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Inequalities for upper bounds of functionals

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Introduction

In the present work we obtain some Kolmogorov type inequalities and apply them to some problems of approximation theory.

Let us introduce the following notations.

(i) L_p $(1 \leqq p < \infty)$ is the space of all 2π-periodic measurable functions $f(x)$ for which $|f(x)|^p$ is integrable over $[0,2\pi]$, the norm being

$$\|f\|_{L_p} = \left\{\int_0^{2\pi} |f(x)|^p\,dx\right\}^{1/p};$$

(ii) C is the space of all continuous 2π-periodic functions with the norm

$$\|f\|_C = \max_x |f(x)|;$$

(iii) $R = C$ or L_p $(1 \leqq p < \infty)$;

(iv) R^r $(r = 1, 2, \ldots;\ R^{(0)} = R)$ is the set of all functions $f \in R$ that have an absolutely continuous $(r-1)$th derivative and $f^{(r)} \in R$;

(v) NW^rR $(r = 0, 1, \ldots)$ is the set of all functions $f \in R^{(r)}$ such that $\|f^{(r)}\|_R \leqq N$ $(W^rR = 1W^rR)$;

(vi) W_n^rR $(r = 0, 1, \ldots;\ n = 1, 2, \ldots)$ is the set of all functions $f \in W^rR$ orthogonal to all trigonometric polynomials of order not exceeding $n-1$.

(vii) We will denote by

$$\omega(f, \delta)_R = \sup_{0 \leqq u \leqq \delta} \|f(x+u) - f(x)\|_R$$

the modulus of continuity of the function $f \in R$;

(viii) $W^rH_R^\omega$ $(r = 0, 1, \ldots)$ is the set of functions such that

$$\omega(f^{(r)}, \delta)_R \leqq \omega(\delta) \quad (0 \leqq \delta \leqq \pi).$$

where $\omega(\delta)$ denotes a given modulus of continuity.

(ix) We will set

$$H_R^\omega = W^0H_R^\omega.$$

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§ 1 Inequalities for the norms of the derivatives of an arbitrary function

The following inequality of Kolmogorov [9] for the norms of the derivatives of a function is well-known:

(1.1)
$$\left\{\frac{\|f^{(r-k)}\|_C}{K_k\|f^{(r)}\|_C}\right\}^{1/k} \leqq \left\{\frac{\|f\|_C}{K_r\|f^r\|_C}\right\}^{1/r}$$

for $f\in C^{(r)}$, $r=2, 3, \ldots$ and $1\leqq k\leqq r$, where

$$K_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(r+1)}}{(2\nu+1)^{r+1}}$$

are the Favard constants $\left(K_0 = 1,\ K_1 = \frac{\pi}{2},\ K_2 = \frac{\pi^2}{8},\ K_3 = \frac{\pi^3}{24}, \ldots\right)$.

Stein [22] established the analogue of this inequality for the metric of L_1:

(1.2)
$$\left\{\frac{\|f^{(r-k)}\|_{L_1}}{K_k\|f^{(r)}\|_{L_1}}\right\}^{1/k} \leqq \left\{\frac{\|f\|_{L_1}}{K_k\|f^{(r)}\|_{L_1}}\right\}^{1/r}$$

for $f\in L_1^{(r)}$, $r=2, 3, \ldots$ and $1\leqq k\leqq r$.

It is also clear that for $f\in L_2^r$ and $0<k<r$ we have

(1.3)
$$\left\{\frac{\|f^{(r-k)}\|_{L_2}}{\|f^{(r)}\|_{L_2}}\right\}^{1/k} \leqq \left\{\frac{\|f\|_{L_2}}{\|f^{(r)}\|_{L_2}}\right\}^{1/r}.$$

Inequality (1.1) becomes an equality if $f(x)=n^{-r}\varphi_r(nx)$, where

(1.4)
$$\varphi_r(x) = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{\sin\left\{(2\nu+1)x-\frac{r\pi}{2}\right\}}{(2\nu+1)^{r+1}}$$

is the rth periodic integral, with zero mean value over the period, of the function $\varphi_0(x)=\operatorname{sign}\sin x$. The inequalities (1.2) and (1.3) are sharp on the sets $L_1^{(r)}$ and $L_2^{(r)}$, respectively.

The analogues of inequalities (1.1)—(1.3) for functions defined on the whole x-axis or on a semi-axis are obtained in the papers [21], [24], and [26].

In the present section we prove some inequalities analogous to (1.1)—(1.3).

The function $\Phi(u)$, $u\geqq 0$ is called an *N-function* if it can be represented as

$$\Phi(u) = \int_0^u p(t)\,dt,$$

where $p(t)$ is a non-decreasing function that is continuous from the right and satisfies the following conditions:

$$p(t) > 0 \quad (t > 0), \quad p(0) = 0, \quad \lim_{t\to\infty} p(t) = \infty.$$

Theorem 1.1. *For the elements* $f \in C^{(r+1)}$ $(r=0, 1, ...)$ *we have the following sharp inequality:*

$$\int_0^{2\pi} \Phi\left(\frac{|f'(x)|}{\|f^{(r+1)}\|_C}\right) dx \leqq \int_0^{2\pi} \Phi\left(\left(\frac{\|f\|_C}{K_{r+1}\|f^{r+1}\|_C}\right)^{r/(r+1)} |\varphi_r(x)|\right) dx, \tag{1.5}$$

where $\Phi(u)$ *is an arbitrary N-function.*

To prove this assertion we need some auxiliary results.

The following lemma can be proved by an almost word-by-word repetition of the arguments of Korneĭčuk (cf. the proof of Lemma 6.1 in [10]), nevertheless we give its proof here for the sake of completeness.

Lemma 1.1. *Let* $g \in W^r C$ $(r=1, 2, ...)$, $\|g\|_C \leqq \alpha^{-r} K_r$,

$$g(\xi) = \alpha^{-r}\varphi_r(\alpha\eta),$$

and suppose that the function $\varphi_r(\alpha t)$ *is monotone on a segment* $[a, b]$ *containing* η. *The following statements are true:*

1. *If* $\varphi_r(\alpha t)$ *increases on* $[a, b]$ *then*

$$\begin{aligned} g(\xi+t) &\leqq \alpha^{-r}\varphi_r(\alpha(\eta+t)) \quad (0 \leqq t \leqq b-\eta), \\ g(\xi-t) &\geqq \alpha^{-r}\varphi_r(\alpha(\eta-t)) \quad (0 \leqq t \leqq \eta-a). \end{aligned} \tag{1.6}$$

If the function $\varphi_r(\alpha t)$ *decreases on* $[a, b]$, *then the inequality signs in* (1.6) *turn the opposite way.*

2. *If*

$$g(\xi_1) = \alpha^{-r}\varphi_r(\alpha\eta_1) \quad (a \leqq \eta_1 \leqq b),$$

then

$$|\xi_1 - \xi| \geqq |\eta_1 - \eta|.$$

Proof. Kolmogorov [9] proved that if $g \in W^r C$ $(r=1, 2, ...)$,

$$\|g\|_C \leqq \alpha^{-r}\|\varphi_r\|_C = \alpha^{-r}K_r,$$

and

$$g(\xi) = \alpha^{-r}\varphi_r(\alpha\xi),$$

then

$$|g'(\xi)| \leqq \alpha^{1-r}|\varphi_{r-1}(\alpha\xi)|.$$

Let us suppose for the sake of concreteness that $\varphi_r(\alpha t)$ increases on $[a, b]$. If in this case we suppose, for example, that

$$g(\xi+t') = \alpha^{-r}\varphi_r(\alpha(\eta+t')) \quad (0 \leqq t' \leqq b-\eta),$$

then we can choose points ξ' and η', $\xi\leqq\xi'\leqq\xi+t'$, $\eta\leqq\eta'\leqq\eta+t'$ so that $g(\xi')=$ $=\alpha^{-r}\varphi_r(\alpha\eta')$ and $|g'(\xi')|>\alpha^{1-r}\varphi_r(\alpha\eta')$, which contradicts Kolmogorov's above mentioned result. This proves statement 1, from which statement 2 follows immediately.

Lemma 1.2. *If* $g\in W^{r+1}C$ $(r=0, 1, \dots)$ *and*

$$\|g\|_C \leqq \alpha^{-r-1}\|\varphi_{r+1}\|_C \leqq K_{r+1}\alpha^{-r-1}, \tag{1.7}$$

then we have

$$\int_0^{2\pi} \Phi(|g'(x)|)\,dx \leqq \frac{1}{\alpha}\min_t \int_0^{2\pi\alpha} \Phi(\alpha^{-r}|\varphi_r(x+t)|)\,dx. \tag{1.8}$$

In the proof of this assertion we will follow the arguments of Taĭkov [25].

Proof. Without loss of generality we may suppose that $g(0)=0$. Set

$$G_g = \{x\in[0, 2\pi]:\ |g(x)| > 0\}.$$

Let us denote by Δ_k those component intervals of the set G_g whose lengths are $\geqq\alpha^{-1}\pi$ and by δ_k those whose lengths are $<\alpha^{-1}\pi$.

If we prove that for every $\delta_k\in G_g$ we have

$$\int_{\delta_k} \Phi(|g'(x)|)\,dx \leqq \min_t \int_{\delta_k} \Phi(\alpha^{-r}|\varphi_r(\alpha x-t)|)\,dx, \tag{1.9}$$

and for every $\Delta_k\in G_g$ we have

$$\int_{\Delta_k} \Phi(|g'(x)|)\,dx \leqq \int_0^{\alpha^{-1}\pi} \Phi(\alpha^{-r}|\varphi_r(\alpha x)|)\,dx, \tag{1.10}$$

then, by adding up these inequalities, we obtain that

$$\int_0^{2\pi} \Phi(|g'(x)|)\,dx = \sum_k \int_{\delta_k} \Phi(|g'(x)|)\,dx + \sum_k \int_{\Delta_k} \Phi(|g'(x)|)\,dx \leqq$$

$$\leqq \min_t \int_0^{2\pi} \Phi(\alpha^{-r}|\varphi_r(\alpha x-t)|)\,dx = \min_t \frac{1}{\alpha}\int_0^{2\pi\alpha} \Phi(\alpha^{-r}|\varphi_r(x-t)|)\,dx,$$

which means that (1.8) is fulfilled. Thus it is enough to prove inequalities (1.9) and (1.10).

