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POLLUTION CONTROL: A DIFFERENTIAL GAME APPROACH

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Abstract

Transnational pollution is formulated as a differential game between two sovereign governments. The symmetric open loop Nash equilibrium is shown to yield more pollution than in a cooperative solution. A model of Stackelberg leadership in pollution control is also investigated. The possibility of limit cycles is illustrated, using bifurcation theory.

Keywords: Pollution control, differential games, bifurcation.

1. Introduction

Theoretical models of pollution control in an intertemporal setting have typically focussed on the case of a single country. Recent concerns over the greenhouse effect and the depletion of the ozone layer indicate that for many types of environmental problems it is necessary to formulate models from an international perspective. This paper is a modest step in this direction.

The paper compares the fully cooperative solution with the alternative non-cooperative equilibria. The open loop Nash equilibrium is considered first because analytically this is the most tractable formulation. It is shown that the resulting steady state stock of pollution is larger than it would be in a fully cooperative solution, that the steady state is stable and that convergence is monotonic. More interesting patterns emerge when one country is the Stackelberg leader and the other is the follower. It is assumed that the leader is able to commit itself to a given path of emission of waste, and that it announces its policy prior to the follower's choice of emission path. In this model, convergence to a steady state may be non-monotonic: spiralling is possible. Furthermore, there exist parameter values such that the optimal path is a closed orbit or converges to a closed orbit. This result is obtained using bifurcation theory and the explicit formula for the calculation of roots of a control system with two state variables.

2. The basic model

There are two countries, indexed by $i = 1, 2$. Each country has a fixed endowment of factors of production and produces a consumption good whose output is denoted by Q^i. The production of this good emits an amount E_i of pollutants. The magnitude of E_i depends on the extent to which factors of production (capital and labor) are applied to emission control activities. This diversion of real resources would of course reduce the output of the consumption good. Following Forster [6,7], we represent the consumption emission trade-off by the reduced form function

$$Q_i = F_i(E_i), \tag{1}$$

where $F_i(E_i)$ is strictly concave and $F_i'(0) > 0$.

The stock of pollution at any time t is denoted by $P(t)$. We assume that the two countries "contribute" to the same stock of pollution. For simplicity, the evolution of stock $P(t)$ is represented by the following linear equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = E_1(t) + E_2(t) - kP(t), \tag{2}$$

where $k > 0$ is a constant rate of decay. (For non-constant decay rates, see Forster [7].)

Pollution is a "public bad" because of its adverse affects on health, quality of life, and also production. We assume that these adverse effects can be represented by having P as an argument of the instantaneous social welfare function W_i, with negative derivative:

$$W_i = W_i(Q_i, P), \tag{3}$$

$$\frac{\partial W_i}{\partial P} < 0. \tag{4}$$

In each country, aggregate social welfare is taken to be the integral of the discounted flow of instantaneous social welfare:

$$V_i = \int_0^\infty \exp(-r_i t) W_i(Q_i(t), P(t))\mathrm{d}t, \tag{5}$$

where $r_i > 0$ is the rate of discount.

For tractability, the function W_i is often assumed to take the separable form:

$$W_i(Q_i, P) = U_i(Q_i) - D_i(P), \tag{6}$$

where $U_i(Q_i)$ may be thought of as the utility of consumption, and $D_i(P)$ as the "disutility" caused by pollution. Following standard practice, we take it that U_i is strictly concave and increasing in Q_i, and that D_i is convex and increasing in P. The possibility that D_i is linear is not ruled out.

