M_p-GROUPS

V. P. Shunkov UDC 519.45

Already in 1945, Chernikov [1, 2] had isolated a very interesting class of groups, namely the p-groups of type $V = S\lambda(a)$, where S is a complete Abelian group, a is an element of order p, with a finite center $Z(V)$; here, as usual, (a) is the cyclic subgroup with generator a. As it turned out, because of their properties, these groups are close to the Frobenius groups and to the dihedral 2-groups. Thus, for example, all cyclic subgroups of the form (ha), $h \in S$, conjugate in V with (a) and the quotients of the upper central series of the group V are finite elementary Abelian groups. In the investigation of groups with various finiteness conditions, in certain situations it is necessary to distinguish as a special case the question of the imbedding of a subgroup of type V into the group. This circumstance has compelled the author to look at the group V from the point of view of its characterization regarding imbeddability in a sufficiently large class of groups. However, the investigation, started in this direction, has exceeded the purpose set initially by the author in connection with the mentioned characterization and has led to the necessity of introducing a new class of groups, the so-called M_p-groups (see definition in Sec. 2). In the present paper we have obtained the characterization of M_p-groups with p-finite handles in the class of groups without involution (the fundamental theorem in Sec. 2). As an application of this theorem we give only one result (the theorem in Sec. 7), but this does not exhaust by far the possibilities of the application of the fundamental theorem to abstract group theory and beyond.

The case when the group contains involution is not considered here (except for a remark in Sec. 6) since it requires a special approach, based on the characterization of Lie-type groups in the class of (periodic) groups.

The notations used in the paper are standard [3, 4].

1. Known Results, Definitions, Auxiliary Propositions

1. Merzlyakov's Theorem. The group $\mathrm{Out}\,G$ of outer automorphisms of an arbitrary Chernikov group G is almost torsion-free ([5]; see also [6, p. 458]). From here it follows, in particular, that the orders of the periodic subgroups of the group $\mathrm{Out}\,G$ are finite and bounded in their totality by the index of the maximal normal torsion-free subgroup [5, 6] (their finiteness has been established in [7]).

2. If an Abelian p-group G has an automorphism of order p with a finite centralizer in G, then G is a Chernikov group [8, 9].

3. Feit–Thompson Theorem. A finite group of odd order is solvable [10].

4. Burnside's Theorem. Let G be a finite group of the form $G = B\lambda A$, where B is an elementary Abelian q-group, A is an elementary Abelian p-group of order p^2 and $q \neq p$.

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Then there exists in A a nontrivial element τ such that $C_G(\tau)\cap B\neq 1$ [11].

5. Let G be a finite group of the form $G=K\leftthreetimes A$, where K is a p'-group, and A is an elementary Abelian p-group of order p^2. Then $K=\mathrm{gr}(C_K(\tau)\mid \tau\in A^{\#})$.

The proof follows from Frattini's lemma and from Proposition 4.

6. Let G be a finite solvable group and let L be the Fitting radical. Then $C_G(L)\leqslant L$ [3].

7. Podufalov's Lemma. Let G be a finite group of the form $G=S\leftthreetimes(Q\leftthreetimes(a))$, where S is a $\{p,q\}'$-group and $C_G(S)\leq S$, Q is a q-group, $q\neq 2$, and a is an element of order $p\neq q$. If $S\leftthreetimes(a)$ is a Frobenius group with complementary factor (a), then $Q\leftthreetimes(a)=Q\times(a)$ [12].

8. Frattini's Lemma. Let G be a finite group; let L be a normal subgroup in G, and let P be a Sylow p-subgroup in L. Then $G=N_G(P)L$ [3, Lemma 17.1.8].

9. If a finite p-subgroup $(p\neq 2)$ has a unique subgroup of order p, then it is a cyclic group [13].

10. A Chernikov p-group is a ZA-group and the normalizer condition is satisfied in it [3, 14].

11. Let G be a p-group of the form $G=P\leftthreetimes A$, where P is an infinite complete Abelian group and A is an elementary Abelian group of order p^2. Then in A there exists an element $t\neq 1$ such that $P\cap C_G(t)$ is infinite [15, 16].

12. Let G be a p-group of the form $G=P\leftthreetimes(a)$, where P is an infinite complete Abelian group and a is an element of order p with finite $C_G(a)$. Then the following assertions hold:

1) $Z(G)$ is an elementary Abelian group, $Z(G)\subset P$;

2) all the elements of the form ha or $ah\,(h\in P)$ are conjugate with a with the aid of elements from P [16].

13. Let G be a group; let L be a complete Abelian p-group, and $L\lhd G$, and let a be an element of order p from G with finite $L\cap C_G(a)$. Then

1) $C_{\bar G}(aL)=C_G(a)L/L$, where $\bar G=G/L$;

2) if K is a finite (a)-invariant subgroup in L, then in $N_G(K)/K=T$ the centralizer $C_T(aK)=RC_G((a))/K$, where R is an elementary Abelian subgroup in L/K.

Proof. The assertions 1) and 2) follow from Proposition 12.

14. Dietzmann's Lemma. A finite invariant set of elements of finite order from an arbitrary group generates a finite normal subgroup [14, 17].

15. Let G be a group; let H be a proper subgroup in it; let a be an element of prime order $p>2$ in G, and assume that for each $g\in G\setminus H\,\mathrm{gr}(a,a^g)$ there exists a Frobenius group with complement (a). Then $G=F\leftthreetimes N_G((a))$, where $F\leftthreetimes(a)$ is a Frobenius group with kernel F [18] (see also [19]).

16. A group G is said to be a Frobenius group with complement H and kernel F if $G=F\leftthreetimes H$, $F^{\#}=G\setminus\bigcup_{x\in G}H^x$, and $H\cap H^g=1$ for any $g\in G\setminus H$.

17. Let G be an infinite complete Abelian p-group satisfying the minimality condition. We introduce the parameter $m_p=m_p(G)=\max\{|S|\ :\ S$ is a Sylow p-subgroup in $\mathrm{Aut}(G)\}$. By Proposition 1, m_p is a finite number and $m_p=p^{\alpha}$. The number m_p is called the p-torsion parameter.

18. Let G be an infinite complete Abelian p-group, satisfying the minimality condition and let $G = A \times B$. Then

$$m_p(G) \geq m_p(A) \cdot m_p(B). \qquad (1)$$

Proof. Obviously, $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ is imbedded in $\mathrm{Aut}(G)$. From here we obtain the inequality (1).

19. Let G be an infinite p-group, being a holomorphic extension of a complete Abelian group B satisfying the minimality condition, with the aid of some group of automorphisms A: $G = B \leftthreetimes A$.

If $Z(G)^{m_p(B)}$ contains elements of order p^2, then $Z(G)$ is an infinite group.

Proof. Obviously, $Z = Z(G) \subset B$. We shall prove the theorem by induction on the rank $\ell_p = \ell_p(B)$. Let c be some element of order p from $Z(A)$. By virtue of Proposition 12, $V = B \cap C_G(c)$ is an infinite group. Let $\tilde{V}$ be its complete part. Obviously, $\tilde{V}$ is an A-invariant subgroup. Since A is a group of automorphisms of the subgroup B, we have $\tilde{V} \neq B$ and, consequently, their ranks $\ell_p(\tilde{V})$, $\ell_p(B)$ satisfy the inequality $\ell_p(\tilde{V}) < \ell_p(B)$. If $C_G(A) > \tilde{V}$, then the assertion of the proposition holds. Let $\tilde{V} \not< C_G(A)$. Obviously, in $G/\tilde{V}$ an element from $\tilde{V}$ induces an almost regular automorphism in $B/\tilde{V}$. From here, in view of Proposition 12, $Z(B/\tilde{V} \leftthreetimes (c\tilde{V}))$ is an elementary Abelian group. But then $Z^p < \tilde{V}$, and since $B = \tilde{V} \times B_1$, it follows, by Proposition 18, that $m_p(\tilde{V}) < m_p(B)$ and thus, $p m_p(\tilde{V}) \leq m_p(B)$. From here $T = Z^{p m_p(\tilde{V})} < \tilde{V}$ and T contains elements of order p^2. By the induction hypothesis, the intersection $\tilde{V} \cap C_G(A)$ is infinite. The proposition is proved.

20. Let G be an infinite Chernikov p-group, and let B be its complete part. If $B \cap (Z(G))^{m_p(B)}$ has elements of order p^2, then $Z(G)$ is infinite.

Proof. The quotient group $G/C_G(B)$ is isomorphic to the group of the automorphisms of the group B, which leaves fixed the subgroup of B containing elements of order p^2. By Proposition 19, $Z(G)$ is an infinite group. The proposition is proved.

21. An element g of the group G is said to be almost regular if $C_G(g)$ is a finite group.

22. Let c, a be elements in the group G. The element c is said to be a-real if $V = \mathrm{gr}(c, a)$ is a Frobenius group with kernel containing the element c and with complement (a). If V is a finite group, then $V = \mathrm{gr}(a, a^c)$ [18, 19].

23. A group G is said to be q-biprimitively finite if, for any finite subgroup H in the quotient group $N_G(H)/H$, any two elements of prime order q generate a finite subgroup.

24. Gorchakov's Lemma. Let G be a locally finite group with Abelian Sylow p-subgroups relative to some p. Then $O^p(G) \cap Z(G)$ is a p'-group [20].

25. Thompson's Theorem. The kernel of a finite Frobenius group is a nilpotent group [22, 11].

26. Let G be a group; let P be a finite p-subgroup of G, and let V be a locally finite normal p'-subgroup of G. Then in $\bar{G} = G/V$ one has the relations

$$N_{\bar{G}}(PV/V) = N_G(P)V/V,$$

$$C_{\bar{G}}(PV/V) = C_G(P)V/V.$$

27. If a locally finite p-group G has an element of order p with a Chernikov centralizer, then G is a Chernikov group [8].

