Annals of Operations Research 37(1992)375–401

OPTCON: AN ALGORITHM FOR THE OPTIMAL CONTROL OF NONLINEAR STOCHASTIC MODELS

Josef MATULKA
Institut für Informationsverarbeitung und -wirtschaft, Wirtschaftsuniversität Wien, Augasse 2-6, A-1090, Vienna, Austria

and

Reinhard NECK
Institut für Volkswirtschaftstheorie und -politik, Wirtschaftsuniversität Wien, Augasse 2-6, A-1090, Vienna, Austria

Abstract

In this paper we describe the algorithm OPTCON which has been developed for the optimal control of nonlinear stochastic models. It can be applied to obtain approximate numerical solutions of control problems where the objective function is quadratic and the dynamic system is nonlinear. In addition to the usual additive uncertainty, some or all of the parameters of the model may be stochastic variables. The optimal values of the control variables are computed in an iterative fashion: First, the time-invariant nonlinear system is linearized around a reference path and approximated by a time-varying linear system. Second, this new problem is solved by applying Bellman's principle of optimality. The resulting feedback equations are used to project expected optimal state and control variables. These projections then serve as a new reference path, and the two steps are repeated until convergence is reached. The algorithm has been implemented in the statistical programming system GAUSS. We derive some mathematical results needed for the algorithm and give an overview of the structure of OPTCON. Moreover, we report on some tentative applications of OPTCON to two small macroeconometric models for Austria.

Keywords: Optimal control, stochastic control, dynamic systems, nonlinear systems, control algorithm, optimal economic policies.

1. Introduction

During the last twenty years it has been increasingly recognized that many economic problems can be viewed as involving the optimization of an intertemporal objective function by a decision-maker who is constrained by a dynamic system subject to various kinds of uncertainties. Such problems arise both on the level of the individual firm, necessitating extensions of operations research and management science to methods of dynamic optimization under uncertainty [14], as well as on the level of policy-making for a national economy [6]. Stochastic optimum control theory has proved to provide a powerful methodology to deal with such problems;

J.C. Baltzer AG, Scientific Publishing Company

see, e.g. [11]. Although the basic developments of stochastic optimum control theory started in the mathematics and engineering literature, economists have also contributed to the analytical and numerical solution of stochastic dynamic optimization problems, such as [1–3,5].

Since stochastic optimum control problems are usually rather complex, for most models (in particular multivariable ones) only numerical solutions under some simplifying assumptions can be obtained for particular values of the parameters. Even then, in most cases only approximations to the true optimum solution can be found at present. This points to the importance of deriving algorithms for stochastic optimum control problems which can be directly applied to given models of the firm or the economy. Although several algorithms for the optimum control of stochastic dynamic economic systems have already been published, they either allow for additive uncertainty only [3], or rule out nonlinear system equations [7, 9,13], or have not been implemented for actual calculations [5, ch. 9]. Thus, there is a need for further algorithmic developments.

In the present paper we report on a new algorithm for the optimum control of nonlinear dynamic models that allows for additive uncertainty as well as for the presence of a stochastic parameter vector in the system equations, to be called OPTCON. The plan of the paper is as follows: In section 2 we state a fairly general class of optimum control problems to which the algorithm OPTCON can be applied. Section 3 introduces some mathematical ingredients and simplifying devices used in the algorithm. In section 4 the main theorem upon which the algorithm is based is proved, and the structure of OPTCON is summarized. In its present version, OPTCON is limited by two simplifications which prevent the solutions obtained to be truly optimal: First, computations of approximately optimal policies are obtained by applying repeated linearizations to the given nonlinear economic model. Second, we exclude any learning about the system parameters. Whereas the first device seems inevitable given the inherent complexities and varieties of general nonlinear models, the second one may be relaxed in a later stage of research. Section 5 reports on some applications of the algorithm performed to ensure its feasibility. Although we confine ourselves to a brief discussion of the results for two small macroeconometric models for Austria, the scope of problems to be solved by OPTCON is much broader and includes models of the firm and non-economic applications as well. Finally, section 6 provides some concluding remarks and indicates directions for further extensions of algorithmic developments. A GAUSS implementation of the algorithm exists which can be obtained on request from the author mentioned first.

2. The optimum control problem

OPTCON can deliver approximate solutions to stochastic optimum control problems with a quadratic objective function and a nonlinear multivariable dynamic model in discrete time under additive and parameter uncertainties. Thus, we consider an intertemporal objective function which is additive in time and can be written as

$$L = \sum_{t=S}^{T} L_t(\mathbf{x}_t, \mathbf{u}_t), \tag{1}$$

with

$$L_t(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2}\begin{pmatrix} \mathbf{x}_t - \tilde{\mathbf{x}}_t \\ \mathbf{u}_t - \tilde{\mathbf{u}}_t \end{pmatrix}' \mathbf{W}_t \begin{pmatrix} \mathbf{x}_t - \tilde{\mathbf{x}}_t \\ \mathbf{u}_t - \tilde{\mathbf{u}}_t \end{pmatrix}. \tag{2}$$

$\mathbf{x}_t$ denotes an n-dimensional vector of state variables, summarizing the information available about the system, $\mathbf{u}_t$ denotes an m-dimensional vector of control variables. The n-dimensional vector $\tilde{\mathbf{x}}_t$ and the m-dimensional vector $\tilde{\mathbf{u}}_t$ denote the given "ideal" levels of the state and control variables, respectively. S denotes the initial and T the terminal period of the finite planning horizon.

The matrix $\mathbf{W}_t$ is defined as

$$\mathbf{W}_t = \begin{pmatrix} \mathbf{W}_t^{xx} & \mathbf{W}_t^{xu} \\ \mathbf{W}_t^{ux} & \mathbf{W}_t^{uu} \end{pmatrix}, \qquad t = S,\ldots,T, \tag{3}$$

where $\mathbf{W}_t^{xx}$, $\mathbf{W}_t^{xu}$, $\mathbf{W}_t^{ux}$, and $\mathbf{W}_t^{uu}$, are $(n \times n)$, $(n \times m)$, $(m \times n)$, and $(m \times m)$ matrices, respectively. Furthermore, we require

$$\mathbf{W}_t = \alpha^{t-1}\mathbf{W}, \qquad t = S,\ldots,T, \tag{4}$$

where $\mathbf{W}$ is a matrix. Without loss of generality it is assumed that $\mathbf{W}$ is symmetric, which entails that

$$\mathbf{W}^{xu} = [\mathbf{W}^{ux}]'. \tag{5}$$

It is easy to see that with

$$\begin{pmatrix} \mathbf{w}_t^x \\ \mathbf{w}_t^u \end{pmatrix} = -\mathbf{W}_t \begin{pmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\mathbf{u}}_t \end{pmatrix}, \tag{6}$$

$$w_t^c = \frac{1}{2}\begin{pmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\mathbf{u}}_t \end{pmatrix} \mathbf{W}_t \begin{pmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\mathbf{u}}_t \end{pmatrix}, \tag{7}$$

(2) can equivalently be written as

$$L_t(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2}\begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{pmatrix}' \mathbf{W}_t \begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{pmatrix} + \begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{pmatrix}' \begin{pmatrix} \mathbf{w}_t^x \\ \mathbf{w}_t^u \end{pmatrix} + w_t^c. \tag{8}$$

The "quadratic tracking form" (2) of the objective function is very common in economic-policy applications of stochastic control theory. It can be interpreted to require deviations of the state variables $\mathbf{x}_t$ and the control variables $\mathbf{u}_t$ from their "ideal" levels $\tilde{\mathbf{x}}_t$ and $\tilde{\mathbf{u}}_t$, respectively, to be punished. The "general quadratic form" (8), however, simplifies notation and computation.

The dynamic system, which may be an econometric model of an economy or of a firm, is assumed to be given by the system of nonlinear difference equations

$$\mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{x}_t, \mathbf{u}_t, \boldsymbol{\theta}, \mathbf{z}_t) + \boldsymbol{\varepsilon}_t, \quad t = S,\dots,T, \tag{9}$$

where $\boldsymbol{\theta}$ denotes a p-dimensional vector of unknown parameters, $\mathbf{z}_t$ denotes an l-dimensional vector of non-controlled exogenous variables, and $\boldsymbol{\varepsilon}_t$ is an n-dimensional vector of additive disturbances. $\boldsymbol{\theta}$ and $\boldsymbol{\varepsilon}_t$, $t = S, \dots, T$, are assumed to be independent random vectors with known expectations ($\hat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$, 0_n for $\boldsymbol{\varepsilon}_t$, $t = S, \dots, T$) and covariance matrices ($\Sigma^{\theta\theta}$ for $\boldsymbol{\theta}$, $\Sigma^{\varepsilon\varepsilon}$ for $\boldsymbol{\varepsilon}_t$, $t = S, \dots, T$). $\mathbf{f}$ is a vector-valued function, where the i-th component of $\mathbf{f}(\dots)$ is denoted by $f^i(\dots)$, $i = 1, \dots, n$.

The assumption of a first-order system of difference equations in (9) is not really restrictive, as higher-order difference equations can be reduced to systems of first-order difference equations by suitably redefining variables as new state variables and augmenting the state vector. Also, the assumption of a quadratic objective function, although of a special form, can be interpreted as a second-order Taylor-series approximation to a more general objective function. Thus, the class of problems to be solved by our algorithm OPTCON is rather broad.

3. Elements of the algorithm

3.1. LINEARIZATION OF THE SYSTEM EQUATIONS

In most optimum control problems in economics and operations research the system dynamics of the form (9) is a convenient starting point. Usually the structural form of an econometric model can easily be transformed into this form. For deriving a control algorithm, however, the state-space representation from control and systems theory is more adequate. The latter does not include $\mathbf{x}_t$ at the right-hand side of (9). Following [2], it is easy to show how one can eliminate this variable in the course of a linearization of the system.

