methods for linear programming: a survey*

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The publication of Karmarkar's paper has resulted in intense research activity into Interior Point Methods (IPMs) for linear programming. Degeneracy is present in most real-life problems and has always been an important issue in linear programming, especially in the Simplex method. Degeneracy is also an important issue in IPMs. However, the difficulties are different in the two methods. In this paper, we survey the various theoretical and practical issues related to degeneracy in IPMs for linear programming.

We survey results, which, for the most part, have already appeared in the literature. Roughly speaking, we shall deal with the effect of degeneracy on the following: the convergence of IPMs, the trajectories followed by the algorithms, numerical performance, and finding basic solutions.

**Keywords:** Linear programming, interior point methods, degeneracy, polynomial algorithms, global and local convergence, basis recovery, numerical performance, sensitivity analysis.

## 1. Introduction

Since Karmarkar [50] published his projective method for solving linear programs, world-wide interest in Interior Point Methods (IPMs) has increased enormously[3]. This resulted in several different classes of IPMs for linear programming. For an introduction and/or survey of IPMs we refer the reader to the excellent

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[3] In Kranich's [59] bibliography there are over 1200 papers listed on this topic.

survey of Gonzaga [36], the IPM section of the article by Goldfarb and Todd in [27], the Ph.D. thesis of Den Hertog [41], Kranich's bibliography [59], and Wright's paper [105]. The IPMs are not only theoretically sound, but several implementations have already shown that some IPMs can outperform the simplex method on large linear programs [12, 49, 61, 62]. The various methods can be divided into four main categories:

(1) projective methods (e.g. [25, 35, 50, 92, 113]);
(2) affine scaling methods (e.g. [8, 9, 17, 18, 33, 96, 99, 101]);
(3) path-following methods (e.g. [31, 36, 41, 44, 54, 77, 80, 84]);
(4) affine potential reduction methods (e.g. [30, 32, 47, 56, 109]).

It is well-known that degeneracy can cause cycling in the simplex method. This motivated many researchers to search for anti-cycling pivot rules. (See the survey of Terlaky and Zhang [90].) But even with such anti-cycling rules implemented, the presence of degeneracy may slow down the computational efficiency of the simplex method.

In this paper we discuss the role of degeneracy in IPMs. At first glance, degeneracy does not seem to be as serious a problem for IPMs as it is for simplex methods. Proofs of polynomiality for IPMs in the first, third and fourth category hold true without any non-degeneracy assumption. However, degeneracy plays a role in these methods when we consider local convergence (see section 3.2.). Degeneracy also plays a role in affine scaling methods (category two), which are believed not to be polynomial [69]. All the known global convergence proofs developed earlier needed some kind of non-degeneracy assumption. It is rather recent that these conditions are removed in a satisfactory way. We will review these results in section 3.1.

The search directions used in IPMs are usually linear combinations of two fundamental search directions, the so-called *affine scaling direction* and the *centering direction* [34, 42]. It is of interest to study the vector fields which are the infinitesimal (continuous) versions of these search directions since the IPMs can be considered as methods for (approximately) following these vector fields. Degeneracy has a strong influence on the shape of these vector fields as well as on their limiting behavior. We especially deal with cases where these continuous trajectories converge in the optimal face. For example, many IPMs follow, in some fashion, the central path which converges to the analytic center of the optimal face.

The numerical performance of the simplex method depends on the degree of degeneracy of the problem. This is seen in terms of the iteration count. Degeneracy also affects IPMs, but as we shall see, the problems and the difficulties are quite different and numerical problems rarely occur in practice [82].

While the simplex method iterates on the vertices of the feasible set, and the final solution is an optimal vertex, the IPMs generate an infinite sequence of points

in the interior of the feasible set. The algorithms stop if the duality gap is sufficiently small, and therefore IPMs never compute an exact optimal solution. From the viewpoint of complexity theory this is not an issue: IPMs produce a solution sufficiently close to an optimal solution, which can then be rounded to an optimal solution. For many practical applications, it suffices to find a point that is merely close to optimal. Sometimes, however, one would like to obtain an optimal basis, e.g. for cutting plane methods in integer programming. It is also important, in practice, that basic solutions have a minimal number of nonzero coordinates (e.g., a manager wants to produce a few products). Degeneracy appears to cause some difficulties in these areas.

All of these aspects will be addressed in this paper. The rest of the paper is organized as follows. In section 2 we introduce the notation and the definitions used in this paper. Some preliminary concepts are also reviewed. Section 3 deals with degeneracy and the global and local convergence of IPMs. In section 4 the influence of degeneracy on continuous and discrete trajectories is discussed. In section 5 we study the influence of degeneracy on the numerical behavior of IPMs. In section 6 we show that degeneracy can cause some problems in finding basic solutions. Theoretical and practical methods for finding an optimal basis are discussed. Finally, some concluding remarks are made in section 7.

## 2. Preliminaries

In this section we introduce the notation and the definitions used in the paper. We also discuss some preliminary concepts.

We first introduce the notation used in the paper. The vector $\mathbf{e}$ denotes the vector of ones and $I$ the identity matrix. Given an $n$-dimensional vector $\mathbf{x}$, we denote by $X$ the $n \times n$ diagonal matrix whose diagonal entries are the coordinates $x_j$ of $\mathbf{x}$; $\mathbf{x}^T$ is the transpose of the vector $\mathbf{x}$, and the same notation applies to matrices. Finally, $\|\cdot\|$ denotes the $l_2$ norm.

We consider the primal linear programming problem in the standard form

$$(\mathbf{P}) \qquad \min \{\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}.$$

Here $A$ is an $m \times n$ matrix, $\mathbf{b}$ is $m$- and $\mathbf{c}$ and $\mathbf{x}$ are $n$-dimensional vectors, respectively. The dual linear program for $(\mathbf{P})$ is

$$(\mathbf{D}) \qquad \max \{\mathbf{b}^T\mathbf{y} : A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq \mathbf{0}\}.$$

A vector $\mathbf{x}$ is called (primal) feasible for $(\mathbf{P})$ if $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}$ is non-negative. We say that $\mathbf{s}$ is (dual) feasible for $(\mathbf{D})$ if there exists a $\mathbf{y}$ such that $(\mathbf{y}, \mathbf{s})$ is feasible for $(\mathbf{D})$. A feasible point $\mathbf{x}$ (or $\mathbf{s}$) is said to be strictly feasible if it is positive. A strictly

feasible point is also called an interior point. We say that $(\mathbf{x}, \mathbf{s})$ is a (strictly) feasible pair for (**P**) and (**D**) if $\mathbf{x}$ is (strictly) feasible for (**P**) and $\mathbf{s}$ is (strictly) feasible for (**D**). A pair $(\mathbf{x}, \mathbf{s})$ is called complementary, if $X\mathbf{s} = \mathbf{0}$ or, equivalently, $\mathbf{x}^T\mathbf{s} = \mathbf{0}$.

We denote by $\mathcal{P} = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, and $\mathcal{D} = \{(\mathbf{y}, \mathbf{s}) : A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \mathbf{s} \geq \mathbf{0}\}$, the set of primal feasible variables and dual feasible variables, respectively; $\mathcal{P}^0$ and $\mathcal{D}^0$ denote the set of strictly primal feasible solutions and strictly dual feasible slacks, respectively. Finally, $\mathcal{P}_*$ and $\mathcal{D}_*$ denote the set of optimal solutions to (**P**) and (**D**), respectively. Note that $\mathcal{P}^0$ and $\mathcal{D}^0$ are the relative interiors of $\mathcal{P}$ and $\mathcal{D}$.

Because of its significance for IPMs, we point out the relationship between the existence of strictly feasible solutions and the boundedness of the level sets (including the optimal solution sets). Assuming that both the primal and the dual programs have feasible solutions, these two concepts are dually related: $\mathcal{P}^0 \neq \emptyset$ if and only if the dual level sets are bounded, and $\mathcal{D}^0 \neq \emptyset$ if and only if the primal level, sets are bounded, see [3] for example. Many IPMs either explicitly or implicitly make the assumption that there exists a strictly feasible pair $(\mathbf{x}, \mathbf{s})$, that is, $\mathcal{P}^0 \neq \emptyset$ and $\mathcal{D}^0 \neq \emptyset$. Both $\mathcal{P}_*$ and $\mathcal{D}_*$ are bounded under this assumption. For convenience, we also assume that $\text{rank}(A) = m$. (This is not a restrictive assumption as it is easy to eliminate the redundant constraints [63].)

The central path or central trajectory of the LP problems (**P**) and (**D**) plays an important role in most IPMs. In order to define the central path, one needs to introduce the logarithmic barrier problems associated with the pair (**P**)–(**D**), which are defined as

$$\min \left\{ \frac{\mathbf{c}^T\mathbf{x}}{\mu} - \sum_{j=1}^{n} \ln x_j : A\mathbf{x} = \mathbf{b} \right\}$$

and

$$\max \left\{ \frac{\mathbf{b}^T\mathbf{y}}{\mu} + \sum_{j=1}^{n} \ln s_j : A^T\mathbf{y} + \mathbf{s} = \mathbf{c} \right\}.$$

The common necessary and sufficient first order optimality conditions for these problems are:

$$\begin{cases} A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, & \mathbf{s} \geq 0, \\ A\mathbf{x} = \mathbf{b}, & \mathbf{x} \geq 0, \\ S\mathbf{x} = \mu\mathbf{e}. \end{cases} \tag{1}$$

Under the above assumptions this system has a unique solution [39, 64] denoted by $(\mathbf{x}(\mu), \mathbf{y}(\mu), \mathbf{s}(\mu))$. The primal and dual central path is defined as the solution set $\mathbf{x}(\mu)$ and $\mathbf{y}(\mu)$ respectively, for $\mu > 0$.