28. Let G be a finite p-solvable group, $p\neq 2$, $S=O_p(G)$, and $C_G(S)\leq S$. If G does not have sections of type $SL(2,3)$, then every Abelian normal divisor from the Sylow p-subgroup of the group G is contained in S.

The proof of the proposition is given in [21, pp. 93-94].

2. M_p-Groups, Examples, Fundamental Result

Definition. Let G be a group; let B be its infinite normal complete Abelian p-subgroup satisfying the minimality condition; let P be an element of order a; and assume that the following conditions hold:

a) the locally finite p-subgroups of $C_G(a)B/B$ are finite;

b) if some complete Abelian p-subgroup C of the group G is contained in the set $\bigcup_{g\in G}$ gr (a, a^g), then $C\leq B$.

The group G from the definition is said to be an M_p-group, while the subgroups B, (a) are called the kernel and the handle, respectively, of the M_p-group G. In an M_p-group G the handle (a) can be of the following types:

1) in $C_G(a)$ the locally finite p-subgroups are finite;

2) in $C_G(a)$ the locally finite p'-subgroups are finite;

3) in $C_G(a)$ the locally finite subgroups are finite.

In accordance with the mentioned demarcations relative to the handle of an M_p-group G, a handle of type 1) will be said to be p-finite, one of type 2) will be called p'-finite, and one of type 3) will be said to be finite.

Examples of M_p-groups (with finite handles):

Example 1. Every Chernikov group having an infinite p-subgroup and an almost regular element of order p is an M_p-group.

Example 2. A holomorphic extension of a group from Example 1 with the aid of any group of outer automorphisms.

Example 3. Every holomorphic extension of an infinite Chernikov p-group with the aid of a group of outer automorphisms is an M_p-group.

The assertion in Example 1 is obvious, while the assertions of Examples 2 and 3 follow from Proposition 1.

Example 4. If G is an M_p-group with a finite handle (a), then $C_G(a)$ may have infinite p-subgroups. For example, it is sufficient to take a direct product of a group from Examples 1 and 2 and of a free periodic Novikov—Adyan group [23].

Example 5. One can give an example of an M_p-group G with handle (a) in which for some element t the subgroup gr $(a,a^t)B/B$ is an infinite periodic group. For example, let P be a p-group of type $P=B\leftthreetimes(C)$, where B is a complete Abelian group, $|C|=p$, $C_B(C)$ is finite, T is a free Novikov—Adyan group of period p and $T=V\leftthreetimes(S)$. We consider the group $H=P\times T$ and in it the subgroup $G=(B\times V)\leftthreetimes(a)$, where $a=cs$. Obviously, G is an M_p-group with handle (a) and kernel B. Making use of the properties of the Novikov—Adyan group [23], one can easily find in V and element t such that $(a, a^t)B/B$ will be an infinite periodic group.

Example 6. The kernel of an M_p-group need not coincide with the maximal complete Abelian p-subgroup of the group. Indeed, let $G = H \times T$, where $H = P \leftthreetimes (c)$, P is an infinite complete Abelian p-group, $|c| = p$, $C_H(c)$ is finite, T is a free product of a quasicyclic p-group S and of a cyclic group (b) of order p. Obviously, G is an M_p-group with kernel P and finite handle (bc), and B is imbedded in the maximal complete Abelian p-subgroup $B \times S \neq B$.

In this paper we shall prove the following:

Fundamental Theorem. Let G be a group without involutions, and let B be its infinite complete Abelian p-subgroup, satisfying the conditions:

1) $H = N_G(B)$ is an M_p-group with kernel B and p-finite handle (a);

2) for every $g \in G \setminus H^*$ the subgroup (a, a^g) is finite;

3) $|C_G(a) : H \cap C_G(a)| < \infty$ and $H \cap C_G(a)$ contains all the p'-elements of finite order from $C_G(a)$;

4) if Q is a finite (a)-invariant q-subgroup from H satisfying $Q \cap C_G(a) \neq 1$ and $q \neq p$, then $N_G(Q) \leq H$.

Then $B \lhd G$.

We show on examples the existence of groups in which condition 4) is automatically satisfied, while each of the conditions 1)-3) is independent from the remaining ones.

Example 7. Let H be the group from Example 1 without involution, and let T be some nontrivial group of involutions. In the periodic (according to S. I. Adyan) product $G = H \bar{*} T$, for the subgroup H and its almost regular element a of order p the conditions 1) and 3) hold, while condition 2) is not satisfied and G is not an M_p-group with handle (a) (regarding the abstract properties of periodic products, see [24]).

Example 8. Let $G = (c) \wr V$, where V is a p-group of type $V = B \leftthreetimes (a)$, B is a complete Abelian group, and a, c are elements of order p. Obviously, $N_G(B)$ and (a) satisfy the conditions 1) and 2), $|C_G(a) : N_G(B) \cap C_G(a)| = \infty$ and G is not an M_p-group.

Example 9. Let G be a locally finite Frobenius group with complementary factor B, a quasicyclic p-group, and let a be some element of order p in B. Conditions 2) and 3) hold for $N_G(B) = B$ and the element a, but B is not an M_p-group with a p-finite handle.

The proof of the fundamental theorem is given in Secs. 3-6, where one assumes that the theorem does not hold, i.e., $H \neq G$. We consider the set $\mathfrak{A}$ of triples of type (T, V, c) where T is a group without involutions, $N_T(V)$ is an M_p-group with kernel V and with a p-finite handle (c), satisfying the conditions 2)-4) of the theorem, and $N_T(V) \neq T$. In the set $\mathfrak{A}$ we consider the subset $\mathfrak{L}$ of triples (G, B, a) with the least rank $\ell_p = \ell(B)$ of the kernel B. In view of condition 1) of the theorem and Proposition 2, ℓ_p is a finite number. But then $m_p = \max\{|S| : S \text{ is a Sylow } p\text{-subgroup of } Aut(B)\}$ is also a finite number (Proposition 1).

3. Subgroups of the Second Kind and Type (*)

LEMMA 1. Let (G, B, α) be a triple in $\mathfrak{L}$. Then the following assertions hold:

1) $(G, B^g, \alpha^g) \in \mathfrak{L}$ for each $g \in G$;

2) $N_G(H) = H$, where $H = N_G(B)$;

3) if S is an (α)-invariant finite subgroup in B and $N_G(S) \not\leq H$, then $(N_G(S), B, \alpha) \in \mathfrak{L}$, $(N_G(S)/S,\ B/S,\ \alpha S) \in \mathfrak{L}$;

4) if S is an (α)-invariant infinite subgroup from B, then $N_G(S) \leq H$;

5) if P is a maximal, locally finite (α)-invariant P-subgroup from H, then P/B is a finite group;

6) $(G, B, \alpha^h) \in \mathfrak{L}$ for any $h \in H$ and, in particular, in view of Proposition 12, $(G, B, r\alpha) \in \mathfrak{L}$ for any $r \in B$;

7) if p is a finite, (α)-invariant P-subgroup in H, $N_G(P) \not\leq H$ and $C_G(P) \cap B$ is infinite, then $B < C_G(P)$ and $(N_G(P), B, \alpha), (C_G(P)\cdot(\alpha), B, \alpha) \in \mathfrak{L}$.

Proof. The statements 1) and 6) are obvious. The statements 3) and 4) follow from Proposition 13, the assumptions of the theorem, and the definition of the set $\mathfrak{L}$. We prove 5). We assume that $\bar{P} = P/B$ is an infinite group. By Proposition 13, $C_{\bar{P}}(\bar{\alpha}) = C_P(\alpha)B/B$, where $\bar{\alpha} = \alpha B$ and, therefore, $|C_{\bar{P}}(\bar{\alpha})| < \infty$. By Proposition 27, $\bar{P}$ is a Chernikov group; let V be its complete part. By Proposition 12, $V < \bigcup_{h \in V} \mathrm{gr}(\bar{\alpha}, \bar{\alpha}^h)$ and we have obtained a contradiction with the definition of an M_p-group [see condition b)]. Consequently, $\bar{P}$ is a finite group and statement 5) is proved. If we had $N_G(H) \neq H$, then in $N_G(H) \setminus H$ there would exist an element x such that $B^x \neq B$, $B^x < H$, and BB^x is an (α)-invariant subgroup in H, while BB^x/B is an infinite group. But then we would obtain a contradiction with the assertion 5) of the lemma, which has already been proved. Consequently, $H = N_G(H)$ and assertion 2) is proved. We prove 7). Let $H_1 = C_G(P)\cdot(\alpha) \cap H$ and let B_1 be the complete part of the intersection $B \cap C_G(P)$. As above, one can easily show that $N_G(H_1) \cap C_G(P)\cdot(\alpha) = H_1$ and the triple $(C_G(P)\cdot(\alpha), B_1, (\alpha))$ satisfies the conditions of the theorem and $\ell_p(B_1) \leq \ell_p$. If we would have $\ell_p(B_1) < \ell_p$, then in view of the definition of $\mathfrak{L}$ we would obtain $B_1 \lhd C_G(P)\cdot(\alpha)$ and, by statement 4) of the lemma, we would have $C_G(P) \leq H$. Since $C_G(P) \lhd N_G(P)$ and $B_1^x \lhd C_G(P) \lhd N_G(P)$, $x \in N_G(P)$, making use of statement 5) and Theorem 18.1.2 of [3], we can see that $N_G(P) \leq H$ in spite of the conditions of the lemma. Consequently, $\ell_p(B_1) = \ell_p$ and $B_1 = B$. But then, obviously, $(N_G(P), B, \alpha) \in \mathfrak{L}, (C_G(P)\cdot(\alpha), B, \alpha) \in \mathfrak{L}$. Statement 7) is proved and thus the proof of the lemma is concluded.

Definition 1. A finite, nontrivial, (α)-invariant subgroup R in B is said to be a subgroup of the second kind if $N_G(R) \not\leq H$; otherwise it is said to be of the first kind.

LEMMA 2. Let K be a finite, (α)-invariant p'-subgroup of the group G. Then $R = N_G(K) \cap B = C_G(K) \cap B$.