For known values of $\mathring{\mathbf{x}}_{t-1}, \mathring{\mathbf{u}}_t, \mathring{\boldsymbol{\theta}}, \mathbf{z}_t$, and $\mathring{\boldsymbol{\varepsilon}}_t$ we can compute a value $\mathring{\mathbf{x}}_t$ such that

$$\mathring{\mathbf{x}}_t = \mathbf{f}(\mathring{\mathbf{x}}_{t-1}, \mathring{\mathbf{x}}_t, \mathring{\mathbf{u}}_t, \mathring{\boldsymbol{\theta}}, \mathbf{z}_t) + \mathring{\boldsymbol{\varepsilon}}_t \tag{10}$$

by straightforward application of a nonlinear equation-solving method such as the well-known Gauss–Seidel approximation algorithm. Then we can linearize the system function $\mathbf{f}(\dots)$ around these reference values and get the following approximative system equations:

$$\mathbf{x}_t \approx \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{c}_t + \xi_t, \quad t = S, \dots, T, \tag{11}$$

where we have defined the $(n \times n)$-matrix $\mathbf{A}_t$, the $(n \times m)$-matrix $\mathbf{B}_t$, and the n-dimensional vectors $\mathbf{c}_t$ and ξ_t, respectively, as

$$\mathbf{A}_t = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \mathbf{F}_{\mathbf{x}_{t-1}}, \tag{12}$$

$$\mathbf{B}_t = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \mathbf{F}_{\mathbf{u}_t}, \tag{13}$$

$$\mathbf{c}_t = \mathring{\mathbf{x}}_t - \mathbf{A}_t \mathring{\mathbf{x}}_{t-1} - \mathbf{B}_t \mathring{\mathbf{u}}_t , \tag{14}$$

$$\xi_t = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \varepsilon_t , \tag{15}$$

and $\mathbf{I}_n$ denotes the $(n \times n)$ identity matrix. Here and in the following we require that the first and second derivatives of the system function with respect to $\mathbf{x}_{t-1}$, $\mathbf{x}_t$, $\mathbf{u}_t$, and $\boldsymbol{\theta}$ exist and be continuous, and we use the following notation:

$\mathbf{F}_{\mathbf{x}_{t-1}}$ denotes an $(n \times n)$-matrix the elements of which are defined by

$$\left(\mathbf{F}_{\mathbf{x}_{t-1}}\right)_{i,j} = \frac{\partial f^i(\ldots)}{\partial x_{t-1,j}}, \quad \begin{array}{l} i = 1,\ldots,n, \\ j = 1,\ldots,n. \end{array} \tag{16}$$

$\mathbf{F}_{\mathbf{x}_t}$ denotes an $(n \times n)$-matrix the elements of which are defined by

$$\left(\mathbf{F}_{\mathbf{x}_t}\right)_{i,j} = \frac{\partial f^i(\ldots)}{\partial x_{t,j}}, \quad \begin{array}{l} i = 1,\ldots,n, \\ j = 1,\ldots,n. \end{array} \tag{17}$$

$\mathbf{F}_{\mathbf{u}_t}$ denotes an $(n \times \mathrm{m})$-matrix the elements of which are defined by

$$\left(\mathbf{F}_{\mathbf{u}_t}\right)_{i,j} = \frac{\partial f^i(\ldots)}{\partial u_{t,j}}, \quad \begin{array}{l} i = 1,\ldots,n, \\ j = 1,\ldots,m. \end{array} \tag{18}$$

$\mathbf{F}_{\boldsymbol{\theta}}$ denotes an $(n \times p)$-matrix the elements of which are defined by

$$\left(\mathbf{F}_{\boldsymbol{\theta}}\right)_{i,j} = \frac{\partial f^i(\ldots)}{\partial \theta_j}, \quad \begin{array}{l} i = 1,\ldots,n, \\ j = 1,\ldots,p. \end{array} \tag{19}$$

Here $x_{t-1,j}$ denotes the jth element of $\mathbf{x}_{t-1}$, etc, The above notation has been chosen to be in accordance with the usual rules of matrix calculus. It has to be kept in mind that all the matrices and vectors defined above are time-dependent functions of the reference path along which they have been evaluated. If this path changes the matrices will also change.

Equation (15) implies for the expectation and the covariance matrix of ξ_t, conditional on the information given at $t-1$, that:

$$\mathrm{E}_{t-1}(\xi_t) = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \mathbf{0}_n = \mathbf{0}_n , \tag{20}$$

$$\mathrm{cov}_{t-1}(\xi_t , \xi_t) = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \Sigma^{\varepsilon\varepsilon} [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}]' . \tag{21}$$

Thus, we have shown how one can approximate the time-invariant nonlinear system $\mathbf{f}(\ldots)$ by time-varying linear system functions.

3.2 COMPUTATION OF PARAMETER COVARIANCES

From the previous definitions it is obvious that the matrices $\mathbf{A}_t, \mathbf{B}_t$ and the vector $\mathbf{c}_t$ are functions of the random parameter vector $\boldsymbol{\theta}$ and are, therefore, random themselves. Of course, both matrices can be written as collections of their column vectors:

$$\mathbf{A}_t = (\mathbf{a}_{t,1} \ \ldots \ \mathbf{a}_{t,n}), \qquad t = S, \ldots, T, \tag{22}$$

$$\mathbf{B}_t = (\mathbf{b}_{t,1} \ \ldots \ \mathbf{b}_{t,m}), \qquad t = S, \ldots, T. \tag{23}$$

Each of these column vectors as well as $\mathbf{c}_t$ are functions of $\boldsymbol{\theta}$. While in general these functions will be nonlinear it is possible to approximate them by linear functions and thus to write

$$\mathbf{a}_{t,i} = \mathbf{D}^{\mathbf{a}_{t,i}}\boldsymbol{\theta} \qquad i = 1,\ldots,n, \ t = S,\ldots,T, \tag{24}$$

$$\mathbf{b}_{t,i} = \mathbf{D}^{\mathbf{b}_{t,j}}\boldsymbol{\theta} \qquad j = 1,\ldots,m, \ t = S,\ldots,T, \tag{25}$$

$$\mathbf{c}_t = \mathbf{D}^{\mathbf{c}_t}\boldsymbol{\theta} \qquad t = S,\ldots,T, \tag{26}$$

where $\mathbf{D}^{\mathbf{a}_{t,i}}, \mathbf{D}^{\mathbf{b}_{t,j}}$, and $\mathbf{D}^{\mathbf{c}_t}$ denote $(n \times p)$-matrices which are defined as follows:

$$\mathbf{D}^{\mathbf{a}_{t,i}} = \begin{bmatrix} \frac{\partial a_{t,1i}}{\partial \theta_1} & \cdots & \frac{\partial a_{t,1i}}{\partial \theta_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_{t,ni}}{\partial \theta_1} & \cdots & \frac{\partial a_{t,ni}}{\partial \theta_p} \end{bmatrix}, \quad \begin{array}{l} i = 1,\ldots,n, \\ t = S,\ldots,T, \end{array} \tag{27}$$

$$\mathbf{D}^{\mathbf{b}_{t,j}} = \begin{bmatrix} \frac{\partial b_{t,1j}}{\partial \theta_1} & \cdots & \frac{\partial b_{t,1j}}{\partial \theta_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_{t,nj}}{\partial \theta_1} & \cdots & \frac{\partial b_{t,nj}}{\partial \theta_p} \end{bmatrix}, \quad \begin{array}{l} j = 1,\ldots,m, \\ t = S,\ldots,T, \end{array} \tag{28}$$

$$\mathbf{D}^{\mathbf{c}_t} = \begin{bmatrix} \frac{\partial c_{t,1}}{\partial \theta_1} & \cdots & \frac{\partial c_{t,1}}{\partial \theta_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial c_{t,n}}{\partial \theta_1} & \cdots & \frac{\partial c_{t,n}}{\partial \theta_p} \end{bmatrix}, \quad t = S,\ldots,T. \tag{29}$$

For the sake of future computation, we reshape these matrices and group them into two matrices and one vector:

$$\mathbf{D}^{\mathbf{a}_t} \equiv [\text{vec}((\mathbf{D}^{\mathbf{a}_{t,1}})'),\ldots,\text{vec}((\mathbf{D}^{\mathbf{a}_{t,i}})'),\ldots,\text{vec}((\mathbf{D}^{\mathbf{a}_{t,n}})')], \tag{30}$$

$$\mathbf{D}^{\mathbf{b}_t} \equiv [\text{vec}((\mathbf{D}^{\mathbf{b}_{t,1}})'),\ldots,\text{vec}((\mathbf{D}^{\mathbf{b}_{t,j}})'),\ldots,\text{vec}((\mathbf{D}^{\mathbf{b}_{t,m}})')], \tag{31}$$

$$\mathbf{d}^{\mathbf{c}_t} \equiv [\text{vec}((\mathbf{D}^{\mathbf{c}_t})')] . \tag{32}$$

In order to compute the matrices introduced above, we have to define second derivatives of the vector-valued system functions with respect to vectors. This is somewhat cumbersome; here we use a notation similar to that introduced in [8]:

$\mathbf{F}_{\mathbf{x}_{t-1},\boldsymbol{\theta}}, \mathbf{F}_{\mathbf{x}_t,\boldsymbol{\theta}}$ and $\mathbf{F}_{\mathbf{u}_t,\boldsymbol{\theta}}$ denote $(n \cdot p \times n)$-matrices which are defined as follows:

$$\mathbf{F}_{\mathbf{x}_{t-1},\theta} = \begin{bmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x_{t-1,1}\partial\theta_1} \\ \vdots \\ \frac{\partial f^1}{\partial x_{t-1,1}\partial\theta_p} \end{pmatrix} & \cdots & \begin{pmatrix} \frac{\partial f^1}{\partial x_{t-1,n}\partial\theta_1} \\ \vdots \\ \frac{\partial f^1}{\partial x_{t-1,n}\partial\theta_p} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} \frac{\partial f^n}{\partial x_{t-1,1}\partial\theta_1} \\ \vdots \\ \frac{\partial f^n}{\partial x_{t-1,1}\partial\theta_p} \end{pmatrix} & \cdots & \begin{pmatrix} \frac{\partial f^n}{\partial x_{t-1,n}\partial\theta_1} \\ \vdots \\ \frac{\partial f^n}{\partial x_{t-1,n}\partial\theta_p} \end{pmatrix} \end{bmatrix}, \tag{33}$$

$\mathbf{F}_{\mathbf{x}_t,\theta}$ is analogous to $\mathbf{F}_{\mathbf{x}_{t-1},\theta}$ save that all occurrences of x_{t-1} are replaced by x_t.

$$\mathbf{F}_{\mathbf{u}_t,\theta} = \begin{bmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial u_{t,1}\partial\theta_1} \\ \vdots \\ \frac{\partial f^1}{\partial u_{t,1}\partial\theta_p} \end{pmatrix} & \cdots & \begin{pmatrix} \frac{\partial f^1}{\partial u_{t,m}\partial\theta_1} \\ \vdots \\ \frac{\partial f^1}{\partial u_{t,m}\partial\theta_p} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} \frac{\partial f^n}{\partial u_{t,1}\partial\theta_1} \\ \vdots \\ \frac{\partial f^n}{\partial u_{t,1}\partial\theta_p} \end{pmatrix} & \cdots & \begin{pmatrix} \frac{\partial f^n}{\partial u_{t,m}\partial\theta_1} \\ \vdots \\ \frac{\partial f^n}{\partial u_{t,m}\partial\theta_p} \end{pmatrix} \end{bmatrix}, \tag{34}$$

We are now in a position to state the following result, which is needed for the approximate computation of the covariances and expectations of the parameters of the linearized system:

THEOREM 1

The matrices introduced above can be evaluated as follows:

$$\mathbf{D}^{\mathbf{A}_t} = [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p][\mathbf{F}_{\mathbf{x}_t,\theta}\mathbf{A}_t + \mathbf{F}_{\mathbf{x}_{t-1},\theta}], \tag{35}$$

$$\mathbf{D}^{\mathbf{B}_t} = [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p][\mathbf{F}_{\mathbf{x}_t,\theta}\mathbf{B}_t + \mathbf{F}_{\mathbf{u}_t,\theta}], \tag{36}$$

$$\mathbf{d}^{\mathbf{c}_t} = \operatorname{vec}\left[((\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\theta})'\right] - \mathbf{D}^{\mathbf{A}_t}\,\mathring{\mathbf{x}}_{t-1} - \mathbf{D}^{\mathbf{B}_t}\,\mathring{\mathbf{u}}_t. \tag{37}$$

Proof

See appendix 1.

