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ADDITIONAL INTEGRALS OF THE GENERALIZED QUANTUM CALOGERO–MOSER PROBLEM

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We consider the quantum integrable Calogero–Moser problem and its generalizations connected with the Coxeter groups. For special values of the coupling constants, this system acquires additional integrals and becomes algebraically integrable. We give an effective description of additional integrals to this quantum problem.

Let us consider the quantum Hamiltonian of the Calogero–Moser problem,

$$H = -\Delta + \sum_{i<j} 2m(m+1)(x_i - x_j)^{-2}, \tag{1}$$

which describes a one-dimensional system of n point-like, pairwise-interacting particles with the interaction potential $u(x) = m(m+1)x^{-2}$, m being an arbitrary parameter. It is well known [1, 2] that this system, together with its classical analog, is integrable, i.e., it possesses n independent involutive integrals of motion. In the quantum case, this means that there exist differential operators $L_1, \ldots, L_n$, $L_i = L_i(x, \partial/\partial x)$, $x = (x_1, \ldots, x_n)$, such that $[L_i, H] = 0$ and $[L_i, L_j] = 0$ for all i, j. The operators L_i can be found explicitly [3], namely,

$$\begin{aligned}
L_1 &= \partial_1 + \ldots + \partial_n, \\
L_2 &= \sum_{i<j} \left(\partial_i \partial_j + m(m+1)(x_i - x_j)^{-2}\right) = \frac{1}{2}(L_1^2 - H), \\
L_3 &= \sum_{i<j<k} \partial_i \partial_j \partial_k + \ldots, \\
&\ldots \\
L_n &= \partial_1 \ldots \partial_n + \ldots,
\end{aligned} \tag{2}$$

where $\partial_i = \partial/\partial x_i$ and the points denote terms of lower orders. Note that all L_i are symmetric in $x_1, \ldots, x_n$.

It was shown in [4] that at integer values of the parameter m, problem (1) possesses additional quantum integrals that are not symmetric in $x_1, \ldots, x_n$ and, hence, cannot be expressed via the integrals $L_1, \ldots, L_n$. Following the terminology of [5, 4], operator (1) at some integer m is an *algebraically integrable* Schrödinger operator. Let us formulate this result more accurately.

For a given natural m, consider the ring of polynomials f of n variables $k_1, \ldots, k_n$, for which

$$\text{the difference } f - s_{ij} f \text{ is divisible by } (k_i - k_j)^{2m} \tag{3}$$

for all $i < j$. Here the operator s_{ij} acts on $f(k_1, \ldots, k_n)$ by permutation of the variables k_i and k_j.

Theorem 1 [4]. *For each $m \in \mathbf{N}$, there exists a function $\psi(k, x)$ $(k, x \in \mathbf{R}^n)$ that satisfies conditions* (3) *w.r.t. variables k, of the form*

$$\psi = P(k, x) \exp(k_1 x_1 + \ldots + k_n x_n), \tag{4}$$

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where P is a polynomial in k (depending on x) with the highest term being $\prod_{i<j}(k_i - k_j)^m$. Then, for each polynomial $f(k)$ satisfying conditions (3), *there exists a differential operator $L_f(x, \partial/\partial x)$ such that $L_f\psi = f(k)\psi$.*

As the function f, one can choose the elementary symmetric polynomial of $k_1, \dots, k_n$; then the corresponding operators L_f coincide with the above-mentioned $L_1, \dots, L_n$. If $f = \prod_{i<j}(k_i - k_j)^{2m+1}$, then the corresponding operator L_f is an antisymmetric operator that commutes with every L_i since they have a common eigenfunction (4). Moreover, as the function f, we can choose an arbitrary polynomial that is divisible by $\prod_{i<j}(k_i - k_j)^{2m}$. Therefore, problem (1) has many additional integrals for the integer m. In the present paper, our purpose is to construct these integrals more explicitly, in comparison with [4, 6], for problem (1) and its generalizations.