Proof. Let L be the nilpotent radical of the subgroup $K \neq 1$. Since K is a solvable group (Proposition 3), we have $L \neq 1$. By Lemma 1, $(G, H, r\alpha)(r \in R)$ is a triple-counterexample. If for some $r_1 \in R$ we would have $L \cap C_G(r_1\alpha) \neq 1$, then, making use of condition 3) of the theorem and of the normalizer condition in nilpotent groups (Theorem 16.2.2 in [3]), as well as of the automorphic admissibility of L in K, we would prove that $K \leq H$, and in this case the assertion of the lemma is obvious. Let $C_G(ra) \cap L = 1$ for some $r \in R$. Then, on the basis of Proposition 4 and of the representability of the subgroup gr (L, R, a) in the form $L \leftthreetimes R \leftthreetimes (a)$, we obtain $R < C_G(L)$. From here and from Proposition 6 there follows that $R \times L \triangleleft K \leftthreetimes R$ and $R < C_G(K)$. The lemma is proved.

LEMMA 3. Let K be a finite, nontrivial, (a)-invariant P'-subgroup from G and let $T = N_G(K) \nless H$.

Then the following assertions hold:

1) if L is the nilpotent radical of the subgroup K, then $L \leftthreetimes (a)$ is a Frobenius group and in K/L the Sylow primary subgroups are cyclic and $(K \leftthreetimes (a)/L) = K/L \times (aL)$;

2) if $L \neq K$, then there exists an (a)-invariant Sylow q-subgroup P of the group K such that $P \nless L$, $T = KN_T(P)$, and $N_T(P) < H$, $L \nless H$.

Proof. The fact that $L \leftthreetimes (a)$ is a Frobenius group has been established in the proof of Lemma 2. From here, in view of Propositions 6 and 7, we obtain

$$(K \leftthreetimes (a))/L = K/L \times (aL).$$

We prove statement 2). Let $L \neq K$ and let P be a Sylow q-subgroup in K, $P \nless L$, and $a \in N_T(P)$. From statement 1) it follows that $P \cap C_G(a) \neq 1$. From here, making use of the normalizer condition in P [3, Theorem 16.2.2], as well as of condition 4) of the theorem, we obtain $N_T(P) < H$. It remains to show that $L \nless H$. Indeed, by Frattini's lemma (Proposition 8), we have $T = KN_T(P)$ while, by statement 1), $K = LC_K(a)$, where $C_K(a) < H$ [condition 3) of the theorem]. From here, if $L < H$, then $K < H$, and, since, according to what has been proved above we have $N_T(P) < H$, we would obtain $T < H$, contradicting the assumption of the lemma. Consequently, $L \nless H$.

We assume that $\bar{P} = PL/L$ is a noncyclic group. In this case, on the basis of Proposition 9 we conclude that $\bar{P}$ has an elementary Abelian subgroup V of order q^2. According to what has been proved above, $V \times (aL)$. Since $L = L_1 \times P_1$, where P_1 is a Sylow q-subgroup in L, it follows that L_1 is a V-invariant subgroup. By Proposition 5, $L_1 = \mathrm{gr}(C_{L_1}(\bar{u}) \mid \bar{u} \in V^{\#})$. Consequently, $C_{L_1}(\bar{u}) = S_u \times (u)$, where u is the q-preimage of the element $\bar{u}$ in K. Obviously, $u \in C_G(a)$. From here, by condition 4) of the theorem, $S_u < H$. But then $L_1 < H$ and thus, also $L < H$, which contradicts the relation $L \nless H$, proved earlier. Consequently, $\bar{P}$ is a cyclic group and the lemma is completely proved.

LEMMA 4. Let P be a finite P-subgroup in G and let $a \in P$. If $R = B \cap P \neq 1$ and $N_G(P) \nless (H)$, then B has subgroups of the second kind.

Proof. Let $\mathfrak{D} = P \cap H$. Obviously, $N_G(\mathfrak{D}) \nless H$ and in $N_G(\mathfrak{D}) \setminus H$ there exists an element c. Then, $L = R \cap Z(\mathfrak{D}) \neq 1$ [3, Theorem 16.2.3]. By virtue of the automorphic admissibility of $Z(\mathfrak{D})$ in $\mathfrak{D}$, we have $L_1 = L^c \leq Z(\mathfrak{D})$ and, in addition, $L_1 < B^c = B_1 < H^c = H$. If $C_G(L_1) \nless H_1$, then, obviously, $C_G(L) \nless H$ and, in this case, L is a subgroup of the second kind. Let $C_G(L_1) < H$. But then $(L, a) < C_G(L_1) < H_1$. We select in L an element $t \neq 1$ and let $S = (t) \times (a)$. We consider the subgroup $B_1 S$. By Proposition 11, in S there exists a nonidentity element r such that $C_G(r) \cap B_1$ is an infinite group. If $r = ha$, where $h \in (t) < B$, then, by virtue of Lemma 1 and condition 3) of the theorem, the intersection $H \cap C_G(r) \cap B_1$ is infinite. But then we obtain a contradiction with condition

1) of the theorem and with Lemma 1. Consequently, $\tau \in (t) < B$. If $C_G(\tau) \nleq H$, then (τ) is a subgroup of the second kind. Let $C_G(\tau) \leq H$. In this case, $B_1 < H$ and since B_1 is an (a)-invariant complete Abelian p-subgroup and $C_{B_1}(a)$ is finite, it follows, by Proposition 12, that $B_1 < \bigcup_{h \in B_1} \mathrm{gr}\,(a, a^h)$. From here and from Lemma 1 [see its statement 5)] it follows that $B^c = B_1 = B$ and $c \in N_G(B) = H$, which is not possible. The obtained contradiction proves the lemma.

Definition 2. A finite, nontrivial, (a)-invariant p-subgroup X in H is said to be a subgroup of type $(*)$ if $N_G(X) \nleq H$.

LEMMA 5. Let P be a subgroup of type $(*)$ and assume that $|N_G(P) : N_G(P) \cap H| < \infty$. Then H has a subgroup of the second kind.

Proof. Since BP is a Chernikov p-group, we have $N_B(P) \cap B = R_1 \neq 1$ (Proposition 10). From the conditions of the lemma it follows that the centralizer R_1 in $G_1 = N_G(P)$ has a finite index in G_1 and by Dietzmann's lemma (Proposition 14), the closure of R_1 in G_1 is finite. Consequently, in G_1 there exists a finite normal subgroup Z_1, containing the subgroup $\mathrm{gr}\,(P, R_1)$. Let S be an (a)-invariant Sylow p-subgroup in Z_1 and let $R_1 < S$. If $S(a) \nleq H$, then, by Lemma 4, B has a subgroup of the second kind. Let $S < H$ and, consequently, $R_1 \lhd S$. From here, on the basis of Proposition 28, we conclude that $R_1 < O_{p'p}(Z_1) = L$. By Lemma 2, $\mathrm{gr}\,(P, R_1) \leq P_1 = O_p(L) \lhd G_1$.

Consequently, $G_2 = N_G(P_1) \nleq H$ and $a \in G_2$. Again by Lemma 4, we obtain that $P_1 < H$. Let $R_2 = B \cap G_2$. By virtue of Proposition 10, $R_2 \nleq P_1$ and thus $R_2 \neq R_1$ and $R_1 < R_2$. As far as the pair (G_2, R_2) is concerned, we proceed in the same way as for (G_1, R_1). Repeating these arguments, we construct in H a strictly increasing chain of finite, (a)-invariant p-subgroups

$$P = P_0 < P_1 < \ldots < P_n < \ldots \tag{2}$$

so that $G_{n+1} = N_G(P_n) \nleq H$, $n = 0, 1, \ldots$

From Lemma 1 it follows that $|P_n / R_n|$, $n = 1, 2, \ldots$, are bounded in their totality. From here it follows at once that for sufficiently large n the subgroup R_n has a nontrivial subgroup, normal in G_{n+1}, i.e., a subgroup of the second kind. If, however, the chain (2) has a finite number of terms, then from the method of its construction there follows the existence of a subgroup of the second kind. The lemma is proved.

LEMMA 6. If H does not have subgroups of the second kind, then the following statements hold:

1) every finite p-subgroup, containing the element a, is contained in H;

2) if P is a subgroup of type $(*)$, $G_1 = N_G(P)$, $H_1 = H \cap G_1$, $\bar{G}_1 = G_1 / P$, $\bar{H}_1 = H_1 / P$, $\bar{a} = aP$, then $C_{\bar{G}_1}(\bar{a}) < \bar{H}_1$;

3) if P is a subgroup of type $(*)$, then $B \cap P = 1$;

4) $a^g \in H \Rightarrow g \in H$ and $C_G(a) < H$.

Proof. Let S be a finite p-subgroup of the group G, containing the element a. We assume that $S_1 = H \cap S \neq S$. In this case, $N_G(S_1) \nleq H$ [3, Theorem 16.2.2]. Since $a \in S$ and $|C_G(a) : H \cap C_G(a)| < \infty$ [condition 3) of the theorem], it follows, obviously, that $|N_G(S_1) : H$

$|N_G(S_1)| < \infty$ and, by Lemma 5, H has subgroups of the second kind, in spite of the assumptions of the lemma. Consequently, $S < H$ and statement 1) is proved. If we would have $C_{\bar{G}_1}(\bar{a}) \nleq \bar{H}_1$, then the complete preimage S_2 of the subgroup $(\bar{a})$ in G would be a subgroup of type $(*)$, containing the element a. Reasoning regarding the subgroup S_2 in the same way as regarding S, we would obtain a contradiction with the conditions of the lemma. Consequently, $C_{\bar{G}_1}(\bar{a}) < \bar{H}_1$ and statement 2) is proved. Statement 3) is proved in a similar manner. Since $a^g \in H$, we have $a \in H^{g^{-1}}$. The subgroup $B^{g^{-1}} \leftthreetimes (a)$ is a locally finite p-group, containing the element a. By statement 1) we have $B^{g^{-1}} < H$ and, in view of Lemma 1, we have $gBg^{-1} = B$; thus, $g \in H$. The lemma is proved.

