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PERFECT DISTRIBUTION OF POINTS ON A SPHERE

by

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§ 1. Circles, dictators and fuel depots

We recall MESCHKOWSKI's [1] interpretation of the problems of the densest packing and the thinnest covering of the sphere by n congruent circles:

How should the residences of n inimical dictators, governing on a planet, be distributed so as to maximize the least distance between any two of them?

How should n fuel depots be arranged on a planet so as to minimize the greatest distance between a point of the planet and the nearest depot?

The dictator problem may be interpreted by fuel depots and vice versa:

How should n fuel depots be arranged on a planet so that an accidental explosion of one of them should least endanger the rest?

How should the residences of n allied dictators, governing on a planet be placed so as to control the planet as well as possible [3]?

These problems have a rather vast literature (see e.g. [2], [3]). However, the only values of n for which the solutions are known, are $n = 2, 3, \ldots, 9, 12, 24$ and $n = 2, 3, \ldots, 7, 10, 12, 14$, respectively.

There are some point systems which are solutions of both problems, namely two antipodal points, the vertices of a regular trigonal tessellation $\{3, k\}$ ($k = 2, 3, 4, 5$) and the vertices and face-centers of $\{3, 2\}$ (or $\{2, 3\}$). So the numbers 2, 3, 4, 5, 6 and 12 are especially favorable for a set of allied dictators not trusting each other, as well as for a set of fuel depots. The starting point of the present investigation was a problem which I was not able to solve: Are there any further such favorable numbers? It may be conjectured that the answer is "No". We try to support this conjecture by proving a weaker statement.

§ 2. Perfect distribution of points

The problem of the densest circle-packing and the problem of the thinnest circle-covering of the sphere may be united in a more general problem: How should the centers of n circles of given radius r be distributed on the sphere so as to maximize the area covered by the circles. It may happen that a set of n points yields the solution of this problem for all values of r. Then we say that the points are *perfectly distributed* or that the set is *perfect.*

A trivial example for a perfect set of points is given by a pair of antipodal points. Furthermore, it is known [4] that the vertices of a regular trigonal tessellation are also perfectly distributed. We claim that this enumeration is complete. This is expressed by the following

THEOREM. *If n points are perfectly distributed on a sphere, then* $n = 2, 3, 4, 6$ *or* 12 *and the points are the vertices of the tessellation* $\{2, 3\}$, $\{3, 2\}$, $\{3, 3\}$, $\{3, 4\}$ *and* $\{3, 5\}$, *respectively.*

The proof follows in §§ 3 and 4.

§ 3. Necessary conditions

Let $U = \{P_1, \ldots, P_n\}$ be a perfect set of $n > 2$ points, a the least spherical distance between pairs of points and g the *graph* of U consisting of the n points and all spherical segments of length a which join pairs of points. The graph g has the obvious property

p. All angles included by adjacent edges of g are greater than $\pi/3$.

We now prove the following condition.

c. The unit vectors issuing from a point of U in the direction of the edges of g are in equilibrium.

Let the edges starting from P_n end in $P_1, P_2, \ldots, P_k$. Let $c_i(r)$ be the closed circular disc of radius r centered at P_i. Since $c_n(a/2)$ is touched by $c_1(a/2), \ldots, c_k(a/2)$ without having a point in common with $c_{k+1}(a/2), \ldots, c_{n-1}(a/2)$, we can choose $r > a/2$ so that $c_n(r)$ is intersected by $c_1(r), \ldots, c_k(r)$ in k disjoint "lenses" without being intersected by $c_{k+1}(r), \ldots, c_{n-1}(r)$. The definition of U implies that the variation of the total area t of these lenses effectuated by a small variation of P_n is never negative.

We choose a point A on the boundary of $c_n(r)$ and introduce polar coordinates with the pole P_n and the initial line P_nA. Let the line P_nA be the equator, so that the point $N = (\pi/2, \pi/2)$ is the north-pole.

