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ON DESTINATION OPTIMALITY IN ASYMMETRIC DISTANCE FERMAT–WEBER PROBLEMS

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Abstract

This paper introduces skewed norms, i.e. norms perturbed by a linear function, which are useful for modelling asymmetric distance measures. The Fermat–Weber problem with mixed skewed norms is then considered. Using subdifferential calculus we derive exact conditions for a destination point to be optimal, thereby correcting and completing some recent work on asymmetric distance location problems. Finally the classical dominance theorem is generalized to Fermat–Weber problems with a fixed skewed norm.

Keywords: Optimality conditions, continuous location problems, asymmetric distance, convex analysis.

1. Introduction

In continuous location problems distance is traditionally measured by way of a norm. The main reasons for this are that one obtains convex optimisation problems for which a large body of theory and methods exist, and that this class of distance measures is sufficiently large to model quite adequately many different situations. Indeed, even if the Euclidean norm was used almost exclusively by pioneers in the field, e.g. Weber [14], many other types of norms have been studied and used in the last thirty years. These include the rectilinear norm (see, e.g. Francis and White [4]), more generally l_p-norms, following the study by Love and Morris [8] of the best norms approximating real road distances (see, e.g. Morris [10]), and block norms (e.g. Ward and Wendell [13]). However, by definition, norms are symmetric and hence the derived distance measures also have the property that distances are equal both ways.

In several real situations this symmetry is violated, either directly, such as shortest path distance on a network containing one-way streets, or when an economics view of time or cost is used. Examples of the latter include transportation in rush-hour traffic, flight in the presence of wind, navigation in the presence of currents, and transportation on an inclined terrain. Perhaps the first theoretical development in this respect was the work of Durier and Michelot [3] developing a geometrical description of the solution set of Fermat–Weber problems in which distances are measured by gauges which are an extension of norms that allow for asymmetry. Hodgson et al. [5] explicitly address the Weber location problem on an inclined plane, extending the classical Weiszfeld [15] method. Finally Drezner and Wesolowsky [2] discuss several asymmetric extensions of rectilinear and Euclidean location problems.

In this paper we focus on the minisum single facility (or Fermat–Weber) problem and introduce a fairly large class of asymmetric distances: the skewed norms. We show first how both the Hodgson et al. [5] and Drezner and Wesolowsky [2] proposals belong to this class. Then we derive necessary and sufficient conditions for one of the destination (or source) points to be the optimal site. This is a fairly common phenomenon in the traditional symmetric Fermat–Weber problems, e.g. in case of dominance (see Witzgall [16]), and the conditions under which it arises are well known since the work of Juel and Love [6]. It seems, however, that in asymmetric distance problems it is less understood: the setting of Durier and Michelot [3] is perhaps somewhat too general for this question and it is not explicitly discussed. Hodgson et al. [5] give an erroneous destination optimality condition which has recently been observed and corrected (without formal proof) by Chen [1]. Drezner and Wesolowsky [2] simply ignore the problem, thereby invalidating the convergence proof of their algorithm. The main aim of this paper is to fill in these gaps, making ample use of convex analysis and subgradient calculus, the value of which in continuous location problems seems to be largely unknown to many scholars working in the field. Therefore we start out in the next section with an overview of the relevant notions and results.

2. Convex analytical preliminaries

Following Minkowski [9] (see also Rockafellar [12]), when B is a convex, closed, bounded set in $\mathbb{R}^n$, the interior of which contains the origin, the gauge of B is defined by

$$\gamma(x) = \inf\{\lambda > 0 \mid x/\lambda \in B\} \text{ for all } x \in \mathbb{R}^n.$$

B is then the unit ball of γ, since

$$B = \{x \in \mathbb{R}^n \mid \gamma(x) \leq 1\}.$$

It is well known that any gauge γ is a convex function on $\mathbb{R}^n$ and that any positively homogeneous, definite and subadditive function is a gauge (see Rockafellar [12]). A distance function d_γ may be derived from γ by

$$d_\gamma(x, y) = \gamma(y - x).$$

It satisfies the following properties:

non-negativity: $d_\gamma(x, y) \geq 0$.

definiteness: $d_\gamma(x, y) = 0$ iff $x = y$.

Triangle inequality: $d_\gamma(x, y) + d_\gamma(y, z) \leq d_\gamma(x, z)$.