LEMMA 7. The subgroup H contains a subgroup of the second kind.

Proof. We assume that the lemma does not hold. First we show that H has subgroups of type $(*)$. We consider the subgroup $L_g = \mathrm{gr}(a, a^g)$, $g \in G \setminus H$. By condition 2) of the theorem it is finite and, by Lemma 6, $a^{g^g} \notin H$; thus, $L_g \nleq H$. If $O_{p'}(L_g) = 1$, then in view of the solvability of L_g (Proposition 3) we have $O_p(L_g) \neq 1$. Since $a \in L_g$, we have that $O_p(L_g) \cdot (a)$ is a finite p-subgroup, containing the element a and, in view of the assumption regarding the absence in H of subgroups of the second kind and by Lemma 6, we have $O_p(L_g) < H$. But then $O_p(L_g)$ is a subgroup of type $(*)$.

Let $F_g = O_{p'}(L_g) \neq 1$. By Frattini's lemma, $L_g = F_g N_{L_g}(Q)$, where Q is a Sylow p-subgroup in $O_{p'p}(L_g)$ and $a \in N_{L_g}(Q)$. If $N_{L_g}(Q) \nleq H$, then Q is a subgroup of type $(*)$. Let $N_{L_g}(Q) < H$; also in this case, $F_g \nleq H$. We assume that the element a is contained in some elementary Abelian p-subgroup R of order p^2 from L_g. We consider the subgroup $F_g \leftthreetimes R$. By Proposition 5, $F_g < \mathrm{gr}(C_G(s) \mid s \in R^{\#})$, and since $F_g \nleq H$, it follows that for some element $b \in R^{\#}$ we have $C_G(b) \nleq H$, $b \in C_G(a) < H$ (Lemma 6). Consequently, (b) is a subgroup of type $(*)$. It remains to consider the case when the Sylow p-subgroup from L_g has a unique subgroup of order p and, in view of the condition $p \neq 2$ and Proposition 9, it is cyclic. On the basis of Lemma 3, the solvability of L_g and the fact that it is generated by two elements of order p, we conclude that $L_g = F_g \leftthreetimes (a)$ is a Frobenius group with complement (a). By Proposition 15, $G = F \leftthreetimes N_G((a))$, where $F \leftthreetimes (a)$ is a Frobenius group. But then $B \cap F = 1$ and thus, $B \leftthreetimes (a) \cap F = 1$, and, in view of the representability of G in the form $F \leftthreetimes N_G((a))$, it would follow from here that in G/F the subgroup $(B \leftthreetimes (a))F/F$ is equal to $BF/F \times (aF)$. However, this is not possible since $(B \leftthreetimes (a))F/F \simeq B \leftthreetimes (a)$ is not an Abelian group. All the considered cases have confirmed the presence of subgroups of type H in $(*)$.

In view of the assumption regarding the absence in G of subgroups of the second kind and by Lemma 6, every subgroup of type $(*)$ intersects B at the identity. From here and from Lemma 1 there follows that, in G, there exists some subgroup P of type $(*)$, not contained in any larger subgroup of type $(*)$.

We introduce the notations

$$G_1 = N_G(P), \quad H_1 = H \cap G_1, \quad S = B \cap G_1, \quad \bar{G}_1 = G_1/P,$$

$$\bar{H}_1 = H_1/P, \quad \bar{S} = SP/P, \quad \bar{a} \doteq aP.$$

In view of Proposition 10, $S \neq 1$ and, by what has been proved above, $\bar{S} \neq 1$. Furthermore, $\bar{a} \in \bar{S}'$. Indeed, if $\bar{a} \in \bar{S}$, then $a = vh$, where $v \in B, h \in P$, and $h = v^{-1}a$. By Proposition 12, the elements h and a are conjugate through some element $x \in B: x^{-1}hx = a$. But then the subgroup P^x would obviously be a subgroup of type (*) and $a \in P^x$. From here and from Lemma 5 it would follow that H has subgroups of the second kind, in spite of the assumption. Consequently, $\bar{a} \notin \bar{S}$. Since $\bar{S} \triangleleft \bar{H}_1$, it follows by Lemma 6 and Theorem 16.2.12 of [3] that the inequality $N_{\bar{G}_1}(\bar{H}_1) \neq \bar{H}_1$ would mean that H_1 has a subgroup $P_1 \geqslant S \neq 1$ of type (*) and this would contradict the assumption that $P_1 \cap B = 1$. Consequently, $N_{\bar{G}_1}(\bar{H}_1) = \bar{H}_1$. Then by Lemma 6, $C_{\bar{G}_1}(\bar{a}) < \bar{H}_1$. Making use of the same considerations that have been applied at the beginning of the proof of the lemma and taking into account that P has been chosen as a subgroup of type (*) of maximal order, it is easy to show that the subgroups of the form $L_g = \text{gr}(\bar{a}, \bar{a}^g)$, where $g \in \bar{G}_1 \setminus \bar{H}_1$, are Frobenius groups with complement $(\bar{a})$. By Proposition 15, $\bar{G}_1 = F \leftthreetimes N_{\bar{G}_1}((\bar{a}))$, where $F \leftthreetimes (\bar{a})$ is a Frobenius group.

Let $\bar{t}$ be an element of order p from $\bar{S}$; let $\bar{t} \in C_{\bar{G}_1}(\bar{a})$ and let t be the preimage of $\bar{t}$ in S. We consider the subgroup $F \leftthreetimes Q$, where $Q = (\bar{t})^{1} \times (\bar{a})$. Since the elements a and at are conjugate in $B \leftthreetimes (a)$ (Proposition 12), it follows that the groups $K_g = \text{gr}(\bar{a}\bar{t}, \bar{a}^g)$, $g \in F \setminus \bar{H}_1$, are finite [condition 2) of the theorem and Lemma 1] and K_g has the form $K_g = \mathcal{D}_g \leftthreetimes \mathcal{Z}$, where $\mathcal{Z}$ is an elementary Abelian p-group of order p^2 and $\bar{a}\bar{t} \in \mathcal{Z}$. By Lemma 1, the cyclic subgroups of the form (ah), $h \in S$, are handles of M_p-groups H and, therefore, a subgroup of the form $F \leftthreetimes (\bar{a}v)$, $v \in \bar{S}$, is also a Frobenius group with complement $(\bar{a}v)$. From here it follows that $Q = \mathcal{Z}$ and, thus, $\bar{t}, \bar{a} \in \mathcal{Z}$. By virtue of Lemmas 2 and 6, $\mathcal{D}_g < C_{\bar{G}_1}(\bar{t})$ and $B \nleq \bar{H}_1$. Obviously, the complete preimage of the subgroup $(\bar{t})$ in G is a subgroup of type (*), having a nontrivial intersection with B. The obtained contradiction concludes the proof of the lemma.

4. Selection of a More Convenient Counterexample

Let Y be a subgroup of the second kind from the lower layer of B and of the largest order in the set of all such subgroups. By Lemma 7, $Y \neq 1$ and, by Lemma 1, $(T_1/Y, B/Y, aY) \in \mathfrak{L}$, where $T_1 = N_G(Y)$. By Lemma 7, there exists in T_1/Y a subgroup $\bar{Y}_1$ of the second kind. If Y_1 is the complete preimage of $\bar{Y}_1$ in T_1, then Y_1 is a subgroup of the second kind in T_1. If $T_2 = N_{T_1}(Y_1)$, then $(T_2/Y_1, B/Y_1, aY_1) \in \mathfrak{L}$ (Lemma 1). Regarding this triple, we proceed in the same way as before. Repeating these reasonings, we construct a strictly increasing chain of subgroups of the second kind

$$Y < Y_1 < Y_2 < \ldots < Y_n < \ldots . \tag{3}$$

To this there will correspond a decreasing chain of subgroups $T_1 \geqslant T_2 \geqslant \ldots \geqslant T_n \ldots$ such that

$$(T_{n+1}/Y_n, B/Y_n, aY_n) \in \mathfrak{L}, \quad n = 1, 2, \ldots .$$

In the chain (3), starting with a certain index n, the subgroup $Y_n^{m_p}$, where m_p is the parameter introduced in Proposition 18, has elements of order p^2. Without loss of generality, we shall assume that already Y_1^{mp} has such elements.

Let $E = C_{T_2}(Y_1)$, $V = gr(E, a)$. By Proposition 12, $V = E \leftthreetimes (a)$.

LEMMA 8. Let P be a subgroup of type (*) and $P < E$. Then $B < C_V(P)$.

Proof. Since $Y_1 < C_V(P)$ and Y_1 has elements of order p^2, it follows, by Proposition 20, that $C_V(P) \cap B$ is an infinite group. From here, Lemma 1, and the definition of a subgroup of type (*), there follows the assertion of the lemma.

In view of Lemmas 1, 7, 8, V has a subgroup K of type (*) in E such that $Y_1 \leq K$ and if K_1 is a subgroup of type (*) in E and $K \leq K_1$, then $K_1 = (B \cap K_1) \cdot K$.

Obviously, $(C_V(K) \leftthreetimes (a), B, a) \in \mathfrak{L}$, and each subgroup of type (*) from $C_V(K)$, containing $\mathfrak{Z}(K)$, is a subgroup of the group $B\mathfrak{Z}(K)$. Therefore, in the sequel we shall assume that K is an Abelian group.

We set

$$V_1 = C_V(K) \leftthreetimes (a),\ Q = H \cap V_1,\ E_1 = E \cap V_1,\ V_1 = E_1 \leftthreetimes (a),\ G_1 = V_1/K,$$
$$H_1 = Q/K,\ \bar{B} = BK/K,\ \bar{E}_1 = E_1/K,\ a_1 = aK.$$

We shall need the following:

Remark. If S is a finite (a_1)-invariant p-subgroup from $H_1 \cap \bar{E}_1$ and $N_{G_1}(S) \nleq H_1$, then $S < \bar{B}$.