Let $B = (r, \beta)$ and $C = (r, \gamma)$ be two points on the boundary of $c_n(r)$. The circular arc BC consists of the points we pass when traveling from B to C in a positive direction. For the moment we suppose that this arc lies on the semicircle $(r, -\pi/2)$ $(r, \pi/2)$. Letting b and c be the distances of B and C from N, we have $\cos b = \sin r \sin \beta$ and $\cos c = \sin r \sin \gamma$. Thus, rotating $c_n(r)$ about N through 2π, the arc BC will sweep over a circular ring of area

$$2\pi(1 - \cos b) - 2\pi(1 - \cos c) = 2\pi \sin r(\sin \gamma - \sin \beta) =$$

$$= 4\pi \sin r \sin \frac{\gamma - \beta}{2} \cos \frac{\beta + \gamma}{2}.$$

This formula continues to hold for any position of B and C, if we choose β and γ so that $0 < \gamma - \beta < 2\pi$ and assign a negative value to the area swept over by an arc (or the part of an arc) lying on the semicircle $(r, \pi/2)$ $(r, -\pi/2)$. Thus the area swept over by the arcs cut off from the boundary of $c_n(r)$ by $c_1(r), \ldots, c_k(r)$ when sliding $c_n(r)$ along the equator P_nA through an infinitesimal distance, is proportional to $\cos\omega_1 + \ldots + \cos\omega_k$, where $\omega_1, \ldots, \omega_k$ denote the polar angles of the midpoints of the arcs. Since this area is equal to the variation of t, we have

$$\cos\omega_1 + \ldots + \cos\omega_k = 0.$$

Replacing $A = (r, 0)$ by the point $(r, \pi/2)$, we obtain

$$\sin\omega_1 + \ldots + \sin\omega_k = 0.$$

This completes the proof of c.

The number of edges issuing from a point is called the *order* of the point. The condition c immediately implies the following ones.

c.1. There is no point of order 1.

c.2. The edges issuing from a point of order 2 are on one line.

c.3. The edges issuing from a point of order 3 end at the vertices of a regular triangle.

c.4. The edges issuing from a point of order 4 are in a centrosymmetric position.

We continue to prove condition

c.5. The edges issuing from a point of order 5 end at the vertices of a regular pentagon.

Let the edges of g issuing from P_n in their cyclic order be $P_nP_1, \ldots, P_nP_5$. Let P_1P_2 be a greatest side of the pentagon $P_1 \ldots P_5$. Let the angles which the directions $P_nP_1, \ldots, P_nP_5$ make with the half-line bisecting the angle $P_1P_nP_2$ be $-\omega, \omega, \omega_3, \omega_4, \omega_5$ (Fig. 1). We claim that $2\omega = \sphericalangle P_1P_nP_2 <$

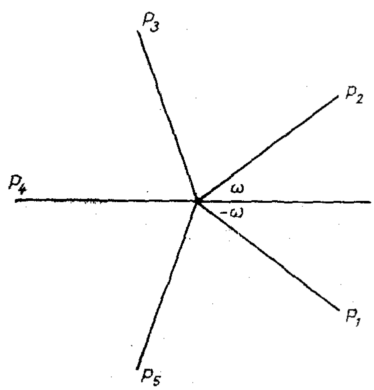

Fig. 1

$< \pi/2$. For, assuming that $\omega \geq \pi/4$, we have in view of p

$$2 \cos \omega + \cos \omega_3 + \cos \omega_4 + \cos \omega_5 \leq 2 \cos 45^\circ + 2 \cos 105^\circ + \cos 165^\circ = \\ = - 0{,}069 \ldots < 0,$$

which contradicts c.

Let R be the least radius such that the boundary of $c_n(R)$ is completely covered by $c_1(R), \ldots, c_5(R)$. As an immediate consequence of the inequality $\sphericalangle P_1P_nP_2 < \pi/2$, observe that $c_n(R)$ has no point in common with $c_6(R), \ldots, c_{n-1}(R)$. This follows from the fact that all of the angles $\sphericalangle P_1P_nP_2, \ldots, \sphericalangle P_5P_nP_1$ are less than an angle of a regular quadrangle of side-length a.

Suppose that among $\sphericalangle P_1P_nP_2, \ldots, \sphericalangle P_5P_nP_1$ there are exactly m greatest angles. Let the half-lines bisecting these angles be $h_1, \ldots, h_m$. Replace R by a smaller radius r so as to obtain on the boundary b of $c_n(r)$ m equal open arcs not covered by $c_1(r), \ldots, c_5(r)$, while the points of b not belonging to these arcs or to their extremities are interior points of one of the circles $t_1(r), \ldots, c_5(r)$. Repeating the argument used in the proof of c, we see that che unit vectors emanating from P_n in the direction of $h_1, \ldots, h_m$ are in q.uilibrium. This implies, along with c that $m = 5$. Thus the angles $\sphericalangle P_1P_nP_2$, et ., $\sphericalangle P_5P_nP_1$ are equal and the pentagon $P_1 \ldots P_5$ is regular.