Thus the gauges are excellent candidates for defining asymmetric distance measure. It was shown by Witzgall [17] that any distance function satisfying the properties above, and such that any function $d(x, \cdot)$ and $d(\cdot, y)$ is convex, is necessarily derived from some gauge. See Durier and Michelot [3] for some examples.
When B is symmetric around the origin, the associated gauge γ also is symmetric, i.e.

$$\gamma(x) = \gamma(-x) \quad \text{for all } x,$$

and thus is a norm, which we will usually denote by N. The derived distance measure is then also symmetric and is thus a classical metric.
For any gauge γ we may define a dual gauge γ^d by

$$\gamma^d(a) = \max\{\langle x, a\rangle \mid \gamma(x) \leq 1\}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product. The unit ball of γ^d is exactly the polar of γ's unit ball (see Rockafellar [12]), denoted by B_γ^0. It is well known that when N is an l_p-norm with $p > 1$, then its dual N^d is the l_q-norm with q defined by $1/p + 1/q = 1$. In particular, the Euclidean norm l_2 (also denoted by $\|\cdot\|$) is self-dual. The block-norms l_1 (rectilinear norm) and l_∞ (Tchebycheff norm) are dual to each other.

The classical Cauchy–Schwarz inequality generalises as follows:

$$\langle x, a\rangle \leq \gamma(x) \cdot \gamma^d(a), \tag{2}$$

where for any fixed x (resp. a) equality is reached for at least one a (resp. x).

If T is a regular linear transformation of $\mathbb{R}^n$, then we may define a new gauge γ_T by $\gamma_T(x) = \gamma(Tx)$. For example, if N is the Euclidean norm, then N_T is an ellipsoidal norm, i.e. a norm for which the unit ball is an ellipsoid centered at the origin.

LEMMA 1

The dual gauge of γ_T is $(\gamma^d)_{(T^{-1})'}$, where $'$ denotes duality of linear transformations, i.e. transposition in terms of matrices.

Proof

By (1) we have:

$$\begin{aligned}\gamma_T^d(a) &= \max\{\langle x, a\rangle \mid \gamma_T(x) \leq 1\}\\ &= \max\{\langle x, a\rangle \mid \gamma(Tx) \leq 1\}\\ &= \max\{\langle T^{-1}z, a\rangle \mid \gamma(z) \leq 1\}\end{aligned}$$

$$= \max\{\langle z, (T^{-1})'a\rangle \mid \gamma(z) \le 1\}$$
$$= \gamma^{\mathrm{d}}((T^{-1})'a)$$
$$= (\gamma^{\mathrm{d}})_{(T^{-1})'}(a). \qquad \square$$

If $f: \mathbb{R}^n \to \mathbb{R}$ is a convex function, we say that $p \in \mathbb{R}^n$ is a *subgradient* of f at $a \in \mathbb{R}^n$ iff

$$\langle x-a, p \rangle + f(a) \le f(x) \quad \text{for any } x \in \mathbb{R}^n.$$

The set of all subgradients of f at a is called the *subdifferential* of f at a, denoted by $\partial f(a)$, and this set is always nonvoid. If f is differentiable at a then the gradient $\nabla f(a)$ of f at a is the only subgradient of f at a, i.e. $\partial f(a) = \{\nabla f(a)\}$, while if f is nondifferentiable then $\partial f(a)$ is a closed convex set.

An important property says that $a \in \mathbb{R}^n$ is a global minimum of the convex function f iff $0 \in \partial f(a)$, thus generalising the classical condition $\nabla f(a) = 0$ in case of differentiability. Furthermore if 0 is an interior point of $\partial f(a)$, then a is the unique global minimum of f.

These notions are standard in convex analysis and we refer the reader to the classical work of Rockafellar [12], in which all necessary results on subdifferential calculus are extensively treated. In particular we will make use of the following properties of the subdifferential operator:

- Linearity: $\partial(f+g)(a) = \partial f(a) + \partial g(a)$ and $\partial(\lambda f)(a) = \lambda \partial f(a)$ for any $\lambda \in \mathbb{R}^+$.
- Chain-rule: If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear function then $\partial(f \circ T)(a) = T'(\partial f(Ta))$ where $\circ$ denotes composition of functions.