The proof of this assertion follows from the definition of the subgroup K and from Lemma 8.

LEMMA 9. The triple $(G_1, \bar{B}, a_1)$ lies in $\mathfrak{L}$; moreover, any subgroup of type (*) from $H_1 \cap \bar{E}_1$ belongs to $\bar{B}$ and $C_{G_1}(a_1) < H_1$.

Proof. If $C_{G_1}(a_1) < H_1$, then, in view of the remark and Lemma 1, the assertion of the lemma holds. Let $C_{G_1}(a_1) \nless H_1$. We set $X_1 = C_{G_1}(a_1) \cap H_1$, $\mathfrak{Z}_1 = C_{G_1}(a_1)$. Obviously, $|\mathfrak{Z}_1 : X_1| < \infty$, $L_1 = \bar{B} \cap \mathfrak{Z}_1 \neq 1$, and $|\mathfrak{Z}_1 : C_{\mathfrak{Z}_1}(L_1)| < \infty$. Consequently, by Dietzmann's lemma (see Sec. 1), the closure S of the subgroup L_1 in $\mathfrak{Z}_1$ is finite. In view of the selection of the subgroup K and Proposition 26, in the complete preimage of the subgroup $O^p(S)$ in V_1, any of its p'-subgroups centralizes the element a. From here and from condition 4) of the theorem, there follows that $N_{G_1}(O^p(S)) \leq H_1$. But $O^p(S) \lhd \mathfrak{Z}_1$ and thus $\mathfrak{Z}_1 < H_1$, in spite of the assumption. Consequently, $O^p(S) = 1$ and S is a finite (a_1)-invariant p-subgroup from $\bar{E}_1$ and $L_1 < S$. If $P = H_1 \cap S \neq S$, then, in view of the normalization condition in S [3, Theorem 16.2.2], we have $N_{G_1}(P) \nleq H_1$ and, according to the remark, $P < \bar{B}$ and $L_1 = P$. If, however, $P = S$, then by similar considerations, $L_1 = S$ and $N_{G_1}(L_1) \nleq H_1$.

We denote $G_2 = N_{G_1}(L_1)$, $\bar{G}_2 = G_2/L_1$, $H_2 = G_2 \cap H_1$, $\bar{H}_2 = H_2/L_1$, $a_2 = a_1 L_1$, $\mathfrak{Z}_2 = C_{\bar{G}_2}(a_2)$, $X_2 = \bar{H}_2 \cap \mathfrak{Z}_2$. Obviously, $\mathfrak{Z}_2 \neq X_2$, $|\mathfrak{Z}_2 : X_2| < \infty$, $\bar{L}_2 = \bar{B}/\bar{L}_1 \cap \mathfrak{Z}_2$, where L_2 is the complete preimage of $\bar{L}_2$ in G_1. If regarding the triple $(\bar{G}_2, \bar{H}_2, a_2)$ and the pair $(X_2, \mathfrak{Z}_2)$ we reason in the same way as in the consideration of the triple (G_1, H_1, a_1) and the pair $(X_1, \mathfrak{Z}_1)$, then we can show that $N_{\bar{G}_2}(\bar{L}_2) \nleq \bar{H}_2$. Returning to the complete preimages in G_1, we obtain $G_3 =$

$N_{G_1}(L_2) \nleq H_1$. Repeating these arguments, we construct a strictly increasing chain of (a_1)-invariant finite subgroups of the group $\bar{B}$

$$1 = L_0 < L_1 < L_2 < \ldots < L_n < \ldots \tag{4}$$

such that

$$G_{n+1} = N_{G_n}(L_n) \nleq H_n,\ G_1 = \bar{G}_1,\ H_1 = \bar{H}_1,\ Z_n \neq X_n,\ n = 1, 2, \ldots .$$

By Proposition 13, the complete preimage of the subgroups Z_n in $\bar{G}_1$ are contained in $\bar{B}Z_1$ and, since $|Z_1 : X_1| < \infty$, where $X_1 = C_{G_1}(a_1) \cap H_1$, it follows that, without loss of generality, one can assume that each subgroup G_n, $n = 1, 2, \ldots$, contains an element $h_n = d_n z_n t$, where $d_n \in \bar{B}$, $z_n \in X_1 = C_{H_1}(a_1)$, t is some element, independent of the index n, from $Z_1 \setminus X_1$, $n = 1, 2, \ldots$.

By the definition of the subgroups L_n we have

$$L_{n-1} = L_{n-1}^{h_n} = L_{n-1}^{d_n z_n t}, \quad n = 1, 2, \ldots .$$

Since $\mathrm{gr}(d_n, L_{n-1}) < \bar{B}$ and $\bar{B}$ is an Abelian group, it follows that

$$L_{n-1} = L_{n-1}^{z_n t}, \quad n = 1, 2, \ldots . \tag{5}$$

If L is the union of the chain (4), then from (5) and from the inclusions $z_n \in H_1$, $n = 1, 2, \ldots$, there follows that $L < \bar{B}^t < H_1^t$. Consequently, in $\bar{B} \cap \bar{B}^t$ there exists an infinite (a_1)-invariant subgroup L. From here, passing to the preimages in the group V and making use of Lemma 1, it is easy to show that $\bar{B}^t = \bar{B}$ and $t \in H_1$ in spite of the assumption. The obtained contradiction concludes the proof of the lemma.

In the following lemma we record a more suitable selection for a counterexample.

LEMMA 10. If the theorem does not hold, then there exists a group G without involution possessing an infinite, complete, Abelian p-subgroup B and an element a of order p, such that the following statements hold:

1) $(G, B, a) \in \mathcal{L}$;

2) $G = E \leftthreetimes (a)$, $N_G(B) = H = \mathcal{D} \leftthreetimes (a)$, $\mathcal{D} < E$;

3) if P is a subgroup of type (*) from $\mathcal{D}$, then $P < B$ and $(N_G(P), B, a) \in \mathcal{L}$, $(N_G(P)/P, B/P, aP) \in \mathcal{L}$;

4) if R is a finite (a)-invariant subgroup of the second kind and $R \triangleleft T$, where T is a finite subgroup from G and T/R is a p'-group, then $R < Z(T)$;

5) if U is a finite subgroup from $\mathcal{D}$, then the intersection $C_G(U) \cap B$ is infinite;

6) $C_G(a) < H$.

Proof. The statements 1)-3) and 6) have been proved above (see Sec. 4). We prove 4). In fact, the group G is a section of the group V_1, introduced above. If S and N are the complete preimages of the groups R and T, respectively, in V_1, then S is an Abelian Sylow p-subgroup in N and, in view of the definition of the subgroup Y and the conditions $Y < Y_1 \leq K \leq S$, the lower layer of the subgroup S is contained in $Z(N)$. But then, according to Proposition 24, $N=S \times N_1$, where N_1 is a p'-group. Obviously, from here there follows the validity of the statement 4). Statement 5) follows from the definition of the group V_1 and from Propositions 20 and 24. The lemma is proved.

5. Structure of Subgroups of Type $L_g=\mathrm{gr}(a,a^g)$, $g\in G\setminus H$

By condition 2) of the fundamental theorem, the subgroups L_g are finite and solvable (Proposition 3). In the sequel, in the consideration of the subgroups L_g one assumes that $g \in G\setminus H$.

LEMMA 11. For G the following statements hold:

1) if P is a finite p-subgroup of the group G and $a\in P$, then $P<H$;

2) if $a^g\in H$, then $g\in H$ and $C_G(a)<H$.

Proof. By Lemma 10, $P=P_1 \leftthreetimes (a)$, where $P_1<E$ and $P_2=P_1\cap \mathcal{D}$. If $(a)\neq P$, then $P_2\neq 1$ [statement 6) of Lemma 10 and Theorem 16.2.3 of [3]]. We assume that $P_2\neq P_1$. Since P_2 is an (a)-invariant subgroup and $P_3=N_G(P_2)\cap P_1 \not< H$ [3, Theorem 16.2.2], it follows that P_2 is a subgroup of type (*) and, by Lemma 10, $P_2\subset B$. Again by Lemma 10, $(N_G(P_2)/P_2, B/P_2, aP_2)\in \mathcal{L}$ and in the subgroup $\bar{P}_3\leftthreetimes(aP_2)$, where $\bar{P}_3=P_3/P_2$, the intersection $C_{\bar{P}_3}(aP_2)\cap \bar{P}_3=\bar{P}_4$ is different from P_2 (Theorem 16.2.3 from [3]). From here and from Lemma 10 it follows that $\bar{P}_4<(\mathcal{D}\cap N_G(P_2))/P_2$. Returning to the complete preimages, we obtain that $P_4<\mathcal{D}$, where P_4 is the complete preimage of $\bar{P}_4$ in G and $P_2\nsubseteq P_4$, which contradicts the definition of the subgroup P_2. Consequently, $P<H$ and statement 1) is proved. As far as statement 2) is concerned, in fact, it has already been proved (see the corresponding assertion from Lemma 6).

LEMMA 12. Let P_g be a Sylow p-subgroup of the group L_g, containing the element a. Then:

1) $F_g=O_{p'}(L_g)$ is the kernel of the Frobenius groups $F_g\leftthreetimes(a)$ and $F_g\leftthreetimes(a^g)$;

2) $L_g=F_g\leftthreetimes P_g$ and $P_g<H$;

3) P_g/R_g $(R_g=B\cap P_g)$ is an elementary Abelian group of order $\leq p^2$;

4) $F_g\leftthreetimes R_g=F_g\times R_g$.

Proof. Since $g\in G\setminus H$, by Lemma 11 we have $a^g\notin H$ and $L_g\not< H$ and, as mentioned above, L_g is a finite solvable group. From here and from Lemma 2 there follows statement 4). We show that the case $F_g<H$ is not possible. Let Q be an (a)-invariant Sylow p-subgroup of the group $O_{p'p}(L_g)$. By Lemma 11, $Q<H$. From here, in view of $F_g<H$, we obtain $O_{p'p}(L_g)<H$. Then, $L_g=O_{p'p}(L_g)N_{L_g}(Q)$ (Frattini's lemma; see Sec. 1) and $L_g\not< H$. Consequently, $N_{L_g}(Q)\not< H$.

