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SOME QUANTUM MECHANICAL PROBLEMS IN LOBACHEVSKY SPACE

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Quantum mechanical problems are considered for potentials satisfying Bertrand's problem in Lobachevsky space. The self-adjointness of the corresponding Schrödinger operators is proved. The energy levels are calculated both from the Schrödinger equation and by means of the Bohr–Sommerfeld method. The effect of the quantum binding of classically infinite motion was discovered and is presented for the first time. It is shown that the quasi-classical limit is equivalent, in a sense, to the Euclidean limit.

1. As is well known, there are two select potentials of the central potentials in the space $\mathbf{E}^n$, the Coulomb and the oscillator, for which all finite trajectories of a classical particle are closed—provided that they exist. One of the three versions of the Bertrand problem [1] consists of obtaining such potentials. An analogous problem on the sphere $\mathbf{S}^3$ was studied in [2], and in the Lobachevsky space $\mathbf{L}^3$ in [3]. It was shown that in each of these cases, there also exist two select potentials (which we call Bertrand-type potentials), for which all finite trajectories of classical particles are closed. In [4], the quantum-mechanical problem was considered for these potentials for the case of the group $SU(2) \simeq \mathbf{S}^3$.

Our aim is the study of quantum mechanical problems with Bertrand-type potentials in the spaces $\mathbf{L}^2$ and $\mathbf{L}^3$. First, we present a new method for solving the Bertrand problem for $n \geq 2$ in spaces $\mathbf{L}^n$ and $\mathbf{S}^n$. In our opinion, this solution is simpler than those presented in [2, 3, 5]; moreover, our approach explicitly reveals the similarity of the two cases to each other, as well as to the case of the space $\mathbf{E}^n$.

Let ds_1^2 be the standard metric on the unit sphere $\mathbf{S}^{n-1}$. For $n \geq 2$, we consider the following two models of simply connected spaces with constant section curvature:

$$\mathbf{L}^n = \{0\} \cup (0,1) \times \mathbf{S}^{n-1},$$

with the metric

$$ds^2 = 4R^2 \frac{dr^2 + r^2 ds_1^2}{(1-r^2)^2}, \qquad r \in [0,1)$$

and the section curvature $\kappa = -R^{-2}$ (the Poincaré model in a ball [6]), and

$$\mathbf{S}^n = \{0,\infty\} \cup (0,\infty) \times \mathbf{S}^{n-1}$$

with the metric

$$ds^2 = 4R^2 \frac{dr^2 + r^2 ds_1^2}{(1+r^2)^2}, \qquad r \in [0,\infty)$$

and the section curvature $\kappa = R^{-2}$ (stereographic projection).

The Lagrange function of a point-like particle with a unit mass propagating in a central potential $U(r)$ is $\mathcal{L} = \frac{1}{2}\dot{s}^2 - U(r)$. One can easily see that the motion is two-dimensional, as in the Euclidean case, i.e., the trajectory of the particle lies on $\mathbf{L}^2 \subset \mathbf{L}^n$ or $\mathbf{S}^2 \subset \mathbf{S}^n$. Now, let $n = 2$, $ds_1 = d\phi$, $0 \leq \phi < 2\pi$, and let

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us introduce the quantity

$$\xi = \frac{1}{2R}\left(r - \frac{1}{r}\right), \qquad -\infty \le \xi \le +\infty, \qquad \text{for} \qquad \mathbf{S}^2,$$
$$\xi = \frac{1}{2R}\left(r + \frac{1}{r}\right), \qquad \frac{1}{R} < \xi \le +\infty, \qquad \text{for} \qquad \mathbf{L}^2,$$

and

$$\xi = \frac{1}{r}, \qquad 0 < \xi \le \infty, \qquad \text{for} \qquad \mathbf{E}^2.$$

Then the equation for the trajectory reads

$$\phi = \int M\Big(2(E - U(r(\xi))) - M^2\xi^2 - \kappa M^2\Big)^{-\frac{1}{2}} d\xi,$$

where M is the momentum integral and κ is the section curvature. Since the term κM^2 can be absorbed into E, in all spaces, one can check whether the trajectories are closed in the same way as was done in [1] and [7] for the Euclidean case. The analysis gives two possible potentials, $U_1 = k\xi$ (analog of the Coulomb potential) and $U_2 = k\xi^{-2}$ (analog of the oscillator potential). To make the description clearer, let us return to the coordinate r.