From the very beginning, we consider a more general situation. A given a set of hyperplanes in $\mathbf{R}^n$, such that the group generated by mirror reflections w.r.t. these planes is a finite group W, is, in this case, called a Coxeter group (see [7]). We call all of the hyperplanes that correspond to all possible reflections from W *mirrors* of the group W. For each mirror, we choose the vector perpendicular to it and let R_+ be the set of these normal vectors. To each $\alpha \in R_+$, we ascribe an integer multiplicity m_α such that $m_\alpha = m_{\alpha'}$ if the corresponding mirrors can be transformed into each other by some transformation from W.

Theorem 2 [4]. 1. *For any Coxeter group W and given W-invariant integer multiplicities $m_\alpha \geq 0$, there exists a unique function $\psi(k, x)$ of form* (4), *which has the polynomial*

$$P(k, x) = \prod_{\alpha \in R_+} (\alpha, k)^{m_\alpha} + \text{ terms of lower orders}$$

and satisfies the following condition:

$$\text{for any } \alpha \in R_+, \quad \psi - s_\alpha\psi \text{ is divisible by } (\alpha, k)^{2m_\alpha}. \tag{5}$$

Here $(s_\alpha\psi)(k, x) = \psi(s_\alpha k, x)$ and s_α is the reflection corresponding to $\alpha \in R_+$. Then any function $\phi(k, x)$ of form (4) *that satisfies condition* (5) *can be obtained from $\psi(k, x)$ by applying some differential operator over the x variable.*

2. *For any polynomial $f(k)$ that satisfies condition* (5), *there exists a differential operator $L_f(x, \partial/\partial x)$ such that $L_f\psi = f(k)\psi$. All of these operators are mutually commuting.*

In the particular case where the group W acts in $\mathbf{R}^n$ by permutations of coordinates, Theorem 2 coincides with Theorem 1. Note that we can always take W-invariant polynomials as f, for instance, $f(k) = -k^2$. Then a simple calculation shows that the operator $H = L_{-k^2}$ reads as follows:

$$H = -\Delta + \sum_{\alpha \in R_+} m_\alpha(m_\alpha + 1)(\alpha, \alpha)(\alpha, x)^{-2}, \tag{6}$$

i.e., it coincides with the generalization of Hamiltonian (1) to the case of an arbitrary Coxeter group proposed by Olshanetsky and Perelomov [3].

Let the subset R'_+ of the set of normal vectors R_+ be chosen in such a way that unification of all hyperplanes corresponding to the normal vectors is a set of mirrors for some Coxeter group $W' \subset W$ and the W'-invariant multiplicities are given for $\alpha \in R'_+$. Then the following Hamiltonian corresponds to the subgroup W':

$$H' = -\Delta + \sum_{\alpha \in R'_+} m'_\alpha(m'_\alpha + 1)(\alpha, \alpha)(\alpha, x)^{-2}. \tag{7}$$

Theorem 3. *There exists a differential operator $D(x, \partial/\partial x)$ that intertwines operators* (6) *and* (7):

$$H \circ D = D \circ H'. \tag{8}$$

Proof. Consider the functions ψ and ψ' corresponding to the given W, m_α and W', m'_α in accordance with Theorem 2. Since the function ψ obviously satisfies condition (5) for any $m'_\alpha \leq m_\alpha$, then, in accordance with Theorem 2, it can be obtained from ψ' by applying an appropriate differential operator. Let $D(x, \partial/\partial x)$ be the differential operator that transforms ψ' into ψ. Then,

$$(H \circ D)\psi' = H\psi = -k^2\psi = -k^2 D\psi' = D(H'\psi').$$

Thus, the operators $H \circ D$ and $D \circ H'$ coincide when acting on $\psi'(k, x)$ for all k and, therefore, are identical.

The main result of the present paper is the following statement.