Thus, we have found a way to compute the linear approximations (24)–(26) and can evaluate the covariances of the parameters of the linearized system equations:

$$\mathrm{cov}_{t-1}(\mathbf{a}_{t,i},\mathbf{a}_{t,k}) = \mathbf{D}^{\mathbf{a}_{t,i}}\,\mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})[\mathbf{D}^{\mathbf{a}_{t,k}}]', \tag{38}$$

$$\mathrm{cov}_{t-1}(\mathbf{a}_{t,i},\mathbf{b}_{t,j}) = \mathbf{D}^{\mathbf{a}_{t,i}}\,\mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})[\mathbf{D}^{\mathbf{b}_{t,j}}]', \tag{39}$$

$$\mathrm{cov}_{t-1}(\mathbf{a}_{t,i},\mathbf{c}_{t}) = \mathbf{D}^{\mathbf{a}_{t,i}}\,\mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})[\mathbf{D}^{\mathbf{c}_{t}}]', \tag{40}$$

$$\mathrm{cov}_{t-1}(\mathbf{b}_{t,q},\mathbf{b}_{t,j}) = \mathbf{D}^{\mathbf{b}_{t,q}}\,\mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})[\mathbf{D}^{\mathbf{b}_{t,j}}]', \tag{41}$$

$$\mathrm{cov}_{t-1}(\mathbf{b}_{t,j},\mathbf{c}_{t}) = \mathbf{D}^{\mathbf{b}_{t,j}}\,\mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})[\mathbf{D}^{\mathbf{c}_{t}}]', \tag{42}$$

$$\mathrm{cov}_{t-1}(\mathbf{c}_{t},\mathbf{c}_{t}) = \mathbf{D}^{\mathbf{c}_{t}}\,\mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})[\mathbf{D}^{\mathbf{c}_{t}}]', \tag{43}$$

for all $i,k = 1,\ldots,n$; $j,q = 1,\ldots,m$.

3.3. EVALUATION OF SOME EXPECTED VALUES

In the process of deriving a control algorithm for this model it becomes necessary to evaluate expectations such as $\mathrm{E}_{t-1}(\mathbf{A}_t'\mathbf{K}_t\mathbf{B}_t)$, where $\mathbf{K}_t$ denotes a deterministic symmetric $(n\times n)$-matrix. Following [5, appendix B] we evaluate $\mathrm{E}_{t-1}(\mathbf{A}_t'\mathbf{K}_t\mathbf{B}_t)$ as

$$\mathrm{E}_{t-1}(\mathbf{A}_t'\mathbf{K}_t\mathbf{B}_t) = \mathrm{E}_{t-1}(\mathbf{A}_t)\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{B}_t)+\Upsilon_t^{AKB}, \tag{44}$$

where

$$\Upsilon_t^{AKB} = \begin{pmatrix} \mathrm{tr}[\mathbf{K}_t\,\mathrm{cov}_{t-1}(\mathbf{b}_{t,1},\mathbf{a}_{t,1})] & \cdots & \mathrm{tr}[\mathbf{K}_t\,\mathrm{cov}_{t-1}(\mathbf{b}_{t,m},\mathbf{a}_{t,1})] \\ \vdots & \ddots & \vdots \\ \mathrm{tr}[\mathbf{K}_t\,\mathrm{cov}_{t-1}(\mathbf{b}_{t,1},\mathbf{a}_{t,n})] & \cdots & \mathrm{tr}[\mathbf{K}_t\,\mathrm{cov}_{t-1}(\mathbf{b}_{t,m},\mathbf{a}_{t,n})] \end{pmatrix}. \tag{45}$$

This means that Υ_t^{AKB} is an $(n\times m)$-matrix the element in the ith row and jth column of which is given by the formula

$$\mathrm{tr}[\mathbf{K}_t\,\mathrm{cov}_{t-1}(\mathbf{b}_{t,j},\mathbf{a}_{t,i})], \quad \begin{matrix} i=1,\ldots,n, \\ j=1,\ldots,m. \end{matrix} \tag{46}$$

If we define the matrices Υ_t^{AKA}, Υ_t^{BKA}, Υ_t^{BKB}, υ_t^{AKc}, υ^{BKc} and υ_t^{cKc} by their elements as

$$[\Upsilon_t^{AKA}]_{i,j} = \mathrm{tr}[\mathbf{K}_t \mathbf{D}^{\mathbf{a}_{t,j}} \mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})(\mathbf{D}^{\mathbf{a}_{t,i}})'], \quad i=1,\ldots,n,\ j=1,\ldots,n, \tag{47}$$

$$[\Upsilon_t^{BKA}]_{i,j} = \mathrm{tr}[\mathbf{K}_t \mathbf{D}^{\mathbf{a}_{t,j}} \mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})(\mathbf{D}^{\mathbf{b}_{t,i}})'], \quad i=1,\ldots,m, j=1,\ldots n, \tag{48}$$

$$[\Upsilon_t^{BKB}]_{i,j} = \mathrm{tr}[\mathbf{K}_t \mathbf{D}^{\mathbf{b}_{t,j}} \mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})(\mathbf{D}^{\mathbf{b}_{t,i}})'], \quad i=1,\ldots,m, j=1,\ldots m, \tag{49}$$

$$[\upsilon_t^{AKc}]_i = \mathrm{tr}[\mathbf{K}_t \mathbf{D}^{\mathbf{c}_t} \mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})(\mathbf{D}^{\mathbf{a}_{t,i}})'], \quad i=1,\ldots,n, \tag{50}$$

$$[\upsilon_t^{BKc}]_i = \mathrm{tr}[\mathbf{K}_t \mathbf{D}^{\mathbf{c}_t} \mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})(\mathbf{D}^{\mathbf{b}_{t,i}})'], \quad i=1,\ldots,m, \tag{51}$$

$$[\upsilon_t^{cKc}] = \mathrm{tr}[\mathbf{K}_t \mathbf{D}^{\mathbf{c}_t} \mathrm{cov}_{t-1}(\boldsymbol{\theta},\boldsymbol{\theta})(\mathbf{D}^{\mathbf{c}_t})'], \tag{52}$$

the expectations $\mathrm{E}_{t-1}(\mathbf{A}_t'\mathbf{K}_t\mathbf{A}_t)$, $\mathrm{E}_{t-1}(\mathbf{B}_t'\mathbf{K}_t\mathbf{A}_t)$, $\mathrm{E}_{t-1}(\mathbf{B}_t'\mathbf{K}_t\mathbf{B}_t)$, $\mathrm{E}_{t-1}(\mathbf{A}_t'\mathbf{K}_t\mathbf{c}_t)$, $\mathrm{E}_{t-1}(\mathbf{B}_t'\mathbf{K}_t\mathbf{c}_t)$, and $\mathrm{E}_{t-1}(\mathbf{c}_t'\mathbf{K}_t\mathbf{c}_t)$, which will be needed later in the derivation of the algorithm, can be evaluated in an analogous way. In the above definitions we have already substituted the approximations derived in the previous section for the covariances of the parameters of the linearized system.

4. Approximative solution of the stochastic optimum control problem by OPTCON

The key idea of our algorithm OPTCON is to use Bellman's principle of optimality:

$$J_t^*(\mathbf{x}_{t-1}) = \min_{\mathbf{u}_t} \mathrm{E}_{t-1}(L_t(\mathbf{x}_t,\mathbf{u}_t) + J_{t+1}^*(\mathbf{x}_t)), \tag{53}$$

where $J_t^*(\mathbf{x}_{t-1})$ denotes the loss which is expected at the end of period $t-1$ for the remaining periods $t, \ldots, T$ if the optimal policy is implemented during these periods. $\mathrm{E}_{t-1}(\cdot)$ denotes conditional expectation; $\mathbf{x}_{k-1}$, $k = S, \ldots, t$, and $\mathbf{u}_{k-1}$, $k = S+1, \ldots, t$, are known at the time when we have to decide about $\mathbf{u}_t$.

Now we are in a position to state the main result upon which the algorithm OPTCON is based:

THEOREM 2

$J_t^*(\mathbf{x}_{t-1})$ can be expressed as a quadratic function of $\mathbf{x}_{t-1}$:

$$J_t^*(\mathbf{x}_{t-1}) = \tfrac{1}{2}\mathbf{x}_{t-1}'\mathbf{H}_t\mathbf{x}_{t-1} + \mathbf{x}_{t-1}'\mathbf{h}_t^x + h_t^c + h_t^s + h_t^p \tag{54}$$

for all periods $t = S, \ldots, T+1$, where

$$\mathbf{H}_t = \boldsymbol{\Lambda}_t^{xx} - \boldsymbol{\Lambda}_t^{xu}(\boldsymbol{\Lambda}_t^{uu})^{-1}\boldsymbol{\Lambda}_t^{ux}, \tag{55}$$

$$\mathbf{h}_t^x = \boldsymbol{\lambda}_t^x - \boldsymbol{\Lambda}_t^{xu}(\boldsymbol{\Lambda}_t^{uu})^{-1}\boldsymbol{\lambda}_t^u, \tag{56}$$

$$h_t^c = \lambda_t^c - \tfrac{1}{2}(\boldsymbol{\lambda}_t^u)'(\boldsymbol{\Lambda}_t^{uu})^{-1}\boldsymbol{\lambda}_t^u, \tag{57}$$
$$h_t^s = \lambda_t^s, \tag{58}$$
$$h_t^p = \lambda_t^p \tag{59}$$

for all $t = S, \ldots, T$ with terminal conditions

$$\mathbf{H}_{T+1} = \mathbf{0}_{n\times n}, \tag{60}$$
$$\mathbf{h}_{T+1}^x = \mathbf{0}_n, \tag{61}$$
$$h_{T+1}^c = 0, \tag{62}$$
$$h_{T+1}^s = 0, \tag{63}$$
$$h_{T+1}^p = 0, \tag{64}$$

and the auxiliary matrices, vectors and scalar variables are defined as follows:

$$\mathbf{K}_t = \mathbf{W}_t^{xx} + \mathbf{H}_{t+1}, \tag{65}$$
$$\mathbf{k}_t^x = \mathbf{w}_t^x + \mathbf{h}_{t+1}^x, \tag{66}$$
$$\boldsymbol{\Lambda}_t^{xx} = \boldsymbol{\Upsilon}_t^{AKA} + \mathrm{E}_{t-1}(\mathbf{A}_t)'\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{A}_t), \tag{67}$$
$$\boldsymbol{\Lambda}_t^{xu} = (\boldsymbol{\Lambda}_t^{ux})', \tag{68}$$
$$\boldsymbol{\Lambda}_t^{ux} = \boldsymbol{\Upsilon}_t^{BKA} + \mathrm{E}_{t-1}(\mathbf{B}_t)'\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{A}_t) + \mathbf{W}_t^{ux}\mathrm{E}_{t-1}(\mathbf{A}_t), \tag{69}$$
$$\boldsymbol{\Lambda}_t^{uu} = \boldsymbol{\Upsilon}_t^{BKB} + \mathrm{E}_{t-1}(\mathbf{B}_t)'\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{B}_t) + 2\mathrm{E}_{t-1}(\mathbf{B}_t)'\mathbf{W}_t^{xu} + \mathbf{W}_t^{uu}, \tag{70}$$
$$\boldsymbol{\lambda}_t^x = \boldsymbol{\upsilon}_t^{AKc} + \mathrm{E}_{t-1}(\mathbf{A}_t)'\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{c}_t) + \mathrm{E}_{t-1}(\mathbf{A}_t)'\mathbf{k}_t^x, \tag{71}$$
$$\boldsymbol{\lambda}_t^u = \boldsymbol{\upsilon}_t^{BKc} + \mathrm{E}_{t-1}(\mathbf{B}_t)'\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{c}_t) + \mathrm{E}_{t-1}(\mathbf{B}_t)'\mathbf{k}_t^x + \mathbf{W}_t^{ux}\mathrm{E}_{t-1}(\mathbf{c}_t) + \mathbf{w}_t^u, \tag{72}$$
$$\lambda_t^s = \tfrac{1}{2}\mathrm{tr}[\mathbf{K}_t \operatorname{cov}_{t-1}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t)] + h_{t+1}^s, \tag{73}$$
$$\lambda_t^p = \tfrac{1}{2}\,\upsilon_t^{cKc} + h_{t+1}^p, \tag{74}$$
$$\lambda_t^c = \tfrac{1}{2}\mathrm{E}_{t-1}(\mathbf{c}_t)'\mathbf{K}_t\mathrm{E}_{t-1}(\mathbf{c}_t) + \mathrm{E}_{t-1}(\mathbf{c}_t)'\mathbf{k}_t^x + w_t^c + h_{t+1}^c. \tag{75}$$