From inequalities (1.1) and (1.7) we obtain

$$\|g'\|_C \leqq K_r\alpha^{-r}. \tag{1.11}$$

Besides this it is clear that

$$\min_t \int_{\delta_k} \Phi(\alpha^{-r}|\varphi_r(\alpha x-t)|)\,dx = \int_{\delta_k} \Phi(\alpha^{-r}|\varphi_r(\alpha x-t_0)|)\,dx, \tag{1.12}$$

where t_0 is chosen so that

$$\varphi_r\left(\frac{1}{2}\alpha(a_k+b_k)-t_0\right) = 0;$$

b_k and a_k are the right and left endpoints of the interval δ_k, respectively. From inequality (1.11), Lemma 1.1, and inequality (1.12) we promptly obtain (1.9).

In [25] it is proved that

$$\int_{\Delta_k} \Phi(|g'(x)|)\,dx = \int_0^{\max_{x\in\Delta_k}|g'(x)|} d\left\{\frac{\Phi(t)}{t}\right\} \int_{E_t} |g'(x)|\,dx, \tag{1.13}$$

where

$$E_t = \{x\in\Delta_k:\ |g'(x)| > t\}.$$

From (1.13) and (1.11) it follows that

$$\int_{\Delta_k} \Phi(|g'(x)|)\,dx \leqq \int_0^{\alpha^{-r}K_r} d\left\{\frac{\Phi(t)}{t}\right\} \int_{E_t} |g'(x)|\,dx.$$

Analogously, we have

$$\int_0^{\alpha^{-1}\pi} \Phi(\alpha^{-r}|\varphi_r(\alpha x)|)\,dx = \int_0^{\alpha^{-r}K_r} d\left\{\frac{\Phi(t)}{t}\right\} \int_{e_t} \alpha^{-r}|\varphi_r(\alpha x)|\,dx,$$

where

$$e_t = \{x\in[0,\alpha^{-1}\pi]:\ |\alpha^{-r}\varphi_r(\alpha x)| > t\}.$$

Hence to prove inequality (1.10), and consequently Lemma 1.2, it suffices to prove that

$$\int_{E_t} |g'(x)|\,dx \leqq \int_{e_t} \alpha^{-r}|\varphi_r(\alpha x)|\,dx.$$

It is clear that

$$\int_{\Delta_k} |g'(x)|\,dx \leqq 2\|g\|_C \leqq 2K_{r+1}\alpha^{-r-1} = \int_0^{\alpha^{-1}\pi} \alpha^{-r}|\varphi_r(\alpha x)|\,dx.$$

On the other hand, if

$$\varphi_r(\alpha\eta_0) = 0,\ |\varphi_r(\alpha\eta)| = t\alpha^r \quad (0 \leqq \eta_0,\ \eta \leqq 2^{-1}\alpha^{-1}\pi),$$

then by Lemma 1.1 (cf. also [10], p. 122) we have

$$\int_{\Delta_k\setminus E_t} |g'(t)|\,dt \geqq 2\int_{\eta_0}^{\eta} \alpha^{-r}|\varphi_r(\alpha x)|\,dx.$$

Consequently,

$$\int_{E_t} |g'(t)|\,dt \leqq \int_0^{\alpha^{-1}\pi} \alpha^{-r}|\varphi_r(\alpha x)|\,dx - 2\int_{\eta_0}^{\eta} \alpha^{-r}|\varphi_r(\alpha x)|\,dx =$$

$$= \int_{e_t} \alpha^{-r}|\varphi_r(\alpha x)|\,dx.$$

This proves Lemma 1.2.

Consequence 1.1. *If $g\in W^rC$ and g is orthogonal to all trigonometric polynomials of order not larger than $n-1$, then we have*

$$\int_0^{2\pi} \Phi(|g(x)|)\,dx \leqq \int_0^{2\pi} \Phi(n^{-r}|\varphi_r(x)|)\,dx.$$

This result is due to TAĬKOV [25].

Proof. Let $G(x)$ be the periodic integral of $g(x)$, the mean value of which over the period is equal to zero. Then the assertion of Consequence 1.1 follows from the Favard inequality

$$\|G\|_C \leqq K_{r+1}n^{-r-1}$$

and from Lemma 1.2.

Consequence 1.2. *If $g\in W^rC$ $(r=0, 1, \ldots)$ and*

$$\|g\|_C \leqq K_{r+1}\alpha^{-r-1},$$

then

$$\int_0^{2\pi} \Phi(|g'(x)|)\,dx \leqq \int_0^{2\pi} \Phi(\alpha^{-r}|\varphi_r(x)|)\,dx.$$

Proof. This assertion follows from Lemma 1.2 and from the following arguments. Let $\varphi_r(x_0)=0$, then for $0\leqq\beta\leqq\pi/2$ we have

$$\frac{\pi}{2}\int_0^{\beta} \Phi(\alpha^{-r}|\varphi_r(x-x_0)|)\,dx \leqq \beta\int_0^{\pi/2} \Phi(\alpha^{-r}|\varphi_r(x-x_0)|)\,dx =$$

$$= \beta\int_0^{\pi/2} \Phi(\alpha^{-r}|\varphi_r(x)|)\,dx,$$

since the function $\frac{1}{\beta}\int_0^{\beta} \Phi(\alpha^{-r}|\varphi_r(x-x_0)|)dx$ increases as β does so, $0\leqq\beta\leqq\pi/2$.

Consequently,

$$\min_t \frac{1}{\alpha} \int_0^{2\pi\alpha} \Phi(\alpha^{-r}|\varphi_r(x-t)|)\,dx \leqq$$

$$\leqq \frac{1}{\alpha}\Big([4\alpha] \int_0^{\pi/2} \Phi(\alpha^{-r}|\varphi_r(x-x_0)|)\,dx + \int_0^{(4\alpha-[4\alpha])\pi/2} \Phi(\alpha^{-r}|\varphi_r(x-x_0)|)\,dx\Big) \leqq$$

$$\leqq \frac{1}{\alpha} 4\alpha \int_0^{\pi/2} \Phi(\alpha^{-r}|\varphi_r(x)|)\,dx = \int_0^{2\pi} \Phi(\alpha^{-r}|\varphi_r(x)|)\,dx.$$

Here $[\gamma]$ denotes the integral part of γ.

Proof of Theorem 1.1. Let $f \in C^{(r+1)}$. Set

$$g(x) = \frac{f(x)}{\|f^{(r+1)}\|_C}.$$

Then $g \in W^{r+1}C$ and by Lemma 1.2 we have

$$\int_0^{2\pi} \Phi(|g'(x)|)\,dx \leqq \int_0^{2\pi} \Phi\left\{\left(\frac{\|g\|_C}{K_{r+1}}\right)^{-r/(r+1)} |\varphi_r(x)|\right\} dx.$$

By substituting $f(x)/\|f^{(r+1)}\|_C$ for $g(x)$ in this inequality we obtain (1.5). To make our proof complete we notice that (1.5) becomes an equality if $f(x)=n^{-r}\varphi_r(nx)$.

Consequence 1.3. *For the elements f of C^{r+1} ($r=0, 1, \ldots$) we have the following sharp inequality:*

$$(1.14) \qquad \left\{\frac{\|f'\|_{L_p}}{\|f^{(r+1)}\|_C \|\varphi_r\|_{L_p}}\right\}^{1/r} \leqq \left\{\frac{\|f\|_C}{K_{r+1}\|f^{(r+1)}\|_C}\right\}^{1/(r+1)} \quad (1 < p < \infty).$$

For the proof ot this inequality it is enough to put $\Phi(t)=t^p$ in Theorem 1.1.

Theorem 1.2. *For the elements f of $C^{(r+1)}$ ($r=0, 1, \ldots$) we have the following sharp inequality:*

$$(1.15) \qquad \left\{\frac{\|f'\|_{L_1}}{4K_{r+1}\|f^{(r+1)}\|_C}\right\}^{1/r} \leqq \left\{\frac{\|f\|_C}{K_{r+1}\|f^{(r+1)}\|_C}\right\}^{1/(r+1)}$$

Proof. As in Theorem 1.1 the inequality (1.15) immediately follows from the following assertion: *If $g \in W^{r+1}C$ and $\|g\|_C \leqq K_{r+1}\alpha^{-r-1}$ ($r=0, 1, 2, \ldots$) then*

$$\|g'\|_{L_1} \leqq \alpha^{-r} \|\varphi_r\|_{L_1}.$$

Following the arguments in the proof of Lemma 1.2 we see that it is enough to show that for $\Phi(t)=t$ we have inequalities (1.9) and (1.10).

Now inequality (1.9) is valid for any non-decreasing function $\Phi(t)$ and (1.10) follows from the following chain of inequalities

$$\int_{\Delta_k} |g'(x)|\,dx \leqq 2\|g\|_C \leqq 2K_{r+1}\alpha^{-r-1} = \int_0^{\alpha^{-1}\pi} |\varphi_r(\alpha x)|\,dx.$$

We remark that inequality (1.15) becomes an equality if $f(x)=n^{-r}\varphi_r(nx)$.

Summing up the assertions of Consequence 1.3 and Theorem 1.2 and using inequality (1.1), we obtain the following result.

Theorem 1.3. *For the elements f of $C^{(r)}$ $(r=1, 2, \ldots)$ we have the following sharp inequality:*

$$\left\{\frac{\|f^{(r-k)}\|_{L_p}}{\|f^{(r)}\|_C\|\varphi_k\|_{L_p}}\right\}^{1/k} \leqq \left\{\frac{\|f\|_C}{K_r\|f^{(r)}\|_C}\right\}^{1/r} \quad (1 \leqq k < r,\ 1 \leqq p < \infty). \tag{1.16}$$

The following assertion gives a slight sharpening of a result of KORNEĬČUK [10].