For $\mathbf{S}^2$, we obtain

$$U_1^s(r) = k\left(r - \frac{1}{r}\right), \qquad U_2^s(r) = \begin{cases} \dfrac{k_1 r^2}{(r^2-1)^2}, & 0 \le r < 1; \\ \dfrac{k_2 r^2}{(r^2-1)^2}, & 1 < r \le \infty, \end{cases} \qquad k_1, k_2 \ge 0.$$

and for $\mathbf{L}^2$,

$$U_1^l(r) = k\left(r + \frac{1}{r}\right), \qquad U_2^l(r) = \frac{kr^2}{(r^2+1)^2}.$$

The $U_2^s(r)$ singularity divides $\mathbf{S}^2$ into two separate domains of motion along the equator $r = 1$. The particle cannot leave either of these domains, which explains the two-parameter form of the solution $U_2^s(k_1, k_2, r)$. For $k_1 = k_2$, one can treat U_2^s as a potential in Riemannian space—the factor space of the sphere $\mathbf{S}^2$ with respect to the central symmetry. In terms of the distance ρ to the point $r = 0$ on the Riemannian sphere,

$$U_1^s(\rho) = -\frac{k}{R}\cot\frac{\rho}{R}, \qquad U_2^s(\rho) = \frac{\omega^2 R^2}{2}\tan^2\frac{\rho}{R}, \qquad \rho = 2R\arctan r,$$
$$U_1^l(\rho) = -\frac{k}{R}\coth\frac{\rho}{R}, \qquad k > 0, \qquad U_2^l(\rho) = \frac{\omega^2 R^2}{2}\tanh^2\frac{\rho}{R}, \qquad \rho = R\log\frac{1+r}{1-r}.$$

Here the constants are chosen in such a way that in the Euclidean limit $R \to \infty$, we obtain $U_1^e = -k/\rho$, $U_2^e = \frac{1}{2}\omega^2\rho^2$. Note that the replacement $R \to iR$ changes U_j^s to U_j^l, $j = 1, 2$, and that $\left[\Delta_3(U_1)\right] = 4\pi k\delta(0)$, as in the Euclidean case. Here Δ_3 is the Laplace–Beltrami operator in $\mathbf{L}^3$ or $\mathbf{S}^3$. However, in contrast to the Euclidean case, $\Delta_3(U_2^{s,l}) \ne \text{const}$.

2. Consider the spaces $\mathbf{L}^2$ and $\mathbf{L}^3$. In the Beltrami model [6], $\mathbf{L}^n = \{0\} \cup (0,1) \times \mathbf{S}^{n-1}$, with the metric

$$ds^2 = R^2\left(\frac{dv^2}{(1-v^2)^2} + \frac{v^2 ds_1^2}{1-v^2}\right), \qquad v \in [0, 1),$$

while $U_1(v) \equiv U_1^l = -k(Rv)^{-1}$, $U_2(v) \equiv U_2^l = \frac{1}{2}R^2\omega^2v^2$. The Hamiltonian of the particle in $\mathbf{L}^2$ reads

$$H_2(p_v, v, p_\phi, \phi) = \frac{1}{2R^2}\left((1-v^2)^2p_v^2 + \frac{1-v^2}{v^2}p_\phi^2\right) + U(v).$$

Consider the case U_1. The trajectory of the particle is

$$v = p(1 + e\cos\phi)^{-1},$$

where

$$p = \frac{M^2}{Rk}, \qquad M = |p_\phi| = \text{const}, \qquad e = \sqrt{1 + \frac{2M^2}{k^2}\left(E + \frac{M^2}{2R^2}\right)}.$$

The conditions for the existence of a motion with energy E are

$$\begin{aligned} E &\geq -\frac{k^2}{2M^2} - \frac{M^2}{2R^2} && \text{for} \quad \frac{M^2}{Rk} < 1, \\ E &> -\frac{k}{R} && \text{for} \quad \frac{M^2}{Rk} \geq 1. \end{aligned}$$

The conditions for the motion to be finite are

$$M < \sqrt{Rk}, \quad E < -\frac{k}{R}. \tag{1}$$

One can see that for $M \geq \sqrt{Rk}$, in contrast to the Euclidean case, finite motion is impossible for any E. The period of rotation along a closed orbit is [3]

$$T = \frac{R\pi}{\sqrt{2}}\left(\left(-\frac{k}{R} - E\right)^{-\frac{1}{2}} - \left(\frac{k}{R} - E\right)^{-\frac{1}{2}}\right).$$