Recall that the problem pair (**P**) and (**D**) is called primal degenerate if there exists a primal feasible $\mathbf{x}$ with less than $m$ positive coordinates, and dual degenerate if there exists a dual feasible $\mathbf{s}$ with less than $n-m$ positive coordinates. The pair $(\mathbf{x},\mathbf{s})$ is called degenerate if it is primal or dual degenerate. A problem (**P**) ((**D**)) is called primal (dual) non-degenerate if it is not primal (dual) degenerate. We note that more restrictive non-degeneracy conditions are used to prove global convergence of some affine scaling algorithms. Sometimes degeneracy definitions apply only to optimal faces, see e.g. [96].

Another important result in linear programming is the existence of a strictly complementary optimal solution, that is, an optimal solution pair $(\mathbf{x}^*,\mathbf{s}^*)$ such that $\mathbf{x}^*+\mathbf{s}^*>0$. It has been known since the early days of linear programming [28] (see also [7, 81]) that such solutions exist in any linear program. It is also well-known that the indices of the positive coordinates are the same for all strictly complementary pairs. We denote by $B\subseteq\{1,2,\ldots,n\}$ the set of indices of the positive coordinates of $\mathbf{x}^*$. Similarly, $N$ denotes the set of indices of the positive coordinates of $\mathbf{s}^*$. We have $B\cup N=\{1,2,\ldots,n\}$ and $B\cap N=\emptyset$, so that $(B,N)$ is a partition of the column indices of $A$. We thus have a column partition $A=(A_B,A_N)$ of $A$.

It is then easy to see that the primal and dual optimal faces are given by

$$\mathcal{P}_*=\{\mathbf{x}: A_B\mathbf{x}_B=b,\mathbf{x}\geq 0,\mathbf{x}_N=\mathbf{0}\}$$

and

$$\mathcal{D}_*=\{(\mathbf{y},\mathbf{s}): \mathbf{s}\geq 0,\mathbf{s}_B=\mathbf{c}_B-A_B^{\mathrm{T}}\mathbf{y}=\mathbf{0},\mathbf{s}_N=\mathbf{c}_N-A_N^{\mathrm{T}}\mathbf{y}\}.$$

We denote the relative interior of $\mathcal{P}_*$ ($\mathcal{D}_*$) by $\mathcal{P}_*^0$ ($\mathcal{D}_*^0$) The importance of $\mathcal{P}_*^0$ and $\mathcal{D}_*^0$ for IPMs is due to the fact that the limit points of various continuous and discrete trajectories (for a definition see [76, p. 216]) for IPMs lie in these sets. Thus, the limiting behavior of these trajectories (how they approach the optimal facet) is interesting only in the degenerate cases.

Projection onto an affine space is a basic operation in IPMs. Given a matrix $A$, the projection matrix onto the null space of $A$ is the matrix

$$P_A=I-A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A.$$

All IPMs start from an initial strictly feasible point and generate strictly feasible solutions. Some algorithms are called primal and generate primal solutions in $\mathcal{P}^0$, e.g., Barnes [8], Dikin [17], Karmarkar [50], Vanderbei et al. [102]. Some are called dual algorithms and generate dual solutions in $\mathcal{D}^0$, e.g., Iri and Imai [47, 46], Adler et al. [1]. The more recent algorithms generate primal and dual solutions in $\mathcal{P}^0\times\mathcal{D}^0$, e.g., the primal–dual IPMs of Kojima et al. [56] and Ye [109]; the path-following methods of Monteiro and Adler [77], Kojima et al. [54] and Roos and

Vial [80]. We refer to any particular IPM simply by specifying its search directions, or giving its author(s).

The IPMs stop when the duality gap is sufficiently small (theoretically smaller than $2^{-2L}$, where $L$ is the length of the input data). In practice, different stopping criteria are used [6, 21, 51, 88, 89, 108]. In general, one has no convergence of the iteration sequence. Only the duality gap converges to zero, and this implies that all limit points of the generated sequence are optimal solutions.

## 3. Convergence of IPMs

In this section we discuss both global and local convergence of IPMs.

### 3.1. GLOBAL CONVERGENCE

As mentioned in the introduction, the methods in categories one, three and four are polynomial without any non-degeneracy assumptions. The best known complexity bound is $O(n^3L)$, with an $O(\sqrt{n}L)$ iteration bound [36, 41].

In the rest of this section we summarize the results concerning methods in category two. The affine scaling method had already been proposed by Dikin [17] in 1967. This simplest IPM is believed not to be polynomial, see Megiddo and Shub [69]. We now explain a version of the method for the primal problem (**P**); the dual case is analogous [17, 1].

Suppose $\mathbf{x}$ is the current iterate, then problem (**P**) is rescaled into

$$(\tilde{\mathbf{P}}) \qquad \min\{\tilde{\mathbf{c}}^{\mathrm{T}}\tilde{\mathbf{x}} : \tilde{A}\tilde{\mathbf{x}} = \mathbf{b}, \tilde{\mathbf{x}} \geq \mathbf{0}\},$$

where $\tilde{A} = AX$, $\tilde{\mathbf{c}} = X\mathbf{c}$. We then replace the non-negativity constraints $\tilde{\mathbf{x}} \geq 0$ by the more restrictive "ball constraint"

$$\|\tilde{\mathbf{x}} - \mathbf{e}\| \leq \beta \leq 1,$$

which makes the problem easy. The solution is explicitly given by

$$\tilde{\mathbf{x}}(\beta) = \mathbf{e} - \beta \frac{P_{\tilde{A}}\tilde{\mathbf{c}}}{\|P_{\tilde{A}}\tilde{\mathbf{c}}\|}.$$

We then unscale to obtain the next iterate for (**P**)

$$\mathbf{x}(\beta) = \mathbf{x} - \beta \frac{XP_{AX}X\mathbf{c}}{\|P_{AX}X_{\mathbf{c}}\|}. \tag{2}$$

It is known [18] that $\mathbf{x}(\beta)$ is also strictly feasible if $\beta \leq 1$, except for that special case where $\mathbf{x}(\beta)$ happens to be on the optimal face with $\beta = 1$.

The dual estimates defined by

$$\mathbf{s}(\mathbf{x}) = X^{-1}P_{AX}X\mathbf{c}, \quad \mathbf{y}(\mathbf{x}) = (AX^2A^{\mathrm{T}})^{-1}AX^2\mathbf{c}$$

play a very important role in the analysis of the affine scaling algorithm. These quantities satisfy $A^{\mathrm{T}}\mathbf{y}(\mathbf{x}) + \mathbf{s}(\mathbf{x}) = \mathbf{c}$ but not necessarily $\mathbf{s}(\mathbf{x}) \geq 0$. The affine scaling algorithm is regarded as a primal-interior dual-exterior point algorithm in the sense that it generates the pair $(\mathbf{x}, \mathbf{s}(\mathbf{x}))$ of primal-interior-feasible solution and dual-feasible/infeasible solution (not necessarily feasible) at each iteration.

To prove global convergence of the algorithm, some non-degeneracy conditions are required in the earlier analyses. Since $\mathbf{x}$ satisfies the equation $A\mathbf{x} = \mathbf{b}$ and $(\mathbf{y}(\mathbf{x}), \mathbf{s}(\mathbf{x}))$ satisfies $A^{\mathrm{T}}\mathbf{y} + \mathbf{s} = \mathbf{c}$, some reasonable conditions for the solutions of these two equations make the analysis easier. We say that the LP problem is strongly primal non-degenerate if every solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{b}$ has at least $m$ nonzero coordinates. It is called strongly dual non-degenerate if every solution $\mathbf{s}$ to the linear system $A^{\mathrm{T}}\mathbf{y} + \mathbf{s} = \mathbf{c}$ has at least $n - m$ nonzero coordinates.

In the primal affine scaling algorithm, the point $\mathbf{x}$ is required to be positive which means that a non-degeneracy condition is needed only for points $\mathbf{x} \geq \mathbf{0}$. This leads to the (usual) primal non-degeneracy condition introduced in section 2. However, the strong dual non-degeneracy condition is needed since we do not have control over the signs of the coordinates of $\mathbf{s}(\mathbf{x})$. Thus, in the primal affine scaling algorithm, the primal non-degeneracy assumption and/or the strongly dual non-degeneracy assumption are relevant for the analysis.

The results of the different convergence proofs for affine scaling methods are summarized in table 1 and discussed below. The great advantage of assuming primal non-degeneracy is that convergence of $\mathbf{x}$ implies convergence of $\mathbf{s}(\mathbf{x})$, while the major advantage of assuming strongly dual non-degeneracy is that the limiting point is confined to a vertex. In order to obtain their global convergence results, Barnes [8] and Vanderbei et al. [102] require primal and strongly dual non-degeneracy. A similar result is presented by Chandru and Kochar [14]. They require primal non-degeneracy and used a perturbation technique to avoid the dual non-degeneracy assumption. Dikin's proof [18] requires primal non-degeneracy, and Tsuchiya's first proof [95] requires a dual non-degeneracy condition that is satisfied if we assume the strong dual non-degeneracy condition. Thus, the symmetry between the non-degeneracy conditions breaks down in the convergence analysis.