Lemma 1.3. *If $g\in W^rC$ $(r=1, 2, \ldots)$, $\|g\|_C \leqq K_r\alpha^r$, then*

$$\omega(g, t) \leqq \omega\left(\alpha^r\varphi_r\left(\frac{x}{\alpha}\right), t\right) = 2\left|\alpha^r\varphi_r\left(\frac{t}{2\alpha}-t_0\right)\right| \quad (0 \leqq t \leqq \alpha\pi),$$

where t_0 is chosen so that $\varphi_r(t_0)=0$.

Proof. This statement can be proved precisely the same way as Lemma 6.2 in the work [10] of KORNEĬČUK. Indeed, if

$$\omega(g, t) = |g(x')-g(x'')|, \quad |x'-x''| \leqq t,$$

then there exist points η' and η'' such that

$$\alpha^r\varphi_r(\alpha^{-1}\eta') = g(x'), \quad \alpha^r\varphi_r(\alpha^{-1}\eta'') = g(x''),$$

and the function $\varphi_r(\alpha^{-1}x)$ is monotone on the interval (η', η''). By Lemma 1.1 we have $|\eta'-\eta''| \leqq |x'-x''| \leqq t$, and the proof of our lemma is finished.

Theorem 1.4. *For the elements f of $C^{(r)}$ $(r=1, 2, \ldots)$ we have the following sharp inequality:*

$$\omega(f, t) \leqq \frac{2\|f\|_C}{K_r\|f^{(r)}\|_C}\left|\varphi_r\left(\left\{\frac{\|f\|_C}{K_r\|f^{(r)}\|_C}\right\}^{-1/r}\frac{t}{2}-t_0\right)\right|. \tag{1.17}$$

Proof. Let $f\in C^{(r)}$ $(r=1, 2, \ldots)$. Set $g(x)=f(x)/\|f^{(r)}\|_C$. Then by Lemma 1.3 we have

$$\omega(g, t) \leqq \frac{2\|g\|_C}{K_r}\left|\varphi_r\left(\left\{\frac{K_r}{\|g\|_C}\right\}^{1/r}\frac{t}{2}-t_0\right)\right|.$$

Substituting $g(x)=f(x)/\|f^{(r)}\|_C$ we obtain inequality (1.17). It only remains to remark that for $f(x)=n^{-r}\varphi_r(nx)$ and for $0 \leqq t \leqq \pi/n$ inequality (1.17) becomes an equality.

§ 2. Estimation of the approximation of one class of functions by another

Theorem 2.1. *For every* $1<p\leqq\infty$ $(L_\infty=C)$ *we have the inequality*

$$A_p^{(r)}(N)=\sup_{f\in W^rL_p}\inf_{\varphi\in NW^{r+1}L_1}\|f-\varphi\|_{L_1}\leqq \tag{2.1}$$
$$\leqq\sup_{\alpha>0}\{\|\varphi_r\|_{L_{p'}}\alpha^r-NK_{r+1}\alpha^{r+1}\},$$

where $p'=p/(p-1)$ *for* $1<p<\infty$ *and* $p'=1$ *for* $p=\infty$, *and the function* $\varphi_r(x)$ *is defined by equation* (1.4).

Proof. From the results of IOFFE and TIHOMIROV (cf. [7, Theorem 3.1]) it follows that

$$A_p^{(r)}(N)=\sup_{g\in W_0^0C}\Big\{\sup_{f\in W^rL_p}\int_{-\pi}^{\pi}f(t)g(t)\,dt-\sup_{f\in NW^{r+1}L_1}\int_{-\pi}^{\pi}f(t)g(t)\,dt\Big\}.$$

It is clear (cf., for example, [11] or [18]) that

$$\sup_{f\in W^rL_p}\int_{-\pi}^{\pi}f(t)g(t)\,dt=\sup_{f\in W_0^0L_p}\int_{-\pi}^{\pi}f(t)g'_{r+1}(t)\,dt=$$
$$=\min_\lambda\|g'_{r+1}(x)-\lambda\|_{L_{p'}}=E_1(g'_{r+1})_{L_{p'}},$$

where g_{r+1} is the rth periodic integral of the function g and $E_1(\varphi)_{L_{p'}}$ is the best approximation, in $L_{p'}$, of the function φ by constants. Analogously,

$$\sup_{f\in NW^{r+1}L_1}\int_{-\pi}^{\pi}f(t)g(t)\,dt=\sup_{f\in NW_0^0L_1}\int_{-\pi}^{\pi}f(t)g_{r+1}(t)\,dt=$$
$$=N\min_\lambda\|g_{r+1}(x)-\lambda\|_C=NE_1(g_{r+1})_C.$$

Consequently,

$$A_p^r(N)=\sup_{g\in W^{r+1}C}\{E_1(g')_{L_{p'}}-NE_1(g)_C\}.$$

From Theorem 1.3 it follows that if

$$E_1(g)\leqq K_{r+1}\alpha^{r+1},$$

then for $1\leqq p'<\infty$ we have

$$\|g'\|_{L_{p'}}\leqq\alpha^r\|\varphi_r\|_{L_{p'}},$$

and even

$$E_1(g')_{L_{p'}}\leqq\alpha^r\|\varphi_r\|_{L_{p'}},$$

which proves inequality (2.1).

Lemma 2.1. *For* $1\leqq k<r$ *we have the inequality*

$$\inf_{\varphi\in NW^rL_1}\|D_k-\varphi\|_{L_1}\leqq\max_{\alpha>0}\{K_k\alpha^k-NK_r\alpha^r\},$$

where

$$D_k(x) = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\cos\left(\nu x - \frac{\pi k}{2}\right)}{\nu^k} \quad (k = 1, 2, \ldots).$$

Proof. From Theorem 3.1 of [7] it follows that

$$\inf_{\varphi \in NW^r L_1} \|D_k - \varphi\|_{L_1} = \sup_{g \in W_0^0 C} \left\{ \int_{-\pi}^{\pi} D_k(t) g(t)\, dt - \sup_{\varphi \in NW^r L_1} \int_{-\pi}^{\pi} \varphi(t) g(t)\, dt \right\}.$$

If $f \in L_1^{(k)}$ then

$$f(x) = \int_{-\pi}^{\pi} D_k(x-t) f^{(k)}(t)\, dt.$$

Consequently,

$$\inf_{\varphi \in NW^r L_1} \|D_k - \varphi\|_{L_1} \leqq \sup_{g \in W^r C} \{g^{(r-k)}(0) - NE_1(g)_C\} \leqq$$

$$\leqq \sup_{g \in W^r C} \{\|g^{(r-k)}\|_C - NE_1(g)_C\}.$$

By using Kolmogorov's inequality (1.1) in the last formula, the proof of our lemma is completed.

Theorem 2.2. *Let $X = C$ or $X = L_1$. For any $1 \leqq k \leqq r$ there exists a linear operator $A_{r,k}(f)$ mapping the class $W^k X$ into the class $NW^r X$ such that*

$$\sup_{f \in W^k X} \|f - A_{r,k}(f)\|_X \leqq \sup_{x > 0} \{K_k x^k - NK_r x^r\}.$$

Proof. From Lemma 2.1 it follows that for every $1 \leqq k < r$ there exists a function $\psi_{r,k}(t)$ such that

$$\int_{-\pi}^{\pi} \psi_{r,k}(t)\, dt = 0, \quad \|\psi_{r,k}^{(r)}\|_{L_1} \leqq 1,$$

and

$$\min_{\lambda} \|D_k - \psi_{r,k} - \lambda\| \leqq \max_{x > 0} \{K_k x^k - NK_r x^r\}.$$

Let us set

$$A_{r,k}(f, x) = \int_{-\pi}^{\pi} f^{(k)}(t) \psi_{r,k}(x-t)\, dt.$$

It is clear that $A_{r,k}(f)$ is a linear operator and, moreover, if $f \in W^k X$ then

$$\|A_{r,k}^{(r)}(f)\|_X \leqq \|\psi_{r,k}^{(r)}\|_{L_1} \|f^{(k)}\|_X \leqq N.$$

On the other hand, for $f \in W^k X$ we have

$$\|f-A_{r,k}(f)\|_X = \left\| \int_{-\pi}^{\pi} \{D_k(x-t)-\psi_{r,k}(x-t)-\lambda\} f^{(k)}(t)\,dt \right\|_X \leqq$$

$$\leqq \min_{\lambda} \|D_k-\psi_{r,k}-\lambda\|_{L_1} \|f^{(k)}\|_X \leqq \max_{x>0} \{K_k x^k - NK_r x^r\}.$$

From this our theorem follows.

In the sequel we will need the following two assertions which give estimation for the approximation of a class by a class.

Theorem 2.3 (Korneĭčuk [11]). *If the modulus of continuity $\omega(t)$ is convex from below, then for all $r=0, 1, \ldots$ we have*

$$\sup_{f \in W^r H_C^\omega} \inf_{\varphi \in N W^{r+1} C} \|f-\varphi\|_C \leqq \max_{x>0} \int_0^x \Theta_{x,r}(t) \{\omega'(t)-N\}\,dt,$$

where

$$\Theta_{x,0}(t) = \begin{cases} \frac{1}{2}, \\ 0, \end{cases} \quad \textit{and for } r = 1,2 \ldots \quad \Theta_{x,r}(t) = \begin{cases} \frac{1}{2} \int_0^{x-t} \Theta_{x,r-1}(u)\,du, & 0 \leqq t \leqq x, \\ 0, & t > x. \end{cases}$$

Theorem 2.4 (Turovec [29]). *If the modulus of continuity $\omega(t)$ is convex from below, then*

$$\sup_{f \in H_C^\omega} \inf_{\varphi \in N W^r C} \|f-\varphi\|_{L_1} \leqq \pi \max_{x>0} \frac{1}{x} \int_0^x \{\omega(t)-Nt\}\,dt.$$

§ 3. Inequalities for upper bounds of semi-norms

Let $\mathcal{U}(t)$ be a linear operator mapping the Banach space R into itself and let $\mathfrak{M}$ be an arbitrary class of elements of R. In the present section we will obtain a series of sharp inequalities between the quantities of the type

$$\sup_{f \in \mathfrak{M}} \|f-\mathcal{U}(f)\|_R. \tag{3.1}$$

The question of sharpness of these inequalities will be considered later (in § 4 and § 5).