For gauges, which are always convex functions, we have a simple description of the subdifferentials:

$$\partial\gamma(0) = \{p \in \mathbb{R}^n \mid \gamma^{\mathrm{d}}(p) \le 1\}$$
$$\partial\gamma(a) = \{p \in \mathbb{R}^n \mid \gamma^{\mathrm{d}}(p) = 1 \text{ and } \langle p, a\rangle = \gamma(a)\}. \tag{3}$$

Please, note that in case of a smooth norm γ, i.e. one which is differentiable everywhere except of course at the origin, the gradient $\nabla\gamma(a)$ at $a \ne 0$ has dual norm $1 : \gamma^{\mathrm{d}}(\nabla\gamma(a)) = 1$.

The first equality in (3) shows that the subdifferential of the gauge γ at the origin is exactly the unit ball of γ's dual γ^{d}, i.e. the polar of γ's unit ball: B_γ^0.

3. Skewed norms and asymmetric distance measures

Let N be a norm on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. We define the *linearly perturbed norm function* $f(N, p)$ by

$$f(N, p)(x) = N(x) - \langle p, x \rangle. \tag{4}$$

This function is evidently positively homogeneous ($f(N, p)(\lambda x) = \lambda f(N, p)$ for $\lambda > 0$) and subadditive ($f(N, p)(x + y) \leq f(N, p)(x) + f(N, p)(y)$). In order to be useful for measuring distances it has to be positive definite, i.e. $f(N, p)(x) > 0$ for all $x \neq 0$.

LEMMA 2

$f(N, p)$ is positive definite iff $N^d(p) < 1$.

Proof

First note that when $p = 0$ the positive definiteness of $f(N, p)$ is guaranteed by that of N. Thus assume further that $p \neq 0$. Suppose first $N^d(p) < 1$. By (2) and the symmetry of N we have, for all $x \neq 0$, $\langle p, x \rangle \leq N^d(p) \cdot N(x) < N(x)$ showing that $f(N, p)(x) > 0$. Conversely, if $f(N, p)$ is positive definite, then for any $x \neq 0$ we must have $\langle p, x \rangle < N(x)$. Choosing y so that $\langle p, y \rangle = N^d(p) \cdot N(y)$, this leads to $N^d(p) < 1$, since $p \neq 0$ guarantees that $y \neq 0$. □

In view of lemma 2 we only consider linearly perturbed norms $f(N, p)$ with $N^d(p) < 1$, and call these *skewed norms*. In view of their properties these functions are gauges. The following theorem gives several characterizations of skewed norms among the much larger class of gauges:

THEOREM 3

For a gauge γ the following properties are equivalent:

(1) γ is a skewed norm.

(2) The function $1/2(\gamma(-x) - \gamma(x))$ is linear.

(3) The unit ball of γ's dual γ^d admits a center of symmetry.

(4) The polar B_γ^0 of γ's unit ball admits a center of symmetry.

Proof

If $\gamma(x) = N(x) - \langle p, x \rangle$ for some N and p, then $1/2(\gamma(-x) - \gamma(x)) = \langle p, x \rangle$ is linear. Conversely for any γ we have $\gamma(x) = 1/2(\gamma(x) + \gamma(-x)) - 1/2(\gamma(-x) - \gamma(x))$. Clearly $N(x) = 1/2(\gamma(x) + \gamma(-x))$ is positively homogeneous, definite, subadditive and symmetric, hence a norm, while the second term $1/2(\gamma(-x) - \gamma(x))$, being linear, equals $\langle p, x \rangle$ for some p. Hence, by definiteness, γ is indeed a skewed norm.

Suppose now again $\gamma(x) = N(x) - \langle p, x \rangle$, then the subdifferential $\partial\gamma(0)$ of γ at 0 is exactly the unit ball B_γ^0 of γ^d. However, $\partial\gamma(x) = \partial N(x) - p$. Hence, $B_\gamma^0 = B_N^0 - p$, i.e. N^d's unit ball translated over $-p$. Since N^d is a norm, it is symmetric and its unit ball is symmetric around the origin. Hence the point $-p$ is a center of symmetry of B_γ^0. Conversely, let q be a center of symmetry B_γ^0. Then $B_\gamma^0 + q$ is symmetric around the origin, hence its gauge is actually a norm N, with unit ball $B_N = B_\gamma^0 + q$.