Barnes [8] shows that, for fixed $\beta < 1$, the method converges if both (**P**) and (**D**) are non-degenerate. Vanderbei et al. [102] allow $\beta$ to be greater than 1, as long as all components $\tilde{x}_i(\beta)$ remain greater than $1 - \gamma > 0$. This corresponds to moving a fraction $\gamma$ of the distance to the boundary of $\mathcal{P}$. This is referred to in the literature as taking large steps.

Dikin [17] proves convergence for the unit step length ($\beta = 1$). Vanderbei and Lagarias [101] clarify Dikin's proof. The proof is extended to large steps by Gonzaga [33] under primal non-degeneracy.

Table 1
Convergence proofs for affine scaling algorithms.

| Author | Step length | Non-degeneracy assumptions |
|---|---|---|
| Dikin [17, 18] | $\beta = 1$ | primal |
| Barnes [8] | $\beta < 1$ | primal and strong dual |
| Vanderbei et al. [102] | large | primal and strong dual |
| Tseng and Luo [94] | $\beta = 2^{-L}$ | – |
| Tsuchiya [95] | $\beta = 1/8$ | strong dual |
| Tsuchiya [96] | $\beta = 1/8$ | – |
| Gonzaga [33] | large | primal |
| Dikin [20] | large 1/2 | – |
| Tsuchiya and Muramatsu [99] | large 2/3 | – |

Tseng and Luo [94] use ergodic convergence theory to show that the affine scaling method converges for all problems if a very small step length is taken (in fact $\beta = 2^{-L}$). Tsuchiya [95] proves convergence for $\beta = 1/8$ under strong dual non-degeneracy, and later in [96] without any non-degeneracy assumptions. We explain here the main ideas of the proof in [95], since the recent long-step convergence proofs without any non-degeneracy assumptions [20, 99] are also based on this approach.

The proofs of global convergence of the affine scaling methods under primal non-degeneracy [18, 8, 102, 33] are based on the fact that the convergence of the iterates immediately implies the convergence of the dual estimates. If the dual estimate is positive, it is a feasible solution to (**D**). The primal solution and the dual estimate satisfy the complementarity condition, but the dual estimate is not necessarily non-negative. Further, the sign of the dual estimate becomes exactly opposite to the sign of the displacement vector of the iterate. It follows from these facts that all components of the dual estimate have to be positive asymptotically. Hence, the main part of the proofs is to show that the iterates converge.

This proof technique breaks down when the primal non-degeneracy assumption is removed. This is because the convergence of the primal iterates does not necessarily imply the convergence of the dual estimates if the primal iterates converge to a point on a primal degenerate face. Thus, the problem is to analyze the behavior of the algorithm in the vicinity of a primal degenerate face. To overcome this difficulty, Tsuchiya's proof [96, 98] (without primal non-degeneracy conditions) uses the observation that the (local) structure of primal degenerate faces is similar to that of homogeneous LP problems. This suggests that the behavior of the affine scaling algorithm near degenerate faces is similar to its behavior when applied to homogeneous LP problems. It is known [11] that the affine scaling algorithm applied to homogeneous LP problems is precisely Karmarkar's algorithm. Thus, we can apply Karmarkar's analysis to study the behavior of the algorithm near degenerate faces. By introducing a local Karmarkar potential function, global convergence can be proved with step size $\beta = 1/8$.

As was mentioned above, this bound was further improved recently. Using his very interesting result on the reduction of Karmarkar's potential function [19], Dikin [20] proved the global convergence of the iterates to an interior point of the optimal face, and the global convergence of the dual estimates to the analytic center of the dual optimal face with large step size (1/2 to the boundary). The best result in this field is due to Tsuchiya and Muramatsu [99]. Motivated by Dikin's [19] work, independently from Dikin's last result they got better results by allowing larger step size (2/3 to the boundary) and they also showed that the asymptotic reduction rate of the objective function value is 1/3. On the basis of Tsuchiya's conjecture, Hall and Vanderbei [40] recently constructed an interesting example to show that the dual sequence cannot be convergent any more if we take any (fixed) step size greater than 2/3. Thus 2/3 is shown to be longest step size for the affine scaling algorithm that guarantees convergence of the primal–dual pair.

All the methods discussed above are either primal or dual methods. There also exists a primal–dual version of the affine scaling method. Taking very small steps, $\beta = (nL \ln n)^{-1}$, Monteiro et al. [78] prove that this method converges without any non-degeneracy assumptions. Their search direction is given by formulae (5) below. In fact, they prove that this primal–dual algorithm is a short-step primal–dual path-following algorithm.

The large-step version of the primal–dual affine scaling algorithms has been implemented in several codes. This method shows good practical behavior even if the problem is primal degenerate. In theory, whether the large step version is

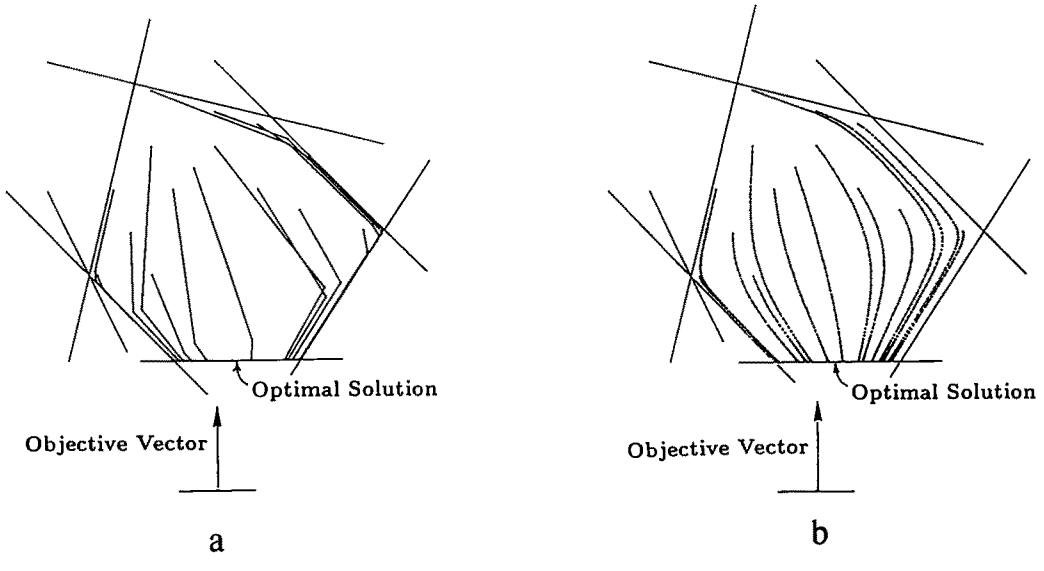


Fig. 1. The iterations of the affine scaling algorithm. The optimal face is a line segment, which is dual degenerate (but primal non-degenerate). (a) The iterates generated by a long-step version where $\gamma = 0.9$. (b) The iterates generated by a short-step version where $\beta = 1/8$. With this step size the global convergence is ensured without requiring any non-degeneracy conditions.

convergent even for both primal and dual non-degenerate problems is still an open question at present.

Figure 1 illustrates some long- and short-step trajectories of the affine scaling algorithm. The optimal face is a line segment, which is dual degenerate (but primal non-degenerate).

### 3.2. LOCAL CONVERGENCE

One may claim that the local convergence properties for IPMs are not very important, as there exist finite termination procedures to obtain an optimal solution from a near optimal solution (see section 6). Since the practical efficiency of these procedures is not fully established, there are good reasons to study local convergence properties of IPMs.

Several recent papers [38, 71, 88, 98, 100, 105, 111, 114, 116] deal with the local convergence of primal–dual path-following and potential reduction methods. The search directions used in these methods are

$$\mathbf{p} = \left(S^{-1} - S^{-1}XA^{\mathrm{T}}\left(AS^{-1}XA^{\mathrm{T}}\right)^{-1}AS^{-1}\right)\left(X\mathbf{s} - \sigma\frac{\mathbf{x}^{\mathrm{T}}\mathbf{s}}{n}\mathbf{e}\right),$$

$$\mathbf{d} = -\left(AS^{-1}XA^{\mathrm{T}}\right)^{-1}AS^{-1}\left(X\mathbf{s} - \sigma\frac{\mathbf{x}^{\mathrm{T}}\mathbf{s}}{n}\mathbf{e}\right),$$

for the primal $\mathbf{x}$ space and the dual $\mathbf{y}$ space, respectively [42, 34]. These are in fact the Newton directions for the Karush–Kuhn–Tucker conditions for the logarithmic barrier problem. The choice of the centering parameter $\sigma$ and the step length are the fundamental issues here.

Zhang et al. [116] give conditions that these choices must satisfy in order to achieve superlinear or quadratic convergence. None of the existing polynomial algorithms satisfies these fast convergence requirements. In [116], a basic assumption for superlinear convergence is the convergence of the iteration sequence, and a basic assumption for quadratic convergence is the non-degeneracy (equivalently the uniqueness) of the primal and dual optimal solutions.