The optimal policy $\mathbf{u}_t$ for each period t is given by the feedback rule

$$\mathbf{u}_t = \mathbf{G}_t\mathbf{x}_{t-1} + \mathbf{g}_t, \tag{76}$$

where

$$\mathbf{G}_t = -(\boldsymbol{\Lambda}_t^{uu})^{-1}\boldsymbol{\Lambda}_t^{ux}, \tag{77}$$
$$\mathbf{g}_t = -(\boldsymbol{\Lambda}_t^{uu})^{-1}\boldsymbol{\lambda}_t^u. \tag{78}$$

Proof

See appendix 2.

As can be seen from the derivation, it would be necessary to evaluate several conditional expectations such as $\mathrm{E}_{t-1}(\mathbf{A}_t)$, $\mathrm{E}_{t-1}(\mathbf{B}_t)$, and $\mathrm{E}_{t-1}(\mathbf{c}_t)$ which are themselves functions of the unknown control variables. In order to be able to derive a feasible algorithm we introduce the following simplifying assumptions:

(1) Each occurrence of $\mathrm{E}_{t-1}(\cdot)$ is substituted by $\mathrm{E}_{S-1}(\cdot)$ and each occurrence of $\mathrm{cov}_{t-1}(\cdot\cdot)$ is substituted by $\mathrm{cov}_{S-1}(\cdot\cdot)$ for all $t = S + 1, \ldots, T + 1$. Thus, we rule out any learning about the parameters of the model.

(2) Although $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{c}_t$ are, in general, nonlinear functions of $\boldsymbol{\theta}$ we will compute their expected values by evaluating eqs. (12), (13), and (14) at the reference values $\mathring{\mathbf{x}}_{t-1}, \mathring{\mathbf{x}}_t, \mathring{\mathbf{u}}_t$, $\mathrm{E}_{S-1}(\boldsymbol{\theta})$, $\mathbf{z}_t$, and $\mathring{\boldsymbol{\varepsilon}}_t = \mathbf{0}_n$, which were true only in case of linear functions.

It must be admitted that these assumptions amount to a drastical simplification of the problem, as they exclude not only "active learning", i.e. exerting influence on the control variables in order to obtain more information about the parameters, but also "passive learning" in the sense of [5], i.e. using a re-estimated model at later periods. Although the latter approach is not formally included within the algorithm OPTCON, it can be pursued in a simple way to some extent by repeatedly applying OPTCON to successively re-estimated models for $t = S + 1, \ldots, T$, and using only the respective first-period controls for actual implementation ("open-loop feedback" control according to [5]). "Active learning" so far cannot be incorporated into our framework; at present we are ignoring the "dual" effect of control. Apart from the linear approximations described in the previous section, this is the main reason why we can only expect to arrive at approximations to the "true optimum" of our stochastic control problem. Computational simplicity is the main justification for this assumption. In the next steps of our research we will gradually introduce various devices of learning along the lines suggested in [5]. Only then will it be possible to tackle the question of the losses which are introduced by the restrictions assumed for OPTCON. This could be done by running Monte Carlo simulations of specific numerical examples under various learning schemes. Apart from the fact that such simulations are methodologically somewhat questionable, previous comparisons of this kind seem to indicate that the relative performance of algorithms employing different learning schemes is problem specific. Therefore one must not expect too much from such exercises. Unfortunately, at present we cannot envisage any possibility of arriving at the optimal solution of a fully stochastic problem with learning by analytical or numerical methods even for extremely simple problems; hence the question of the amount of suboptimality of OPTCON's solutions has to remain open.

We conclude this section by describing the algorithm OPTCON in a schematic way. This can serve as an overview of the way in which it has been implemented in the programming system GAUSS.

INPUT OF THE ALGORITHM

system function	$\mathbf{f}(\ldots)$
initial values of state variables	$\mathbf{x}_{S-1} \equiv \mathring{\mathbf{x}}_{S-1} \equiv \mathbf{x}^*_{S-1}$
tentative path of control variables	$(\mathring{\mathbf{u}}_t)_{t=S}^T$
path of exogenous variables not subject to control	$(\mathbf{z}_t)_{t=S}^T$
expected values of system parameters	$\hat{\boldsymbol{\theta}}$
covariance matrix of system parameters	$\boldsymbol{\Sigma}^{\theta\theta}$
covariance matrix of system noise	$\boldsymbol{\Sigma}^{\varepsilon\varepsilon}$
weighting matrices of objective function	$\mathbf{W}^{xx}$, $\mathbf{W}^{ux}$, $\mathbf{W}^{uu}$
discount rate of objective function	α
target path for state variables	$(\tilde{\mathbf{x}}_t)_{t=S}^T$
target path for control variables	$(\tilde{\mathbf{u}}_t)_{t=S}^T$

OUTPUT OF THE ALGORITHM

expected optimal path of state variables	$(\mathbf{x}^*_t)_{t=S}^T$
expected optimal path of control variables	$(\mathbf{u}^*_t)_{t=S}^T$
expected optimal welfare loss	J^*_S

DESCRIPTION OF THE ALGORITHM

(1) *Compute a tentative state path*: Use the Gauss–Seidel algorithm, the tentative policy path $(\mathring{\mathbf{u}}_t)_{t=S}^T$, and the system equation $\mathbf{f}(\ldots)$ to calculate the tentative state path $(\mathring{\mathbf{x}}_t)_{t=S}^T$.

(2) *Nonlinearity loop*: Repeat the steps (a) to (e) until convergence is reached (i.e., until the optimal control and state variables calculated do not change more than a prespecified small number from one iteration to the next) or the number of iterations is larger than a prespecified number.

(a) *Initialization for backward recursion.*

$$\mathbf{H}_{T+1} = \mathbf{0}_{n\times n}, \tag{79}$$

$$\mathbf{h}^x_{T+1} = \mathbf{0}_n, \tag{80}$$

$$h^c_{T+1} = 0, \tag{81}$$

$$h^s_{T+1} = 0, \tag{82}$$

$$h^p_{T+1} = 0. \tag{83}$$

(b) *Backward recursion*:

Repeat the following steps (i) to (vii) for $t = T, \ldots, S$.

(i) *Compute the expected values of the parameters of the linearized system equation*:

$$\mathbf{A}_t = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{x}_{t-1}}, \tag{84}$$

$$\mathbf{B}_t = (\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{u}_t}, \tag{85}$$

$$\mathbf{c}_t = \mathring{\mathbf{x}}_t - \mathbf{A}_t\,\mathring{\mathbf{x}}_{t-1} - \mathbf{B}_t\,\mathring{\mathbf{u}}_t, \tag{86}$$

$$\Sigma_t^{\xi\xi} = \text{cov}_{S-1}(\xi_t,\xi_t) = (\mathbf{I}_n - \mathbf{F}_{x_t})^{-1}\boldsymbol{\Sigma}^{\varepsilon\varepsilon}\,[(\mathbf{I}_n - \mathbf{F}_{x_t})^{-1}]', \tag{87}$$

where all derivatives are evaluated at the reference values $\mathring{\mathbf{x}}_{t-1}$, $\mathring{\mathbf{x}}_t$, $\mathring{\mathbf{u}}_t$, $\hat{\boldsymbol{\theta}}$, $\mathbf{z}_t$, and $\mathring{\boldsymbol{\varepsilon}}_t = \mathbf{0}_n$.

(ii) *Compute the derivatives of the parameters of the linearized system with respect to* $\boldsymbol{\theta}$:

$$\mathbf{D}^{\mathbf{A}_t} = [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p][\mathbf{F}_{\mathbf{x}_t,\boldsymbol{\theta}}\mathbf{A}_t + \mathbf{F}_{\mathbf{x}_{t-1},\boldsymbol{\theta}}], \tag{88}$$

$$\mathbf{D}^{\mathbf{B}_t} = [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p][\mathbf{F}_{\mathbf{x}_t,\boldsymbol{\theta}}\mathbf{B}_t + \mathbf{F}_{\mathbf{u}_t,\boldsymbol{\theta}}], \tag{89}$$

$$\mathbf{d}^{\mathbf{c}_t} = \text{vec}[((\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\boldsymbol{\theta}})'] - \mathbf{D}^{\mathbf{A}_t}\mathring{\mathbf{x}}_{t-1} - \mathbf{D}^{\mathbf{B}_t}\mathring{\mathbf{u}}_t, \tag{90}$$

where all derivatives are evaluated at the same reference values as above.