Let us introduce some notations which will be used in this section. Let $\Psi(f)$ be an arbitrary semi-norm given on the space R, i.e., a functional such that

$$\Psi(0) = 0, \quad \Psi(f) \geqq 0, \quad \Psi(f_1+f_2) \leqq \Psi(f_1)+\Psi(f_2), \quad \Psi(\lambda f) = |\lambda| \Psi(f).$$

Set

(3.2) $$l_{\nu,m}(R) = \sup_{f \in W_m^\nu R} \Psi(f) \quad (\nu, m = 0, 1, \dots),$$

(3.3) $$l_\nu(R) = \sup_{f \in W^\nu R} \Psi(f) \quad (\nu = 0, 1, \dots),$$

(3.4) $$l_\nu^\omega(R) = \sup_{f \in W^\nu H_R^\omega} \Psi(f) \quad (\nu = 0, 1, \dots).$$

In the author's paper [14] the following results are proved:

Theorem 3.1. *For $m=0, 1, \dots$, $r=2, 3, \dots$ and $1 \leqq k < r$ we have the inequality*

(3.5) $$\left\{\frac{l_{k,m}(C)}{K_k l_{0,m}(C)}\right\}^{1/k} \leqq \left\{\frac{l_{r,m}(C)}{K_r l_{0,m}(C)}\right\}^{1/r},$$

where

(3.6) $$K_r = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(r+1)}}{(2\nu+1)^{r+1}}.$$

Theorem 3.2. *For all $m=0, 1, \dots$, $r=2, 3, \dots$ and for all $1 \leqq k < r$ we have the following inequality*

(3.7) $$\left\{\frac{l_{k,m}(L_1)}{K_k l_{0,m}(L_1)}\right\}^{1/k} \leqq \left\{\frac{l_{r,m}(L_1)}{K_r l_{0,m}(L_1)}\right\}^{1/r}.$$

For $m=0$, $r=2, 3, 4, 5$ and $1 \leqq k < r$ inequality (3.5) was earlier obtained by Žuk [30]. We will prove some further inequalities of the type (3.5) or (3.7).

The following result is quite simple, however, for the sake of completeness we include it here.

Theorem 3.3. *For all $m=0, 1, \dots, 0<k<r$ we have the inequality*

(3.8) $$\left\{\frac{l_{k,m}(L_2)}{l_{0,m}(L_2)}\right\}^{1/k} \leqq \left\{\frac{l_{r,m}(L_2)}{l_{0,m}(L_2)}\right\}^{1/r}.$$

Proof. Let $f \in L_2^{(k)}$ and

$$f(t) \sim \sum_{\nu=m}^{\infty} (a_\nu \cos \nu t + b_\nu \sin \nu t)$$

be the Fourier expansion of this function. Let us set

$$\varphi(f, t) = \sum_{\nu=m}^{\infty} \varrho_\nu (a_\nu \cos \nu t + b_\nu \sin \nu t),$$

where the numbers $\varrho_\nu = \varrho_\nu(h)$ are defined in the following way: $\varrho_\nu = h\nu^{k-r}$ if $h \leqq m^{r-k}$ and

$$\varrho_\nu = \begin{cases} 1, & m \leqq \nu \leqq h^{(r-k)^{-1}}, \\ h\nu^{r-k}, & h^{(r-k)^{-1}} < \nu < \infty, \end{cases}$$

if $h > m^{r-k}$. It is not hard to see that $\varphi(f,t)$ and $\varphi^{(r)}(f,t)=\varphi(f^{(r)},t)\in L_2$. Using the sub-additivity of the functional $\Psi(f)$ and the definition of the quantity $l_{\nu,m}(L_2)$ we obtain:

$$\Psi(f) \leqq l_{0,m}(L_2)\,\|f-\varphi(f)\|_{L_2}+l_{r,m}(L_2)\,\|\varphi(f^{(r)})\|_{L_2}$$

or

$$\Psi(f) \leqq l_{0,m}(L_2)\left\{\sum_{\nu=m}^{\infty}(1-\varrho_\nu)^2(a_\nu^2+b_\nu^2)\right\}^{1/2}+l_{r,m}(L_2)\left\{\sum_{\nu=m}^{\infty}(\varrho_\nu \nu^r)^2(a_\nu^2+b_\nu^2)\right\}^{1/2}$$

for every function $f\in L_2^{(k)}, f\perp t_{m-1}$. From this we get

$$\Psi(f) \leqq \left\{l_{0,m}(L_2)\max_{m\leqq\nu<\infty}\left|\frac{1-\varrho_\nu}{\nu^k}\right|+l_{r,m}(L_2)\max_{m\leqq\nu<\infty}|\varrho_\nu \nu^{r-k}|\right\}\|f^{(k)}\|_{L_2}$$

or

$$l_{k,m}(L_2) \leqq l_{0,m}(L_2)\max_{m\leqq\nu<\infty}\left|\frac{1-\varrho_\nu}{\nu^k}\right|+l_{r,m}(L_2)\max_{m\leqq\nu<\infty}|\varrho_\nu \nu^{r-k}|. \tag{3.9}$$

Since we have

$$\max_{m\leqq\nu<\infty}\left|\frac{1-\varrho_\nu}{\nu^k}\right| \leqq \max_{h^{1/(r-k)}\leqq x<\infty}\left(\frac{1}{x^k}-\frac{h}{x^r}\right)=\frac{r-k}{r}\left(\frac{k}{rh}\right)^{k/(r-k)}$$

and

$$\max_{m\leqq\nu<\infty}|\varrho_\nu h^{r-k}| = h,$$

from (3.9) it follows easily that for any $h>0$ the inequality

$$l_{k,m}(L_2) \leqq \frac{r-k}{r}\left(\frac{k}{rh}\right)^{k/(r-k)} l_{0,m}(L_2)+h\,l_{r,m}(L_2)$$

holds true. Setting

$$h=\frac{k}{r}\,l_{0,m}(L_2)^{(r-k)/r}\,l_{r,m}(L_2)^{k/r},$$

we obtain that

$$l_{k,m}(L_2) \leqq l_{0,m}(L_2)^{(r-k)/r}\,l_{r,m}(L_2)^{k/r},$$

which is equivalent to (3.8).

Theorem 3.4. *Let $\Psi(f)$ vanish on the constant functions. Then for $1<p\leqq\infty$ ($L_\infty=C$), $r=2, 3, \ldots$ and $1\leqq k<r$ we have the inequality*

$$\left\{\frac{l_k(L_p)}{l_0(L_1)\,\|\varphi_k\|_{L_{p'}}}\right\}^{1/k} \leqq \left\{\frac{l_r(L_1)}{K_r\,l_0(L_1)}\right\}^{1/r}, \tag{3.10}$$

where $p'=p/(p-1)$ if $1<p<\infty$ and $p'=1$ if $p=\infty$. The function $\varphi_k(x)$ is determined by means of equation (1.4).

Proof. Let f be an arbitrary function from the class W^kL_p $(1<p\leqq\infty)$ and let $\varphi=\varphi(f)$ be the function from the class $NW^{k+1}L_1$ which approximates the function f the best in the metric of L_1. The sub-additivity of the functional $\Psi(f)$ implies that

$$\Psi(f)\leqq\Psi(f-\varphi)+\Psi(\varphi)\leqq l_0(L_1)\,\|f-\varphi\|_{L_1}+Nl_{k+1}(L_1).$$

From this and from Theorem 2.1 it follows that

$$\Psi(f)\leqq l_0(L_1)\max_{\alpha\geqq 0}\{\|\varphi_k\|_{L_{p'}}\alpha^k-NK_{k+1}\alpha^{k+1}\}+Nl_{k+1}(L_1)$$

or

$$l_k(L_p)\leqq l_0(L_1)\max_{\alpha\geqq 0}\{\|\varphi_k\|_{L_{p'}}\alpha^k-NK_{k+1}\alpha^{k+1}\}+Nl_{k+1}(L_1).$$

Given an arbitrary α_0, we can find an $N_0=N(\alpha_0)$ such that

$$\max_{\alpha\geqq 0}\{\|\varphi_k\|_{L_{p'}}\alpha^k-N_0K_{k+1}\alpha^{k+1}\}=\|\varphi_k\|_{L_{p'}}\alpha_0^k-N_0K_{k+1}\alpha_0^{k+1}.$$

Consequently, choosing α_0 satisfy the equation

$$K_{k+1}l_0(L_1)\alpha_0^{k+1}=l_{k+1}(L_1),$$

that is

$$\alpha_0=\left\{\frac{l_{k+1}(L_1)}{K_{k+1}l_0(L_1)}\right\}^{1/(k+1)},$$

we obtain that

$$l_k(L_p)\leqq l_0(L_1)\,\|\varphi_k\|_{L_{p'}}\left\{\frac{l_{k+1}(L_1)}{K_{k+1}l_0(L_1)}\right\}^{k/(k+1)}$$

or

$$\left\{\frac{l_k(L_p)}{\|\varphi_k\|_{L_{p'}}l_0(L_1)}\right\}^{1/k}\leqq\left\{\frac{l_{k+1}(L_1)}{K_{k+1}l_0(L_1)}\right\}^{1/(k+1)} \tag{3.11}$$

Now, from Theorem 3.2 it follows that if $\Psi(f)$ is a semi-norm vanishing on the constant functions, then for $1\leqq k<r$ we have

$$\left\{\frac{l_k(L_1)}{K_kl_0(L_1)}\right\}^{1/k}\leqq\left\{\frac{l_r(L_1)}{K_rl_0(L_1)}\right\}^{1/r},$$

which together with (3.11) furnishes the proof of our theorem.