Ye et al. [114] study the "predictor–corrector" method of Mizuno et al. [74], which takes $\sigma = 1$ and $\sigma = 0$ alternately. They prove the superlinear convergence of the algorithm under the condition that the solution sequence converges. They also prove the quadratic convergence of the duality gap to zero, while maintaining the global $O(\sqrt{n}L)$ iteration bound, under the uniqueness of the primal and dual optimal solutions. Finally, Ye et al. [111] and Mehrotra [72] obtain the same results without any assumptions.

In a different development, Iri and Imai [47] prove that the iterates in their method converge quadratically if (**P**) is non-degenerate and exact line searches

are performed. The search direction used is the Newton direction of the multiplicative barrier function (4). Recently, Tsuchiya [97] has removed the non-degeneracy assumption and has shown that most of the generated sequences converge to vertices of the optimal face.

## 4. Continuous trajectories

The main purpose of studying the limiting behavior of IPMs is to investigate the properties of the resulting optimal solutions. This analysis also reveals the rich structure of the continuous trajectories. Here we review the asymptotic properties of the trajectories from these standpoints. We first discuss the primal (and dual) IPMs, and then the primal–dual IPMs.

### 4.1 CONTINUOUS TRAJECTORIES OF THE PRIMAL (AND DUAL) IPMs

It is now well-known [34, 42] that the search directions of the various primal IPMs for (**P**) can be written as linear combinations of the affine scaling search direction (see (2))

$$\mathbf{p}_{\mathrm{AFS}} = -XP_{AX}X\mathbf{c},$$

and the Newton direction for the problem $\max\{\sum_{i=1}^{n} \ln x_i : A\mathbf{x} = \mathbf{b}\}$, which is also called the centering direction [42, 34, 11, 106]

$$\mathbf{p}_{\mathrm{C}} = XP_{AX}\mathbf{e}.$$

We refer to the vector fields defined by $\mathbf{p}_{\mathrm{AFS}}$ and $\mathbf{p}_{\mathrm{C}}$ as the affine scaling vector field and the centering vector field, respectively.

After reviewing the asymptotic behavior of the trajectories of these two fundamental vector fields, we explain the asymptotic behavior of the continuous versions of some of the primal IPMs: the path-following algorithm [84, 79, 80, 29], Karmarkar's projective scaling algorithm [50], the Iri–Imai algorithm [47, 46], and Gonzaga's affine scaling potential reduction algorithm [30].

It has been observed by several authors (e.g., [69]) that both the affine scaling and the centering vector fields can be smoothly extended to the boundary of $\mathcal{P}$. The extended vector field on a face $\mathcal{U}$ is exactly the same as the vector field defined for the subproblem obtained by restricting the feasible region to $\mathcal{U}$. This property holds true even when (**P**) is degenerate.

The singular set of a vector field is the set of points where it vanishes, that is, where it is equal to zero. It is known [87, 96] that the singular set of the affine scaling vector field consists of the dual degenerate faces (including vertices) on which the objective function is constant. Moreover, the singular set of the centering vector field is known to be the analytic centers of the individual faces [69, 87, 97].

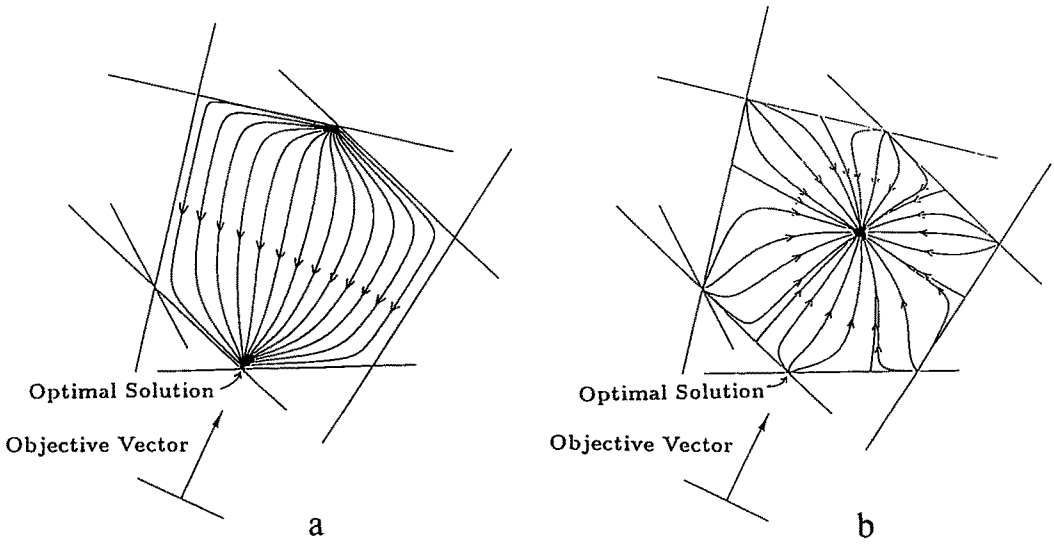


Fig. 2. The two fundamental vector fields associated with IPMs. (a) The affine scaling vector field in the dual non-degenerate case. (b) The centering vector field.

An illustration of the affine scaling and the centering vector fields is shown in fig. 2.

*4.1.1. Continuous trajectories of the affine scaling direction*

The trajectories of the affine scaling vector field are by definition the solutions to the system of ordinary differential equations (ODE)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -XP_{AX}X\mathbf{c}.$$

The limiting behavior of these trajectories is studied by Megiddo and Shub [69] under primal and strong dual non-degeneracy. In this case, the global convergence of the trajectories is easily deduced from the global convergence proof of the original discrete algorithm under the same assumptions [8, 102]. Megiddo and Shub focus attention on the behavior of the trajectories near the optimal vertex. They prove that all the trajectories share the same limiting direction, namely the limiting tangent direction of the central path of the problem.

It is not trivial to extend these global and local convergence results to the general case without making some non-degeneracy assumptions. To analyze the problem, Adler and Monteiro [3] (see also [76, 104]) consider the trajectories consisting of the minimizers $\mathbf{x}_p(\mu)$ of the following one-parameter convex programming problem

$$\min\left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} - \mu\left(\mathbf{p}^{\mathrm{T}}\mathbf{x} + \sum_{i=1}^{n} \ln x_i\right) : A\mathbf{x} = \mathbf{b}\right\}. \tag{3}$$

They show that for suitable $p$ the trajectories $\{\mathbf{x}_p(\mu) : \mu \geq 0\}$ are the same as the continuous trajectories of the affine scaling algorithm. Once an initial interior point is specified, the vector $\mathbf{p}$ and the initial value of $\mu$ are easily computed from the Karush–Kuhn–Tucker conditions $A\mathbf{x} = \mathbf{b}, \mathbf{x} > \mathbf{0}, \mathbf{c} - A^{\mathrm{T}}\mathbf{y} - \mu X^{-1}\mathbf{e} = \mu\mathbf{p}$. The limiting behavior of the trajectories of system (3) can be analyzed by letting $\mu \to 0$. Based on this idea, Adler and Monteiro [3, theorem 3.2] prove the following result: initiated at any interior point of the feasible region, the affine scaling continuous trajectory converges to an interior point of $\mathcal{P}_*$, which depends on the initial point. Further, the limiting tangent directions exist. The $N$-component of all the limiting directions is the same, but the $B$-component does depend on the initial points. It is interesting to note that $B$- and $N$-components of both the limiting points and the limiting directions of the trajectories exhibit quite different behavior.

It is worth noting that this result requires neither non-degeneracy assumptions nor the boundedness of $\mathcal{P}_*$. (In [3], Adler and Monteiro require boundedness of $\mathcal{P}_*$, and Monteiro removes this assumption in [76] to make use of their result in analyzing the limiting behavior of the continuous trajectory of Karmarkar's projective scaling algorithm.)

Some continuous affine scaling trajectories are shown in fig. 2(a) (dual non-degenerate case) and fig. 4(d) (dual degenerate case).

### *4.1.2. Continuous trajectories of the (negative) centering direction*

The trajectories of the negative centering direction are the solutions to the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -XP_{AX}\mathbf{e}.$$

It is known [4] that every trajectory of this ODE coincides with the central path of some LP problem. From the properties of the central trajectories (studied in detail because of its theoretical importance, e.g. [11]), we have the following result: every continuous trajectory of the negative centering vector field is well-defined when $t \to \infty$, and converges to the analytic center of one of the faces of $\mathcal{P}$. The vertices of $\mathcal{P}$ are the only stable limit points, the remaining limit points being unstable [87, 97].

It is worth commenting on how these negative centering trajectories approach their associated limiting points, see [87]. Let $\mathcal{U}$ be a face of $\mathcal{P}$, and let $\mathbf{x}_{\mathcal{U}}$ denote the components of a vector $\mathbf{x}$ which are always active on $\mathcal{U}$. Interestingly, $\mathbf{x}_{\mathcal{U}}$ behaves as if it were the vector field of the Newton method to find a point $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}_{\mathcal{U}} = \mathbf{0}$ in a sufficiently small neighborhood of $\mathcal{U}$. (Note that we say nothing about the behavior of the remaining components of the vector field.) This property even holds when

$\mathcal{U}$ is a degenerate face. In particular, if $\mathcal{U}$ is a vertex $\mathbf{x}^*$, then the negative centering vector at an interior point $\hat{\mathbf{x}}$ near $\mathbf{x}^*$ is approximated well by the displacement vector from $\hat{\mathbf{x}}$ to $\mathbf{x}^*$. Thus, the negative centering vector field has a property similar to that of the Newton direction. This is an interesting feature of the negative centering vector field, which is quite different from the limiting behavior of the continuous affine scaling trajectories.