For U_2, the trajectory has the form

$$v^2 = \frac{M^2}{R^2}\left(E + \frac{M^2}{2R^2} - \sqrt{\left(E + \frac{M^2}{2R^2}\right)^2 - \omega^2M^2\sin 2\phi}\right)^{-1}.$$

The existence conditions for a motion with energy E are

$$\begin{aligned} E &\geq M\omega - \frac{M^2}{2R^2} && \text{for} \quad \frac{M}{\omega R^2} < 1, \\ E &> \frac{\omega^2R^2}{2} && \text{for} \quad \frac{M}{\omega R^2} \geq 1. \end{aligned}$$

The finiteness conditions for the motion are

$$\frac{M^2}{2R^2} < E < \frac{\omega^2R^2}{2}. \tag{2}$$

At $M \geq \omega R^2$, finite motion does not exist for any E. The period of rotation along a closed orbit is [3] $T = 2R\pi(\omega^2R^2 - 2E)^{-\frac{1}{2}}$.

3. The Bohr–Sommerfeld quantization. For $\mathbf{L}^3$, the Hamiltonian has the form

$$H_3(p_v, v, p_\psi, \psi, p_\phi, \phi) = \frac{1}{2R^2}\left((1-v^2)^2 p_v^2 + \frac{1-v^2}{v^2}\left(\frac{p_\phi^2}{\sin^2\psi} + p_\psi^2\right)\right) + U(v),$$
$$0 \le \psi \le \pi, \qquad 0 \le \phi \le 2\pi.$$

The integrals of motion are $M^2 = p_\psi^2 + p_\phi^2 \sin^{-2}\psi$, $L = p_\phi$, and $E = H_3$. The basic cycles on the three-dimensional torus are chosen analogously to the Euclidean case [8] and have the same Maslov indexes.

Let us consider U_1. The quantization conditions have the form

$$\frac{2}{\pi\hbar}\int_0^{2\phi} L\,d\phi = \frac{4L}{\hbar} = 4n_1, \qquad n_1 = 0. \pm 1, \pm 2 \dots,$$
$$\frac{4}{\pi\hbar}\int_{\psi_1}^{\psi_2} \sqrt{M^2 - \frac{L^2}{\sin^2\psi}}\,d\psi = \frac{4}{\hbar}(M - |L|) = 2 + 4n_2, \qquad n_2 = 0, 1, 2 \dots,$$

where $\psi_1 = \arcsin\frac{|L|}{M}$ and $\psi_2 = \pi - \psi_1$, and

$$\frac{4}{\pi\hbar}\int_{v_1}^{v_2}\left(\frac{2R^2}{(1-v^2)^2}\left(E + \frac{k}{Rv}\right) - \frac{M^2}{(1-v^2)v^2}\right)^{\frac{1}{2}} dv =$$
$$= \frac{2\sqrt{2}R}{\hbar}\left(-\sqrt{-E - \frac{k}{R}} + \sqrt{-E + \frac{k}{R}}\right) - \frac{4M}{\hbar} = 2 + 4n_3, \quad n_3 = 0, 1, 2 \dots, \tag{3}$$

where v_1 and v_2 are the roots of the last integrand. Thus, taking into account the finiteness conditions, we obtain $L = n_1\hbar$, $M = \hbar(\frac{1}{2} + n_2 + |n_1|)$,

$$E_n = -\frac{n^2\hbar^2}{2R^2} - \frac{k^2}{2n^2\hbar^2}, \qquad 1 \le n < \frac{\sqrt{Rk}}{\hbar}, \tag{4}$$

where $n := 1 + n_3 + n_2 + |n_1|$. The degeneracy of the energy levels is n^2, as in the Euclidean case. Note that the Bohr quantization, which takes into account only circular orbits, leads to the same result. Also note that E_n grows monotonously in the range of n described above.