(iii) *Compute the influence of the stochastic parameters*: Compute all the matrices the cells of which are defined by

$$[\boldsymbol{\Upsilon}_t^{AKA}]_{i,j} = \text{tr}[\mathbf{K}_t\mathbf{D}^{\mathbf{a}_{t,j}}\boldsymbol{\Sigma}^{\boldsymbol{\theta\theta}}(\mathbf{D}^{\mathbf{a}_{t,i}})'], \quad i = 1,\dots,n,\ j = 1,\dots,n, \tag{91}$$

$$[\boldsymbol{\Upsilon}_t^{BKA}]_{i,j} = \text{tr}[\mathbf{K}_t\mathbf{D}^{\mathbf{a}_{t,j}}\boldsymbol{\Sigma}^{\boldsymbol{\theta\theta}}(\mathbf{D}^{\mathbf{b}_{t,i}})'], \quad i = 1,\dots,m,\ j = 1,\dots,n, \tag{92}$$

$$[\boldsymbol{\Upsilon}_t^{BKB}]_{i,j} = \text{tr}[\mathbf{K}_t\mathbf{D}^{\mathbf{b}_{t,j}}\boldsymbol{\Sigma}^{\boldsymbol{\theta\theta}}(\mathbf{D}^{\mathbf{b}_{t,i}})'], \quad i = 1,\dots,m,\ j = 1,\dots,m, \tag{93}$$

$$[\boldsymbol{\upsilon}_t^{AKc}]_i = \text{tr}[\mathbf{K}_t\mathbf{D}^{\mathbf{c}_t}\boldsymbol{\Sigma}^{\boldsymbol{\theta\theta}}(\mathbf{D}^{\mathbf{a}_{t,i}})'], \quad i = 1,\dots,n, \tag{94}$$

$$[\boldsymbol{\upsilon}_t^{BKc}]_i = \text{tr}[\mathbf{K}_t\mathbf{D}^{\mathbf{c}_t}\boldsymbol{\Sigma}^{\boldsymbol{\theta\theta}}(\mathbf{D}^{\mathbf{b}_{t,i}})'], \quad i = 1,\dots,m, \tag{95}$$

$$[\upsilon_t^{cKc}] = \text{tr}[\mathbf{K}_t\mathbf{D}^{\mathbf{c}_t}\boldsymbol{\Sigma}^{\boldsymbol{\theta\theta}}(\mathbf{D}^{\mathbf{c}_t})']. \tag{96}$$

(iv) *Convert the objective function from "quadratic-tracking" to "general quadratic" format*:

$$\mathbf{W}_t^{xx} = \alpha^{t-1}\mathbf{W}^{xx}, \tag{97}$$

$$\mathbf{W}_t^{ux} = \alpha^{t-1}\mathbf{W}^{ux}, \tag{98}$$

$$\mathbf{W}_t^{uu} = \alpha^{t-1}\mathbf{W}^{uu}, \tag{99}$$

$$\mathbf{w}_t^x = -\mathbf{W}_t^{xx}\tilde{\mathbf{x}}_t - \mathbf{W}_t^{xu}\tilde{\mathbf{u}}_t\,, \tag{100}$$

$$\mathbf{w}_t^u = -\mathbf{W}_t^{ux}\tilde{\mathbf{x}}_t - \mathbf{W}_t^{uu}\tilde{\mathbf{u}}_t\,, \tag{101}$$

$$w_t^c = \tfrac{1}{2}\tilde{\mathbf{x}}_t'\mathbf{W}_t^{xx}\tilde{\mathbf{x}}_t + \tilde{\mathbf{u}}_t'\mathbf{W}_t^{ux}\tilde{\mathbf{x}}_t + \tfrac{1}{2}\tilde{\mathbf{u}}_t'\mathbf{W}_t^{uu}\tilde{\mathbf{u}}_t\,. \tag{102}$$

(v) *Compute the parameters of the function of expected accumulated loss*:

$$\mathbf{K}_t = \mathbf{W}_t^{xx} + \mathbf{H}_{t+1}, \tag{103}$$
$$\mathbf{k}_t^x = \mathbf{w}_t^x + \mathbf{h}_{t+1}^x, \tag{104}$$

$$\Lambda_t^{xx} = \Upsilon_t^{AKA} + \mathbf{A}_t'\mathbf{K}_t\mathbf{A}_t, \tag{105}$$
$$\Lambda_t^{xu} = (\Lambda_t^{ux})', \tag{106}$$
$$\Lambda_t^{ux} = \Upsilon_t^{BKA} + \mathbf{B}_t'\mathbf{K}_t\mathbf{A}_t + \mathbf{W}_t^{ux}\mathbf{A}_t\,, \tag{107}$$
$$\Lambda_t^{uu} = \Upsilon_t^{BKB} + \mathbf{B}_t'\mathbf{K}_t\mathbf{B}_t + 2\mathbf{B}_t'\mathbf{W}_t^{xu} + \mathbf{W}_t^{uu}, \tag{108}$$

$$\boldsymbol{\lambda}_t^x = \boldsymbol{\upsilon}_t^{AKc} + \mathbf{A}_t'\mathbf{K}_t\mathbf{c}_t + \mathbf{A}_t'\mathbf{k}_t^x, \tag{109}$$

$$\boldsymbol{\lambda}_t^u = \boldsymbol{\upsilon}_t^{BKc} + \mathbf{B}_t'\mathbf{K}_t\mathbf{c}_t + \mathbf{B}_t'\mathbf{k}_t^x + \mathbf{W}_t^{ux}\mathbf{c}_t + \mathbf{w}_t^u, \tag{110}$$

$$\lambda_t^s = \tfrac{1}{2}\operatorname{tr}\left[\mathbf{K}_t\Sigma_t^{\xi\xi}\right] + h_{t+1}^s, \tag{111}$$

$$\lambda_t^p = \tfrac{1}{2}\,\upsilon_t^{cKc} + h_{t+1}^p, \tag{112}$$

$$\lambda_t^c = \tfrac{1}{2}\mathbf{c}_t'\mathbf{K}_t\mathbf{c}_t + \mathbf{c}_t'\mathbf{k}_t^x + w_t^c + h_{t+1}^c. \tag{113}$$

(vi) *Compute the parameters of the feedback rule*:

$$\mathbf{G}_t = -(\Lambda_t^{uu})^{-1}\Lambda_t^{ux}, \tag{114}$$
$$\mathrm{g}_t = -(\Lambda_t^{uu})^{-1}\boldsymbol{\lambda}_t^u. \tag{115}$$

(vii) *Compute the parameters of the function of minimal expected accumulated loss*:

$$\mathbf{H}_t = \Lambda_t^{xx} - \Lambda_t^{xu}(\Lambda_t^{uu})^{-1}\Lambda_t^{ux}, \tag{116}$$

$$\mathbf{h}_t^x = \boldsymbol{\lambda}_t^x - \Lambda_t^{xu}(\Lambda_t^{uu})^{-1}\boldsymbol{\lambda}_t^u, \tag{117}$$

$$h_t^c = \lambda_t^c - \tfrac{1}{2}(\boldsymbol{\lambda}_t^u)'(\Lambda_t^{uu})^{-1}\boldsymbol{\lambda}_t^u, \tag{118}$$

$$h_t^s = \lambda_t^s, \tag{119}$$

$$h_t^p = \lambda_t^p. \tag{120}$$

(c) *Forward projection*:

Repeat the following steps (i) and (ii) for $t = S, \ldots, T$.

(i) *Compute the expected optimal policy*:

$$\mathbf{u}_t^* = \mathbf{G}_t \mathbf{x}_{t-1}^* + \mathbf{g}_t. \tag{121}$$

(ii) *Compute the expected optimal state*: Use the Gauss–Seidel algorithm to compute $\mathbf{x}_t^*$ such that

$$\mathbf{x}_t^* = \mathbf{f}(\mathbf{x}_{t-1}^*, \mathbf{x}_t^*, \mathbf{u}_t^*, \hat{\boldsymbol{\theta}}, \mathbf{z}_t). \tag{122}$$

(d) *Set the new tentative paths for the next iteration*:

$$(\mathring{\mathbf{x}}_t)_{t=S}^T = (\mathbf{x}_t^*)_{t=S}^T, \tag{123}$$

$$(\mathring{\mathbf{u}}_t)_{t=S}^T = (\mathbf{u}_t^*)_{t=S}^T. \tag{124}$$

(e) *Compute the expected welfare loss*:

$$J_S^* = \mathbf{x}_{S-1}' \mathbf{H}_S \mathbf{x}_{S-1} + \mathbf{x}_{S-1}' + \mathbf{h}_{S-1}^x + h_{S-1}^c + h_{S-1}^p + h_{S-1}^s. \tag{125}$$

5. Applications to small econometric models for Austria

In principle, the algorithm OPTCON can be applied to any discrete-time intertemporal optimization problem under stochastic uncertainty, provided that the objective can be expressed as (or approximately expressed by) a quadratic function and the system dynamics fulfill the assumptions stated for f(. . .). Many examples of stochastic optimum control problems of this kind for management and economic systems are given in [14]. For instance, in production–inventory problems, the control variables may be production and orders and the state variables may be inventories; in queueing problems, a control may be the service rate and a state may be the length of a waiting line; in maintenance problems, the controls may be maintenance and replacement policies and the state may be a machine's productivity; in portfolio selection problems, a control variable may be investment and a state may be wealth; in quality control problems, the control may be the sampling rate and a state may be the number of failures; in advertising problems, the control may be advertising expenditures and the state may be sales; etc. In all these cases the objective may be defined in terms of expected profits or costs.

Unfortunately, for management applications frequently necessary data are not available outside the respective firm for scientific research. Therefore we choose a macroeconomic policy problem to illustrate the applicability of our algorithm OPTCON. In this case, the controls are fiscal and monetary policy variables, the states are macroeconomic target variables, and the objective may express "social welfare" or a policy-maker's (planner's) objectives. We report on some applications of OPTCON to two small macroeconometric models of the Austrian economy. The primary purpose of this exercise is to test the algorithm and its implementation in the programming system GAUSS. More detailed information about the results of these and related control experiments is given in [10] and [12].

5.1. THE MODELS

We consider two small demand-side macroeconometric models for Austria, to be called AUSTRIA1 and AUSTRIA2. AUSTRIA1 is a nonlinear model estimated by ordinary least squares, whereas AUSTRIA2 is a linear model estimated by three stage least squares. The estimation period has been 1965 to 1988.

Tables 1 and 2 show the results of the estimations and the identities of the models AUSTRIA1 and AUSTRIA2, respectively, together with the statistical characteristics of the regressions. Values in brackets below regression coefficients denote estimated standard deviations. R^2 is the coefficient of determination, R_c^2 the coefficient of determination adjusted for the degrees of freedom, SE is the estimated standard error of the equation, MAPE is the mean absolute percentage error, DW is the Durbin–Watson statistic for serial correlation and ρ is the estimated first-order autocorrelation coefficient of the residuals.

The following abbreviations are used for the model variables:

CR_t real private consumption,
IR_t real fixed investment,
MR_t real imports of goods and services,
R_t nominal rate of interest,
YR_t real gross domestic product at market prices,
VR_t real total aggregate demand,
PV_t general price level,
$PV\%_t$ rate of inflation,
$T\%_t$ net tax rate,
TR_t real net tax receipts,
GR_t real public consumption,
$M1_t$ nominal stock of money supply M1,
$M1R_t$ real stock of money supply,
PM_t import price level,
PMV_t relative price of imports,
AR_t real autonomous expenditures,
PY_t domestic price level.

Model AUSTRIA1 has eight endogenous or state variables (CR_t, IR_t, MR_t, R_t, YR_t, VR_t, PV_t, $PV\%_t$), three control variables ($T\%_t$, GR_t, $M1_t$), and three exogenous non-controlled variables (PM_t, AR_t, PY_t). Model AUSTRIA2 has six endogenous variables (CR_t, IR_t, MR_t, R_t, YR_t, VR_t), three control variables (TR_t, GR_t, $M1R_t$), and two exogenous non-controlled variables (PMV_t, AR_t). As can be seen from tables 1 and 2, the two models have a very similar structure.

Table 1

Model AUSTRIA1.