Some continuous centering trajectories are shown in fig. 2(b).

### *4.1.3. Continuous trajectory of path-following algorithms*

Since path-following algorithms are homotopy methods which follow the central path [84, 79, 80, 29], their continuous trajectories coincide with the central path, which ends at the analytic center of the optimal face $\mathcal{P}_*$. It is easy to see that the affine scaling and the centering trajectories coincide with the central path if they are initiated at a point on the central path.

We show some central trajectories, with different objective functions, in fig. 3. The central path ends up in the analytic center of the optimal face in the degenerate case, and at the optimal vertex in the non-degenerate case [39, 64].

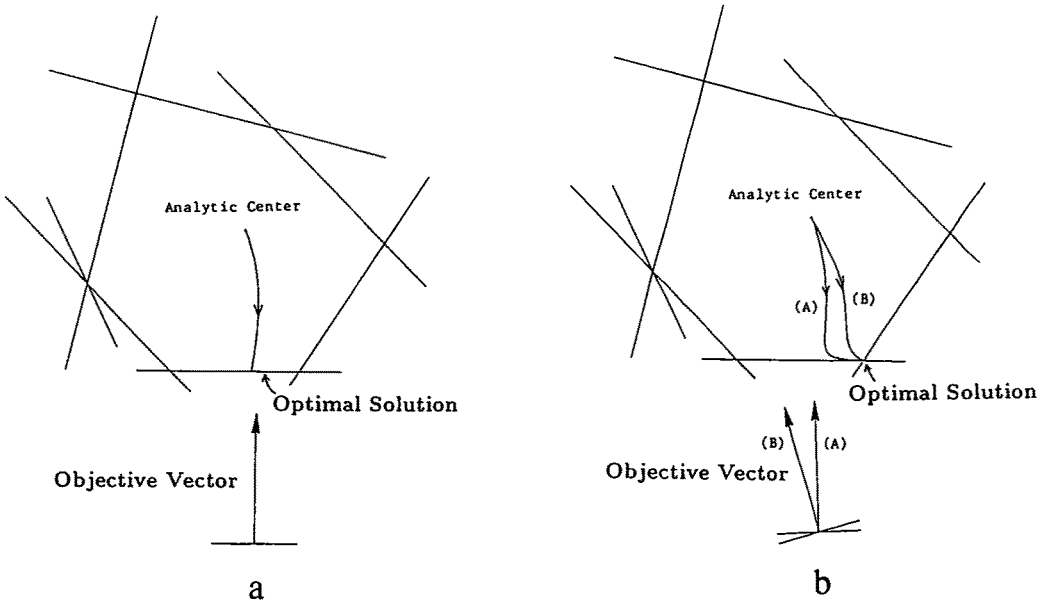


Fig. 3. Central trajectories, with different objective functions. The central path ends up in the analytic center of the optimal face. (a) The degenerate case. (b) The almost degenerate and non-degenerate cases. In the almost degenerate case the central path comes close to the boundary, which illustrates that the analytic center is not a geometric concept.

### 4.1.4. Continuous trajectories of Karmarkar's projective scaling algorithm

Bayer and Lagarias [11] prove that the iterates of the projective scaling algorithm [50] are obtained as the conical projections of the affine scaling iterates for the homogenized LP problem. Hence, the results on the limiting behavior of the affine scaling continuous trajectories can be used to analyze the limiting behavior of the projective scaling continuous trajectories. Using this idea, Monteiro [76, theorem 4.1] obtains the following result: the continuous trajectories of the projective scaling algorithm applied to LP problems in Karmarkar's canonical form converge to an interior point of $\mathcal{P}_*$. Further, the projective dual estimates defined by Todd and Burrell [92] converge to the analytic center of $\mathcal{D}_*$.

We note that analogous results hold true for the variants of the projective scaling algorithm for the general standard form problems, e.g., [25, 113], even without requiring the boundedness of the optimal face [98].

Before presenting the results on the Iri–Imai algorithm and Gonzaga's algorithm, we introduce the potential functions used in these methods. Let $z$ be the optimal value of (**P**). We assume below that $z$ is known in advance and that $\mathcal{P}_*$ is bounded. The potential function and its multiplicative version are

$$\begin{aligned} f_q(\mathbf{x}) &= q \ln (\mathbf{c}^{\mathrm{T}}\mathbf{x} - z) - \sum_{i=1}^{n} \ln x_i, \\ F_q(\mathbf{x}) &= \exp (f_q(\mathbf{x})) = \frac{(\mathbf{c}^{\mathrm{T}}\mathbf{x} - z)^q}{\prod_{i=1}^{n} x_i}, \end{aligned} \tag{4}$$

respectively, where $q$ is a non-negative parameter that depends on the algorithm used. Under some reasonable assumptions, the value of the potential function is shown to diverge to $-\infty$ only if $\mathbf{x}$ approaches $\mathcal{P}_*$. Hence, solving (**P**) reduces to minimizing the potential function.

### 4.1.5. Continuous trajectories of the Iri–Imai algorithm

Imai [45] proves that the potential function $F_q(\mathbf{x})$ above, which they call the multiplicative barrier function [47, 46], is strictly convex if $q \geq n + 1$, or if $q = n$ and $\mathcal{P}$ is bounded. They propose to solve (**P**) using the Newton method to minimize $F_q(\mathbf{x})$. Their search direction is also a linear combination of the affine scaling and the centering directions (see e.g. [42, 106]). Tsuchiya [97] analyzes the discrete trajectories of this algorithm in the vicinity of the optimal solution set, demonstrates its quadratic convergence, and observes that the search direction approaches the negative centering direction in the limit. Using the same type of analysis, he obtains the following result: the limiting point of a continuous trajectory of the Iri–Imai algorithm is the analytic center of one of the faces (including vertices) of $\mathcal{P}_*$. Further, the only stable limit points are the vertices, and the remaining limiting points are

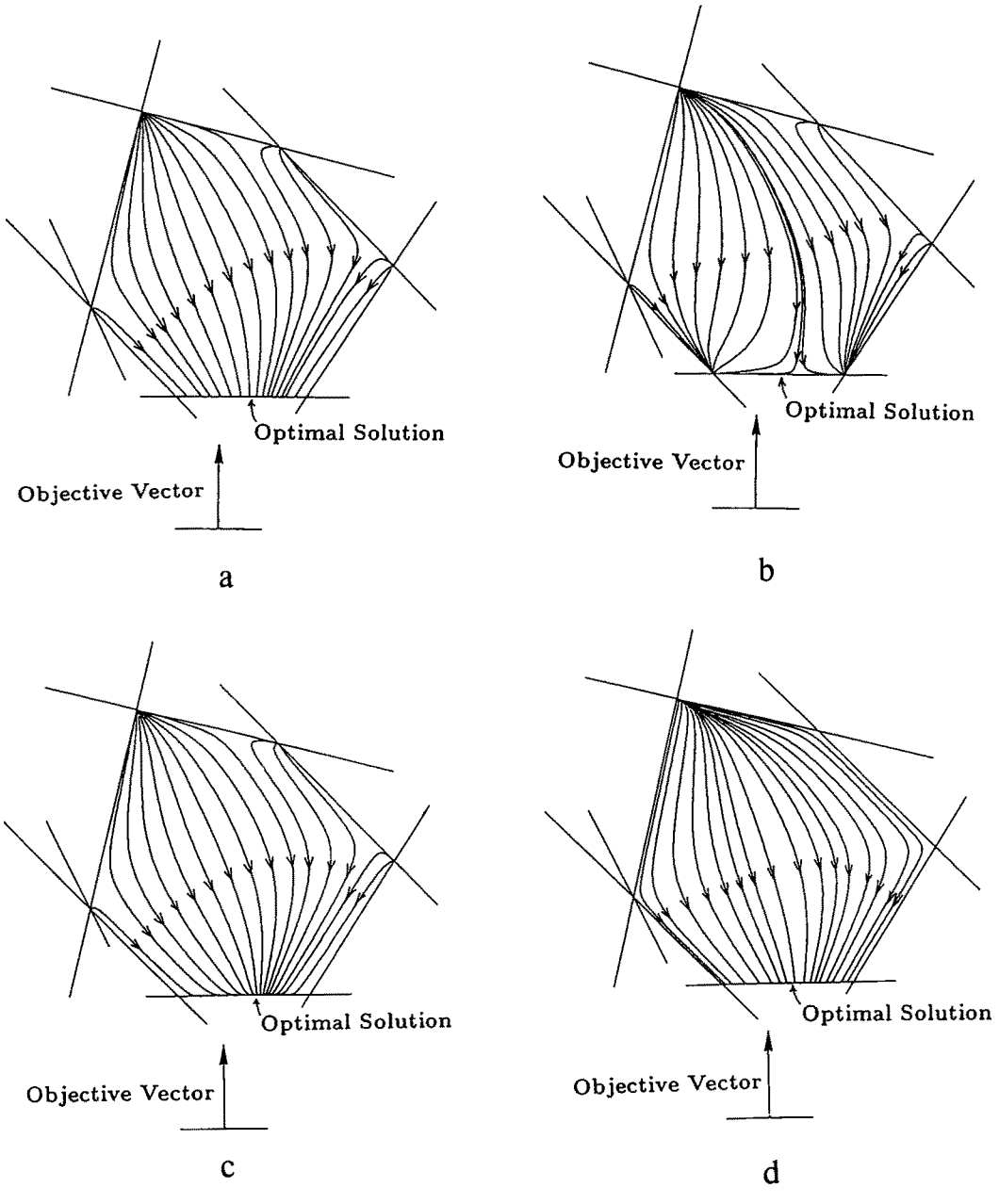


Fig. 4 Continuous trajectories for various IPMs. (a) The continuous projective scaling trajectories. Although the trajectories terminate in the relative interior of the optimal face, they do not approach the analytic center of this face. (b) The continuous trajectories of the Iri–Imai algorithm. Most of the trajectories are pulled towards one of the two vertices of the optimal face as they approach it. (c) The continuous trajectories of Gonzaga's affine scaling potential reduction algorithm. This figure illustrates the case where $q > \tau$. Every trajectory is seen to approach the analytic center of the optimal face from a direction tangential to the optimal face. (d) The continuous trajectories of the affine scaling method. The trajectories end up in the relative interior of the optimal face.

unstable. This is similar to the limiting behavior of the negative centering direction mentioned earlier.