Given that each country would want to have the highest possible aggregate social welfare V_i, what is the course of action that each should take? The answer depends on the extent to which the two countries can cooperate. If the two countries have the same rate of discount r (i.e. $r_1 = r_2$), it would seem natural that a fully cooperative solution would involve the maximization of the sum (or some weighted average) of V_1 and V_2. The solution for this is standard. There would be a common "shadow price" q, of the stock of pollution, which would serve to guide the optimal policy. This shadow price would be negative, and at each instant the marginal rate of trade-off between consumption and environmental control (the derivative $F_i'(E_i)$) must be equated with the absolute value of the shadow price expressed in terms of the consumption good:

$$F_i'(E_i(t)) = \frac{-q(t)}{U_i'(Q_i)}. \tag{7}$$

Furthermore, $q(t)$ must change over time; its rate of change would follow the equation

$$\frac{dq}{dt} = (r+k)q + D_1'(P) + D_2'(P). \tag{8}$$

There would exist an optimal steady state stock of pollution, P , and a corresponding steady state shadow q^* such that

$$-q^* = \frac{D_1'(P^*) + D_2'(P^*)}{r+k} \tag{9}$$

and the optimal steady state emission rates would be E_1^* and E_2^*, where

$$U_i'(F_i(E_i^*))F_i'(E_i^*) = -q^* \tag{10}$$

and

$$E_1^* + E_2^* = kP^*. \tag{11}$$

Equations (9) and (10) give the rule for the optimal long run supply of a "public bad":

$$U_i'(F_i(E_i^*))F_i'(E_i^*) = \frac{D_1'(P^*) + D_2'(P^*)}{r+k}. \tag{12}$$

In economic terms, eq. (12) states that the marginal contribution of waste emission to the utility of consumption must be equated to the sum of the marginal costs of pollution, measured in present value. This rule is the intertemporal counterpart of the familiar Lindahl–Samuelson rule for the optimal provision of a public good.

The fully cooperative outcome is unlikely to be achievable in the real world, because of the costs of coordination. Equation (12) is best thought of as a benchmark against which other alternative outcomes can be compared. Two of these alternatives will be considered in this paper. Firstly, one can compute the open loop Nash equilibrium, in which each country forecasts the entire time path of emission of its neighbour, and optimizes with respect to its own emission rates. Secondly, one can study the open loop Stackelberg equilibrium: the home country (country 1) knows that the foreign country will react to its time path of emission $E_1(t)$, and is able to pre-commit itself to any particular path $E_1(t)$; its problem is to find the best path. The open loop Stackelberg solution yields a higher level of welfare to the home country, as compared with the open loop Nash solution: This is because the latter solution is a feasible choice for the home country.

3. The open loop Nash equilibrium

Suppose country i believes that country j will follow a given time path $E_j(t)$, regardless of what $E_i(t)$ might be. Country i's problem is then to choose $E_i(t)$ that maximizes the integral of its discounted flow of social welfare, as specified by (5) and (6). The solution of country i's problem may be denoted as

$$E_i(\cdot) = \phi_i\{E_j(\cdot); P_0\}. \tag{13}$$

Following Basar and Olsder [3], we define an open loop Nash equilibrium as a pair of time paths $\{E_1^N(\cdot); E_2^N(\cdot)\}$ such that

$$E_1^N(\cdot) = \phi_1\{E_2^N(\cdot); P_0\} \tag{14}$$

and

$$E_2^N(\cdot) = \phi_2\{E_1^N(\cdot); P_0\}. \tag{15}$$

To determine the properties of an open loop Nash equilibrium of our pollution game, we first set up country i's Hamiltonian function associated with the control problem (5):

$$H_i = U_i(F_i(E_i)) - D_i(P) + q_i(E_1 + E_2 - kP). \tag{16}$$

The Maximum Principle yields:

$$U_i' F_i' = -q_i, \tag{17}$$

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = (r_i + k)q_i + D_i'(P), \tag{18}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = E_1 + E_2 - kP. \tag{19}$$

To these, we add the transversality condition:

$$\lim_{t\to\infty} \exp(-r_i t)q_i(t)P(t) = 0. \tag{20}$$

For simplicity of notation, let us define

$$\psi_i(E_i) = U_i(F_i(E_i)). \tag{21}$$

Equation (17) becomes

$$\psi_i'(E_i) = -q_i. \tag{22}$$

From (22) and (18) we obtain the differential equation

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} = \frac{(r_i + k)\psi_i'(E_i) - D_i'(P)}{\psi_i''(E_i)}. \tag{23}$$

From (23), with $i = 1, 2$, and (19), we have three differential equations in E_1, E_2 and P. These and the boundary conditions

$$P(0) = P_0, \tag{24}$$

$$\lim_{t\to\infty} \exp(-r_i t)\psi_i'(E_i)P(t) = 0 \tag{25}$$

determine the open loop Nash equilibrium.