1. Private consumption, real

$$CR_t = \underset{(0.14364)}{0.3061} CR_{t-1} + \underset{(0.13237)}{0.63312} YR_t(1-\tfrac{T\%_t}{100}) - \underset{(0.74461)}{1.81043} (R_t - PV\%_t) + \underset{(5.904)}{5.27457}$$

$$R^2 = 0.996 \quad R_c^2 = 0.996 \quad SE = 5.55399$$
$$MAPE = 0.99 \quad DW = 1.972 \quad \rho = 0.01$$

2. Fixed investment, real

$$IR_t = \underset{(0.03425)}{0.93547} IR_{t-1} + \underset{(0.04793)}{0.2359} (VR_t - VR_{t-1}) - \underset{(0.59133)}{0.42742} (R_t - PV\%_t) + \underset{(6.18576)}{8.95606}$$

$$R^2 = 0.975 \quad R_c^2 = 0.972 \quad SE = 5.40704$$
$$MAPE = 2.19 \quad DW = 1.559 \quad \rho = 0.22$$

3. Imports of goods and services, real

$$MR_t = \underset{(0.14035)}{0.21599} MR_{t-1} + \underset{(0.05414)}{0.28844} VR_t - \underset{(0.55423)}{0.93284} \left(\frac{PM_t}{PV_t} \cdot 100\right) + \underset{(69.10811)}{13.43473}$$

$$R^2 = 0.993 \quad R_c^2 = 0.992 \quad SE = 8.66294$$
$$MAPE = 2.92 \quad DW = 1.163 \quad \rho = 0.35$$

4. Rate of interest, nominal

$$R_t = \underset{(0.1604)}{0.792} R_{t-1} - \underset{(0.01755)}{0.01857} \left(\frac{M1_t}{PV_t} \cdot 100\right) + \underset{(0.00228)}{0.00169} YR_t + \underset{(1.31912)}{2.76811}$$

$$R^2 = 0.616 \quad R_c^2 = 0.558 \quad SE = 0.67843$$
$$MAPE = 5.64 \quad DW = 1.323 \quad \rho = 0.32$$

5. GDP at market prices, real

$$YR_t = CR_t + IR_t + GR_t + AR_t - MR_t$$

6. Total aggregate demand, real

$$VR_t = YR_t + MR_t$$

7. Deflator of total demand

$$PV_t = \frac{YR_t}{VR_t} PY_t + \frac{MR_t}{VR_t} PM_t$$

8. Rate of inflation

$$PV\%_t = (PV_t - PV_{t-1})/PV_{t-1} \cdot 100$$

Table 2

Model AUSTRIA2.

1. Private consumption, real

$$CR_t = \underset{(0.13749)}{0.39064} \; CR_{t-1} + \underset{(0.12351)}{0.53807} \; (YR_t - TR_t) + \underset{(5.88465)}{8.80957}$$

$$R^2 = 0.995 \quad R_c^2 = 0.995 \quad SE = 6.19276$$

$$MAPE = 1.22 \quad DW = 1.527 \quad \rho = 0.23$$

2. Fixed investment, real

$$IR_t = \underset{(0.03786)}{0.97107} \; IR_{t-1} + \underset{(0.04955)}{0.17498} \; (VR_t - VR_{t-1}) - \underset{(1.43829)}{3.16947} \; R_t + \underset{(10.11194)}{28.691}$$

$$R^2 = 0.974 \quad R_c^2 = 0.970 \quad SE = 5.58826$$

$$MAPE = 2.49 \quad DW = 1.723 \quad \rho = 0.11$$

3. Imports of goods and services, real

$$MR_t = \underset{(0.12478)}{0.15719} \; MR_{t-1} + \underset{(0.04812)}{0.31204} \; VR_t - \underset{(0.49042)}{0.87247} \; PMV_t - \underset{(61.13881)}{1.56625}$$

$$R^2 = 0.993 \quad R_c^2 = 0.992 \quad SE = 8.7048$$

$$MAPE = 2.98 \quad DW = 0.977 \quad \rho = 0.45$$

4. Rate of interest, nominal

$$R_t = \underset{(0.14261)}{0.71444} \; R_{t-1} - \underset{(0.01546)}{0.01967} \; M1R_t + \underset{(0.002)}{0.00201} \; YR_t + \underset{(1.17251)}{3.29059}$$

$$R^2 = 0.615 \quad R_c^2 = 0.557 \quad SE = 0.68262$$

$$MAPE = 5.75 \quad DW = 1.204 \quad \rho = 0.37$$

5. GDP at market prices, real

$$YR_t = CR_t + IR_t + GR_t + AR_t - MR_t$$

6. Total aggregate demand, real

$$VR_t = YR_t + MR_t$$

5.2 THE OPTIMIZATION EXPERIMENTS

For the optimum control experiments we have to specify an intertemporal objective function of a hypothetical policy-maker. Here we assume the quadratic tracking function (1) with (2). The planning horizon for the control experiments has been chosen as $S = 1971$ to $T = 1988$. For the "ideal" values of the state and control variables ($\tilde{\mathbf{x}}_t$ and $\tilde{\mathbf{u}}_t$, respectively), we assume for 1970 historical values for all variables to be given and postulate growth rates of 3.5% p.a. for the years 1971 to 1988 for all real variables. In model AUSTRIA1 a growth rate of 2% p.a. is considered as "ideal" for the price level PV_t and a constant value of 2 is hence assumed as "ideal" rate of inflation $PV\%_t$. The rate of interest R_t has an "ideal" constant value of 7 for all periods. In the model AUSTRIA1 the historical value of 1970 is used as constant "ideal" value for $T\%_t$, and an "ideal" growth rate of 5.5% p.a. from the 1970 historical value is assumed for $M1_t$.

We assume a discount factor $\alpha = 1$. This implies $\mathbf{W}_t = \mathbf{W}$ for all t. All off-diagonal elements of this matrix are set equal to zero, and the main diagonal elements get the following weights for the model where the respective variable appears:

variable	CR_t	IR_t	MR_t	R_t	YR_t	VR_t	PV_t	$PV\%_t$	$T\%_t$	TR_t	GR_t	$M1_t$	$M1R_t$
weight	5	5	5	2.5	10	0	5	0	5	5	5	1	1

Thus real GDP at market prices is regarded as the main objective variable.

For the model AUSTRIA1, three different control experiments are performed: In experiment 1 all parameters of the model are regarded as known with certainty. The only stochastic influences considered are the additive error terms in the behavioral equations. We assume the covariance matrix of the additive error terms to be a diagonal matrix with the squared estimated standard errors of the behavioral equations in the main diagonal.

For experiments 2 and 3, also performed with AUSTRIA1, we tentatively introduce stochastic parameters. Here we assume the covariance matrix of the parameters to be diagonal and select only some diagonal elements of this matrix to be non-zero. In experiment 2 those parameters whose estimates have the lowest t-values are regarded as stochastic. These are the coefficient of R_t in the investment equation and the coefficients of $M1_t$ and YR_t in the interest rate equation. In experiment 3 the marginal propensity to consume (the second coefficient in the consumption equation) is added to the above ones as fourth stochastic parameter, again taking the estimated coefficient and its estimated standard deviation as the first and second moments of that parameter, respectively.

For model AUSTRIA2 two experiments have been run. Experiment 4 is analogous to experiment 1 for model AUSTRIA1 in assuming all parameters to be known for certain. Here the entire estimated covariance matrix of the behavioral equations' additive disturbances is available as the model has been estimated by 3SLS. Finally, in experiment 5 the stochastic nature of all the parameters of the model (the coefficients and the constants) is taken into account by assuming the estimated values of the parameters to be their expected values and the covariance matrix of the coefficients of the model to be the covariance matrix of the parameters.

The five optimum control experiments described above were carried out on an IBM compatible 12 MHz PC-AT with an 80287 mathematical coprocessor. The

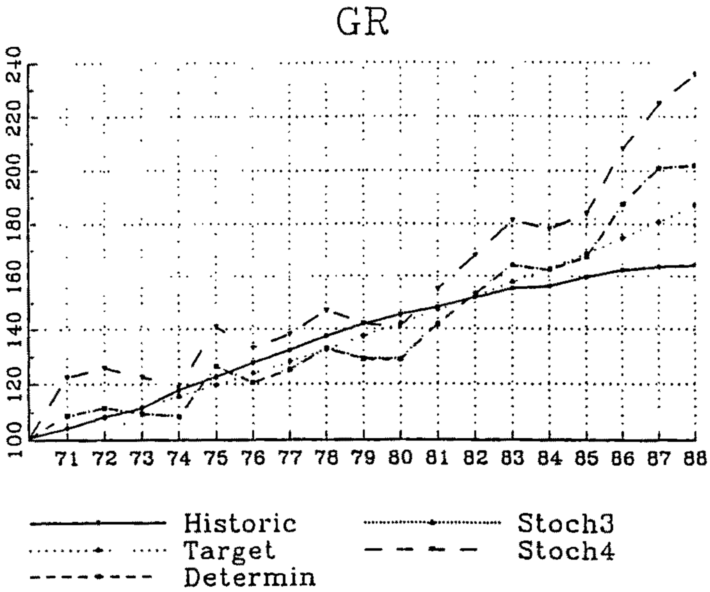

Fig. 1. GR_t in model AUSTRIA1.

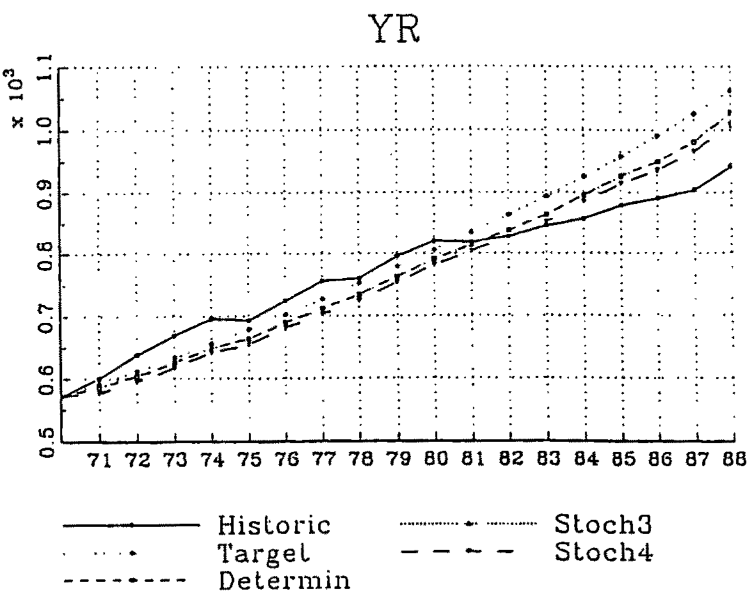

Fig. 2. YR_t in model AUSTRIA1.

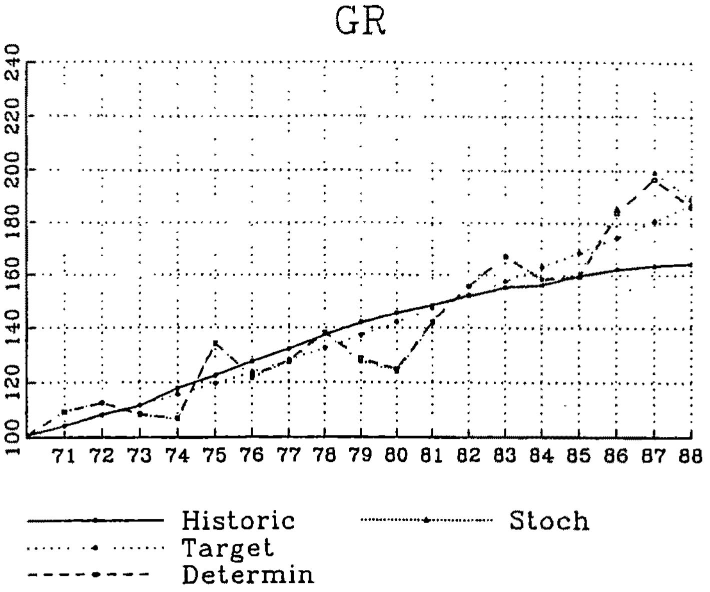

Fig. 3. GR_t in model AUSTRIA2.