The continuous trajectories of many IPMs end up in the relative interior of $\mathcal{P}_*$, whereas almost all the trajectories of the Iri–Imai method lead to vertices. This unusual property deserves further investigation.

#### 4.1.6. *Continuous trajectories of Gonzaga's affine scaling potential reduction algorithm*

Gonzaga [30] proposes a steepest descent method (with respect to the metric used in the affine scaling algorithm) for minimizing $f_q$ $(q \geq n)$, and proves its polynomiality. This steepest descent direction is a linear combination of the affine scaling and centering directions [42]. The limiting behavior of the continuous version of this algorithm is analyzed in detail by Monteiro [75]. He uses the idea of characterizing the trajectory as the set of minimizers of the one-parameter convex programming problems as in [3]. He proves the following interesting results under the assumption of bounded feasible region: the continuous trajectories of Gonzaga's affine scaling potential reduction algorithm converge to the analytic center of $\mathcal{P}_*$, and the limiting direction depends on the parameter $q$ of the potential function. There exists a threshold value $\tau$ such that if $q < \tau$, then the limiting direction of every trajectory coincides with the limiting direction of the central path. If $q > \tau$, the limiting direction of the trajectories is tangential to the optimal face $\mathcal{P}_*$. In the remaining case $(q = \tau)$, the limiting directions (which depend on the initial point) of the trajectories are not parallel to $\mathcal{P}_*$.

In fig. 4 some trajectories of the following methods can be found: projective (fig. 4(a)), Iri–Imai (fig. 4(b)), potential reduction (fig. 4(c)), and affine scaling (fig. 4(d)).

### 4.2. CONTINUOUS TRAJECTORIES OF PRIMAL–DUAL IPMs

As in the primal (or dual) IPMs, there are several standard primal–dual IPMs such as the primal–dual affine scaling algorithm, path-following algorithms, and the potential reduction algorithms. The search directions for these algorithms can be shown to be linear combinations of the search directions of the primal–dual affine scaling algorithm. A recent survey of search directions is given by Den Hertog and Roos [42]. Also see Gonzaga [34] for an earlier survey on search directions. The primal–dual affine scaling directions are

$$\mathbf{p}^*_{\mathrm{AFS}} = -DP_{AD}(XS)^{1/2}\mathbf{e}, \quad \mathbf{d}^*_{\mathrm{AFS}} = -D^{-1}(I - P_{AD})D(XS)^{1/2}\mathbf{e}, \tag{5}$$

where $D = X^{1/2}S^{-1/2}$, and the primal–dual centering directions are

$$\mathbf{p}^*_{\mathrm{C}} = -DP_{AD}(XS)^{-1/2}\mathbf{e}, \quad \mathbf{d}^*_{\mathrm{C}} = -D^{-1}(I - P_{AD})(XS)^{-1/2}\mathbf{e}.$$

While the vector fields associated with the primal IPMs have a rich structure near or on the boundary of the feasible region, as demonstrated above, the corresponding properties are not yet known for the primal–dual vector fields. This is an interesting topic for further research.

One of the important tools in analyzing the continuous trajectories of the primal–dual IPMs is the following map $T$ from $\mathcal{P}^0 \times \mathcal{D}^0$ to the interior of $\mathbb{R}^n_+$ given by

$$T(\mathbf{x}, \mathbf{s}) = XS\mathbf{e} = \mathbf{t}.$$

This map is studied by McLinden [64] in a more general context and is shown to be a homeomorphism (i.e., $T$ is one to one and onto, and $T$ and $T^{-1}$ are continuous) between the two sets. He also shows that the inverse map $T^{-1}$ is differentiable almost everywhere. The boundary behavior of mapping $T$ is studied for monotone complementarity problems in [65, 37].

*4.2.1. Continuous trajectories of the primal–dual affine scaling algorithm*

The primal–dual affine scaling algorithm can be regarded as the Newton method applied to the Karush–Kuhn–Tucker necessary conditions for (**P**) and (**D**). Hence, in the **t**-space introduced above, the trajectory is the line connecting the initial point and the origin. The limiting behavior of this trajectory is studied by McLinden [64], Megiddo [66], and Kojima et al. [53]. The continuous trajectory of the primal–dual affine scaling algorithm, initiated at the point $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$, converges to the weighted primal–dual analytic centers of the optimal faces $\mathcal{P}_*$ and $\mathcal{D}_*$, which are the unique minimizers of the weighted logarithmic barrier function,

$$\min_{\mathbf{x},\mathbf{s}} \left\{ -\sum_{i \in B} \hat{t}_i \ln x_i - \sum_{i \in N} \hat{t}_i \ln s_i \right\},$$

where $\hat{\mathbf{t}} = \hat{X}\hat{\mathbf{s}}$ and $(B, N)$ is the partition discussed in section 2.

*4.2.2. The primal–dual potential reduction algorithm*

The primal–dual potential function

$$\left(n + \sqrt{n}\right) \ln \mathbf{x}^{\mathrm{T}}\mathbf{s} - \sum_{i=1}^{n} \ln x_i s_i = \left(n + \sqrt{n}\right) \ln \mathbf{t}^{\mathrm{T}}\mathbf{e} - \sum_{i=1}^{n} \ln t_i$$

was introduced independently by Tanabe [85] and Todd and Ye [93]. The primal–dual potential reduction algorithm is the steepest descent method for the primal–dual potential function with respect to the affine scaling metric in the **t**-space, see [85, 56, 55].

It is shown [86, 55] that the continuous trajectories of this algorithm can be expressed explicitly in t-space. Using this result, we see that every trajectory approaches zero in the t-space from the direction **e**. Together with the result explained above, this implies the following: the continuous trajectories of the primal–dual potential reduction algorithm converge to the analytic centers of $\mathcal{P}_*$ and $\mathcal{D}_*$.

## 5. Effects of degeneracy on numerical performance

The main computational step in all IPMs is solving a linear equation system

$$AD^2A^{\mathrm{T}}\mathbf{u} = \mathbf{v}, \tag{6}$$

for some $\mathbf{v}$, where $D$ is a diagonal matrix with positive diagonal elements $d_i$, $i = 1, \ldots, n$. The matrix $D$ depends on the IPM. The primal methods usually use $D = X$, the dual methods $D = S^{-1}$, and the primal–dual methods use $D = X^{1/2}S^{-1/2}$.

It is shown in Güler and Ye [38] that if any algorithm for linear programming generates interior points $(\mathbf{x}^k, \mathbf{s}^k)$ satisfying the condition

$$\frac{\min(X^k\mathbf{s}^k)}{(\mathbf{x}^k)^{\mathrm{T}}\mathbf{s}^k} \geq \xi, \tag{7}$$

for some constant $\xi > 0$, then any limit point $(\mathbf{x}^*, \mathbf{s}^*)$ of the iterates $(\mathbf{x}^k, \mathbf{s}^k)$ is in $\mathcal{P}_*^0 \times \mathcal{D}_*^0$. In fact, relation (7) implies that there exists a constant $\gamma$, where $0 < \gamma < 1$, such that the relations

$$\gamma \leq x_j^k \leq 1/\gamma \quad \text{for } j \in B, \tag{8}$$

$$\gamma \leq s_j^k \leq 1/\gamma \quad \text{for } j \in N, \tag{9}$$

are satisfied for all $k \geq 0$. Here $(B, N)$ is the partition defined in section 2. To the best of our knowledge, except for the Iri and Imai method, methods like Karmarkar's original algorithm where the optimal value is assumed to be known, and the primal–dual potential reduction algorithms with exact line search, all the known polynomial IPMs satisfy condition (7) and hence conditions (8) and (9).

Note that $\mathbf{x}_N^k \to 0$ and $\mathbf{s}_B^k \to 0$ in any convergent IPM. Therefore, the limiting behavior of $A_B(D_B^k)^2A_B^{\mathrm{T}}$ determines the asymptotic behavior of the linear systems (6).