By Lemma 10 we have $Q_1=Q\cap E\lhd N_{L_g}(Q)$ and, therefore, Q_1 is a subgroup of type (*) for $Q_1\neq 1$ and, by Lemma 10, $Q_1\leq R_g$. If we would have $Q=Q_1\leq R_g$, then on the basis of Lemma 10 [see statement 5) in it] and the definition of the subgroups $O_{p'p}(L_g)$ we would conclude that $L_g=O_{p'p}(L_g)$. Since, according to what has been proved above, $O_{p'p}(L_g)<H$, we would also have $L_g<H$, which is not possible. Consequently, $Q_1\neq Q$ and $|Q:Q_1|=p$. But then also in this case, in view of the representation $G=E\leftthreetimes(a)$ we obtain $L_g=O_{p'p}(L_g)$ which, as mentioned above, leads to a contradiction with the assumption $L_g\not< H$.

Consequently, $F_g \nless H$. We show that $R_g \lhd L_g$. By Lemma 2, $R_g < C_{L_g}(F_g) \lhd L_g$. Let Q be a Sylow (a)-invariant p-subgroup from $C_{L_g}(F_g) \cap E = W$. By Lemma 11, $Q < H$ and since $F_g \nless H$, it follows that Q is a subgroup of type (*) for $Q \neq 1$. By Lemma 10, $Q < B$. But $R_g < W$ and thus $Q = R_g$. From here and from the Abelian property of the group R_g there follows that R_g is a Sylow subgroup in $O_{p'p}(W)$

By Lemma 2, $O_{p'p}(W) = O_{p'}(W) \times R_g$ and since $O_{p'p}(W)$ is automorphically admissible in W and $W \lhd L_g$, we have $R_g \lhd L_g$.

We introduce the notations:

$$\bar{L}_g = L_g/R_g,\ \bar{P}_g = P_g/R_g,\ \bar{F}_g = F_g R_g/R_g,\ \bar{S}_g = S_g/R_g,$$

where S_g is a component of the subgroup P_g in E, $\bar{a} = aR_g$. Let X be a maximal elementary Abelian subgroup of the group $C_{\bar{S}_g}(\bar{a})$. We consider separately two cases.

1) $|X| \geq p^2$. By Proposition 5,

$$\bar{F}_g = \mathrm{gr}\ (C_{\bar{F}_g}(z) \mid z \in X^{\#}). \tag{6}$$

The complete preimage of any cyclic subgroup (z), $z \in X^{\#}$, is an (a)-invariant p-subgroup of the group $\mathcal{D}$ and, as follows from Lemma 10 and the condition $R_g \neq S_g$, it is not a subgroup of type (*). But then, obviously, from (6) we obtain $F_g < H$, in spite of the above-proved relation $F_g \nless H$. Consequently, one has the second case:

2) $|X| \leq p$. If $X = 1$, then $\bar{S}_g = 1$ and, obviously, $\bar{L}_g = \bar{F}_g \leftthreetimes (\bar{a})$. Making use of Lemma 3 [see statement 1) in it] and taking into account that the group $\bar{L}_g$ is generated by two elements of order p, it is easy to show that it has the Frobenius property with kernel $\bar{F}_g$. Obviously, from here it follows that $F_g \leftthreetimes (a)$ and $F_g \leftthreetimes (a^g)$ are Frobenius groups with kernel F_g.

Let $|X| = p$. We consider in $\bar{P}_g$ the subgroup $A = (z) \times (\bar{a})$, where $(z) = X$. We show that $\bar{a} \in Z(P_g)$. Obviously, $z \in Z(\bar{P}_g)$. If $C_{\bar{P}_g}(A) \neq \bar{P}_g$, then, in view of the nilpotency of the group P_g in $N_{\bar{P}_g}(A) \setminus C_{\bar{P}_g}(A)$, there exists an element h such that $\bar{a}^{b} = \bar{a}z$. From here we obtain that all elements of the form $\bar{a}z^{\beta}$, $1 \leq \beta < p$, are conjugate with $\bar{a}$. If T_g is the nilpotent radical of the subgroup F_g, then, in view of Lemma 3 and the condition $F_g \nless H$, the subgroups $T_g \nless H$ and $T_g \leftthreetimes (a^u)$, $u \in L_g$, are Frobenius groups. By Proposition 4, in A there exists an element $c \neq 1$ such that $C_{\bar{L}_g}(c) \cap \bar{T}_g \neq 1$, where $\bar{T}_g = T_g R_g / R_g$. If for some β_1 we would have $c \in \mathrm{gr}(az^{\beta_1})$, then, obviously, the subgroups $T_g \leftthreetimes (a^u)$ would not be Frobenius groups for each $u \in L_g$, in spite of what has been proved above. Consequently, $c \in (z)$ and, moreover, $\bar{T}_g < C_{\bar{L}_a}(z)$. However, the last inclusion would contradict the condition $T_g \nless H$, since the complete preimage of the subgroup (z) in G would be a subgroup of type (*), not belonging to B, in spite of Lemma 10. Consequently, $C_{\bar{P}_g}(A) = \bar{P}_g$ and $\bar{a} \in Z(\bar{P}_g)$. But then S_g is a cyclic group (Proposition 9) and, obviously, $\bar{L}_g = \bar{F}_g \leftthreetimes \bar{P}_g$, where $\bar{P}_g$ is an elementary Abelian group of order p^2. From here, from Lemma 3, and from the fact that the group L_g is generated by the elements a, a^g, there follows that $F_g \leftthreetimes (a)$, $F_g \leftthreetimes (a^g)$ are Frobenius groups with complementary sets (a), (a^g). All the statements of the lemma are proved.

LEMMA 13. Let A be a subgroup of type $A=(\tau)\times(a)$, where τ is an element of order p from $\mathcal{D}$, $d=\tau^{\alpha}a$. If $C_G(d)\not< H$, then the intersection $C_G(d)\cap B$ is finite.

Proof. We assume that $C_G(d)\cap B$ is infinite. By Lemma 1, $(C_G(d), S, a)\in \mathcal{Y}$ where S is the complete part of the intersection $C_G(d)\cap B$. But then $B<C_G(d)$. On the other hand, by Lemma 10 [see statement 5) in it], $C_B(\tau^{\alpha})$ is infinite. Consequently, $N=B\cap C_G(\tau^{\alpha})\cap C_G(d)$ is infinite, and $N<C_G(a)$, which is not possible. The obtained contradiction proves the lemma.

LEMMA 14. Let d be the element from Lemma 13, and let $R=B\cap C_G(d)$. Then $R\lhd C_G(d)$.

Proof. Let $T=C_G(d)$, $V=T\cap H$, $X=N_T(R)$. Assume first that $|T:X|<\infty$. By Dietzmann's lemma (see Sec. 1), the closure U of the subgroup R in T is finite. Since U is generated by some subgroups, conjugate with R in T, on the basis of Lemmas 1 and 2 we conclude that $O_{p'}(U)<Z(U)$. From here it follows that if Q is a Sylow p-subgroup of the group $O_{p'p}(U)$, then $Q\lhd U\lhd T$. Obviously, $U<E$ and, in view of Lemma 11, we have $Q<\mathcal{D}$. By Lemma 10 [see statement 3) in it] we have $Q<B$ and $Q=R$. From here and from the definition of the subgroup U there follows that $U=R$. But then $R\lhd X$. Now it is clear that if $R<Y$, where Y is a finite subgroup of the group G and $T<N_G(Y)$, then $R\lhd C_G(d)$.

We assume that $R\ntriangleleft X$. Then one can assume that $N_T(R_1)\leq X$ for each nontrivial (a)-invariant subgroup R_1 of the group K. Indeed, this can be easily proved taking into account the beginning of the proof, the second statement of Proposition 13, and the fourth statement of Lemma 10.

Now we consider the situation in $C_G(d)/(d)$. For the sake of the simplicity of the reasoning, we preserve the same notations. By Lemma 11, $C_T(a)<V$. First we probe that $a^b\in X\Rightarrow b\in X$ $(b\in T)$. The implication $a^b\in V\Rightarrow b\in V\leq X$ is true by virtue of Lemma 11. We assume that $b\in T\setminus X$ and $a^b\in X$. We consider the subgroup $L_b=\mathrm{gr}(a,a^b)$. Since $V\leq X$ and $b\in T\setminus X\leq T\setminus V$, by condition 2) of the theorem, the group L_b is finite. Taking into account that $T=C_G(d)/(d)$, with the aid of Lemma 12 we obtain that $L_b=(O_{p'}(L_b)\times S_b)\lambda(a)$, where $S_b\leq R$. For some element $h\in L_b$ we have $a^{bh}\in S_b\lambda(a)$ and, by Lemma 11, $bh=\tau\in V$. From here, $b=\tau h^{-1}$. But $V\leq X$ and $h^{-1}\in X$ and, therefore, $b\in X$, in spite of the assumption. Thus, if $b\in T\setminus X$, then $L_b\not\leq X$. If for some element $c\in T\setminus X$ we would have $S_c\neq 1$, then, in view of the above-made remark regarding the nontrivial (a)-invariant subgroups from R, we would obtain $L_c<N_T(S_c)\leq X$, which is not possible. Consequently, for any $b\in T\setminus X$ the group L_b is a Frobenius group with complement (a). By Proposition 15, $T=F\lambda C_T(a)$, where $F\lambda(a)$ is a Frobenius group with complement (a).

