SOME REMARKS AND PROBLEMS ON THE COLOURING OF INFINITE GRAPHS AND THE THEOREM OF KURATOWSKI[1]

By

JAN MYCIELSKI (Wrocław)

(*Presented by* G. HAJÓS)

1. We consider the following propositions:

T. The topological product of any number of bicompact Hausdorff spaces is bicompact.[2]

T*. The topological product of any number of non-empty bicompact Hausdorff spaces is non-empty and bicompact.

I. In every Boolean algebra A there is a maximal ideal different from A.

R. Every Boolean algebra is isomorphic to a field of sets.

M. Every consistent elementary theory has a model.[3]

T_n. The topological product af any number of Hausdorff spaces, each having exactly n points, is non-empty and bicompact.[4]

S_n. Let M be any set of disjoint sets each having exactly n elements and $R(x, y)$ is a symmetric relation defined between the elements belonging to different elements of M. Suppose that for any finite set $F \subseteq M$ there exists an $f \in \underset{X \in F}{\mathsf{P}} X$ such that $R(f(X_1), f(X_2))$ holds for any $X_1, X_2 \in F$. Then there exists an $f \in \underset{X \in M}{\mathsf{P}} X$ such that $R(f(X_1), f(X_2))$ holds for any $X_1, X_2 \in M$.[5]

P_n. Every graph, each finite subgraph of which can be coloured with n colours, can be coloured with n colours.[6]

C_n. The Cartesian product of any number of sets, each having exactly n elements, is non-empty.

It is known that the axiom of choice implies each of the above propo-

[1] This is a lecture delivered on the Colloquium on the Theory of Graphs in Dobogókő, 22 October 1959.

[2] Here and farther *any number* means any positive finite or infinite cardinal number.

[3] We do not suppose that the number of symbols and statements of the theories is denumerable.

[4] Here and farther n is running over positive integers.

[5] P denotes the Cartesian product operator.

[6] A colouring of a graph with n colours is a partition of the set of vertices into n classes such that no two vertices in one class are joint by an edge.

sitions, but the following logical relations can be proved without the use o this axiom:

(1) $\mathrm{T} \leftrightarrow \mathrm{T}^* \leftrightarrow \mathrm{I} \leftrightarrow \mathrm{R} \leftrightarrow \mathrm{M} \leftrightarrow \mathrm{T}_m \leftrightarrow \mathrm{S}_n$ for $m = 2, 3, \ldots,\ n = 4, 5, \ldots;$

(2) $\mathrm{S}_4 \rightarrow \mathrm{S}_3 \rightarrow \mathrm{S}_2;$

(3) $\mathrm{S}_n \rightarrow \mathrm{P}_n \rightarrow \mathrm{C}_n$ for $n = 2, 3, \ldots;$

(4) $\mathrm{P}_{n+1} \rightarrow \mathrm{P}_n$ for $n = 2, 3, \ldots;$

(5) $\mathrm{P}_2 \leftrightarrow \mathrm{C}_2.$

It would be interesting to know any further implication between these propositions. Some implications and independences between the propositions C_n are known, e. g. $\mathrm{C}_2 \leftrightarrow \mathrm{C}_4$, $\mathrm{C}_{mn} \rightarrow \mathrm{C}_m$ and others (see [8], [10], [11]). The equivalences (1) and implications (2) are proved in the papers [4], [5], [6], [7]. Other interesting propositions which may be added to the equivalences (1) are given in [9].

Let us prove (3), (4) and (5):

$\mathrm{S}_n \rightarrow \mathrm{P}_n$ is obvious (compare the proof of P_n given in [2]).

$\mathrm{P}_n \rightarrow \mathrm{C}_n$. Let K be a set of disjoint n-element sets. We treat $\bigcup_{X \in K} X$ as a set of vertices of a graph, two vertices being joint if and only if they belong to the same X. By P_n it is easy to see that this graph can be coloured with n colours. Take all the vertices of one colour, this clearly defines a selection from K as required in C_n.

$\mathrm{P}_{n+1} \rightarrow \mathrm{P}_n$. Let G be a graph each finite subgraph of which can be coloured with n colours. We add a new vertex and join it to all vertices of G. Using P_{n+1} we easily see that the new graph can be coloured with $n+1$ colours. Removing the additional vertex we obtain n-colourings of G as needed in P_n.