In the primal or dual affine scaling methods, it seems difficult to obtain detailed information about the asymptotic behavior of $D_B^k$ and hence to obtain information about the asymptotic rank of the matrices $A_B(D_B^k)^2A_B^{\mathrm{T}}$. Of course, in the primal affine scaling method of Dikin and its variants, if (**P**) is non-degenerate, then the matrix $A_B(D_B^k)^2A_B^{\mathrm{T}}$ converges to the matrix $A_B(X_B^*)^2A_B^{\mathrm{T}}$, which is

non-singular. If (**P**) is not near-degenerate, then this matrix is well-conditioned and numerical problems are absent. However, the algorithm might have trouble if the starting interior point $\mathbf{x}^0$ is very close to the boundary of the non-negative orthant. The same considerations apply to the dual affine scaling methods.

The situation is much clearer for path-following methods [30–32, 43, 52, 54, 79, 80, 77], and primal–dual potential-reduction algorithms [56, 93, 109]. As shown in [38], all these algorithms either explicitly generate a primal–dual solution sequence $(\mathbf{x}^k, \mathbf{s}^k)$ satisfying condition (7), or can generate such a sequence. It turns out that we can obtain more information about the matrices $D_B^k$ in this case, as we shall now explain.

Assume for the moment that $(D^k)^2 = X^k(S^k)^{-1}$. Then, for any $i$,

$$\frac{x_i}{s_i} = \frac{x_i^2}{\mathbf{x}^T\mathbf{s}} \cdot \frac{\mathbf{x}^T\mathbf{s}}{x_i s_i}.$$

If $i \in B$, it follows from relations (7), (8), and (9) that

$$\frac{\gamma^2}{\mathbf{x}^T\mathbf{s}} \leq \frac{x_i}{s_i} \leq \frac{1}{\xi\gamma^2(\mathbf{x}^T\mathbf{s})}. \tag{10}$$

This shows that the condition numbers of the matrices $D_B^k$ are uniformly bounded and bounded away from zero. Thus, when matrix $A_B$ has full rank, then matrices $A_B(D_B^k)^2A_B^T$ also have full rank, and the condition numbers of the latter matrices are uniformly bounded. It is not hard to verify that similar results hold true for $D_B^k$ in the remaining cases. We summarize our conclusions below.

(1) If (**P**) and (**D**) are both non-degenerate on their respective optimal faces, then both programs have unique solutions and the matrices $A_B$ and $A_B(D_B^k)^2A_B^T$ are all non-singular. The linear systems (6) are well conditioned, at least when $\xi$ and $\gamma$ are not too small in (10).

(2) If (**P**) is degenerate and (**D**) is non-degenerate, matrix $A_B$ has less than $m$ columns and so rank $A_B < m$ and $A_B(D_B^k)^2A_B^T$ is singular. This means that the linear system (6) is ill-conditioned. Numerical problems caused by this ill-conditioning are reported by Gill et al. [26]. Shanno [82] reports that in his experience with OB1, degeneracy does not seem to cause great problems under these conditions. In fact, IPMs achieve great advantages over the simplex method in precisely these situations.*

(3) If (**P**) is non-degenerate and (**D**) is degenerate, then matrix $A_B$ has more than $m$ columns and so rank $A_B = m$ and $A_B(D_B^k)^2A_B^T$ is non-singular. This implies that the linear systems (6) will be well-conditioned. It is rather interesting that

*Recently our attention was called to the paper of Stuart, where he proves that the norm of the matrix $(AD^2A^T)^{-1}AD^2$ is bounded uniformly, independent of the scaling matrix $D$. This might clarify the computational robustness of IPMs. [G.W. Stuart, On scaling projections and pseudoinverses, Lin. Alg. Appl. 112 (1989) 189–193.]

if (**P**) is non-degenerate, the degeneracy status of (**D**) matters little from the numerical point of view.

(4) If both (**P**) and (**D**) are degenerate, then not much can be said about the partition $(A_B, A_N)$. It is possible that $A_B$ has full rank $m$ so that the resulting matrices $A(D^k)^2A^T$ might have a chance of being well-conditioned. However, ill-conditioned matrices cannot be ruled out.

One final issue here is the selection of the starting point. A well centered initial point $(\mathbf{x}^0, \mathbf{s}^0)$ and repeated centering during the algorithm is essential for good numerical performance [82]. A bad initial point $(\mathbf{x}^0, \mathbf{s}^0)$ (meaning close to the boundary, but neither close to an optimal face nor an optimal vertex) can cause immediate problems for any IPM. This point is emphasized by various authors, e.g. [10, 62, 63, 82, 83]. To the best of our knowledge, it is not known how an IPM behaves numerically when it is initiated from a degenerate (primal and/or dual) point very close to the optimal set.

## 6. Finding basic solutions

### 6.1. FROM AN OPTIMAL INTERIOR SOLUTION TO AN OPTIMAL BASIS

The simplex method solves the LP problem by moving from basis to basis, while the objective value changes monotonically (in general, not strictly). Due to degeneracy problems, it is impossible to guarantee strict monotonicity of the objective value in pivot methods, and this implies cycling and stalling problems. As a compensation, pivot methods always provide an optimal basic solution, and this is important for several reasons. A basic solution is necessary for cutting plane methods in mixed integer programming, and methods for sensitivity analysis and parametric programming are relatively inexpensive when an optimal basic solution is at hand. Basic solutions have a minimal number of non-zero coordinates, which is also important in practice.

These advantages provide sufficient motivation for generating an optimal basic solution from an optimal or near-optimal solution obtained by an IPM. It is evident that this question occurs only in the case of degeneracy, since otherwise the primal and dual optimal solutions are unique and are also basic solutions. As we have seen in section 3, in the case of degeneracy, most of the IPMs (except Iri–Imai's method) converge to the interior of the optimal face, and hence provide an optimal solution with a maximal number of nonzero coordinates in both the primal and dual problem (see e.g. Güler and Ye [38]). The existence of strictly complementary primal–dual optimal solutions has been proved first by Goldman and Tucker [28]. Balinski and Tucker [7] propose a (non-polynomial) algorithm to generate such a strictly complementary pair. In contrast, as we will see below, an optimal basis can be obtained from an optimal primal–dual solution pair in strongly polynomial time.

Since IPMs provide a maximal complementary optimal solution pair, a description of the optimal face is immediately at hand. This seems to be useful, additional information for which one has to do work in pivot methods. It was believed for a long time that for postoptimal analysis an optimal basis solution is necessary. Until recently there was no method known for postoptimal analysis without an optimal basis at hand. For surveys of the traditional methods, see Gal [22, 23] and Ward and Wendell [103]. Without first obtaining basic solutions, Adler and Monteiro [2] present a method for right hand side parametric analysis. The amount of work involved in these methods is substantial. In contrast, the familiar simplex postoptimal analysis techniques are inexpensive. Unfortunately, in the degenerate case, the methods implemented in simplex based packages provide unreliable information [48]. To get the correct information one has to find all the primal optimal and all the dual optimal bases [48, 103]. In theory this is an exponential procedure, and in practice at least as expensive as the algorithm in Adler and Monteiro [2]. Jansen et al. [48] present a method for postoptimal analysis based on the solution obtained from an IPM. The IPM and simplex based approaches are compared as well. They show that all the information concerning postoptimal and parametric analysis can be obtained by using IPMs, with the same or frequently better computational complexity. In conclusion we may say that if the optimal solution is nondegenerate then there is no difference between the IPM and simplex based postoptimal analysis approaches. In case of degeneracy, the IPM approach is theoretically better since the computational cost of obtaining the complete information is polynomial.

Megiddo [67] presents a strongly polynomial algorithm for finding an optimal basis, provided optimal solutions are available to both (**P**) and (**D**). Due to its theoretical and practical importance we present the algorithm.

## ALGORITHM TO FIND AN OPTIMAL BASIS

- *Initialization*:
  Suppose that $\mathbf{x}$ primal and $\mathbf{y}$, $\mathbf{s}$ dual optimal solutions are available. Let $A = [A_1, A_2, A_3]$, $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{s} = (s_1, s_2, s_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ where index 1 refers to the positive coordinates of $\mathbf{x}$, index 2 refers to the zero coordinates of both $\mathbf{x}$ and $\mathbf{s}$, and index 3 refers to the positive coordinates of $\mathbf{s}$. Then we have $A_1\mathbf{x}_1 = \mathbf{b}$, $\mathbf{x}_1 > 0$, $\mathbf{x}_2 = 0$, $\mathbf{x}_3 = 0$, and $A_1^{\mathrm{T}}\mathbf{y} = \mathbf{c}_1$, $A_2^{\mathrm{T}}\mathbf{y} = \mathbf{c}_2$, $A_3^{\mathrm{T}}\mathbf{y} < \mathbf{c}_3$.
- *Reduce the positive part of x*:
  **While** the columns of $A_1$ are dependent **do**
  **begin**
  Find (e.g., by pivoting) a vector $\mathbf{t}$ such that $A_1\mathbf{t} = 0$ (this implies $\mathbf{c}_1^{\mathrm{T}}\mathbf{t} = 0$). Using $\mathbf{t}$, eliminate a positive coordinate (say $j$) from $\mathbf{x}_1$, while preserving the non-negativity of $\mathbf{x}_1$ (ratio test). Remove column $\mathbf{a}_j$ from $A_1$ and add it to $A_2$.

**end**

Let $B = A_1$. (Note that the columns of $B$ are independent at this step.)

- *Extend B to a basis*:

Extend $B$ using $A_2$:

**While** $\text{rank}(B) < \text{rank}[A_1, A_2]$ **do**

**begin**

If $\text{rank}[A_1, A_2] > \text{rank}(B)$ and a column $\mathbf{a}_j$ of $[A_1, A_2]$ is independent from $B$, add $\mathbf{a}_j$ to $B$.