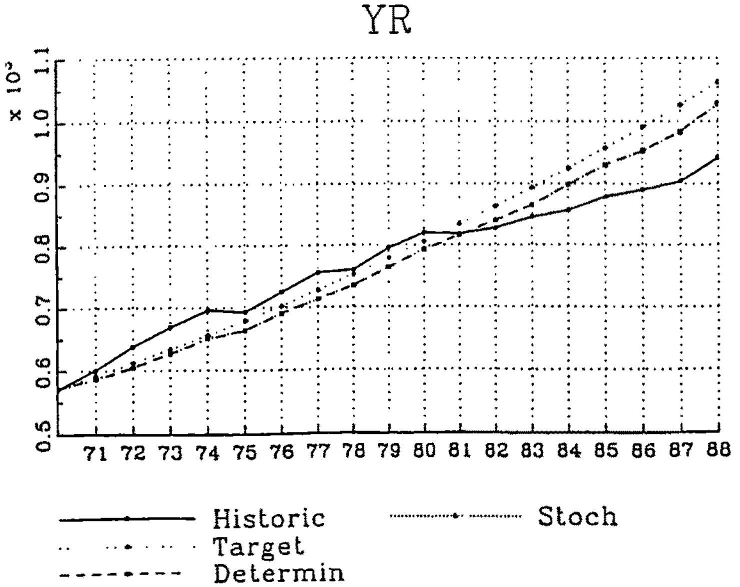

Fig. 4. YR_t in model AUSTRIA2.

running time of the GAUSS program of OPTCON ranged from 5 min 5 sec in experiment 4 to 24 min 37 sec in experiment 3. The results will be only briefly summarized; see also [12].

Figures 1 to 4 show the results of the experiments for one selected control variable (GR_t) and one state variable (YR_t). The time paths denoted as "Historic" and "Target" show the historical and the "ideal" values, respectively. In figs. 1 and 2, approximately optimal values from experiments 1, 2 and 3 are denoted by "Determin", "Stoch3" and "Stoch4", respectively. In figs. 3 and 4, approximately optimal values from experiments 4 and 5 are denoted by "Determin" and "Stoch", respectively.

All the experiments indicate that fluctuations of the main objective variables can be stabilized to some extent by optimal policies. The optimal values of the control variables, especially those of fiscal policies, exhibit counter-cyclical behavior. Optimal values of most real variables are lower than historical ones until the end of the seventies and higher during the eighties, showing that optimal policies in this model can to some extent overcome the lower historical growth rates of the period after the two oil price shocks.

Introducing stochastic parameters has the following effects on optimal policies for model AUSTRIA1: In experiment 2, the differences to the results of experiment 1 are minor. On the other hand, for experiment 3 the differences to experiment 1 are more pronounced. It seems that optimal fiscal policies become more active (i.e., the absolute deviations from the "ideal" values of these variables become larger on average) if their effects are uncertain, which is somewhat counterintuitive.

From the experiments with model AUSTRIA2 we get the following results: In experiment 4, the optimal values of most variables appearing in both models are rather close to the optimal values from experiment 1, which shows that both models embody similar trade-offs. Experiment 5 gives the somewhat surprising result that optimal values for all objective variables are very close to those obtained in experiment 4. Control variables show slightly more active behavior, but in general it seems that in this model taking fully account of stochastic parameters does not have a strong influence upon the optimal policies. It remains an open question whether this result generalizes to models with different kinds of nonlinearities.

6. Concluding remarks

In this paper we have presented an algorithm for the optimal control of nonlinear dynamic macroeconometric models with stochastic additive error terms and stochastic parameters under a quadratic intertemporal objective function. This algorithm has been implemented in the programming language GAUSS and applied to two small econometric models of the Austrian economy in order to show the feasibility of the algorithm. The optimization experiments show that optimal policies may lead to a considerable stabilization of the time paths of the main objective variables of the models. Several experiments with different kinds of stochastic parameters have demonstrated the influence of parameter uncertainty on optimal policies.

Several directions of further research may be suggested. More optimization experiments are required in order to study results under a greater variety of stochastic parameter patterns and different economic models. Also, alternative objective functions should be applied, and the numerical sensitivity of the optimal policies to the choice of the "ideal" values of the objective variables and the elements of the weighting matrix in the objective function has to be examined. For the algorithm itself, there exist several possible extensions. By adding updating equations for the stochastic parameters such as the one used in [9], it could be expanded into a passive-learning algorithm in the sense of [5]. Another interesting extension to be considered in the future will be the examination of the effects of decentralized policy-making; here results of decentralized control theory (dynamic team theory) and dynamic game theory will have to be incorporated.

Appendix 1: Proof of theorem 1

MacRae [8] defines differentiation of a $(p \times q)$-matrix $\mathbf{Y}$ whose elements are functions of an $(m \times n)$-matrix $\mathbf{X}$ as follows:

$$\frac{\partial \mathbf{Y}}{\partial \mathbf{X}} = \mathbf{Y} \otimes \frac{\partial}{\partial \mathbf{X}}, \tag{126}$$

where $\otimes$ denotes the Kronecker product, $\partial/\partial\mathbf{X}$ is a matrix of derivative operators the cells of which are defined by

$$\left(\frac{\partial}{\partial \mathbf{X}}\right)_{i,j} = \frac{\partial}{\partial x_{i,j}}, \quad i = 1,\ldots,m, \;\; j = 1,\ldots,n, \tag{127}$$

and premultiplication of a matrix element by a derivative operator is understood to denote differentiation.

In her paper MacRae gives theorems which can be used to evaluate more complex expressions. Two of them, the product rule and the inverse rule, are repeated here, as we need them for the derivation of theorem 1:

$$\frac{\partial \mathbf{Y}^{-1}}{\partial \mathbf{X}} = -(\mathbf{Y}^{-1} \otimes \mathbf{I}_m)\left(\frac{\partial \mathbf{Y}}{\partial \mathbf{X}}\right)(\mathbf{Y}^{-1} \otimes \mathbf{I}_n \qquad \text{(inverse rule)}, \tag{128}$$

where $\mathbf{Y}$ is a non-singular matrix whose elements are functions of the $(m \times n)$ matrix $\mathbf{X}$,

$$\frac{\partial \mathbf{YZ}}{\partial \mathbf{X}} = \frac{\partial \mathbf{Y}}{\partial \mathbf{X}}(\mathbf{Z} \otimes \mathbf{I}_n) + (\mathbf{Y} \otimes \mathbf{I}_m)\frac{\partial \mathbf{Z}}{\partial \mathbf{X}} \qquad \text{(product rule)}, \tag{129}$$

where **Y** and **Z** are matrices whose elements are functions of the $(m \times n)$-matrix **X** and the product **YZ** is well-defined.

Using this matrix differential calculus it is possible to derive the derivatives of the parameters of the linearized system equation with respect to $\boldsymbol{\theta}$ as given in theorem 1. In order to prove these formulas, it is convenient first to derive an intermediate result. Applying the inverse rule we obtain:

$$\frac{\partial[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}]}{\partial\boldsymbol{\theta}} = -[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\frac{\partial(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})}{\partial\boldsymbol{\theta}}(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}. \tag{130}$$

As $\mathbf{I}_n$ does not depend on $\boldsymbol{\theta}$ this can be simplified to:

$$\frac{\partial[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}]}{\partial\boldsymbol{\theta}} = -[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\frac{\partial\mathbf{F}_{\mathbf{x}_t}}{\partial\boldsymbol{\theta}}(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}. \tag{131}$$

Now we turn to the derivation of $\mathbf{D}^{\mathbf{A}_t}$. First we have

$$\mathbf{D}^{\mathbf{A}_t} = \frac{\partial\mathbf{A}_t}{\partial\boldsymbol{\theta}} = \frac{\partial[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{x}_{t-1}}]}{\partial\boldsymbol{\theta}}. \tag{132}$$

Straightforward application of the product rule gives:

$$\frac{\partial\mathbf{A}_t}{\partial\boldsymbol{\theta}} = \frac{\partial[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}]}{\partial\boldsymbol{\theta}}\mathbf{F}_{\mathbf{x}_{t-1}} + [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\frac{\partial\mathbf{F}_{\mathbf{x}_{t-1}}}{\partial\boldsymbol{\theta}}. \tag{133}$$

If we insert (131) we can write:

$$\begin{aligned}\frac{\partial\mathbf{A}_t}{\partial\boldsymbol{\theta}} &= [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\mathbf{F}_{\mathbf{x}_t,\boldsymbol{\theta}}(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{x}_{t-1}} \\ &\quad + [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\mathbf{F}_{\mathbf{x}_{t-1},\boldsymbol{\theta}}.\end{aligned} \tag{134}$$

If we recall the definition for $\mathbf{A}_t$ we can further simplify this to

$$\frac{\partial\mathbf{A}_t}{\partial\boldsymbol{\theta}} = [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p][\mathbf{F}_{\mathbf{x}_t,\boldsymbol{\theta}}\mathbf{A}_t + \mathbf{F}_{\mathbf{x}_{t-1},\boldsymbol{\theta}}]. \tag{135}$$

Next comes the derivation of $\mathbf{D}^{\mathbf{B}_t}$:

$$\mathbf{D}^{\mathbf{B}_t} = \frac{\partial\mathbf{B}_t}{\partial\boldsymbol{\theta}} = \frac{\partial[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{u}_t}]}{\partial\boldsymbol{\theta}}. \tag{136}$$

Again, we apply the product rule to get:

$$\frac{\partial\mathbf{B}_t}{\partial\boldsymbol{\theta}} = \frac{\partial[(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}]}{\partial\boldsymbol{\theta}}\mathbf{F}_{\mathbf{u}_t} + [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\frac{\partial\mathbf{F}_{\mathbf{u}_t}}{\partial\boldsymbol{\theta}}, \tag{137}$$

which can be further simplified (after insertion for the inverse) to:

$$\frac{\partial \mathbf{B}_t}{\partial \boldsymbol{\theta}} = [(\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1} \otimes \mathbf{I}_p]\,[\mathbf{F}_{\mathbf{x}_t,\boldsymbol{\theta}}\mathbf{B}_t + \mathbf{F}_{\mathbf{u}_t,\boldsymbol{\theta}}]. \tag{138}$$

Finally, we prove the formula for $\mathbf{d}^{c_t}$:

$$\mathbf{d}^{c_t} = \frac{\partial \mathbf{c}_t}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{x}_t}{\partial \boldsymbol{\theta}} - \frac{\partial[\mathbf{A}_t\mathbf{x}_{t-1}]}{\partial \boldsymbol{\theta}} - \frac{\partial[\mathbf{B}_t\mathbf{u}_t]}{\partial \boldsymbol{\theta}}. \tag{139}$$