§ 4. Admissible graphs

We say that an edge of g is of *type* (kl) if it joins vertices of order k and l. By $c.1$ and p we have $2 \leq k, l \leq 5$. We claim that if g has an edge of type (kl) than all edges are of this type. We will show this by scrutinizing the various types of edges in a special order.

22. Preserving the notations of the proof of c, we assume that $k = 2$. Supposing that $a < \pi/2$, the arc of the lenses $c_1(r) \cap c_n(r)$ and $c_2(r) \cap c_n(r)$ would decrease by sliding $c_n(r)$ perpendicularly to the segment P_1P_2 through a small distance. Since this is impossible, we must have $a \geq \pi/2$. If $a = \pi/2$, then P_1 and P_2 are antipodal points, say, the nort- and south-pole, and we have exactly three further points P_3, P_4, P_5, all lying on the equator. By $c.3$ the equatorial points must be the vertices of a regular triangle. Now we can replace P_3 by another equatorial point without changing the total area T covered by $c_1(r), \ldots, c_5(r)$, but disturbing the condition of equilibrium in P_1 and P_2. Thus T could be increased by a small variation of P_1 or P_2, showing that the original distribution of $P_1, \ldots, P_5$ could not be perfect. Consequently we have $a > \pi/2$. This implies immediately that $n = 3$, so that we have three edges of type (22). At the same time we see that g cannot contain edges of type (23), (24) or (25).

45. We call a connected set of edges of g belonging to the boundary of a convex polygon a *convex arc*. If A and B are adjacent vertices of order

5 and 4, respectively, then g contains a convex arc $XABY$ such that $A = \sphericalangle XAB = 2\pi/5$ and $B = \sphericalangle ABY \leq \pi/2$. Since the distance XY is clearly less than a, X and Y are identical points (Fig. 2). Therefore the image of the edge $BY \equiv BX$ reflected in AB is an edge of g, which contradicts the assumption that B is of order 4. This excludes the possibility of an edge of type (45).

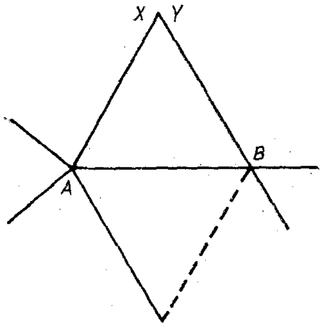

Fig. 2

55. If the edge AB is of type (55), then the argument used in **45** shows the existence of a point X of g such that ABX is an equilateral triangle. By **22**, **45** and c.3 the point X can only be of order 5. It follows that all points of g are of order 5.

44. Let BC be an edge of type (44). Let A and D be the images of C and B reflected in B and C, respectively. The edges other than AB, BC and CD issuing from B and C cannot lead to four points all different from one another. Therefore there is a point E such that BE and CE are edges of g (Fig. 3). It follows that CF and FD are also edges of g, where F is the image

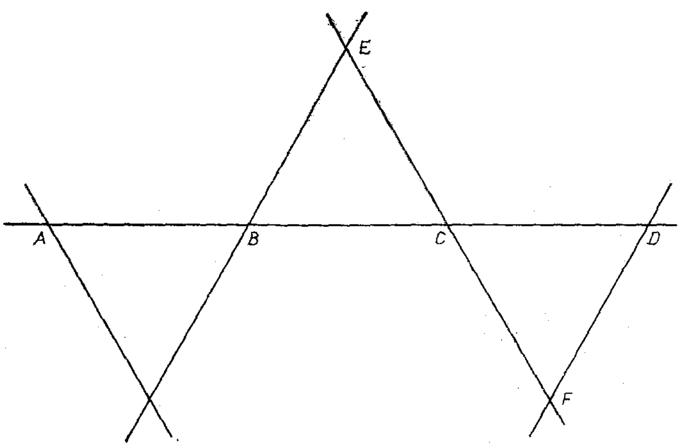

Fig. 3

of E reflected in C. Owing to **22, 45** and $c.3$ the points E, F and D are of order 4, showing that all edges are of type (44).

35. Let AB be an edge joining the point A of order 5 with the point B of order 3. By $c.5$. there is another edge AC such that $\sphericalangle BAC = 2\pi/5$. By **22, 45** and **55** the point C is also of order 3. Therefore g contains a convex arc $XBACY$ such that $B = C = 2\pi/3$ (Fig. 4).