Consider $U = U_2$ on $\mathbf{L}^3$. In this case, Eq. (3) is replaced by the condition

$$\frac{2}{\hbar}\left(\omega R^2\left(1 - \sqrt{1 - \frac{2E}{\omega^2 R^2}}\right) - M\right) = 2 + 4n_3, \qquad n_3 = 0, 1, 2 \dots.$$

As a result, substituting $n := 2n_3 + n_2 + |n_1|$, we obtain

$$E_n^{(3)} = \hbar\omega\left(n + \frac{3}{2}\right) - \frac{\hbar^2(n + \frac{3}{2})^2}{2R^2}, \qquad 0 \le n < \frac{\omega R^2}{\hbar} - \frac{3}{2}. \tag{5}$$

The degeneracy of the levels is the same as in the Euclidean case, namely, $(n+1)(n+2)/2$. Similarly, for $U = U_2$ on $\mathbf{L}^2$, we have

$$E_n^{(2)} = \hbar\omega(n+1) - \frac{\hbar^2(n+1)^2}{2R^2}, \qquad 0 \le n < \frac{\omega R^2}{\hbar} - 1. \tag{6}$$

The degeneracy is $n + 1$. Evidently, at $R \to \infty$, the expressions for E_n, $E_n^{(3)}$, $E_n^{(2)}$ become the known formulas for the Euclidean case.

4. Now let us consider the corresponding spectral problems for the Schrödinger equation

$$-\frac{\hbar^2}{2}\Delta\phi + U\phi = E\phi,$$

where Δ is the Laplace–Beltrami operator on $\mathbf{L}^3$ or $\mathbf{L}^2$.

4.1. First, we formulate some results relating to the self-adjointness of the operators $-\Delta + U$ in $\mathbf{L}^n$. All spaces of the summable functions are understood w.r.t. the invariant measure in $\mathbf{L}^n$. We present $\mathbf{L}^n$ in the form $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n, x^n > 0\}$, $ds^2 = (x^n)^{-2}\sum_{i=1}^n (dx^i)^2$, $d\mu(x) = (x^n)^{-n} dx^1 \wedge \cdots \wedge dx^n$,

$$\Delta = (x^n)^2 \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2} - (n-2)x^n \frac{\partial}{\partial x^n}.$$

It is clear that the operator Δ is elliptic.

Let $V(x) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbf{L}^n)$, $V(x) \geq 0$, be a real-valued function. Let us define $\hat{A} = -\Delta + V(x)$ on $C_0^\infty(\mathbf{L}^n)$ and let

$$q(\phi, g) := (\hat{A}\phi, g) = \int_{\mathbf{R}^n_+} \left(\frac{1}{(x^n)^{n-2}} \sum_{i=1}^n \frac{\partial \phi}{\partial x^i}\frac{\partial \bar{g}}{\partial x^i} + \frac{V\phi\bar{g}}{(x^n)^n} \right) dx^1 \wedge \cdots \wedge dx^n, \qquad \phi, g \in C_0^\infty(\mathbf{L}^n).$$

Obviously, $\hat{A}$ is a positive-symmetric operator on $C_0^\infty(\mathbf{L}^n) \subset \mathcal{L}^2(\mathbf{L}^n)$. Thus, by virtue of Theorem X.23 on p. 200 of [9], its Friedrichs extension A is self-adjoint. At the same time, $D(A) \subset D(q)$, where $D(q) = \{f \in \mathcal{L}^2(\mathbf{L}^n) \mid q(f,f) < \infty\}$.

Almost all of the results of [9] obtained for the Euclidean case can be applied to the case of spaces $\mathbf{L}^n$ (Theorems 1 and 2). More generally, they can be applied to any infinite-volume complete Riemannian space without boundaries. The only difference consists in the necessity of making calculations in terms of local (e.g., normal [10]) coordinates.

Theorem 1. *The following formula holds*:

$$D(A) = \{\phi \in \mathcal{L}^2(\mathbf{L}^n) | V\phi \in \mathcal{L}^1_{\mathrm{loc}}(\mathbf{L}^n), -\Delta\phi + V\phi \in \mathcal{L}^2(\mathbf{L}^n)\}, \qquad A\phi = -\Delta\phi + V\phi.$$

(Hereafter, derivatives are understood as distributions.)

Theorem 2. *Let $V \in \mathcal{L}^2_{\mathrm{loc}}(\mathbf{L}^n)$, $V \geq 0$; then $-\Delta + V$ is self-adjoint in $C_0^\infty(\mathbf{L}^n)$.*

The next quite-standard result ensures that the second partial derivatives of functions from $D(A)$ are locally summable.