$\mathrm{P}_2 \leftrightarrow \mathrm{C}_2$. Owing to (3) it remains to prove $\mathrm{C}_2 \rightarrow \mathrm{P}_2$. If G is a connected graph, each finite subgraph of which can be coloured with 2 colours, then it is easy to see that, putting two vertices in the same class if and only if there exists a path from one to the other with an odd number of edges, we obtain a two-colouring of G. Now if G is not connected, using C_2 we select one of these classes for each component of G. We consider the partition of the vertices of G into 2 classes: the union of the selected classes and the remaining vertices. It is easy to see that it is a two-colouring of G.

REMARK (due to C. RYLL-NARDZEWSKI). The proposition P_n restricted to denumerable graphs can be proved without using the axiom of choice.

2. We consider the following properties of a graph G (by a graph we mean here a one-dimensional simplicial complex with the natural topology, we do not suppose that it is locally finite and the cardinality of the set of vertices of G is arbitrary):

(i) G does not contain topologically any one of KURATOWSKI's two graphs (Fig. 1).

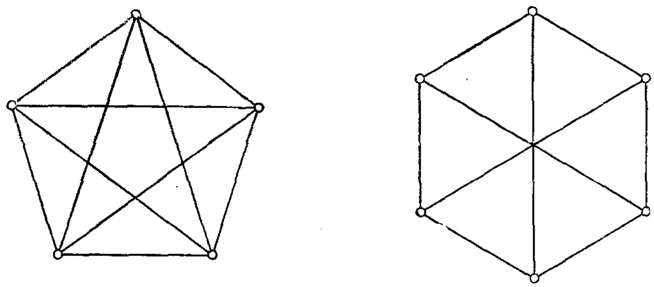

Fig. 1

(ii) Every finite subgraph of G is homeomorphically imbeddable in the plane R^2.

(iii) There exists a system of homeomorphisms $\{h_F(x)\}$ where F runs over all finite subgraphs of G such that h_F maps homeomorphically F into R^2 and for any F_1 and F_2

(*) $$h_{F_1}|F_1 \cap F_2 \text{ is homotopical to } h_{F_2}|F_1 \cap F_2.$$ [7]

(iv) One can define for every circuit C of G a partition of the set $|G| \setminus |C|$ [8] into two classes Int (C), Ext (C) such that two vertices belonging to different classes are not joint by an edge and

$$\text{if } |C_1| \subset |C_2| \cup \text{Int } (C_2), \text{ then Int } (C_1) \subset |C_2| \cup \text{Int } (C_2);$$
$$\text{if } |C_1| \subset |C_2| \cup \text{Ext } (C_2), \text{ then Ext } (C_1) \subset |C_2| \cup \text{Ext } (C_2).$$

THEOREM. *The properties* (i), (ii), (iii), (iv) *are equivalent.*

PROOF. (i)→(ii) by the well-known theorem of KURATOWSKI [3].

(ii)→(iii). We denote by S_F the set of homotopy types of homeomorphical applications of F into R^2 (F runs over the finite subgraphs of G). S_F is finite; we treat it as a discrete topological space. By the statement T (Section 1 of this paper) the topological product $\mathop{\mathrm{P}}_F S_F$ is bicompact.

For any $t_1 \in S_{F_1}$ and $t_2 \in S_{F_2}$ we put $t_1 \sim t_2$ if and only if (*) holds for some h_{F_1} of type t_1 and h_{F_2} of type t_2. Let $F_1, \ldots, F_m$ be any finite set of

[7] $f|X$ denotes the mapping f with domain restricted to X.