We now verify that the steady state equilibrium $(E^{\mathrm{N}}, P^{\mathrm{N}})$ has the saddle-point property. The Jacobian matrix is given by

$$J = \begin{bmatrix} -k & 1 & 1 \\ -D_1''/\psi_1'' & r_1 + k & 0 \\ -D_2''/\psi_2'' & 0 & r_2 + k \end{bmatrix}. \tag{26}$$

From this, trace $J > 0$ and det $J < 0$. Therefore, there exists a negative eigenvalue, implying saddle-point stability.

Consider the special case in which the two countries are identical. The symmetric Nash equilibrium yields the steady state solution (E_1^N, E_2^N, P^N), where $E_1^N = E_2^N$ and $P^N = (2/k)E_1^N$,

$$\psi_1'(E_1^N) = \frac{D_1'((2/k)E_1^N)}{r+k}. \tag{27}$$

Equation (27) may be compared with its counterpart of the fully cooperative solution, i.e. eq. (12) which is reproduced below for the symmetric case

$$\psi_1'(E^*) = \frac{2D_1'((2/k)E_1^*)}{r+k}. \tag{28}$$

Since ψ_1' is decreasing in E_1 and D_1' is increasing in E_1, we conclude that the steady state emission in the open loop Nash equilibrium is higher than that of the fully cooperative solution (which is, of course, essentially the one-player case).

4. The open loop Stackelberg equilibrium

In the preceding section, it was assumed that each country behaved as if it must choose the whole time path $E_i(t)$ at the beginning of the game. An open loop Nash equilibrium is a pair of time paths of emission rates such that, given one country's time path, the other country's time path is optimal from its own point of view. If each country believes that at no stage would its neighbour deviate from its chosen time path, then clearly there is no incentive for it to deviate either. The open loop Nash equilibrium is thus time-consistent (though in general it is not subgame perfect). We now study a different kind of solution, called the open loop Stackelberg solution, which has the potential of being time-inconsistent, and which is therefore credible only if the "leader" can pre-commit itself to the plan it announces at the outset.

Assume that the home country (country 1) is the "leader". It knows that for any time path $E_1(t)$ that it commits itself to, country 2 would find the time paths $E_2(t)$ and $q_2(t)$ that satisfy eqs. (17) to (20), with $i = 2$. The home country could of course commit itself to the open loop Nash equilibrium path $E_1^N(t)$, in which case country 2 would choose $E_2^N(t)$ and would be back to the previous section. However, the home country can do better. From (17), we know that E_2 is a function of q_2:

$$E_2 = (\psi_2')^{-1}(-q_2) = Z(q_2). \tag{29}$$

Since

$$\psi_1'(E_2)\mathrm{d}E_2 = -\mathrm{d}q_2 \tag{30}$$

$$\frac{\mathrm{d}z}{\mathrm{d}q_2} = \frac{-1}{\psi_2''(E_2)} > 0. \tag{31}$$

Equation (31) implies that when q_2 becomes less negative (i.e. closer to zero), the foreign country will allow a higher rate of emission. So country 1 can indirectly control country 2's emission by suggesting a time path of shadow price $q_2(t)$. This suggestion would be accepted by country 2 if $q_2(t)$ satisfies the following differential equations:

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = (r_2 + k)q_2 + D_2'(P), \tag{32a}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = Z(q_2) + E_1(t) - kP, \tag{32b}$$

and

$$\lim_{t\to\infty} \exp(-r_2 t)q_2(t)P(t) = 0. \tag{32c}$$

Conditions (32a) to (32c), regarded as conditions on the time path of q_2, may be terms of "the incentive compatibility conditions" of principal-agent problem. If country 1 can suggest a time path $q_2(t)$ and project the time path $E_1(t)$ and $P(t)$ such that conditions (32a) to (32c) are satisfied, country 2's optimization problem is solved. Country 1's problem then is to choose the best path $E_1(t)$ and indirectly $q_2(t)$. This is an optimal control problem involving two state variables, $P(t)$ and $q_2(t)$. Note that while $q_2(t)$ is a co-state variable in country 2's problem, it is a state variable in country 1's problem.

Country 1 seeks $E_1(t)$, $q_2(t)$ and $P(t)$ that maximize V_1, subject to (32). Let Θ and γ be country 1's co-state variables associated with the state variables P and q_2. The Hamiltonian of country 1's optimal control problem is:

$$H = \psi_1(E_1) - D_1(P) + \Theta[E_1 + Z(q_2) - kP] + \gamma[(k + r_2)q_2 + D_2'(P)].$$

The necessary conditons are

$$\psi_1'(E_1) = -\Theta, \tag{33}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = E_1 + Z(q_2) - kP, \tag{34}$$

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = (k + r_2)q_2 + D_2'(P), \tag{35}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = (r_1 + k)\Theta + D_1'(P) - \gamma D_2''(P), \tag{36}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = (r_1 - r_2 - k)\gamma - \Theta Z'(q_2). \tag{37}$$

From (33), E_1 is a function of Θ alone. We may also write

$$E_1 = G(\Theta), \tag{38}$$

With

$$G'(\Theta) = -\frac{1}{\psi_1''} > 0, \tag{39}$$

substituting (38) into (36), we obtain from (34) to (36) four autonomous differential equations. To characterize the equilibrium of this system, we first note that at the equilibrium

$$\gamma[(r_1 - r_2 - k)(r_1 + k) - Z'D_2''] = -D_1'Z', \tag{40}$$

$$\Theta = \frac{(r_1 - r_2 - k)\gamma}{Z'}, \tag{41}$$

$$kP = G(\Theta) + Z(q_2), \tag{42}$$

$$q_2 = \frac{-D_2'(P)}{r_2 + k}. \tag{43}$$

Thus, the equilibrium stock of pollution must satisfy the following equation

$$G\left[\frac{(r_1 - r_2 - k)D'(P)}{Z'D_2''(P) - (r_1 - r_2 - k)(r_1 + k)}\right] + Z\left[\frac{-D'_2(P)}{r_2 + k}\right] - kP = 0. \tag{44}$$

Let $\Omega(P)$ denote the left-hand side of (44). If $\Omega(0) > 0$ and $\Omega(P)$ tends to minus infinity as P tends to infinity, then (44) has a solution $P^* > 0$. Sufficient conditions for this are:

$$D_2'(0) = D_1'(0) = 0, \tag{45a}$$

$$Z(0) > 0, G(0) > 0, \tag{45b}$$

$$\lim_{\Theta \to -\infty} G(\Theta) = \lim_{q_2 \to -\infty} Z(q_2) = 0, \tag{45c}$$

$$r_1 - r_2 - k < 0. \tag{45d}$$

Condition (45a) says that when the stock of pollution is zero, the detrimental effects of a small increase in pollution are negligible. Condition (45b) is plausible:

when the shadow price of the "public bad" is zero, emissions are positive. Conditions (45c) and (45d) are also easily interpretable.