Theorem 3.5. *If the modulus of continuity $\omega(t)$ is convex from below and $\Psi(t)$ vanishes on the constant functions, then for all $0\leqq k<r$ we have the inequality*

$$l_k^{\omega}(C)\leqq l_0(C)\int_0^{\lambda_r}\omega'(t)\,\Theta_{\lambda_r,k}(t)\,dt, \tag{3.12}$$

where

$$\lambda_r=\pi\left\{\frac{l_r(C)}{K_rl_0(C)}\right\}^{1/r},$$

the K_r are the Favard constants and $\Theta_{\alpha,r}$ are the functions defined in Theorem 2.3.

Proof. Let $f\in W^kH_C^\omega$ and denote by $\varphi(\in NW^{k+1}C)$ the function which differs, in the metric of C, from the function f the least among all functions in the class $NW^{k+1}C$. From the sub-additivity of the functional $\Psi(f)$ and from the definition of the quantities $l_\nu(C)$ it follows that

$$\Psi(f) \leqq \Psi(f-\varphi)+\Psi(\varphi) \leqq l_0(C)\,\|f-\varphi\|_C+Nl_{k+1}(C). \tag{3.13}$$

From this and from Theorem 2.3 it follows for every function $f\in W^kH_C^\omega$ $(k=0, 1, \dots)$ that

$$\Psi(f) \leqq l_0(C)\max_{\alpha\geqq 0}\int_0^\alpha \Theta_{\alpha,k}(t)\,\{\omega'(t)-N\}\,dt+Nl_{k+1}(C).$$

This implies that for $k=0, 1, \dots$ we have

$$l_k^\omega(C) \leqq l_0(C)\max_{\alpha\geqq 0}\int_0^\alpha \Theta_{\alpha,k}(t)\,\{\omega'(t)-N\}\,dt+Nl_{k+1}(C). \tag{3.14}$$

In [11] it is proved that for an arbitrary $\alpha_0>0$ we can find $N_0=N(\alpha_0)$ such that

$$\max_{\alpha\geqq 0}\int_0^\alpha \Theta_{\alpha,r}(t)\,\{\omega'(t)-N_0\}\,dt = \int_0^{\alpha_0} \Theta_{\alpha_0,r}(t)\,\{\omega'(t)-N_0\}\,dt.$$

Setting $\alpha_0=\lambda_{r+1}$ in (3.14) we obtain that

$$l_k^\omega(C) \leqq l_0(C)\int_0^{\lambda_{k+1}} \Theta_{\lambda_{k+1},k}(t)\,dt+N(\lambda_{k+1})l_{k+1}(C)- \tag{3.15}$$

$$-N(\lambda_{k+1})l_0(C)\int_0^{\lambda_{k+1}} \Theta_{\lambda_{k+1},k}(t)\,dt.$$

As is proved in [11],

$$\int_0^\alpha \Theta_{\alpha,k}(t)\,dt = K_{k+1}\left(\frac{\alpha}{\pi}\right)^{k+1}$$

Using this equality in inequality (3.15) we obtain

$$l_k^\omega(C) \leqq l_0(C)\int_0^{\lambda_{k+1}} \Theta_{\lambda_{k+1},k}(t)\,\omega'(t)\,dt.$$

If $\Psi(f)$ vanishes on the constant functions, then by Theorem 1.1 we have for $1\leqq k\leqq r$ that

$$\left\{\frac{l_k(C)}{K_k l_0(C)}\right\}^{1/k} \leqq \left\{\frac{l_r(C)}{K_r l_0(C)}\right\}^{1/r}. \tag{3.16}$$

Consequently, for $1\leqq k<r$ we have

$$\lambda_k \leqq \lambda_r. \tag{3.17}$$

On the other hand, it is clear that for $0 \leqq a \leqq b$, $k=0, 1, \dots$ we have

$$\int_0^a \Theta_{a,k}(t)\omega'(t)\,dt \leqq \int_0^b \Theta_{b,k}(t)\omega'(t)\,dt. \tag{3.18}$$

Comparing inequalities (3.15), (3.17), and (3.18) we obtain the assertion of our theorem.

For $k=0$ and $r=1$ inequality (3.12) has the from

$$l_0^\omega(C) \leqq \frac{l_0(C)}{2}\,\omega\left(\frac{2l_1(C)}{l_0(C)}\right). \tag{3.19}$$

In particular, if $\mathscr{U}_n(f)$ is a linear operator mapping C into the set of trigonometric polynomials of order not greater than $n-1$ and such that $\mathscr{U}_n(\lambda)=\lambda$, then, setting

$$\Psi(f) = \|f-\mathscr{U}_n(f)\|_C$$

in (3.10), and taking into account that in this case

$$l_0(C) = \sup_{f\in W^0C} \|f-\mathscr{U}_n(f)\|_C \leqq 2,$$

we obtain the following assertion.

Consequence 3.1. *If the modulus of continuity $\omega(t)$ is convex from below, then*

$$\sup_{f\in H_C^\omega} \|f-\mathscr{U}_n(f)\|_C \leqq \omega\big(\sup_{f\in W^1C} \|f-\mathscr{U}_n(f)\|_C\big).$$

Theorem 3.6. *If the modulus of continuity $\omega(t)$ is convex from below and $\Psi(f)$ vanishes on the constant functions, then*

$$l_0^\omega(C) \leqq \pi l_0(L_1)\frac{\pi l_0(L_1)}{2l_1(C)} \int_0^{\frac{2l_1(C)}{\pi l_0(L_1)}} \omega(t)\,dt.$$

Proof. Let $f\in H_C^\omega$ and denote by $\varphi(f)$ ($\in NW^1C$) the function which approximates, in the metric of L_1, the function f the best among all functions belonging to the class NW^1C. Then we have

$$\Psi(f) \leqq \Psi(f-\varphi(f))+\Psi(\varphi(f)) \leqq l_0(L_1)\,\|f-\varphi(f)\|_{L_1}+Nl_1(C).$$

From this and from Theorem 2.4 we obtain for all $f\in H_C^\omega$ that

$$\Psi(f) \leqq \pi l_0(L_1)\max_{\alpha>0}\left\{\frac{1}{\alpha}\int_0^\alpha \omega(t)\,dt-\frac{N\alpha}{2}\right\}+Nl_1(C)$$

or

$$l_0^\omega(C) \leqq \pi l_0(L_1) \max_{\alpha>0} \left\{ \frac{1}{\alpha} \int_0^\alpha \omega(t)\,dt - \frac{N\alpha}{2} \right\} + N l_1(C).$$

If the function $\omega(t)$ is convex from below, then the function

$$\omega_1(\alpha) = \frac{1}{\alpha} \int_0^\alpha \omega(t)\,dt$$

is also convex from below with respect to α, consequently, for an arbitrary $\alpha_0 > 0$ we can find $N_0 = N(\alpha_0)$ such that

$$\max_{\alpha>0} \left\{ \frac{1}{\alpha} \int_0^\alpha \omega(t)\,dt - \frac{1}{2} N_0 \alpha \right\} = \frac{1}{\alpha_0} \int_0^{\alpha_0} \omega(t)\,dt - \frac{1}{2} N_0 \alpha_0 .$$

Setting

$$\alpha_0 = \frac{2 l_1(C)}{\pi l_0(L_1)},$$

we obtain the statement of our theorem.

§ 4. Application of the inequalities for upper bounds of semi-norms to some problems of approximation theory

Let H_n be an arbitrary n-dimensional subspace of the space R and suppose that H_n contains the constant functions. Set

$$E_{H_n}(f)_R = \inf_{\varphi \in H_n} \|f - \varphi\|_R$$

and

$$E_{H_n}(\mathfrak{M})_R = \sup_{f \in \mathfrak{M}} E_{H_n}(f)_R$$

for any class of functions $\mathfrak{M} \subset R$. Since

$$E_{H_n}(W_0^0 R)_R = E_{H_n}(W^0 R)_R \leqq 1,$$

then, substituting in Theorems 3.1—3.3 $m=0$ and

$$\Psi(f) = E_{H_m}(f)_R \quad (R = C,\ L_1,\ \text{or}\ L_2),$$

we obtain the following results.

Consequence 4.1. *For any subspace $H_n \subset X$ and for $1 \leqq k < r$ we have the inequality*

$$\left\{ \frac{E_{H_n}(W^k X)_X}{K_k} \right\}^{1/k} \leqq \left\{ \frac{E_{H_n}(W^r X)_X}{K_r} \right\}^{1/r}, \tag{4.1}$$

where $X = C$ or $X = L_1$.

Consequence 4.2. *For any subspace $H_n \subset L_2$ and for $0<k<r$ we have the inequality*

$$E_{H_n}(W^k L_2)_{L_2}^{1/k} \leqq E_{H_n}(W^r L_2)_{L_2}^{1/r}. \tag{4.2}$$

Analogously, from Theorems 3.4—3.6 we obtain the following consequences.