Since $R<C_T(a)$ and the elements a and τa for $\tau\in R$ are conjugate in G (more exactly, some of their preimages in $C_G(d)$), it follows that the subgroups $L_g=\mathrm{gr}(\tau a, a^g)$ are finite for $g\in F\setminus X\cap F$. Let s be some element from $R^{\#}$ and let b be some element from $F\setminus X\cap F$. By Lemmas 1 and 12, $W=\mathrm{gr}(sa,a^b)=(O_{p'}(W)\times R_b)\lambda(sa)$, where $R_b\leq R$ and $R_b\neq 1$. According to what has been proved above, we have $a^b\in X$ and thus, $W\not\leq X$. But, as mentioned above, $N_T(R_b)\leq X$ and, since $W<N_T(R_b)\leq X$ (this can be seen from the

above-indicated structure of the subgroup W), we obtain a contradiction with the relation $W \ntriangleleft X$. The lemma is proved.

LEMMA 15. For each $g \in G \setminus H$, the group L_g has the form $L_g = (O_{p'}(L_g) \times R_g) \lambda (a)$, where R_g is a subgroup from B .

Proof. We assume that for some $c \in G \setminus H$ the subgroup L_c does not have the form indicated in the lemma. By Lemma 12, $R_c \triangleleft L_c$ and in L_c / R_c the Sylow p-subgroup is an elementary Abelian subgroup of order p^2. By Lemma 10, $(N_G(R_c)/R_c, B/R_c, aR_c) \in \mathfrak{L}$ and, therefore, without loss of generality, we can assume that already $L_c = F_c \lambda A$, where

$$A = (\tau) \times (a), \quad \tau^p = 1 \text{ and } \tau \in \mathfrak{D} \setminus B.$$

Furthermore, $A < H$, $F \nless H$, and F is nilpotent as the kernel of a Frobenius group (Proposition 25). From here, making use of Proposition 4 and of the normalizer condition in F [3, Theorem 16.2.2], we prove the existence in A of an element $d \neq 1$ such that $C_G(d) \cap F \nless$ Obviously, $R_1 = B \cap C_G(d) \neq 1$. By virtue of Lemmas 13 and 14, R_1 is a finite group and $C_G(d) < N_G(R_1)$. By Lemma 10,

$$(N_G(R_1)/R_1, \ B/R_1, aR_1) \in \mathfrak{L}$$

and in $W = N_G(R_1)/R_1$ all the conditions of Lemmas 13 and 14 regarding the element dR_1 are preserved, i.e., $R_2/R_1 = B/R_1 \cap C_W(dR_1) \neq R_1$ and R_2/R_1 is a finite group; moreover, $C_W(dR_1) \nless W \cap H/R_1$. By Lemma 14, $R_2/R_1 \triangleleft C_W(dR_1)$ and since $C_G(d)R_1/R_1 < C_W(dR_1)$, we have $C_G(d) < N_G(R_2)$ where R_2 is the complete preimage of the subgroup R_2/R_1 in G . Regarding the quotient group $N_G(R_2)/R_2$ we proceed as before. Repeating these reasonings, we construct in B a strictly increasing chain of finite (a) -invariant subgroups

$$R_1 < R_2 < \ldots < R_n < \ldots \tag{7}$$

such that $C_G(d) < N_G(R_n)$, $n = 1, 2, \ldots$ If R is the union of the chain (7), then $C_G(d) < N_G(R)$. But then, by Lemma 1, $N_G(R) \leqslant H$ and, consequently, $C_G(d) < H$, in spite of the assumption. The lemma is proved.

6. Conclusion of the Proof of the Fundamental Theorem

LEMMA 16. Let b be an a-real element from $E \setminus \mathfrak{D}$. Then, without loss of generality, one can assume that the subgroups $L_t = \mathrm{gr}(a, a^{tb})$, $t \in B$, are finite Frobenius groups with complement (a).

Proof. By Lemma 15,

$$L_t = (O_{p'}(L_t) \times S_t) \lambda (a),$$

where $O_{p'}(L_t) \lambda (a)$ is a Frobenius group with complement (a) and $S_t < B$. We show that $b \in N_G(S_t)$. Since $S_t \lambda (a)$ is a Sylow p-subgroup of the group L_t, it follows, by Lemma 10, that $a^{tb} \notin S_t$ and in $O_{p'}(L_t)$ there exists an element c such that $a^{tbc^{-1}} = ha$, where $h \in S_t < B$. By Proposition 12, for some element $l \in B$ we have $ha = a^l$ and $a^{tbc^{-1}} = a^l$. From here, $tbc^{-1}l^{-1} = \tau \in C_G(a) < H$ (Lemma 10) or $tb = \tau l c$, $\tau^{-1} b = t^{-\tau} l c$. Since $t^{-\tau} l \in B$, $S_t < B$, $c \in C_G(S_t)$ and B is an Abelian group, we have $\tau^{-1} b \in C_G(S_t)$ and, in addition, $a \in N_G(S_t)$. From here we obtain that $a, \tau^{-1} b \in N_G(S_t)$. But then

$$N_G(S_t) > \mathrm{gr}\,(a, a^{r^{-1}b}) = \mathrm{gr}\,(a, a^b) = \mathrm{gr}\,(a, b)$$

(Proposition 22). Consequently, $b \in N_G(S_t)$.

Thus, we have proved that if in the set of the subgroups of type L_t, $t \in B$, there exists at least one subgroup which is not a Frobenius group, then B has a subgroup S_1 of the second kind, normalized by the element b, i.e., $b \in N_G(S_1) = G_1$. The triple $(\bar{G}_1, B/S_1, aS_1)$ lies in $\mathcal{L}$ (Lemma 10), where $\bar{G}_1 = G_1/S_1$. Applying to this triple reasonings similar to the previous ones, we show that either in $\bar{G}_1$ all the subgroups are of the type $L_t = \mathrm{gr}\,(\bar{a}, \bar{a}^{t\bar{b}})$, where $t \in B/S_1$, $\bar{a} = aS_1$, $\bar{b} = bS_1$, is a Frobenius group with complement $(\bar{a})$, or B/S_1 has a subgroup $S_2/S_1 \neq S_1$ of the second kind, normalized by the element $\bar{b}$, where S_2 is the complete preimage S_2/S_1 in B. Reasoning in this manner, we construct a strictly increasing chain of subgroups of the second kind

$$S_1 < S_2 < \ldots < S_n < \ldots \tag{8}$$

so that $b \in N_G(S_n)$, $n = 1, 2, \ldots$ If the chain (8) would not terminate at a finite index, then its union S would be infinite and $b \in N_G(S)$. But then by Lemma 1 we would obtain $b \in N_G(S) \leq H$, which is not possible. Consequently, the chain (8) terminates at the finite index n and for the triple $(N_G(S_n)/S_n, B/S_n, aS_n) \in \mathcal{L}$ the assertion of the lemma holds. The lemma is proved.

In the sequel we assume that for the triple $(G, B, a) \in \mathcal{L}$ the assertion of Lemma 16 holds, i.e., for some a-real element $b \in E \setminus \mathcal{D}$ and any element $t \in B$, the subgroups $L_t = \mathrm{gr}\,(a, a^{tb})$ are finite Frobenius groups with complement (a).

<u>LEMMA 17.</u> Let t be an element in B. Then $tb = rc$, where $r \in C_G(a) \cap B$, c is an a-real element in $E \setminus \mathcal{D}$.

<u>Proof.</u> By Lemma 16, $L_t = \mathrm{gr}\,(a, a^{tb})$ is a Frobenius group with complement (a): $L_t = F_t \leftthreetimes (a)$. In view of Proposition 16 and Lemma 10, there exists in F_t an element c such that $a^{tbc^{-1}} = a$. From here we obtain that $tbc^{-1} = r \in C_G(a) < H$ (Lemma 10) and $tb = rc$, and since $t, b, c \in E$, we have $r \in \mathcal{D}$.

We consider the equality

$$atba = arca. \tag{9}$$

Obviously, at $at \in H$ and at, a are conjugate in H (Proposition 12). The elements a and ba are also conjugate in G since they are contained in the Frobenius group (a, b) with complement (a) and kernel containing the element b (Proposition 16). In addition, $ba \notin H$ and, in view of condition 2) of the fundamental theorem and of Lemma 15,

$$L = \mathrm{gr}\,(at, ba) = F \leftthreetimes (ba),$$

where $F = O_{p'}(L) \times S$ and $S < B$.

From (9) it is clear that $arca \in L$. Since $a^2 \neq 1$ and $rc \in E$, we have, obviously, $arca \notin E$. From here and from $arca \in L$ there follows that $arca \in L \setminus F$. But all the elements from $L \setminus F$ are conjugate with some elements from $(a)^{\#}$. In this case, by condition 2) of the theorem, the subgroup $Z = \mathrm{gr}(a, arca)$ is finite and $Z \not< H$, while $\mathrm{gr}\,(a, rc) =$

Z. But then $Z \geq \mathrm{gr}(a, a^{x}) = \mathrm{gr}(a, a^{c}) = \mathrm{gr}(a, c)$ (Proposition 22) and $\tau \in Z$. Since $\tau \in \mathcal{D}$, making use of Lemma 15 regarding Z, we obtain that $\tau \in B$. The lemma is proved.

We proceed directly to the proof of the fundamental theorem. If the theorem were not true, then all the preceding lemmas were true and, in particular, by virtue of Lemmas 16 and 17, $E \setminus \mathcal{D}$ would contain an a-real element b such that

$$tb = \tau_t c_t \quad \text{for any} \quad t \in B, \tag{10}$$

where $\tau_t \in B \cap C_G(a)$, c_t is an a-real element from $E \setminus \mathcal{D}$. We multiply Eq. (10) on the right by the element a and we consider the subgroup $X_t = \mathrm{gr}(c_t a, ba)$. From (10) it is clear that $\tau_t^{-1} t \in X_t$ and $X_t = \mathrm{gr}(\tau_t^{-1} t, ba)$. Since $(a, b) = F \leftthreetimes (a)$ is a Frobenius group with kernel F and $b \in F$, it follows that a and ba are conjugate through some element d from F (Proposition 16): $a^d = ba$. The triple (G^d, B^d, ba) lies in $\mathcal{L}$ and all the preceding lemmas hold for it (Lemma 1). We show that $B < \mathfrak{M}^d$, where $\mathfrak{M} = \bigcup_{g \in H} \mathrm{gr}(a, a^g)$.