If $\Omega'(P) < 0$ then the steady state pollution stock P^* is unique. Differentiating the left-hand side of (44) with respect to P, and bearing in mind (45d) and the properties of G, Z, D_1 and D_2, we see that a sufficient condition for uniqueness is:

$$Z'D_2''' - \frac{Z''(D_2'')^2}{r_2 + k} \leq 0. \tag{46}$$

Assuming uniqueness, can we determine whether the level P^* of this problem is higher than P^N, the steady state pollution level in an open loop Nash equilibrium? As the present level of generality, it does not seem possible to obtain an unambiguous answer. Let us consider the special case where the two countries have identical preferences and technologies, with

$$\psi_i(E_i) = AE_i - \tfrac{1}{2}E_i^2, \tag{47}$$

$$D_i(P) = \frac{s}{2} P^2, \tag{47b}$$

$$r_1 = r_2 (= r). \tag{47c}$$

It is then easy to show, using (26) and (44), that

$$P^N = \frac{A}{(k/2) + (s/r+k)} \tag{48a}$$

and

$$P^* = \frac{A}{\frac{k}{2} + \frac{1}{2}\left(\frac{s}{r+k}\right) + \frac{1}{2}\left(\frac{s}{r+k+(s/k)}\right)}. \tag{48b}$$

Therefore, $P^* > P^N$ in this case. We now show that *in this more polluted environment country 1 (the leader) emits more waste, and country 2 emits less, as compared with their steady state emission levels in the open loop Nash equilibrium.* In order to see this, note that in a steady state

$$E_2 = A + q_2 = A - [sP/(r+k)].$$

Therefore, as $P^* > P^N$, we must have $E_2^* < E_2^N$, as is clear from the above equation. Finally, since $kP^N = 2E_2^N = 2E_1^N$ and $kP^* = E_1^* + E_2^*$, E_1^* must exeed E_1^N (recall that $P^* > P^N$ and $E_2^* < E_2^N$).

It remains to investigate the stability properties of the equilibrium P^*. Consider the system of eqs. (34) to (37), with $E_1 = G(\Theta)$. Linearizing this system at the steady state, we obtain a Jacobian matrix with the following pattern

$$J = \begin{bmatrix} A & B \\ C & r_1 I - A^{\mathrm{T}} \end{bmatrix},$$

where B and C are symmetric matrices. This pattern is common to all optimal control problems. In our problem,

$$\begin{bmatrix} a & b & d & e \\ g & h & e & i \\ j & m & r_1 - a & -g \\ m & n & -b & r_1 - h \end{bmatrix}, \tag{49}$$

where

$$a = \frac{\partial \dot{P}}{\partial P} = -k, \qquad b = \frac{\partial \dot{P}}{\partial q_2} = Z',$$

$$d = \frac{\partial \dot{P}}{\partial \Theta} = G, \qquad e = \frac{\partial \dot{P}}{\partial \gamma} = 0,$$

$$g = \frac{\partial \dot{q}_2}{\partial P} = D_2'', \qquad h = \frac{\partial \dot{q}_2}{\partial q_2} = k + r_2,$$

$$i = \frac{\partial \dot{q}_2}{\partial \gamma} = 0, \qquad j = \frac{\partial \dot{\Theta}}{\partial P} = D_1'' - \gamma D_2''',$$

$$m = \frac{\partial \dot{\Theta}}{\partial q_2} = 0, \qquad n = \frac{\partial \dot{\gamma}}{\partial q_2} = -\Theta Z''.$$

Let

$$w = r_1 a + r_1 h - a^2 - h^2 - 2bg - 2em - dj - ni. \tag{50}$$

As is shown in Dockner [4] and also in appendix 1 of Kemp et al. [9], the four roots of matrix j are

$$\lambda_{1,2,3,4} = \frac{r_1}{2} \pm \left[\frac{r_1^2}{4} - \frac{w}{2} \pm \frac{1}{2} \Delta^{1/2} \right]^{1/2}, \tag{51}$$

where

$$\Delta \equiv w^2 - 4\det J. \tag{52}$$

It follows that there are two positive real roots and two negative real roots if and only if the following conditions are satisfied:

$$\Delta \geq 0, \tag{53a}$$

$$w < 0, \tag{53b}$$

$$\det J > 0. \tag{53c}$$

Consider the plane (w, $\det J$), where w is measured along the horizontal axis and $\det J$ along the vertical axis. Conditions (53a) to (53c) define a unique region S_1 of this plane in which we have two positive real roots and two negative real roots. Consider next the region above the parabola $\det J = (w/2)^2$. In this region Δ is negative, and we have four complex roots. Kemp et al. showed that the real parts of these complex roots may vanish only along the curve defined by*

$$\det J = (w/2)^2 + r_1^2(w/2), \quad w > 0, \tag{54}$$

The curve depicted by (54), where $w > 0$, is called the bifurcation locus.