Consequence 4.3. *For any subspace $H_n \in L_p$ $(1<p\leqq\infty)$ and for $1\leqq k<r$ we have the inequality*

$$\left\{\frac{E_{H_n}(W^k L_p)_{L_1}}{\|\varphi_k\|_{L_{p'}}}\right\}^{1/k} \leqq \left\{\frac{E_{H_n}(W^r L_1)_{L_1}}{K_r}\right\}^{1/r}. \tag{4.3}$$

Consequence 4.4. *If the modulus of continuity $\omega(t)$ is convex from below and $0\leqq k<r$, then for all $H_n \subset C$ we have the inequality*

$$E_{H_n}(W^k H_C^{\omega})_C \leqq \int_0^{\gamma_r} \omega'(t)\,\Theta_{\gamma_r,r}(t)\,dt, \tag{4.4}$$

where

$$\gamma_r = \pi\left\{\frac{E_{H_n}(W^r C)_C}{K_r}\right\}^{1/r}. \tag{4.5}$$

Consequence 4.5. *If the modulus of continuity $\omega(t)$ is convex from below, then for every subspace $H_n \subset L_1$ we have the inequality*

$$E_{H_n}(H_C^{\omega})_{L_1} \leqq \pi\lambda_n^{-1}\int_0^{\lambda_n} \omega(t)\,dt, \tag{4.6}$$

where

$$\lambda_n = \frac{2}{\pi} E_{H_n}(W^1 C)_{L_1}.$$

Let $\hat{E}_{H_n}(f)_R$ be the best approximation of f from below in the metric of R by elements of $H_n(\subset C)$ (cf., for example, [1], p. 384), i. e.,

$$\hat{E}_{H_n}(f)_R = \inf_{\substack{f\in H_n \\ \varphi\leqq f}} \|f-\varphi\|_R$$

and

$$\hat{E}_{H_n}(\mathfrak{M})_R = \sup_{f\in\mathfrak{M}} \hat{E}_{H_n}(f)_R.$$

Since

$$\hat{E}_{H_n}(W_0^0 C)_{L_1} \leqq \sup_{f\in W_0^0 C} \int_{-\pi}^{\pi} \{f(x)-\lambda\}\,dx \leqq 2\pi$$

and for $\nu=1, 2, \ldots$

$$\hat{E}_{H_n}(W_0^{\nu} C)_{L_1} = \hat{E}_{H_n}(W^{\nu} C)_{L_1},$$

substituting, in Theorem 3.1, $m=0$ and

$$\Psi(f) = \hat{E}_{H_n}(f)_{L_1},$$

we obtain the following result.

Consequence 4.6. *For any subspace $H_n \subset C$ and for $1 \leqq k < r$ we have the inequality*

$$\left\{\frac{\hat{E}_{H_n}(W^k C)_{L_1}}{2\pi K_k}\right\}^{1/k} \leqq \left\{\frac{\hat{E}_{H_n}(W^r C)_{L_1}}{2\pi K_r}\right\}^{1/r}. \tag{4.7}$$

If $H_n = T_{2m-1}$, the subspace of trigonometric polynomials of order not greater than $m-1$, then as follows from the papers [2], [5], [10], and [27], inequalities (4.1)—(4.4) and (4.7) become equalities.

In the sequel we will consider the problem of determining diameters of sets in Banach spaces and the problem of finding the best subspaces for these sets.

The n-dimensional diameter of the set $\mathfrak{M}$ in the space R is, by definition, the quantity

$$d_n(\mathfrak{M}, R) = \inf_{H_n \subset R} E_{H_n}(\mathfrak{M})_R, \tag{4.8}$$

where inf is taken over all subspaces H_n of dimension n.

The n-dimensional subspace H_n is called a best subspace for the set $\mathfrak{M}$ in the space R if

$$E_{H_n}(\mathfrak{M})_R = d_n(\mathfrak{M}, R). \tag{4.9}$$

The problem of diameters is due to Kolmogrov. It was he who obtained the first sharp results concerning the calculation of diameters. He proved [8], for example, that for $r>0$ and $n=1, 2, \dots$ we have

$$d_{2n}(W^r L_2, L_2) = d_{2n-1}(W^r L_2, L_2) = \frac{1}{n^r}. \tag{4.10}$$

Tihomirov ([27] and [28]) proved that

$$d_{2n}(W^r C, C) = d_{2n-1}(W^r C, C) = \frac{K_r}{n^r} \quad (n, r = 1, 2, \dots). \tag{4.11}$$

Korneĭčuk [10] proved that for any modulus of continuity $\omega(t)$ that is convex from below and for $r=0, 1, \dots, n=1, 2, \dots$ we have

$$d_{2n-1}(W^r H_C^\omega, C) = \int_0^{\pi/n} \Theta_{\pi/n, r}(t)\,\omega'(t)\,dt, \tag{4.12}$$

where $\Theta_{a,r}(t)$ are the same functions as in Theorem 2.3.

RUBAN [19] proved that if the modulus of continuity $\omega(t)$ is convex from below, then

(4.13) $$d_{2n}(W^1 H_C^\omega, C) = d_{2n-1}(W^r H_C^\omega, C)$$

and

(4.14) $$d_{2n-1}(W^1 H_C^\omega, L_1) = 4n \int_0^{\pi/n} \Theta_{\pi/n,2}(t)\omega'(t)\,dt.$$

MAKAVOZ ([16] and [17]) found that

(4.15) $$d_{2n-1}(W^r C, L_1) = \frac{4K_{r+1}}{n^r} \quad (n, r = 1, 2, \ldots)$$

and

(4.16) $$d_{2n}(W^1 C, L_1) = d_{2n-1}(W^1 C, L_1) \quad (n = 1, 2, \ldots).$$

SUBBOTIN [24] proved that

(4.17) $$d_{2n-1}(W^r L_1, L_1) = \frac{K_r}{n^r} \quad (n, r = 1, 2, \ldots).$$

We recalled here only those results on the precise calculation of diameters which will be necessary in the sequel.

Theorem 4.1. *For $n, r=1, 2, \ldots$ we have*

(4.18) $$d_{2n}(W^r L_1, L_1) = d_{2n-1}(W^r L_1, L_1) = \frac{K_r}{n^r}.$$

Proof. From the obvious inequality

$$d_{2n}(W^r L_1, L_1) \leqq d_{2n-1}(W^r L_1, L_1)$$

and from relation (4.17) it follows that it is enough to prove the inequality

(4.19) $$d_{2n}(W^r L_1, L_1) \geqq \frac{K_r}{n^r}.$$

From Consequence 4.3 for $k=1$ and $p=\infty$ we obtain that

$$\frac{d_{2n}(W^1 C, L_1)}{4K_2} \leqq \left\{\frac{d_{2n}(W^r L_1, L_1)}{K_r}\right\}^{1/r}$$

for all $n, r=1, 2, \ldots$. Comparing this with equality (4.16), we obtain (4.19) and thus the assertion of the theorem.

From equality (4.11) and Consequence 4.1 we can infer the following

Consequence 4.7. *For $1 \leqq k < r$ in the space C every n-dimensional subspace, extremal for the class $W^r C$ is extremal for the class $W^k C$, as well.*

From Consequences 4.1 and 4.2 and from equalities (4.18) and (4.10) we obtain the following results.

Consequence 4.8. *For $1 \leqq k < r$ in the space L_1 every n-dimensional subspace, extremal for the class $W^r L_1$ is extremal also for the class $W^k L_1$.*

Consequence 4.9. *For $0 < k < r$ in the space L_2 every n-dimensional subspace, extremal for the class $W^r L_2$ is extremal also for the class $W^k L_2$.*

From Consequence 4.3 for $p = \infty$ and from equalities (4.15) and (4.17) we obtain the following result.

Consequence 4.10. *For $1 \leqq k < r$ in the space L_1 every $2n-1$-dimensional subspace, extremal for the class $W^r L_1$ is extremal for the class $W^k C$, as well.*

From Consequence 4.3 for $p = \infty$ and $k = 1$, and from equalities (4.16) and (4.18) we can infer the following assertion.

Consequence 4.11. *For $r \geqq 1$ in the space L_1 any n-dimensional subspace, extremal for the class $W^r L_1$ is extremal also for the class $W^1 C$.*

From Consequences 4.4 and 4.5, and from equalities (4.12)—(4.14) we obtain:

Consequence 4.12. *Suppose that the modulus of continuity $\omega(t)$ is convex from below. Then for $0 \leqq k < r$ in the space C every n-dimensional subspace, extremal for the class $W^r C$ is extremal also for the class $W^k H^\omega_C$.*

For $n = 2m-1$ and $r = k+1$ this assertion was obtained by Korneĭčuk [12].

Consequence 4.13. *Suppose that the modulus of continuity $\omega(t)$ is convex from below. Then for $r \geqq k$ in the space L_1 every $(2n-1)$-dimensional subspace, extremal for the class $W^r C$ is extremal for the class H^ω_C, as well.*

The quantity

$$\hat{d}_n(\mathfrak{M}, R) = \inf_{H_n \in R} \hat{E}_{H_n}(\mathfrak{M})_R$$

is called the *one-sided (lower) diameter* of the class $\mathfrak{M}$ in the space R. Then from Consequence 4.6 it follows that if $1 \leqq k < r$, then

$$\left\{\frac{\hat{d}_n(W^k C, L_1)}{2\pi K_k}\right\}^{1/k} \leqq \left\{\frac{\hat{d}_n(W^r C, L_1)}{2\pi K_r}\right\}^{1/r} \tag{4.20}$$

Consequence 4.14. *Suppose that $g\in W^{k+1}L_1$ $(r=0, 1, \ldots)$ and the modulus of continuity $\omega(t)$ is convex from below. Then we have*

$$\sup_{f\in W^k H_C^\omega} \int_{-\pi}^{\pi} f(t)g'(t)\,dt \leqq \int_0^{\xi_k} \omega'(t)\Theta_{\xi_k,k}(t)\,dt,$$

where the functions $\Theta_{a,r}$ are defined according to Theorem 2.2 and

$$\xi_k = \pi\{K_{k+1}^{-1}\|g\|_{L_1}\}^{1/(k+1)}.$$

Proof. Let $g\in W^{k+1}L_1$. Let us substitute in Theorem 3.4

$$\Psi(f) = \int_{-\pi}^{\pi} f(t)g^{(k+1)}(t)\,dt$$

and $r=k+1$. Taking into account that in this case

$$l_0(C) = \sup_{\|f\|_C\leqq 1} \int_{-\pi}^{\pi} f(t)g^{(k+1)}(t)\,dt = \|g^{(k+1)}\|_{L_1} \leqq 1,$$

$$l_{k+1}(C) = \sup_{f\in W^{k+1}C} \int_{-\pi}^{\pi} f(t)g^{(k+1)}(t)\,dt \leqq \sup_{\|f\|_C\leqq 1} \int_{-\pi}^{\pi} f(t)g(t)\,dt = \|g\|_{L_1},$$

and

$$l_k^\omega(C) = \sup_{f\in W^k H_C^\omega} \int_{-\pi}^{\pi} f(t)g'(t)\,dt,$$

we obtain the desired estimation.