We assume that for some element $u \in B$ one has the relation $X_u \not< H^d$. Since the subgroup X_u is generated by two elements $c_u a$, ba, conjugate with certain elements from $(a)^{\#}$ by condition 2) of the theorem, and in view of the assumption that $c_u a \notin H^d$, the group X_u has the form

$$X_u = (O_{p'}(X_u) \times S_u) \leftthreetimes (ba),$$

where $S_u < B^d$ (Lemma 15). But $\tau_u^{-1} u \in X_u \cap B$ and, therefore, $\tau_u^{-1} u \in S_u$ and, in addition, $X_u = \mathrm{gr}(\tau_u^{-1} u, ba)$. From here it follows that X_u is a finite p-group, containing the element ba and, by Lemma 11, $X_u < H^d$, in spite of the assumption. Consequently,

$$\tau_t^{-1} t \in \mathrm{gr}(c_t a, ba) < H^d \quad \text{for any} \quad t \in B. \tag{11}$$

In view of Proposition 12, $|\tau_t| \leq p$ and, therefore, $t^p \in X_t < H^d$. But $B^p = B$ and from (11) there follows that $B < \mathfrak{M}^d$. On the basis of the last inclusion and the definition of an M_p-group, we conclude that $B = B^d$ and $d \in N_G(B) = H$. But then $b \in H$, in spite of the assumption. The obtained contradiction concludes the proof of the theorem.

COROLLARY. Let G be a group without involution; let B be an infinite complete Abelian p-subgroup; let a be an element of order p of the group G; let V be a subgroup from $B \cap C_G(a)$ of finite index in $B \cap C_G(a)$, assume that the triple $(G, N_G(B), a)$ satisfies the conditions 2)-4) of the fundamental theorem; and let $H = N_G(B)$ be an M_p-group with kernel B and handle (a). Then $N_G(V) \leq H$.

Proof. We denote $T = N_G(V), H_1 = H \cap T$. It is easy to show that H_1 is an M_p-group with handle (a) and kernel B, while $N_T(H_1) = H_1$ and, in view of Propositions 12 and 13, the triple $(T/V, H_1/V, aV)$ satisfies, obviously, all the conditions of the fundamental theorem. But then $T/V = H_1/V$ and $T = H_1$. The corollary is proved.

We note that condition 2) in the fundamental theorem can be relaxed to the following one: the subgroups $\mathrm{gr}(a, a^g)$, $g \in G \setminus H$, are finite if $G \setminus H$ is nonempty and any finite subgroup, containing $\mathrm{gr}(a, a^g)$, is a p-solvable group $(p > 2)$ having no sections of type $SL(2,3)$. Such a group G may have involutions. As a matter of fact, taking into account this remark, we have characterized the M_p -groups with p-finite handles $(p>2)$ in the class of all infinite groups.

7. An Application of the Fundamental Theorem

LEMMA 18. Let G be a group and let a be an almost regular element of prime order p in G, satisfying the condition:

$$\text{all the subgroups gr } (a, a^g), g \in G, \text{ are finite.} \tag{12}$$

Then one of the following statements holds:

1) the Sylow p-subgroups in G, containing the element a, are finite and conjugate among themselves, and the number of such subgroups is finite;

2) G has an infinite, (a)-invariant, complete, Abelian p-subgroup.

Proof. Let

$$P_1, P_2, \dots, P_n, \dots \tag{13}$$

be an infinite sequence of distinct finite p-subgroups and let $a \in P_n$, $n=1,2,\dots$. Since $C_G(a)$ is finite and $Z(P_n) < C_G(a)$, $n=1,2,\dots$, it follows that the sequence (13) has an infinite subsequence

$$P_{i_1}, P_{i_2}, \dots, P_{i_n}, \dots$$

such that

$$Z_1 = Z(P_{i_1}) = Z(P_{i_2}) = \dots = Z(P_{i_n}) = \dots \text{ and } Z_1 \neq 1.$$

We consider $G_1 = N_G(Z_1)$. In $\bar{G}_1 = G_1/Z_1$ the number of subgroups of type P_{i_n}/Z_1 is infinite, $aZ_1 \in P_{i_n}/Z_1$ and, obviously, $C_{\bar{G}_1}(aZ_1)$ is finite and, therefore, as shown above, there exists in $\bar{G}_1$ a finite p-subgroup $\bar{Z}_2 \neq Z_1$ such that $aZ_1 \in N_{\bar{G}_1}(\bar{Z}_2)$ and the number of finite p-subgroups containing the element aZ_1 is infinite. If G_2, Z_2 are the complete preimages of the subgroups $N_{\bar{G}_1}(\bar{Z}_2)$, $\bar{Z}_2$ in G_1 then $G_2 = N_{G_1}(Z_2)$. Regarding G_2/Z_2 we proceed similarly to the previous case. Reasoning in this way, we construct a strictly increasing chain of finite p-subgroups

$$Z_1 < Z_2 < \dots < Z_n < \dots, \tag{14}$$

which does not terminate at a finite index and $a \in N_G(Z_n)$, $n=1,2,\dots$. If Z is the union of the chain (14), then Z is an infinite, locally finite p-subgroup and $a \in N_G(Z)$. Obviously, $L = Z \cdot (a)$ is also an infinite, locally finite group. From here, based on Proposition 27 and on the finiteness of the centralizer $C_L(a)$, it is easy to show that L has an infinite, (a)-invariant, complete, Abelian subgroup.

Assume now that G does not have an infinite set of finite p-subgroups, containing the element a. We show that, in this case, the assertion 1) of the lemma holds. We assume that some Sylow p-subgroup S from G, containing the element a, is infinite. Since $C_S(a)$ is finite and S is infinite, it follows that the set of elements of the form a^h, $h \in S$, is infinite. But then the set of finite p-subgroups of the form (a, a^h), $h \in S$, is infinite [condition (12)], in spite of the assumption regarding the finiteness of the set of such subgroups. Consequently, the Sylow p-subgroups containing the element a are finite, and their number is finite. Let $\mathfrak{Y}$ be the set of such subgroups, and assume that for some pair S, $R \in \mathfrak{Y}$, the subgroups S and R are not conjugate in G and, from all the subgroups from $\mathfrak{Y}$, not conjugate with S in G, the intersection $\mathcal{D} = S \cap R$ has the largest order for R. We consider $V = N_G(\mathcal{D})$. Since $a \in \mathcal{D}$, it follows that V is a finite group. We denote $S_1 = S \cap V$, $R_1 = R \cap V$. In view of the normalizer condition in S and R [3, Theorem 16.2.2], we have $S_1 \neq \mathcal{D}$, $R_1 \neq \mathcal{D}$, and $\mathcal{D} = S_1 \cap R_1$. If S_2, R_2 are Sylow p-subgroups from V and $S_1 \leq S_2$, $R_1 \leq R_2$, then, by Sylow's theorem, S_2 and R_2 are conjugate in V, i.e., for some element $t \in V$ we have $S_2 = R_2^t$. But then $\mathcal{D} < S_1 < R_2^t$ and $R_2^t \leq L \in \mathfrak{Y}$. By virtue of the definition of the subgroup R, we obtain $L = S$ since $L \cap S \geq S_1 > \mathcal{D}$ and $S_1 \neq \mathcal{D}, L \in \mathfrak{Y}$. On the other hand, $\mathcal{D} < R_1^t < L = S$ and $R_1^t \neq \mathcal{D}$. Consequently, $R^t \cap S \geq R_1^t > \mathcal{D}$ and, again, in view of the definition of the subgroup R, we obtain that $R^t = S$, which contradicts the assumption on the nonconjugacy of R and S in G. The obtained contradiction means that the subgroups from $\mathfrak{Y}$ are conjugate in G. The lemma is proved.

THEOREM. Let G be a group without involution; let a be an element of prime order p in G with centralizer $C_G(a)$, which is a finite p-subgroup satisfying condition (12). Then G has a complete, Abelian, normal p-subgroup B such that, in G/B, the Sylow p-subgroups containing the element aB are finite and conjugate and their number is finite.

Proof. If in G every Sylow p-subgroup, containing a, is finite, then, in view of Lemma 18, the assertion of the theorem holds and, in this case, $B = 1$. Assume that the element a is contained in an infinite p-subgroup. By Lemma 18, G has an infinite, (a)-invariant, complete, Abelian p-subgroup. By Zorn's lemma, in the set of such subgroups we select as B a maximal subgroup and we consider $H = N_G(B)$. Since $a \in H$, it follows that condition (12) holds in $\bar{H} = H/B$ (for the element $\bar{a} = aB$) and, in view of Proposition 13, the centralizer $C_{\bar{H}}(\bar{a})$ is finite. If $\bar{H}$ would have an infinite, $(\bar{a})$-invariant, complete, Abelian p-subgroup $\bar{V}$, then, in view of Proposition 27 and Theorem 21.1.4 of [3], its complete preimage V in G would be an (a)-invariant, complete, Abelian p-subgroup such that $B \leq V$ and $B \neq V$, but this is not possible in view of the selection of B as a maximal (a) invariant, complete, Abelian p-subgroup. Consequently, statement 1) of Lemma 18 holds for $\bar{H}$ and the element $\bar{a}$. From here and from the finiteness of the centralizer $C_G(a)$ there follows, obviously, that the triple $(G, B, (a))$ satisfies the conditions of the fundamental theorem and, by this theorem, $B \lhd G$. The theorem is proved.

Example 10. In the last theorem, condition (12) is essential. Indeed, this can easily be seen in the example of the free product of an infinite Chernikov p-group, possessing an almost regular element of order $p (p \neq 2)$, and some nontrivial periodic group without involution. If, however, one takes the periodic product of the mentioned groups, then we obtain an example of a periodic group without involution in which all the conditions of the theorem hold with the exception of condition (12), while the assertion of the theorem does not hold for such a group.

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