Theorem 4. *Let $W' \subset W$ be a Coxeter subgroup of the Coxeter group W, $R'_+ \in R_+$ be the normal vectors to its mirrors, and W'-invariant multiplicities m'_α be chosen such that $m'_\alpha \leq m_\alpha$. Furthermore, let L'_f be a quantum integral of problem* (7) *that corresponds to some polynomial f satisfying condition* (5) *for m'_α, $\alpha \in R'_+$. Then the operator*

$$L = D \circ L'_f \circ D^* \tag{9}$$

is a quantum integral to problem (6), *where D is the intertwining operator from* (8) *and D^* is the operator formally conjugate to D, i.e.,*

$$\int_{\mathbf{R}^n} (Du)v \, dx = \int_{\mathbf{R}^n} u(D^* v) \, dx$$

for any (*finite*) *functions $u(x)$, $v(x)$.*

Proof. Since operators (6) and (7) are self-adjoint, let us conjugate both sides of relation (8),

$$D^* \circ H = H' \circ D^*.$$

Hence, for operator (9), we have

$$L \circ H = D \circ L'_f \circ D^* \circ H' = D \circ L'_f \circ H' \circ D^* = D \circ H' \circ L'_f \circ D^* = H \circ D \circ L'_f \circ D^* = H \circ L,$$

i.e., L commutes with Hamiltonian (6).

Let us prove that L commutes with each integral L_g of problem (6). For this, consider the commutator $I = [L, L_g]$. This operator is, again, an integral of motion, i.e., it commutes with Hamiltonian (6). However, one can easily show that operators L and L_g have constant coefficients at $x = \infty$ (similar to operator (6), which is equal to $-\Delta$ as $x \to \infty$). This follows from the procedure of finding the ψ-function and the integrals L_f described in [4]. As a consequence, their commutator I is equal to zero at infinity and, thus, it vanishes identically by virtue of the following lemma.

Lemma [8]. *If the commutator of a differential operator I with the Laplace operator in $\mathbf{R}_n$ is of an order not exceeding the order of I, then the coefficients of its higher powers are polynomials.*

Thus, $[L, L_g] = 0$. Q.E.D.

To illustrate Theorem 4, let us consider some examples.

Example 1. Type A_1. Here R_+ consists of a single vector α with multiplicity $m_\alpha = m$. If we identify α with the unity in $\mathbf{R}^1$, then the corresponding function $\psi(k,x)$ $(k, x \in \mathbf{R})$ should satisfy the condition

$$\psi(k,x) - \psi(-k,x) \quad \text{is divisible by} \quad k^{2m}$$

and have the following form [6]:

$$\psi(k,x) = D(e^{kx}), \qquad D = D(x, d/dx) = (d/dx - m/x) \circ \ldots \circ (d/dx - 1/x).$$

Thus, the operator

$$A = D \circ d/dx \circ D^*,$$

where $D^* = (-1)^m (d/dx + 1/x) \circ \ldots \circ (d/dx + m/x)$ is an operator of order $2m+1$ that commutes with $L = -d^2/dx^2 + m(m+1)x^{-2}$. This means that L is a doubly degenerate finite-gap operator (see [9]).

Example 2. Type A_2. $R_+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\} \subset \mathbf{R}^3$, $m_\alpha = 1$. The corresponding ψ-function, which was found in [6], is

$$\psi(k,x) = \left[D \circ (\partial_{12} - 2/x_{12})\right] e^{kx},$$

where the operator D looks as follows:

$$D = \partial_{13}\partial_{23} - 2x_{13}^{-1}\partial_{23} - 2x_{23}^{-1}\partial_{13} + 4x_{13}^{-1}x_{23}^{-1} + 2x_{12}^{-2}, \tag{10}$$

and $\partial_{ij} = \partial/\partial x_i - \partial/\partial x_j$, $x_{ij} = x_i - x_j$.