Analogously, from Theorem 2.6 we obtain the following assertion.

Consequence 4.15. *If $g\in W^1C$ and the modulus of continuity $\omega(t)$ is convex from below, then we have*

$$\sup_{f\in H_C^\omega} \int_{-\pi}^{\pi} f(t)g'(t)\,dt \leqq \frac{\pi^2}{2\|g\|_{L_1}} \int_0^{(2/\pi)\|g\|_{L_1}} \omega(t)\,dt.$$

§ 5. Sharp estimates of the best approximation by splines on the classes of periodic functions

Let the system of nodes $x_k=k\pi/n$ $(k=0, 1, 2, \ldots, 2n)$ be given. The 2π-periodic function $s_{2n,\mu}$ $(\mu=0, 1, 2, \ldots;\ n=1, 2, \ldots)$ is called a *spline function* of order μ and of defect 1 (or simply a *spline*), if on each subinterval $(x_{\nu-1}, x_\nu)$ $(\nu=1, 2, \ldots, 2n)$ it is an algebraic polynomial of order not greater than μ and for $\mu=1, 2, \ldots$ we have $s_{2n,\mu}\in C^{(\mu-1)}$.

The spline $s_{2n,\mu}(x)$ will be said to be *interpolating* for the function f, and then denoted by $s_{2n,\mu}(f,x)$ if we have

$$s_{2n,\mu}(f, x_\nu) = f(x_\nu) \quad (\mu = 1, 3, 5, \dots;\ \nu = 1, 2, \dots, 2n)$$

and

$$s_{2n,\mu}\left(f, \frac{1}{2}(x_\nu + x_{\nu+1})\right) = f\left(\frac{1}{2}(x_\nu + x_{\nu+1})\right) \quad (\mu = 0, 2, 4, \dots;\ \nu = 1, 2, \dots, 2n).$$

For any μ and n there is a unique interpolating spline (cf. [3] and [24]).

Let us denote by $S_{2n,\mu}$ the set of functions $s_{2n,\mu}$ for fixed n and μ. The set $S_{2n,\mu}$ is a $2n$-dimensional subspace of the space W^rC.

Tihomirov [28] proved that for $r=1, 2, \dots$ we have

$$\sup_{f\in W^rC} \|f - s_{2n,r-1}(f)\|_C = E_{S_{2n,r-1}}(W^rC)_C = \frac{K_r}{n^r}. \tag{5.1}$$

From this and from Consequences 4.5 and 4.12, and from equalities (4.12), (4.13) we obtain the following assertions.

Theorem 5.1. *For $r \geqq k-1$ $(k, n=1, 2, \dots)$ we have the equalities*

$$E_{S_{2n,r}}(W^kC)_C = \frac{K_k}{n^k}. \tag{5.2}$$

Theorem 5.2. *Suppose that the modulus of continuity $\omega(t)$ is convex from below. Then for $r \geqq k$, $k=0, 1, \dots$ and $n=1, 2, \dots$ we have the equality*

$$E_{S_{2n,r}}(W^kH_C^\omega)_C = \int_0^{\pi/n} \Theta_{\pi/n,k}(t)\,\omega'(t)\,dt, \tag{5.3}$$

where $\Theta_{a,r}(t)$ are the functions defined in Theorem 2.3.

For $r=k$ this result was obtained by Korneĭčuk [12].

The following theorem shows that there exists a linear method of approximation by splines of order r that gives the same error on the class W^kC $(r>k-1>0)$ as the best approximation by splines of order $k-1$ (and, consequently, of order μ, $\mu \geqq k-1$).

Theorem 5.3. *For $r>k-1>0$ there exists a linear operator $s_{2n,r,k}(f)$ mapping C into the subspace $S_{2n,r}$ and such that*

$$\sup_{f\in W^kC} \|f - s_{2n,r,k}(f)\|_C = \frac{K_k}{n^k}. \tag{5.4}$$

Proof. Let the spline $s_{2n,r,k}(f)$ of order r interpolate the function $A_{r,k}(f)$ occurring in Theorem 2.2, i.e., let

$$s_{2n,r,k}(f,t) = s_{2n,r}(A_{r,k}(f),t).$$

From Theorem 2.2 and equality (5.1) it follows that

$$(5.5)\qquad \|f-s_{2n,r,k}(f)\|_C \leqq \|f-A_{r,k}(f)\|_C+\|A_{r,k}(f)-s_{2n,r}(A_{r,k}(f))\|_C \leqq$$
$$\leqq \max_{x>0}\{K_k x^k - NK_r x^r\}+\frac{K_r}{n^r}\|A_{r,k}^{(r)}(f)\|_C \leqq \max_{x>0}\{K_k x^k - NK_r x^r\}+N\frac{K_r}{n^r}.$$

For any $x_0>0$ we can choose $N_0=N(x_0)$ so that

$$\max_{x>0}\{K_k x^k - N_0 K_r x^r\} = K_k x_0^k - N_0 K_r x_0^r.$$

Putting $x_0=n^{-r}$ in (5.5) we obtain for $f\in W^kC$ that

$$\|f-s_{2n,r,k}(f)\|_C \leqq \frac{K_k}{n^k}.$$

The rest follows from Theorem 5.1.

As follows from the results of Nikolskiĭ ([18], equality (2.9)), we have

$$E_{S_{2n,r}}(f)_{L_1} = \sup_{\substack{\|g\|_C\leqq 1\\ g\perp S_{2n,r}}} \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

Since the subspace $S_{2n,r}$ contains the constant functions, we have

$$\int_{-\pi}^{\pi} g(t)\,dt = 0.$$

Consequently, for $f\in W^{r+1}L_1$ we obtain that

$$E_{S_{2n,r}}(f)_{L_1} = \sup_{\substack{g\in W^rC\\ g\perp S_{2n,0}}} \int_{-\pi}^{\pi} f^{(r)}(t)g(t)\,dt$$

or

$$E_{S_{2n,r}}(f)_{L_1} = \sup_{\substack{g\in W^{r+1}C\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \int_{-\pi}^{\pi} f^{(r+1)}(t)g(t)\,dt.$$

From this it follows that

$$\sup_{f\in W^{r+1}L_{p'}} E_{S_{2n,r}}(f)_{L_1} = \sup_{f\in W^{r+1}L_{p'}} \sup_{\substack{g\in W^{r+1}C\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \int_{-\pi}^{\pi} f^{(r+1)}(t)g(t)\,dt \leqq$$
$$\leqq \sup_{\substack{g\in W^{r+1}C\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \|g\|_{L_p},$$

where $p'=1$ if $p=\infty$ ($L_\infty=C$), $p'=\infty$ if $p=1$, and $p'=p/(p-1)$ if $1<p<\infty$. Consequently, we have

$$E_{S_{2n,r}}(W^{r+1}L_{p'})_{L_1} \leqq \sup_{\substack{g\in W^{r+1}C \\ g(x_\nu)=0\ (\nu=1,2,\ldots,2n)}} \|g\|_{L_p}. \tag{5.6}$$

Lemma 5.1. *If* $g\in C^{(r+1)}$, $\|g^{(r+1)}\|_C<1$, *and* $g(x_\nu)=0$ $(\nu=1,2,\ldots)$, *then the difference*

$$\Delta_1(x) = g(x)-n^{-r}\varphi_r\left(n\left(x-\frac{1+(-1)^{r+1}}{4n}\pi\right)\right)$$

and the difference

$$\Delta_2(x) = g(x)+n^{-r}\varphi_r\left(n\left(x-\frac{1+(-1)^{r+1}}{4n}\pi\right)\right)$$

change signs at the points x_k $(k=1,2,\ldots,2n)$ *and only at these points.*

It is clear that $\Delta_1^{(r)}(x)$ and $\Delta_2^{(r)}(x)$ have at most $2n$ zeros. Consequently, by Rolle's theorem $\Delta_1(x)$ and $\Delta_2(x)$ have at most $2n$ zeros. Taking into account that $\Delta_1(x_\nu)=\Delta_2(x_\nu)=0$ $(\nu=1,2,\ldots,2n)$ we obtain the assertion of the lemma.

From relation (5.1) it follows that if $\|g^{(r+1)}\|_C\leqq 1$ and $g(x_\nu)=0$ $(\nu=1,2,\ldots,2n)$, then

$$\|g\|_C = \|g-s_{2n,r}(g)\|_C \leqq \frac{K_{r+1}}{n^{r+1}}.$$

From this and from Lemma 5.1 it follows that if $\|g^{(r+1)}\|_C<1$ and $g(x_\nu)=0$ $(\nu=1,2,\ldots,2n)$, then

$$|g(x)| \leqq n^{-r}\left|\varphi_r\left(n\left(x-\frac{1+(-1)^{r+1}}{4n}\pi\right)\right)\right|$$

for all x, $-\pi\leqq x\leqq\pi$. Hence for all $g\in W^{r+1}C$ such that $g(x_\nu)=0$ $(\nu=1,2,\ldots,2n)$ the following inequality holds:

$$|g(x)| \leqq n^{-r}\left|\varphi_r\left(n\left(x-\frac{1+(-1)^{r+1}}{4n}\pi\right)\right)\right| \quad (-\pi\leqq x\leqq\pi).$$

From this we obtain

Lemma 5.2. *For all* $g\in W^{r+1}C$ *such that* $g(x_\nu)=0$ $(\nu=1,2,\ldots,2n)$ *we have the inequality*

$$\|g\|_{L_p} \leqq n^{-(r+1)}\|\varphi_{r+1}\|_{L_p} \quad (1\leqq p\leqq\infty).$$

From the last formula and from equality (5.6) we get

$$E_{S_{2n,r}}(W^{r+1}L_{p'})_{L_1} \leqq n^{-(r+1)}\|\varphi_{r+1}\|_{L_p}. \tag{5.7}$$

In particular,

$$E_{S_{2n,r}}(W^{r+1}L_1)_{L_1} \leqq \frac{K_{r+1}}{n^{r+1}}.$$

From this and from Theorem 4.1 and Consequence 4.1 we obtain the desired result.