Since the roots are continuous in w and $\det J$, it follows from the above observations that in the sets S_2 and S_3 defined below, we have four complex roots, two with negative real parts and two with positive real parts:

$$S_2 = \{(w, \det J)\colon w \leq 0 \text{ and } \det J > (w/2)^2\},$$

$$S_3 = \{(w, \det J)\colon w > 0 \text{ and } \det J > (w/2)^2 + r_1^2(w/2)\}.$$

It follows that a sufficient condition for the steady state to have two roots with negative real parts (or two negative real roots) is that *both* $\det J$ and ($-w$) are positive. Now

$$-w = r_2^2 + 2kr_2 + 2k^2 - r_1 r_2 + 2D_2'' Z' + G' D''^1 - \frac{D_1' Z' G' D_2'''}{Z' D_2'' - (r_1 - r_2 - k)(r_1 + k)}. \tag{55}$$

Therefore, ($-w$) is positive provided that

$$r_1 < r_2 + k \tag{56a}$$

and

$$D_2''' \leq 0\,. \tag{56b}$$

*In Dockner [4], it was stated in part (iv) of theorem 2 that pure imaginary roots are not possible if $r_1 > 0$. Professor Dockner has acknowledged in private correspondence that part (iv) of theorem 2 is incorrect.

The expression for $\det J$ is slightly more complicated:

$$\det J = (k^2 + kr_2 + Z'D_2'')\,[Z'D_2'' - (r_1 - r_2 - k)(r_1 + k)] \\ + G'(r_2 + k - r_1)\{D_1''(k + r_2) + \gamma[(Z''/Z')(D_2'')^2 - (k + r_2)D_2''']\}\,. \tag{57}$$

If we assume (56a) and (56b), then $\det J$ is positive provided that the expression inside the last square brackets is positive. This would be the case if Z were linear.

We conclude that under certain mild restrictions, the steady state is stable in the sense that convergence can be assured by suitable choices of initial values of q_2 and Θ. It is possible, however, to find parameter values at which bifurcation takes place, so that there exists an optimal path that perpetually orbits around the steady state. This holds only if:

$$w > 0 \tag{58a}$$

and

$$\det J = (w/2)^2 + r_1^2(w/2). \tag{58b}$$

(See Kemp et al. [9] for a proof.) *Therefore, bifurcation is possible in this model only if* $D_2''' > 0$ *or* $r_1 \geq r_2 + k$ *or both. In particular if* D_2 *is quadratic and* $r_1 = r_2$ *then there is no bifurcation.*

Let us construct an example where there are bifurcations. From Kemp et al. we know that for a pair of pure imaginary roots, we only need to find an angle β and a triplet $(r_1, w, \det J)$ such that the following equations are satisfied simultaneously:

$$\cos\frac{\beta}{2} = (r_1/2)[(r_1/2)^4 - w(r_1/2)^2 + \det J]^{-1/4}\,, \tag{59a}$$

$$\cos\beta = [(r_1/2)^2 - (w/2)][(r_1/2)^4 - w(r_1/2)^2 + \det J]^{-1/2}\,. \tag{59b}$$

Each of the following values does the trick:

$$(\beta, r_1, w, \det J) = \left(\frac{\pi}{2}; \sqrt{2};\ 1;\ 1.25\right), \tag{60}$$

$$(\beta, r_1, w, \det J) = \left(\frac{2\pi}{3};\ 1;\ 1.50;\ 1.3125\right). \tag{61}$$