Theorem 3. *Let $V \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbf{L}^n)$, then $D(A) \subset W^{2,2}_{\mathrm{loc}}$.*

Proof. Let $f \in D(A)$ and let Ω be a domain in $\mathbf{L}^n$ with a compact closure. Then $(Af)|_\Omega \in \mathcal{L}^2(\Omega)$, whence, by virtue of the claim on p. 177 of [11], $f|_{\Omega'} \in W^{2,2}(\Omega')$, $\forall \Omega' \subseteq \Omega$. Since Ω is arbitrary, we obtain the required result.

The formulation of Theorem 4 is taken from [9], but its proof is slightly altered since one cannot use a Fourier transformation to prove the basic estimate. The proof relies on the transitivity of the isometry group $\mathbf{L}^n$.

Theorem 4. *Let $V \in \mathcal{L}^\infty(\mathbf{L}^3) + \mathcal{L}^2(\mathbf{L}^3)$, V being a real function. Then A is self-adjoint in $C_0^\infty(\mathbf{L}^3)$ and is self-adjoint in*

$$D(-\Delta) = \{\phi \in \mathcal{L}^2(\mathbf{L}^3) | -\Delta\phi \in \mathcal{L}^2(\mathbf{L}^3)\}.$$

Proof. By virtue of Theorems X.12 and X.15 from [9], it is sufficient to prove the following estimate: $\forall a > 0 \ \ \exists b > 0$ such that $\forall \phi \in D(-\Delta)$, the inequality $\|\phi\|_\infty \le a\|\Delta\phi\|_2 + b\|\phi\|_2$ holds, where $\|\cdot\|_2$ is the norm in $\mathcal{L}^2(\mathbf{L}^3)$ and $\|\cdot\|_\infty$ is the norm in $\mathcal{L}^\infty(\mathbf{L}^3)$. Let $\Omega \subseteq \mathbf{L}^3$ be a domain with compact closure. According to Theorem 8.24 from [11], the set

$$P(B) = \{\phi \in D(-\Delta),\ \|\Delta\phi\|_{2,\Omega} + \|\phi\|_{2,\Omega} < B\}$$

is a compact inclusion in $C(\bar{\Omega}')$, $\forall \Omega' \subseteq \Omega$, $\forall B > 0$. Here $\|\cdot\|_{2,\Omega}$ is the norm in $\mathcal{L}^2(\Omega)$. Let us show that $\forall a > 0 \ \ \exists b > 0$ such that $\forall \phi \in D(-\Delta)$, the inequality $\|\phi\|_{C(\bar{\Omega}')} \le a\|\Delta\phi\|_{2,\Omega} + b\|\phi\|_{2,\Omega}$ holds. We assume the inverse. This means that $\exists a > 0$, $\{b_n\} \to \infty$, $\{\phi_n\} \subset D(-\Delta)$ such that

$$\|\phi_n\|_{C(\bar{\Omega}')} > a\|\Delta\phi_n\|_{2,\Omega} + b_n\|\phi_n\|_{2,\Omega} =: \alpha_n.$$

Without loss of generality (in proper normalization), we can assume that $\|\Delta\phi_n\|_{2,\Omega} + \|\phi_n\|_{2,\Omega} = 1$. Since $P(2)$ is compact in $C(\bar{\Omega}')$, then $\exists \phi_{n_k} \to \psi$ in $C(\bar{\Omega}')$ and, hence, in $\mathcal{L}^2(\Omega')$. The sequence α_{n_k} is restricted and (since $b_{n_k} \to \infty$) $\|\phi_{n_k}\|_{2,\Omega} \to 0$. However, this implies $\|\phi_{n_k}\|_{2,\Omega'} \to 0, \psi = 0$, and $\|\Delta\phi_{n_k}\|_{2,\Omega} \to 1$. Passing to the limit, we come to a contradiction: $0 = \|\psi\|_{C(\bar{\Omega}')} \ge a > 0$. Thus, $\forall a > 0 \ \ \exists b > 0$ such that $\forall \phi \in D(-\Delta)$, the following inequality holds:

$$\|\phi\|_{C(\bar{\Omega}')} \le a\|\Delta\phi\|_{2,\Omega} + b\|\phi\|_{2,\Omega} \le a\|\Delta\phi\|_2 + b\|\phi\|_2.$$