Setting $R'_+ = R_+$ with $m_\alpha \equiv 0$, we find that operator (7) is the Laplace operator $-\Delta$ in $\mathbf{R}^3$. From Theorem 4, for any operator L_0 with constant coefficients, the operator L of the form

$$L = D \circ \left(\partial_{12} - \frac{2}{x_{12}}\right) \circ L_0 \circ \left(\partial_{12} + \frac{2}{x_{12}}\right) \circ D^*$$

is a quantum integral of the Calogero–Moser problem (1) for the case of three particles with $m = 1$. Then L reads as

$$L = \partial_{13}^2 \partial_{23}^2 \partial_{12}^2 L_0 + \text{ lower terms.}$$

Example 3. R_+ and m_α are the same as in the previous example, but $R'_+ = \{e_1 - e_2\}$ with $m'_\alpha = 1$. Then $\psi'(k,x) = (k_{12} - 2/x_{12})e^{kx}$ and $\psi(k,x) = D\psi'(k,x)$, where D is given by formula (10). Therefore, in accordance with Theorem 4, the operator $L = D \circ L_0 \circ D^*$ is a quantum integral of the three-particle Calogero–Moser problem (1) with $m = 1$ if L_0 is an integral of motion for the problem with the Hamiltonian $H_0 = -\Delta + 2(x_1 - x_2)^{-2}$. Choosing L_0 equal to unity and ∂_3, we obtain the following quantum integrals of problem (1) with $m = 1$ for the case $n = 3$:

$$L_1 = D \circ D^* = (\partial_{13}\partial_{23})^2 + \text{ lower terms,}$$
$$L_2 = D \circ \partial_3 \circ D^* = (\partial_{13}\partial_{23})^2 \partial_3 + \text{ lower terms.}$$

One can analogously obtain integrals with the higher symbols

$$(\partial_{12}\partial_{23})^2, (\partial_{12}\partial_{13})^2, (\partial_{12}\partial_{23})^2\partial_2, (\partial_{12}\partial_{13})^2\partial_1.$$

Example 4. $R_+ = \{e_i - e_j\}_{i<j\le n+1} \subset \mathbf{R}^{n+1}$ with $m_\alpha \equiv 1$ and we choose R'_+ equal to a subsystem $R'_+ = \{e_i - e_j\}_{i<j\le n}$ with $m'_\alpha \equiv 1$. The corresponding functions $\psi(k,x)$ and $\psi'(k,x)$ are connected by the relation $\psi = D\psi'$; the explicit form of the differential operator from [6] is

$$D = \prod_{i\le n}(\partial_i - \partial_{n+1}) + \text{ lower terms.}$$

Therefore, the operator $L = D \circ L_0 \circ D^*$ is a quantum integral of problem (1) if we choose L_0 to be equal to unity or to any integral (2).

Note 1. In [4], the trigonometric version of Theorem 2 is presented. There, in condition (5), the polynomial $(\alpha,k)^{2m_\alpha}$ should be replaced by $\prod_{s=1}^{m_\alpha}\big((\alpha,k)^2 - s^2(\alpha,\alpha)\big)$. Then R_+ is the positive part of an arbitrary root system in $\mathbf{R}^n$ and the group W is the corresponding Weyl group. In this case, the literal analogs of Theorems 1–4 are valid and the corresponding operator H has the form

$$H = -\Delta + \sum_{\alpha\in R_+} m_\alpha(m_\alpha+1)(\alpha,\alpha)\sin^{-2}(\alpha,x).$$

Formulas of Examples 2–4 also have trigonometric analogs that determine the integrals of the Sutherland system (10) with the Hamiltonian

$$H = -\Delta + \sum_{i<j} 4\sin^{-2}(x_i - x_j).$$

Note 2. Using Theorem 4, we can construct the quantum integrals L_f to problem (6) for f of the form

$$f(k) = f_0(k) \prod_{\alpha\in R_+\setminus R'_+} (\alpha,k)^{2m_\alpha}, \qquad (11)$$

where R'_+ corresponds to a Coxeter subgroup W' and f_0 is an arbitrary polynomial that is invariant w.r.t. this subgroup. However, as Volchenko and Kozachko demonstrated in [11], already for the system R_+ of type B_2, there exist polynomials that satisfy condition (5) but are not expressible via polynomials of type (11). This demonstrates that not all integrals L_f of problem (6) can be obtained by formula (9). This concerns the trigonometric case as well.

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