Applying the product rule twice yields:

$$\frac{\partial \mathbf{c}_t}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{x}_t}{\partial \boldsymbol{\theta}} - \mathbf{D}^{\mathbf{A}_t}\mathbf{x}_{t-1} - (\mathbf{A}_t \otimes \mathbf{I}_p)\frac{\partial \mathbf{x}_{t-1}}{\partial \boldsymbol{\theta}} - \mathbf{D}^{\mathbf{B}_t}\mathbf{u}_t - (\mathbf{B}_t \otimes \mathbf{I}_p)\frac{\partial \mathbf{u}_t}{\partial \boldsymbol{\theta}}. \tag{140}$$

As $\mathbf{x}_{t-1}$ and $\mathbf{u}_t$ do not depend on $\boldsymbol{\theta}$ this can be shortened to:

$$\frac{\partial \mathbf{c}_t}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{x}_t}{\partial \boldsymbol{\theta}} - \mathbf{D}^{\mathbf{A}_t}\mathbf{x}_{t-1} - \mathbf{D}^{\mathbf{B}_t}\mathbf{u}_t. \tag{141}$$

$\mathbf{x}_t$ does depend on $\boldsymbol{\theta}$ via the system function. Using the implicit function theorem we can establish this derivative and then state:

$$\frac{\partial \mathbf{c}_t}{\partial \boldsymbol{\theta}} = \mathrm{vec}[((\mathbf{I}_n - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\boldsymbol{\theta}})'] - \mathbf{D}^{\mathbf{A}_t}\mathbf{x}_{t-1} - \mathbf{D}^{\mathbf{B}_t}\mathbf{u}_t, \tag{142}$$

where the vectorization is necessary to reshape the first derivatives of the system equation (for which we have used the more common notation) to conform to the way derivatives are defined by MacRae. In the above all functions have been assumed to be evaluated along the reference path.

Appendix 2: Proof of theorem 2

First, we show that

$$J_t^*(\mathbf{x}_{t-1}) = \tfrac{1}{2}\mathbf{x}_{t-1}'\mathbf{H}_t\mathbf{x}_{t-1} + \mathbf{x}_{t-1}'\mathbf{h}_t^x + h_t^c + h_t^s + h_t^p \tag{143}$$

for all periods $t = S, \ldots, T+1$. Obviously, this is true for $t = T+1$; we do not count any losses after the end of the planning period, hence (60)–(64) hold by definition. As the second step of the induction proof, we now have to show that if $J_{t+1}^*(\mathbf{x}_t)$ can be expressed as a quadratic function of $\mathbf{x}_t$, then $J_t^*(\mathbf{x}_{t-1})$ will also be quadratic in $\mathbf{x}_{t-1}$. Thus, we presume that

$$J_{t+1}^*(\mathbf{x}_t) = \tfrac{1}{2}\mathbf{x}_t'\mathbf{H}_{t+1}\mathbf{x}_t + \mathbf{x}_t'\mathbf{h}_{t+1}^x + h_{t+1}^c + h_{t+1}^s + h_{t+1}^p. \tag{144}$$

Then, from (8) and (144) we get, using the definitions (65) and (66):

$$L_t(\mathbf{x}_t, \mathbf{u}_t) + J_{t+1}^*(\mathbf{x}_t) = \frac{1}{2}\begin{pmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{pmatrix}'\begin{pmatrix}\mathbf{K}_t & \mathbf{W}_t^{xu}\\ \mathbf{W}_t^{ux} & \mathbf{W}_t^{uu}\end{pmatrix}\begin{pmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{pmatrix} + \begin{pmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{pmatrix}'\begin{pmatrix}\mathbf{k}_t^x\\ \mathbf{w}_t^u\end{pmatrix} + w_t^c + h_{t+1}^c + h_{t+1}^s + h_{t+1}^p. \tag{145}$$

Substituting the linearized system eq. (11) for $\mathbf{x}_t$ we have after collection of terms:

$$
\begin{aligned}
L_t(\mathbf{x}_t,\mathbf{u}_t)+J^*_{t+1}(\mathbf{x}_t) = {} & \tfrac{1}{2}\mathbf{x}'_{t-1}\mathbf{A}'_t\mathbf{K}_t\mathbf{A}_t\mathbf{x}_{t-1}+\tfrac{1}{2}\mathbf{x}'_{t-1}[\mathbf{B}'_t\mathbf{K}_t\mathbf{A}_t+\mathbf{W}^{ux}_t\mathbf{A}_t]'\mathbf{u}_t \\
& +\tfrac{1}{2}\mathbf{u}'_t[\mathbf{B}'_t\mathbf{K}_t\mathbf{A}_t+\mathbf{W}^{ux}_t\mathbf{A}_t]\mathbf{x}_{t-1} \\
& +\tfrac{1}{2}\mathbf{u}'_t[\mathbf{B}'_t\mathbf{K}_t\mathbf{B}_t+2\mathbf{W}^{ux}_t\mathbf{B}_t+\mathbf{W}^{uu}_t]\mathbf{u}_t \\
& +\mathbf{x}'_{t-1}[\mathbf{A}'_t\mathbf{K}_t\mathbf{c}_t+\mathbf{A}'_t\mathbf{K}_t\xi_t+\mathbf{A}'_t\mathbf{k}^x_t] \\
& +\mathbf{u}'_t[\mathbf{B}'_t\mathbf{K}_t\mathbf{c}_t+\mathbf{B}'_t\mathbf{K}_t\xi_t+\mathbf{B}'_t\mathbf{k}^x_t+\mathbf{W}^{ux}_t\mathbf{c}_t+\mathbf{W}^{ux}_t\xi_t+\mathbf{w}^u_t] \\
& +\tfrac{1}{2}\mathbf{c}'_t\mathbf{K}_t\mathbf{c}_t+\mathbf{c}'_t\mathbf{K}_t\xi_t+\tfrac{1}{2}\xi'_t\mathbf{K}_t\xi_t+\mathbf{c}'_t\mathbf{k}^x_t+\xi'_t\mathbf{k}^x_t \\
& +w^c_t+h^c_{t+1}+h^s_{t+1}+h^p_{t+1}. \qquad (146)
\end{aligned}
$$

Next we calculate the function of expected accumulated loss, namely

$$J_t(\mathbf{x}_{t-1},\mathbf{u}_t)=\mathrm{E}_{t-1}(L_t(\mathbf{x}_t,\mathbf{u}_t)+J^*_{t+1}(\mathbf{x}_t)). \qquad (147)$$

If we assume that $\mathbf{H}_{t+1}$, $\mathbf{h}^x_{t+1}$, h^c_{t+1}, h^s_{t+1}, and h^p_{t+1} are non-stochastic or known after $\mathbf{x}_{t-1}$ has been realized, then using (20), (21), (44) and the analogous expressions listed following (52) in section 3.3, together with (67)–(75), we can see that

$$J_t(\mathbf{x}_{t-1},\mathbf{u}_t)=\frac{1}{2}\begin{pmatrix}\mathbf{x}_{t-1}\\ \mathbf{u}_t\end{pmatrix}'\begin{pmatrix}\Lambda^{xx}_t & \Lambda^{xu}_t\\ \Lambda^{ux}_t & \Lambda^{uu}_t\end{pmatrix}\begin{pmatrix}\mathbf{x}_{t-1}\\ \mathbf{u}_t\end{pmatrix}+\begin{pmatrix}\mathbf{x}_{t-1}\\ \mathbf{u}_t\end{pmatrix}'\begin{pmatrix}\boldsymbol{\lambda}^x_t\\ \boldsymbol{\lambda}^u_t\end{pmatrix}+\lambda^c_t+\lambda^s_t+\lambda^p_t. \qquad (148)$$

Minimizing this function with respect to $\mathbf{u}_t$ and assuming Λ^{uu}_t to be symmetric and positive definite yields the feedback rule (76) with (77) and (78). By substituting the feedback rule for $\mathbf{u}_t$ into $J_t(\mathbf{x}_{t-1},\mathbf{u}_t)$ we can derive $J^*_t(\mathbf{x}_{t-1})$, the function of minimal expected accumulated loss, as

$$J^*_t(\mathbf{x}_{t-1})=\tfrac{1}{2}\mathbf{x}'_{t-1}\mathbf{H}_t\mathbf{x}_{t-1}+\mathbf{x}'_{t-1}\mathbf{h}^x_t+h^c_t+h^s_t+h^p_t. \qquad (149)$$

Thus, it has been proved by induction that $J^*_t(\mathbf{x}_{t-1})$ is a quadratic function of $\mathbf{x}_{t-1}$ for all periods $t=S,\dots,T$. It can be easily verified that Λ^{uu}_t and hence $\mathbf{H}_t$ are symmetric which completes the proof of theorem 2.

Acknowledgements

Financial support from the "Fonds zur Förderung der wissenschaftlichen Forschung" (research project no. P 5846) is gratefully acknowledged. Thanks are due to four anonymous referees for several useful comments upon an earlier version. The usual disclaimer applies.

References

[1] M. Aoki, *Optimal Control and System Theory in Dynamic Economic Analysis* (North-Holland, New York, 1976).

[2] G.C. Chow, *Analysis and Control of Dynamic Economic Systems* (Wiley, New York, 1975).

[3] G.C. Chow, *Econometric Analysis by Control Methods* (Wiley, New York, 1981).

[4] P. Coomes, PLEM: A computer program – passive learning stochastic control for reduced-form econometric models, Discussion paper 85-4, Center for Economic Research, University of Texas, Austin (1985).

[5] D.A. Kendrick, *Stochastic Control for Economic Models* (McGraw–Hill, New York, 1981).

[6] D.A. Kendrick, *Feedback: A New Framework for Macroeconomic Policy* (Kluwer, Dordrecht, 1988).

[7] D.A. Kendrick and P. Coomes, DUAL, A program for quadratic–linear stochastic control problems, Discussion paper 84-15, Center for Economic Research, University of Texas, Austin (1984).

[8] E.C. MacRae, Matrix derivatives with an application to an adaptive linear decision problem, Ann. Statist. 2(1974)337.

[9] E.C. MacRae, An adaptive learning rule for multiperiod decision problems, Econometrica 43(1975)893.

[10] J. Matulka and R. Neck, Optimal control of nonlinear stochastic macroeconometric models: An algorithm and an economic example, in: *Econometric Decision Models: New Methods of Modeling and Applications*, ed. J. Gruber (Berlin, 1991) p. 57.

[11] R. Neck, Stochastic control theory and its applications to models of operations research and economics, in: *Symulacyjne Modele Przedsiebiorstw*, ed. L. Piecuch (Cracow, 1989) p. 137.

[12] R. Neck and J. Matulka, Stochastic optimum control of macroeconometric models using the algorithm OPTCON, Working Paper, Vienna University of Economics and Business Administration, Vienna, Austria (1991).

[13] A.L. Norman, First order dual control, Ann. Econ. Social Measurement 5(1976)311.

[14] C.S. Tapiero, *Applied Stochastic Models and Control in Management* (North-Holland, Amsterdam, 1988).