Let the point $x \in \mathbf{L}^3$ be arbitrary. There exists an γ-isometry mapping x into $x' \in \Omega'$. Then

$$\|\Delta(\phi \circ \gamma)\|_2 = \|\Delta\phi\|_2, \qquad \|\phi \circ \gamma\|_2 = \|\phi\|_2,$$

and, thus,

$$\|\phi\|_\infty \le a\|\Delta\phi\|_2 + b\|\phi\|_2.$$

4.2. Further, let us solve the Schrödinger equation with the potential U_1 in $\mathbf{L}^3$ and the potential U_2 in $\mathbf{L}^3$ and $\mathbf{L}^2$. It follows from the theory of elliptic equations that the eigenfunctions are infinitely smooth at the points where the equation coefficients are smooth. It also follows that in our spherically symmetric case, the eigenfunctions can be found by separating the variables. Consider $U = U_1$ in $\mathbf{L}^3$. The replacement $u := v^{-1}$, $1 < u \le \infty$ gives

$$\Delta = \frac{1}{R^2}\left((u^2-1)^2\frac{\partial^2}{\partial u^2} + (u^2-1)\Delta_s\right), \qquad \omega_3 = \frac{R^3 du \wedge \omega_s^2}{(u^2-1)^2},$$

where ω_s^2 is the volume form on the unit sphere $\mathbf{S}_1^2$ with a standard metric. Let $\psi = W(u)\Phi_{l,m}(x)$, $x \in \mathbf{S}_1^2$, $\Delta_s\Phi_{l,m} = -l(l+1)\Phi_{l,m}$, where $\Phi_{l,m}$ is a spherical harmonic. The equation for W has the form

$$W''(u) + \frac{a + bu + cu^2}{(u^2-1)^2}W(u) = 0, \tag{7}$$

with $a = 2R^2E\hbar^{-2} + l(l+1)$, $b = 2Rk\hbar^{-2}$, $c = -l(l+1)$. Note that in order to pass to the Euclidean limit, one should make the replacement $r = R/u$ and let $R \to \infty$, keeping r constant, which becomes the Euclidean radius.

Equation (7) is a Riemannian equation [12] with three singular points $\pm1, \infty$, where its solution as a function of the complex variable has branching points. The critical exponents at these points are

$$\rho_1^{(1)} = \frac{1 + \sqrt{1-a-b-c}}{2}, \qquad \rho_2^{(1)} = \frac{1 - \sqrt{1-a-b-c}}{2},$$
$$\rho_1^{(-1)} = \frac{1 + \sqrt{1-a+b-c}}{2}, \qquad \rho_2^{(-1)} = \frac{1 - \sqrt{1-a+b-c}}{2},$$
$$\rho_1^{(\infty)} = l, \qquad \rho_2^{(\infty)} = -1-l.$$

In accordance with Theorem 4, we are interested in solutions for which

$$\int_1^\infty \frac{|W|^2 du}{(u^2-1)^2} < \infty \qquad \text{and} \qquad \Delta\left[W\left(\frac{1}{v}\right)\Phi_{l,m}(x)\right] \in \mathcal{L}^2(\mathbf{L}^3). \tag{8}$$

The first condition in (8) excludes the $\rho_2^{(1)}$ asymptotic behavior at $u \to 1$ and imposes the restriction

$$1 - a - b - c > 0. \tag{9}$$

It also excludes the $\rho_2^{(\infty)}$ asymptotic behavior at $u \to \infty$, $l > 0$. The second condition in (8) eliminates $\rho_2^{(\infty)}$ at $l = 0$. Thus, the eigenfunctions should have the asymptotic behavior $(u-1)^{\rho_1^{(1)}}$ as $u \to 1$ and u^{-l} as $u \to \infty$.

Using the general theory of the Riemannian equation [12], let us bring Eq. (7) to a hypergeometric form by means of the replacement $y(\xi) = W(u)(u-1)^{-\rho_1^{(1)}}(u+1)^{-\rho_2^{(-1)}}$, $\xi = \frac{1}{2}(1-u)$. Then

$$\xi(1-\xi)y'' + \big(\gamma - \xi(\alpha+\beta+1)\big)y' - \alpha\beta y = 0.$$