Thus, if we set

$$r_1 = 1;\ r_2 = 0.6;\ k = 0.6;\ Z' = 1;\ G' = 1;\ D_1' = 1;\ D_1'' = 0,$$

$$D_2''(P^*) = 1.0648,\quad D_2'''(P^*) = 6.6880,$$

then (61) is satisfied. These values imply

$$\Theta^* = -0.1444,\quad \gamma^* = 0.7221.$$

Furthermore, if we assume the following explicit functional form

$$D_2(P) = 1.1147\ P^3,$$

then

$$P^* = 0.1592,\ q_2^* = -0.0726.$$

We have therefore constructed an example of bifurcation, with the home country's disutility function $D_1(P)$ being linear and the foreign country's $D_2(P)$ being cubic, and $r_1 > r_2$. The steady state pair (q_2^*, P^*) is surrounded by an orbit.

The conditions that lead to limit cycles in this model can be related to a result in Brock and Sheinkman [2]. Let $\lambda = (\Theta, \gamma)$ and $x = (P, q_2)$. Then Brock and Sheinkman state that if trace $H_{\lambda x}$ is negative, then limit cycles can be ruled out globally. In our model,

$$\text{trace}\,(H_{\lambda x}) = r_2 > 0.$$

Thus, the trace condition is violated.

5. Concluding remarks

In this paper we have studied the possible outcome of a pollution game between two neighbouring countries. In section 3, both countries move simultaneously and the moves are chosen at the outset. The Stackelberg equilibrium was considered in section 4, where a variety of possible outcomes was displayed. Cyclical behaviour was shown to be possible. Both models require the ability of the government to make credible commitment. When such commitment is not possible, the appropriate equilibrium concept would be Markov perfect equilibrium, as in Fershtman and Kamien [5].

Our approach has relied on certain symmetry between the two countries. It would be interesting to explore the implications of a relaxation of this symmetry. For example, in the case of global warming, some countries may gain from global atmospheric pollution. Another extension of the model would be to allow for investment in production capacity and pollution control technology.

Finally, we have restricted our analysis to local properties of steady states. It would seem desirable to investigate the global dynamics of the system.

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References

[1] T. Basar and B.J. Olsder, *Dynamic Non-Cooperative Game Theory* (Academic Press, 1982).

[2] W.A. Brock and J. Sheinkman, The global asymptotic stability of optimal control with applications to dynamic economic theory, in: *Applications of Control Theory to Economic Analysis*, ed. J.D.Pitchford and S.J. Turnovsky (North-Holland, Amsterdam, 1977).

[3] J.M. Conrad and L.J. Olson, Polluted control and resource management, Working Paper, Department of Agricultural Economics, Cornell University, New York (1990).
[4] E. Dockner, Local stability analysis in optimal control problems with two state variables, in: *Optimal Control Theory and Economic Analysis 2*, ed. G. Feichtinger (North-Holland, Amsterdam, 1985).
[5] C. Fershtman and M. Kamien, Dynamic duopolistic competition with sticky prices, Econometrica 55(1987)257.
[6] B.A. Forster, Optimal consumption planning in a polluted environment, Econ. Record 49(1973)534.
[7] B.A. Forster, Optimal pollution control with a non-constant exponential rate of decay, J. Environ. Econ. Manag. 2(1975)1.
[8] J. Guckenheimer and P. Holmes, *Non-Linear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields* (Springer, 1983).
[9] M.C. Kemp, N.V. Long and K. Shimomura, Cyclical and non-cyclical redistributive taxation, Working Paper, Department of Economics, University of New South Wales, Australia (1990).
[10] C.G. Plourde, A model of waste accumulation and disposal, Can. J. Econ. 5(1972)115.
[11] C.G. Plourde and D. Yeung, A model of industrial pollution in a stochastic environment, J. Environ. Econ. Manag. 17(1989)100.