**end**

Extend $B$ using $A_3$:

**While** $\text{rank}(B) < m$ **do**

**begin**

Find (e.g., by pivoting) a vector $\mathbf{u}$ such that $B^T\mathbf{u} = \mathbf{0}$ (this implies $A_1^T\mathbf{u} = \mathbf{0}$, $A_2^T\mathbf{u} = \mathbf{0}$) and $A_3^T\mathbf{u} \neq \mathbf{0}$. Note that $\mathbf{u}$ satisfies $\mathbf{b}^T\mathbf{u} = 0$ (since $\mathbf{u}^T A_1^T \mathbf{x}_1 = \mathbf{b}^T\mathbf{u}$). Using $\mathbf{u}$, eliminate a positive coordinate (say $j$) from $\mathbf{s}_3$, while preserving the dual feasibility of $\mathbf{s}$ (ratio test). Remove $\mathbf{a}_j$ from $A_3$, and add it to $A_2$ and $B$.

**end**

(We now have an optimal complementary pair $(\mathbf{x}, \mathbf{s})$, where $\text{rank}(B) = m$. Using the formulae $B\mathbf{x}_B = \mathbf{b}$ and $B^T\mathbf{y} = \mathbf{c}_B$, we see that basis $B$ is optimal.)

We remark that only Gaussian elimination steps (pivoting) are necessary to perform this algorithm. The amount of work involved depends on the degree of degeneracy of the LP problem. In the worst case $n$ pivots (the dimension of the space) are necessary to identify an optimal basis. The algorithm uses both primal and dual information and generates optimal basic solutions both to (**P**) and (**D**). In this sense Megiddo's result is quite unique since most authors concentrate on getting just a primal or dual optimal basic solution, which is much simpler.

Further, Megiddo [67] presents the following three problems:

**Problem 1.** Find an optimal basis for the LP problem or conclude that no such basis exists.

**Problem 2.** If it is known that optimal solutions exist, find an optimal basis, i.e., a basis which is optimal for both (**P**) and (**D**).

**Problem 3.** Having an optimal solution to (**P**), find an optimal basis, i.e., a basis which is optimal for both (**P**) and (**D**).

Problems 1 and 2 are equivalent to solving the LP problem. Problem 3 asks for an optimal basis (optimal both for the primal and dual problems) provided a primal optimal solution is available. This last problem seems to be easier than the other two since we have an optimal solution to (**P**). But Megiddo also proves the surprising result that the complexity of the above three problems is the same. This result shows that if we have only partial information (just a primal or dual optimal solution), then identifying an optimal basis is equivalent to solving an LP problem.

These results indicate that we need a primal–dual approximate optimal solution if our purpose is to obtain a primal–dual optimal basis. Primal–dual and path-following methods generate primal and dual solutions in the course of the algorithm; the other methods do not. In theory, many of the primal (and dual) algorithms including the affine-scaling algorithm and Karmarkar's algorithm are shown to generate a dual (and primal) optimal solution in the final stage of the iterations. However, from the practical point of view, at present it seems fair to state that primal–dual algorithms have an advantage in producing approximate primal–dual optimal solutions.

If we were only interested in an optimal basic feasible solution to (**P**), problem 3 can be relaxed.

**Problem 4.** Having an optimal solution to (**P**), find an optimal basic feasible solution to (**P**), i.e., a basis which is optimal for the primal, but not necessarily feasible for (**D**).

The solution to this problem can be obtained by modifying Megiddo's algorithm. The algorithm can be sketched as follows: if the column vectors of the positive coordinates are dependent, then one can eliminate the dependent columns (one Gauss elimination for a dependent vector) while the objective does not change. Then the obtained independent system can easily be extended to a basis. Therefore, we conclude that there exists a strongly polynomial algorithm for generating a primal optimal basis from a primal optimal solution.

Based on an old paper of Charnes and Kortanek [15], Kortanek and Zhu [58] present some "purification algorithms" which solve problem 4. From an interior solution, they generate a basic feasible solution which has at least the same objective value. Conversely, having a basic feasible solution, they generate an interior solution with at least the same objective value as the basic solution. This purification algorithm, like Megiddo's algorithm, is based on Gaussian elimination (pivoting). It is a strongly polynomial algorithm for identifying an optimal solution from a $2^{-2L}$ approximate optimal solution, which demonstrates its theoretical importance.

Mehrotra [70] chooses another approach. His algorithm generates "controlled random perturbations" of the LP problem. The objective vector is perturbed with a small positive vector $\mathbf{r}$ with components $r_i = \epsilon RAND(1,2)/(40nx_i)$ for all $i$. Here $RAND(1,2)$ is a function generating a random variable uniformly distributed in $(1,2)$. If the primal–dual predictor–corrector method ends at a basis, the algorithm is stopped; otherwise another perturbation is generated. There is no guarantee of success in this method; theoretically an exact basis identification method (see above) might still be necessary. However, Mehrotra reports that in his experiments the perturbation method always produced an optimal basis. This algorithm is different from the Megiddo and Kortanek–Zhu approaches. Instead of postoptimal basis generation, the problem is modified to obtain an optimal basis.

### 6.2. SWITCHING OVER TO A SIMPLEX METHOD

Another interesting (reported to be efficient for very large problems) approach to generate an optimal basis is to switch over from an interior point method to a simplex method. Such approaches are discussed below. The theoretical disadvantages of these methods are obvious. Polynomial time bounds are lost and without degeneracy handling methods cycling (stalling in practice) is possible.

The first such approach is presented by Kojima and Tone [57] for Karmarkar's projective algorithm. If the duality gap is small, they use the value of the variables to predict the optimal basis, and initiate the simplex method from this basis.

The recent experiments carried out by Bixby et al. [12], Lustig [60] and Ye and Kaliski [112] use such an approach. They report that this approach is superior (on an extremely large practical problem) to both the simplex method and the IPMs. After generating a feasible basis, the simplex method seems to work well. Bixby et al. [12] and Lustig [60] use Megiddo's strongly polynomial optimal basis identification technique to switch over to the simplex method. This algorithm does not suffer from degeneracy problems during the generation of an almost optimal interior solution. Ye and Kaliski [112, 49] project the actual interior solution onto the predicted optimal facet. If this does not give satisfactory results due to numerical problems, they switch over to the simplex method by using some purification algorithm. Another difference between the above mentioned two approaches is that Bixby et al. switch over to the simplex method earlier than Ye and Kaliski. In both approaches, after generating an optimal basis, the simplex method seems to work well.

Megiddo [68] also considers a similar approach and switches from a primal–dual IPM to an appropriately parametrized variant of Dantzig's parametric self-dual simplex algorithm [16]. Here any iterate of the primal–dual IPM provides an appropriate parameter vector and an initial solution to the self-dual parametric simplex algorithm, which terminates at an optimal basic solution.

Bixby and Saltzman [13] use a different approach. They do not use the dual information provided by the IPM, but only the actual primal solution. Therefore their method is theoretically not polynomial. Another difference is that unlike Megiddo's [67] strongly polynomial algorithm they construct a numerically stable full basis at the start (not step by step as Megiddo) and then they use the standard simplex steps to recover the solution. Both the standard Phase I and Phase II procedures of CPLEX are used.

The purification algorithm of Kortanek and Zhu [58] can also be regarded as a combination of IPMs and simplex methods. Here, given an interior solution, a primal feasible basis with an improved objective value is identified. Then the optimality of the primal basis is checked. If it is not optimal, then a step is taken back into the interior of the feasible set. In case of degeneracy cycling prevention algorithms are needed.

### 6.3. DETECTING POSITIVE AND ZERO VARIABLES AT THE OPTIMUM

In the third class of methods one tries to identify an optimal basis during the process of an IPM. Such methods usually assume non-degeneracy, but in some cases the non-degeneracy condition can be relaxed. Using inscribed and circumscribed ellipsoids, Todd [91], Ye [107], and Ye and Todd [115] give criteria for identifying active and inactive constraints at the optimum. Asic et al. [5] and Tapia and Zhang [88] use an indicator associated with the current iterate which converges at the same rate as the square of the measure associated with the iterate. Hence, this indicator converges much faster and identifies the active and inactive variables at the optimum. This indicator approach breaks down if both the primal and dual problems are degenerate.

Gay [24], Ye [110] and Mehrotra and Ye [73] have also developed some techniques to identify optimal faces of LP problems. We do not discuss these methods here, for two reasons. On the one hand, these are not basis identification, but optimal face identification methods. On the other hand, there is no guarantee that active or inactive variables can be identified in every case.

## 7. Conclusions

Although IPMs go through the interior of the feasible region, degeneracy still has a role to play in IPMs. Global convergence proofs are dependent on non-degeneracy only for affine scaling algorithms, which might suggest that IPMs do not suffer from degeneracy problems. Unfortunately, this is not the case, but the problems here are different from the cycling and stalling problems occurring in the simplex method.

Degeneracy and redundancy affect the central path, which most IPMs aim to follow. Numerical performance of the algorithms may suffer from problems of numerical instability and ill-conditioning if the optimal solutions are degenerate or near-degenerate. Degeneracy does not seem to cause serious problems in other areas.

In case of degeneracy, most of the IPMs provide a strict complementary solution pair. Generating an optimal basis from this solution is strongly polynomial, but the computational complexity depends on the degree of degeneracy.

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