The critical exponents of function y are α and β at the infinity point, 0 and $1-\gamma$ at the zero point, and 0 and $\gamma - \alpha - \beta$ at unity. Thus,

$$\alpha = \rho_1^{(\infty)} + \rho_1^{(1)} + \rho_2^{(-1)} = l + 1 + \frac{1}{2}\left(\sqrt{1-a-b-c} - \sqrt{1-a+b-c}\right),$$

$$\beta = \rho_2^{(\infty)} + \rho_1^{(1)} + \rho_2^{(-1)} = -l + \frac{1}{2}\left(\sqrt{1-a-b-c} - \sqrt{1-a+b-c}\right),$$

$$1-\gamma = \rho_2^{(1)} - \rho_1^{(1)} = -\sqrt{1-a-b-c}, \qquad \gamma = 1 + \sqrt{1-a-b-c}, \qquad \alpha - \beta = 2l+1.$$

The required function $y(\xi)$ should have the following asymptotic form: const at 0 and $\xi^{-\alpha}$ at $\xi \to -\infty$. Thus,

$$y(\xi) = F(\alpha,\beta,\gamma,\xi) \qquad \text{and} \qquad \lim_{\xi\to-\infty} F(\alpha,\beta,\gamma,\xi)\xi^{\beta} = 0.$$

Since ([13]),

$$\lim_{\xi\to-\infty} F(\alpha,\beta,\gamma,\xi)\xi^{\beta} = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}$$

and $\gamma > 0$, $\alpha - \beta > 0$, and

$$\gamma - \beta = 1 + l + \frac{1}{2}\left(\sqrt{1-a-b-c} + \sqrt{1-a+b-c}\right) > 0,$$

it follows that $\alpha = -m+1$, $m = 1, 2, 3 \dots$, i.e.,

$$-\sqrt{1-a-b-c} + \sqrt{1-a+b-c} = 2(m+l) = 2n, \qquad n = 1, 2, 3 \dots, \tag{10}$$

whence

$$\sqrt{1-a-b-c} + \sqrt{1-a+b-c} = \frac{b}{n}. \tag{11}$$

Summation and subtraction of (10) and (11) give, accounting for (9),

$$\sqrt{1-a+b-c} = \frac{b}{2n} + n, \qquad \sqrt{1-a-b-c} = \frac{b}{2n} - n > 0.$$

Thus, substituting the values of a, b, and c, we obtain

$$E_n = \frac{\hbar^2}{2R^2} - \frac{n^2\hbar^2}{2R^2} - \frac{k^2}{2n^2\hbar^2}, \qquad 1 \le n < \frac{\sqrt{Rk}}{\hbar}. \tag{12}$$

Note that at these values of n, the energy E_n grows monotonously. As before, its levels are n^2-degenerate. Equation (12) differs from Eq. (4) by an additional term, $\frac{\hbar^2}{2R^2}$. We now show that this term is responsible for an interesting effect—the quantum mechanical binding of a classically infinite motion.

The second condition for a classical motion to be finite, Eq. (1), may not be satisfied for E_n from (12). Indeed, choosing Rk such that $\sqrt{Rk}\,\hbar^{-1} = n_1 + \varepsilon$, $\varepsilon > 0$, one can make $E_{n_1} > -\frac{k}{R} = U_1^{\max}$; in addition, the inequality $E_n - U_1^{\max} < \frac{\hbar^2}{2R^2}$ always holds. Note that the operator of the square of the total angular momentum is $-\hbar^2\Delta_s$ and its eigenvalues are $\hbar^2 l(l+1) = \hbar^2(n-m)(n-m+1) \le \hbar^2(n-1)n < Rk$. This satisfies the first condition from (1).

Now let us turn to the radial eigenfunction $W_{n,l}$, which corresponds to the energy E_n. For $\alpha = 1-m$, $\beta = -m-2l$, and $\gamma = 1 + \frac{Rk}{n\hbar^2} - n$, $y(\xi) = F(\alpha,\beta,\gamma,\xi)$ is a polynomial of degree exactly $(m-1)$. Its explicit form is

$$y(\xi) = P_{m-1}(\xi) = 1 + \sum_{p=1}^{m-1} \frac{(\alpha)_p(\beta)_p}{p!(\gamma)_p}\xi^p,$$

where $(\alpha)_p := \alpha(\alpha+1)\dots(\alpha+p-1)$. Since $\rho_1^{(1)} = \frac{1}{2}\left(1 + \frac{Rk}{n\hbar^2} - n\right)$ and $\rho_2^{(-1)} = \frac{1}{2}\left(1 - \frac{Rk}{n\hbar^2} - n\right)$, we obtain

$$W_{n,l}(u) = P_{n-l-1}\left(\frac{1-u}{2}\right)(u^2-1)^{\frac{1-n}{2}}\left(\frac{u-1}{u+1}\right)^{\frac{Rk}{2n\hbar^2}}.$$

In order to pass to the Euclidean limit, one should first set $u = R/r$ and multiply $W_{n,l}$ by R^{n-1}, subsequently letting $R \to \infty$. Then r becomes the Euclidean radius.