[8] $|H|$ denotes the set of vertices of the graph H. $\setminus$ denotes the set-theoretical difference.

finite subgraphs of G. We put $K_{F_1,\ldots,F_m}=\{f: f\in \mathop{\mathsf{P}}_F S_F,\ f(F_i)\sim f(F_j)$ for $i,j=1,\ldots,m\}$. Of course, the sets $K_{F_1,\ldots,F_m}$ are closed subsets of $\mathop{\mathsf{P}}_F S_F$. They are also non-empty, since if F is a finite subgraph of G such that $F_1,\ldots,F_m$ are subgraphs of F and h_F is a homeomorphism $h_F: F\to R^2$ (it exists by (ii)), then one can take for $f\in K_{F_1,\ldots,F_m}$ any function $f\in \mathop{\mathsf{P}}_F S_F$ such that $f(F_i)$ is the homotopy type of $h_F|F_i$. The finite intersections of the sets $K_{F_1,\ldots,F_m}$ are also non-empty, since

$$K_{F_1^{(1)},\ldots,F_m^{(1)}}\cap K_{F_1^{(2)},\ldots,F_n^{(2)}}\supset K_{F_1^{(1)},\ldots,F_m^{(1)},F_1^{(2)},\ldots,F_n^{(2)}}.$$

It follows that there exists an f_0 such that

$$f_0\in \bigcap_{m=1}^{\infty}\ \bigcap_{F_1,\ldots,F_m} K_{F_1,\ldots,F_m}$$

and clearly any system $\{h_F\}$, such that the homotopy type of h_F is $f_0(F)$ satisfies (iii); q. e. d.

(iii)$\to$(iv). A system $\{h_F\}$ being given, for every circuit C and every vertex $v\in|G|\setminus|C|$ we put $v\in \operatorname{Int}(C)$ if the homeomorphism $h_{C\cup(v)}$ maps v inside the domain bounded by the image of C and $v\in \operatorname{Ext}(C)$ in the other case. It is easy to verify that our definition satisfies (iv).

(iv)$\to$(i). Clearly a subgraph of a graph satisfying (iv) satisfies (iv) One can prove by a direct verification that no one of the Kuratowski graphs satisfies (iv); and our implication follows.

COROLLARY. (DIRAC and SCHUSTER [1].) *A denumerable graph satisfying* (i) *has a continuous* 1—1 *mapping into* R^2.

PROOF. By the theorem the graph satisfies (iii) and one can construct the mapping by an easy induction.

REMARK. The equivalence (ii)$\leftrightarrow$(iii) remains valid if one replaces in these statements R^2 by any bicompact 2-manifold.

PROBLEM. Does there exist a finite set of finite graphs such that any finite graph G can not be homeomorphically imbedded in a given bicompact 2-manifold (e. g. the projective plane) if and only if G contains a subgraph homeomorphic to one of them?

INSTITUTE OF MATHEMATICS,
UNIVERSITY OF WROCŁAW, POLAND

(Received 15 January 1960)

References

[1] G. A. Dirac and S. Schuster, A theorem of Kuratowski, *Indag. Math.*, **16** (1954), pp. 343—348.

[2] N. G. de Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in the theory of relations, *Indag. Math.*, **13** (1951), pp. 371—373.

[3] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.*, **15** (1930), pp. 271—283.

[4] J. Łoś, Sur le théorème de Gődel pour les théories indénombrables, *Bull. Acad. Polon. Sci. Cl. III*, **2** (1954), pp. 319—320.

[5] J. Łoś, Remarks on Henkin's paper: Boolean representations through propositional calculus, *Fund. Math.*, **44** (1957), pp. 82—83.

[6] J. Łoś and C. Ryll-Nardzewski, On the application of Tychonoff's theorem in mathematical proofs, *Fund. Math.*, **38** (1951), pp. 233—237.

[7] J. Łoś and C. Ryll-Nardzewski, Effectiveness of the representation theory for Boolean algebras, *Fund. Math.*, **41** (1954), pp. 49—56.

[8] A. Mostowski, Axiom of choice for finite sets, *Fund. Math.*, **33** (1945), pp. 137—168.

[9] H. Rubin and D. Scott, Some topological theorems equivalent to the Boolean prime ideal theorem, *Bull. Amer. Math. Soc.*, **60** (1954), p. 398.

[10] W. Sierpiński, L'axiome du choix pour les ensembles finis, *Le Mathematiche*, **10** (1955), pp. 92—99.

[11] W. Szmielew, On choices from finite sets, *Fund. Math.*, **34** (1947), pp. 75—80.