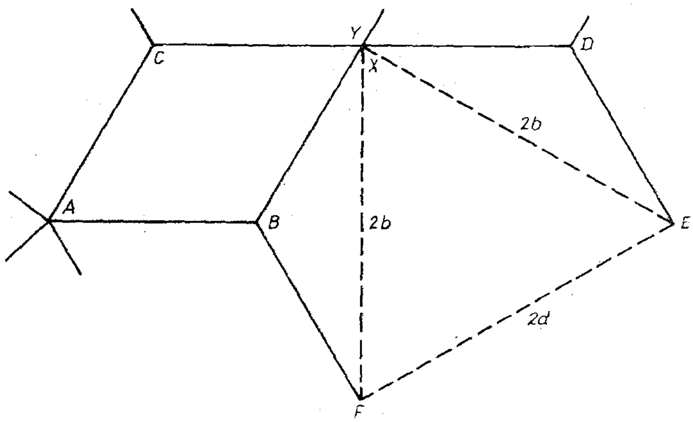

Fig. 4

Varying the angle $A = \sphericalangle BAC$ under these conditions the distance XY will vary too, and a simple computation shows that in case of $A \leq \alpha = 60° + 2\text{arccosec}\sqrt{12} \approx 93°33'$ we have $XY < a$. Since for the original arc $XBACY$ we have $A = 2\pi/5 < \alpha$, the points X and Y must coincide. Thus $ABXC$ is a rhombus of side-length a and angles $2\pi/5$ and $2\pi/3$, showing that

$$\cos a = \cos \pi/5 \cot \pi/3, \quad a \approx 37°22'.$$

We claim that X has order 5. Since $c.3$ rules out the possibility of the order 3, we only have to investigate the case that X is of order 4. Then, by $c.4$, the image XD of XC reflected in X is also an edge of g, and by **45** and **44** D is of order 3. Thus we have the convex arc $EDXBF$ with angles $D = B = 2\pi/3$ and $X = \pi - 2\pi/5 = 3\pi/5$. Because of the congruence of the triangles XDE, XBF and ABX we have $\sphericalangle EXD = \sphericalangle FXB = \pi/5$, and consequently $\sphericalangle FXE = 2(54° - 36°) = 2 \cdot 18°$. Thus, with the notation $XE = XF = 2b$ and $EF = 2d$, we have

$$\sin b = \sin a \sin 60°, \ \sin d = \sin 2b \sin 18°,$$

whence

$$0 < 2d \approx 32° < a.$$

This being impossible, we conclude that X is in fact of order 5, and so all edges of g are of type (35).

34. Let AB be an edge joining the point A of order 4 with the point B of order 3. Let AC be another edge of g such that $\sphericalangle BAC \leq \pi/2$. In view of **22, 44** and **45** the point C is also of order 3. Since $\sphericalangle BAC \leq 90° < \alpha \approx \approx 93°33'$, there are two edges BD and CD starting from B and C, respec-

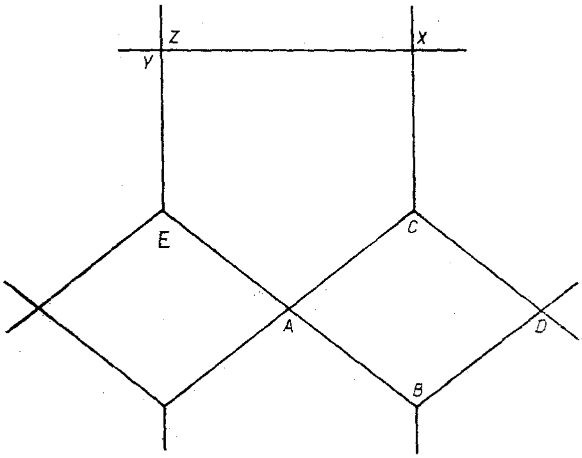

Fig. 5

tively, and meeting in one point D other than A. By **22, 35** and $c.3$ D is of order 4. So we have a set of points of order 4 equally spaced on the "equator" AD. The edges issuing from these points meet in points of order 3 from which further edges start heading to the north- and south-pole, respectively (Fig. 5). Thus we have a convex arc $XCAEY$ such that $C = E = 2\pi/3$ and $\pi/2 \leq A < < 2\pi/3$.