The equation for the radial eigenfunctions of U_2 can also be brought to the form of a Riemannian equation with an independent variable $t = u^2$. We present only the results. The eigenvalues corresponding to the potential U_2 in $\mathbf{L}^3$ are

$$E_n^{(3)} = \frac{\hbar^2}{2R^2}\left(\frac{3}{4} - \left(n+\frac{3}{2}\right)^2\right) + \omega\hbar\left(n+\frac{3}{2}\right)\sqrt{1+\frac{\hbar^2}{4\omega^2R^4}},$$
$$0 \le n < \frac{\omega R^2}{\hbar}\sqrt{1+\frac{\hbar^2}{4\omega^2R^4}} - \frac{3}{2}.$$

The degeneracy of the energy levels is $(n+1)\times(n+2)/2$, as in the Euclidean case. The eigenfunctions corresponding to $E_n^{(3)}$ have the form

$$\psi_{n,l,m}(u,x) = (u^2-1)^{-\frac{\sigma+n}{2}}u^\sigma P_{n,l}^{(3)}(u^2)\Phi_{l,m}(x),$$

where

$$\sigma = \frac{1}{2}\left(1 - \sqrt{1+\frac{4\omega^2R^4}{\hbar^2}}\right), \qquad x \in \mathbf{S}^2,$$
$$P_{n,l}^{(3)}(t) = 1 + \sum_{j=1}^{k}\frac{(-k)_j(-k-l-\frac{1}{2})_j}{j!(-\sigma-n)_j}(1-t)^j, \qquad n = 2k+l.$$

For the potential U_2 in $\mathbf{L}^2$,

$$E_n^{(2)} = -\frac{\hbar^2}{2R^2}(n+1)^2 + \omega\hbar(n+1)\sqrt{1+\frac{\hbar^2}{4\omega^2R^4}},$$
$$0 \le n < \frac{\omega R^2}{\hbar}\sqrt{1+\frac{\hbar^2}{4\omega^2R^4}} - 1.$$

The degeneracy of the energy levels is $n+1$, as in the Euclidean case. We see that, in contrast to the Euclidean case, the exact eigenvalues E_n for U_1 and U_2 differ from the corresponding quasi-classical values from (5) and (6). The eigenfunctions corresponding to $E_n^{(2)}$ have the form

$$\psi_{n,m}(u,\phi) = (u^2-1)^{-\frac{\sigma+n}{2}} u^\sigma P_{n,m}^{(2)}(u^2)\exp(im\phi),$$

where

$$P_{n,m}^{(2)}(t) = 1 + \sum_{j=1}^{k} \frac{(-k)_j(-k-|m|)_j}{j!(-\sigma-n+\frac{1}{2})_j}(1-t)^j, \qquad n = 2k+|m|.$$

Note that at $R \to \infty$, in all cases, these eigenfunctions (under proper normalization) become the known eigenfunctions of the corresponding problems in $\mathbf{E}^2$ and $\mathbf{E}^3$.

One can easily see that the energies $E_n^{(3)}$ and $E_n^{(2)}$ grow monotonously as the level number n increases within the range pointed out above. Similar to the case U_1, the finiteness conditions for the classical motion (2) can be violated for maximum n. Meanwhile, as before, $E_n^{(3)}$ exceeds $U_2^{\max}$ by no more than $\frac{\hbar^2}{2R^2}$, while $E_n^{(2)}$ exceeds $U_2^{\max}$ by no more than $\frac{\hbar^2}{8R^2}$.

This result initiates the following problem. Let M be a Riemann manifold of infinite volume and $-\Delta + U(x)$ be the Schrödinger operator on $M \ni x$ such that $U_{\max} = \sup U(x) < \infty$. It is required to find the upper bound for $E_n - U_{\max}$, where E_n are the eigenvalues of this operator.

All of the above formulas for the eigenvalues demonstrate that the quasi-classical limit is equivalent to the Euclidean limit in the sense that for $\hbar \to 0$ and for $R \to \infty$, the leading term in the eigenvalue asymptotic expansions is given by the corresponding classical formula in Euclidean space.

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