Theorem 5.4. *For $r \geqq k-1$ ($k, n=1, 2, \ldots$) we have the equality*

$$E_{S_{2n,r}}(W^k L_1)_{L_1} = \frac{K_k}{n^k}.$$

Theorem 5.5. *For $r \geqq k-1$ ($k, n=1, 2, \ldots$) and $1<p\leqq\infty$ we have the equality*

$$E_{S_{2n,r}}(W^k L_{p'})_{L_1} = \frac{\|\varphi_k\|_{L_p}}{n^k}.$$

Proof. The estimation, from above, of the quantity $E_{S_{2n,r}}(W^k L_{p'})_{L_1}$ follows from inequality (5.7) and from Consequence 4.3. The estimation from below can be obtained exactly the same way as that in [27].

Theorem 5.6. *If the modulus of continuity $\omega(t)$ is convex from below and $r \geqq 0$, then we have the equalities*

$$E_{S_{2n,r}}(H_C^{\omega})_{L_1} = n \int_0^{\pi/n} \omega(t)\,dt.$$

Proof. From Theorem 5.5 for $k=1$ and $p=\infty$ and from Consequence 4.5 we obtain that

$$E_{S_{2n,r}}(H_C^{\omega})_{L_1} \leqq n \int_0^{\pi/n} \omega(t)\,dt.$$

On the other hand, let $f_n(x)$ be the function with period $2\pi/n$ that is equal to $2^{-1}\omega(2x)$ for $0 \leqq x \leqq (2n)^{-1}\pi$ and to $2^{-1}\omega\big((2n)^{-1}\pi - 2x\big)$ for $(2n)^{-1}\pi \leqq x \leqq n^{-1}\pi$, and set

$$f_{n,r}(x) = f_n\left(x + \frac{1+(-1)^r}{4n}\pi\right).$$

Then we have

$$E_{S_{2n,r}}(f_{n,r})_{L_1} = \|f_n\|_{L_1} = n \int_0^{\pi/n} \omega(t)\,dt.$$

It is well-known that if $f \in C^{(r)}$, then for $r=0, 1, \ldots$ we have

$$E_{S_{2n,r}}(f)_C \leqq \frac{C_r}{n^r}\,\omega\left(f^{(r)}, \frac{\pi}{n}\right)_C, \tag{5.10}$$

where the quantities C_r depend only on r.

The following result shows that the smallest possible constant in inequality (5.10) equals $2^{-1}K_r$ ($r=0, 1, \ldots$).

Theorem 5.7. *For all $r=0, 1, \ldots$ and $n=1, 2, \ldots$ we have the equalities*

$$\sup_{\substack{f\in C^{(r)} \\ f\neq \mathrm{const}}} \frac{E_{S_{2n,r}}(f)_C}{\omega\left(f^{(r)}, \frac{\pi}{n}\right)_C} = \frac{K_r}{2n^r}. \tag{5.11}$$

The proof of this assertion follows arguments due to KORNEĬČUK [10].

Proof. The function $\varphi(x)$ will be called *simple*, if $|\varphi(x)|>0$ for $\alpha<x<\beta$, $\varphi(x)=0$ for $x\leqq\alpha$ and $x\geqq\beta$; and for any y, $0<y<\max_x |\varphi(x)|$, the equation $|\varphi(x)|=y$ has exactly two solutions. Let $\varphi(x)$ be a simple function with intervals of monotonicity (α, α') and (β, β'), $\alpha<\alpha'\leqq\beta'<\beta$. Let us define the function $\varrho(x)$ by the equality

$$\varphi(x) = \varphi(\varrho(x)), \quad \alpha \leqq x \leqq \alpha', \quad \beta' \leqq \varrho(x) \leqq \beta. \tag{5.12}$$

Then

$$\int_0^{2\pi} f(x)\varphi'(x)\,dx = \int_\alpha^{\alpha'} f(x)\varphi'(x)\,dx + \int_{\beta'}^{\beta} f(x)\varphi'(x)\,dx =$$

$$= \int_\alpha^{\alpha'} f(x)\varphi'(x)\,dx - \int_\alpha^{\alpha'} f(\varrho(x))\varphi'(\varrho(x))\varrho'(x)\,dx.$$

From (5.12) it follows that

$$\varphi'(\varrho(x))\varrho'(x) = \varphi'(x).$$

Consequently,

$$\int_0^{2\pi} f(x)\varphi'(x)\,dx = \int_\alpha^{\alpha'} \{f(x)-f(\varrho(x))\}\varphi'(x)\,dx \leqq \tag{5.13}$$

$$\leqq \omega(f, \beta-\alpha)_C \int_\alpha^{\alpha'} |\varphi'(x)|\,dx = \frac{1}{2}\omega(f, \beta-\alpha)_C \int_0^{2\pi} |\varphi'(x)|\,dx.$$

In [10] it is proved that any function $g\in L_1^{(1)}$ can be represented as a sum of simple functions $\varphi_k(x)$, moreover, if (α, β) is the longest segment, where the function $g(x)$ does not change sign, then $\max(\beta_k-\alpha_k)\leqq\beta-\alpha$. Furthermore,

$$\sum_k \|\varphi_k'\|_{L_1} = \|g'\|_{L_1}.$$

Taking this into account, from inequality (5.13) we obtain that for any function $g\in L_1^{(1)}$

$$\int_{-\pi}^{\pi} f(x)g'(x)\,dx \leqq \frac{1}{2}\omega(f, \beta-\alpha)_C \|g'\|_{L_1}, \tag{5.14}$$

where $\beta-\alpha$ is the length of the longest segment on which the function $g(x)$ does not change sign.

Let $g\in W^{r+1}L_1$ and $g(x_\nu)=0$ $(\nu=1, 2, \dots, 2n; r=0, 1, \dots)$. Then the length of the longest segment on which the function $g(x)$ does not change sign does not exceed π/n. Consequently, for $f\in C$ we have

$$\int_{-\pi}^{\pi} f(t)g'(t)\,dt \leqq \frac{1}{2}\omega\left(f, \frac{\pi}{n}\right)_C \|g'\|_{L_1}.$$

From this and from the equalities

$$E_{S_{2n,r}}(f)_C = \sup_{\substack{\|g\|_{L_1}\leqq 1\\ g\perp S_{2n,r}}} \int_{-\pi}^{\pi} f(t)g(t)\,dt =$$

$$= \sup_{\substack{g\in W^{r+1}L_1\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \int_{-\pi}^{\pi} f^{(r)}(t)g'(t)\,dt,$$

which follow from equality (2.9) in [18] we obtain that

$$E_{S_{2n,r}}(f)_C \leqq \frac{1}{2}\omega\left(f^{(r)}, \frac{\pi}{n}\right)_C \sup_{\substack{g\in W^{r+1}L_1\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \|g'\|_{L_1}. \tag{5.15}$$

On the other hand, from the same equality (2.9) of [18] it follows that

$$\sup_{f\in W^{r+1}C} E_{S_{2n,r}}(f)_C = \sup_{f\in W^{r+1}C} \sup_{\substack{g\in W^{r+1}L_1\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \int_0^{2\pi} f^{(r+1)}(t)g(t)\,dt =$$

$$= \sup_{\substack{g\in W^{r+1}L_1\\ g(x_\nu)=0\ (\nu=1,2,\dots,2n)}} \min_\lambda \|g(x)-\lambda\|_{L_1}.$$

From this and from (5.1) it follows that if $g\in W^{r+1}L_1$, $g(x_\nu)=0$ $(\nu=1, 2, \dots, 2n)$, then

$$\min_\lambda \|g(x)-\lambda\|_{L_1} \leqq \frac{K_{r+1}}{n^{r+1}}.$$

From this and from Stein's inequality (1.2) we can derive that if $g\in W^{r+1}L_1$, $g(x_\nu)=0$ $(\nu=1, 2, \dots, 2n)$, then

$$\|g'\|_{L_1} \leqq \frac{K_r}{n^r}.$$

From this and from inequality (5.15) it follows that for $f\in C^{(r)}$ we have

$$E_{S_{2n,r}}(f)_C \leqq \frac{K_r}{2n^r}\omega\left(f^{(r)}, \frac{\pi}{n}\right)_C.$$

To make the proof complete it remains to notice that on account of Theorem 5.1 for $f\in C^{(r)}$ we have

$$\sup_{\substack{f\in C^{(r)}\\ f\neq\text{const}}} \frac{E_{S_{2n,r}}(f)_C}{\omega\left(f^{(r)},\dfrac{\pi}{n}\right)_C} \leqq \frac{1}{2}\sup_{\substack{f\in C^{(r)}\\ f\neq\text{const}}} \frac{E_{S_{2n,r}}(f)_C}{\|f^{(r)}\|_C} =$$

$$= \frac{1}{2}\sup_{f\in W^r C} E_{S_{2n,r}}(f)_C = \frac{K_r}{2n^r}.$$

Thus Theorem 5.7 is completely proved.

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Неравенства для верхних граней функционалов

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В работе устанавливаются некоторые точные неравенства типа неравенств А. Н. Колмогорова между нормами последовательных производных произвольной функции. Дано их применение к некоторым задачам теории приближения, и в частности, к задачам сплайн аппроксимации.

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УЛ. ДНЕПРОСТРОЕВСКАЯ 2
ИНДУСТРИАЛЬНЫЙ ИНСТИТУТ