Suppose that the points X and Y do not coincide. If both X and Y are of order 3, we would have two edges issuing from X and Y crossing each other. Since this is impossible, and since by **22** and **35** no point can be of order 2 or 5, one of the points X and Y, say X, must be of order 4. This implies the existence of a convex arc $ZXCAEY$ such that $\pi/3 < X < 2\pi/3$. It easily follows that the distance ZY is less than a, so that $Z \equiv Y$. Thus the segment XY, as well as its image reflected in X are edges of g. But this is impossible because together with X and Y there are further points of g equally spaced on the parallel circle XY. This contradiction shows that the points X, Y, ... must coincide with the north- and south-pole, respectively.

If the poles were of order 3, we would have $\sphericalangle CAE = 2\pi/3$, i.e $\sphericalangle BAC = \pi/3$, which is impossible. Thus the poles are of order 4, showing that all edges are of type (34).

33. If g contains an edge of type (33), then by **22**, **34** and **35** all edges are of type (33).

To recapitulate our results, we denote by $v\{p, q\}$, $e\{p, q\}$ and $f\{p, q\}$ the set of the vertices, the set of the edge-midpoints and the set of the face-centers of the tessellation $\{p, q\}$. Again, we write $(v+f)\{p, q\}$ for the union of $v\{p, q\}$ and $f\{p, q\}$. Note that $v\{p, q\} = f\{q, p\}$, $e\{p, q\} = e\{q, p\}$ and $(v+f)\{p, q\} = (v+f)\{q, p\}$. The following table gives the point-systems compatible with our conditions for the various types of edges:

(22)	$v\{3, 2\}$
(33)	$v\{3, 3\}$ $v\{4, 3\}$ $v\{5, 3\}$
(34)	$(v+f)\{3, 4\}$
(35)	$(v+f)\{3, 5\}$
(44)	$e\{3, 3\}$ $e\{3, 4\}$ $e\{3, 5\}$
(55)	$v\{3, 5\}$.

Observe that the circles of radius a centered at the points of any of these point systems cover the sphere completely. Therefore the graph g cannot contain points of order 0.

To conclude the proof of our theorem we must only remark that $v\{4, 3\}$ $v\{5, 3\}$, $(v+f)\{3, 4\}$, $(v+f)\{3, 5\}$, $e\{3, 4\}$ and $e\{3, 5\}$ are not solutions of the problem of 8, 20, 14, 32, 12 and 30 inimical dictators, respectively. Thus the only configurations coming into consideration as sets of perfectly distributed points are $v\{3, 2\}$, $v\{3, 3\}$, $e\{3, 3\} = v\{3, 4\}$ and $v\{3, 5\}$. The fact that these sets of points are really perfectly distributed follows from a general theorem referred to in §2.

§ 5. Remarks

An interesting problem is to enumerate the sets of perfectly distributed points in the elliptic plane. Representing the points of the elliptic plane as pairs of antipodal points on a sphere, the sets $v\{2, 3\}$, $v\{4, 2\}$, $v\{3, 4\}$ and $v\{3, 5\}$ obviously represent sets of 1, 2, 3 and 6 perfectly distributed points. Since the sets $v\{4, 3\}$, $v\{5, 3\}$, $e\{3, 4\}$ and $e\{3, 5\}$ can be easily ruled out by counterexamples, the only further possibilities for sets of perfectly distributed points are given by $(v+f)\{3, 4\}$ and $(v+f)\{3, 5\}$. Both sets have some chances of representing perfect sets in the elliptic plane.

The notion of perfect sets may be extended to the Euclidean or hyperbolic plane and also to spaces of higher dimension. Let $S(r)$ be a set of balls with radius r having nowhere accumulating centers. Let s be a finite subset of S and $V(s, r)$ the volume of that part of the space which is covered by the balls of s without being covered by the rest of the balls. If $V(s, r)$ cannot be increased by rearranging the balls of s in any way, no matter how we choose s and r, then the centers are said to be perfectly distributed. It can be proved [5] that the vertices of $\{3, 6\}$, $\{3, 7\}, \ldots$ are perfectly distributed in the Euclidean and hyperbolic plane, respectively. In the Euclidean plane there is no perfect set other than $v\{3, 6\}$. As to the hyperbolic plane, it seems likely that there are infinitely many semi-regular sets of perfectly distributed points. The sets which have the best chance to be perfect, are $(v+f)\{7, 3\}$ and $(v+f)\{2q \pm 1, q\}$, $q = 4, 5, \ldots$. Note that the Euclidean 3-space does not admit a perfect distribution of points. For, here the problems of the densest sphere-packing and the thinnest sphere-covering, though not solved as yet, certainly lead to completely different sets of centers.

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