Now for any x we have

$$\begin{aligned}\gamma(x) &= \max\{\langle s, x\rangle \mid s \in B_\gamma^0\}\\ &= \max\{\langle s, x\rangle \mid s + q \in B_\gamma^0 + q\}\\ &= \max\{\langle t - q, x\rangle \mid t \in B_N\}\\ &= \max\{\langle t, x\rangle \mid t \in B_N\} - \langle q, x\rangle\\ &= N^{\mathrm{d}}(x) - \langle q, x\rangle,\end{aligned}$$

showing that γ is a skewed norm. □

COROLLARY 4

γ is a gauge with ellipsoidal unit ball iff γ is a skewed norm of the form $f(N_T, p)$ with N the Euclidean norm.

The proof is easy using theorem 1 and the fact that the polar of an ellipsoid also is an ellipsoid.

We now define a "distance" measure in the classical way by

$$d(x, y) = f(N, p)(y - x).$$

One easily sees that this measure satisfies all properties of a metric, except the symmetry (if $p \neq 0$) due to the linear perturbation, and therefore is an adequate candidate for an asymmetric distance measure.

EXAMPLE 1

Hodgson et al. [5] derived the following expression for the work expended when moving a block of mass m from a point $a = (a_1, a_2)$ to point $x = (x_1, x_2)$ along a plane inclined at angle Θ in the direction of the second coordinate axis:

$$W(a, x) = mg\{\mu[(x_1 - a_1)^2 \cos^2\Theta + (x_2 - a_2)^2]^{1/2} + (x_2 - a_2)\tan\Theta\},$$

where g is the gravitational acceleration constant and μ the coefficient of sliding friction. One easily sees that $W(a, x) = mf(N_T, p)(x - a)$ where $N = l_2$ is the Euclidean norm, T is defined by the regular transformation matrix

$$T = \begin{pmatrix} g\mu\cos\Theta & 0 \\ 0 & g\mu \end{pmatrix} \text{ and } p = \begin{pmatrix} 0 \\ -g\tan\Theta \end{pmatrix}.$$

Since

$$(T^{-1})' = \begin{pmatrix} \frac{1}{g\mu \cos \Theta} & 0 \\ 0 & \frac{1}{g\mu} \end{pmatrix},$$

the condition of lemma 2 (and using lemma 1) reduces to $(\tan\Theta)/\mu < 1$, or $\tan\Theta < \mu$, pointed out by Hodgson et al. [5, p. 223] in order to avoid situations in which gravity would cause blocks to slip, in which case the model would not apply.

EXAMPLE 2

As a second application, consider the time necessary to fly from one point to another in the presence of a steady wind. Let us choose an orthogonal coordinate system with second axis in the wind's direction. Let $v \geq 0$ denote the drift velocity due to the wind and $s > 0$ the airplane's speed relative to the surrounding air. Suppose one starts off from the origin in some direction $y = (y_1, y_2)$ and always keeps the same flight direction. Then by the time t in which y would have been reached without any wind, i.e. $st = \| y \|$, one will in fact have reached some point $x = (x_1, x_2) = (y_1, y_2 + tv)$. It is quite clear that this point x cannot be reached in a quicker way. Hence, the shortest possible time $D(x)$ to reach x from the origin is given by

$$D(x) = \frac{1}{s} \| y \|,$$

where y is chosen such that $y_1 = x_1$ and $x_2 - y_2 = (v/s)\| y \|$. By squaring this last equality we find that y_2 should be the root of the quadratic equation $(s^2 - v^2)y_2^2 - 2s^2x_2y_2 + s^2x_2^2 - v^2x_1^2 = 0$ that is not greater than x_2. The discriminant being $(s^2 - v^2)v^2x_1^2 + v^2s^2x_2^2$, this root always exists as soon as $s > v$. Indeed, it is only when the airplane's speed exceeds the wind velocity that all points of the plane may be reached; when $s \leq v$ only downwind points will be reachable! We then immediately find

$$\begin{aligned} D(x) &= \frac{1}{s} \| y \| = \frac{1}{v}(x_2 - y_2) \\ &= \frac{1}{s^2 - v^2}((s^2 - v^2)x_1^2 + s^2x_2^2)^{1/2} - \frac{v}{s^2 - v^2} x_2 . \end{aligned}$$

After translation of the origin to some point a, the time necessary to reach some destination point x, starting from a, is

$$D(x - a) = f(N_T, p)(x - a),$$

where $N = l_2$ is the Euclidean norm, T is defined by the regular matrix

$$T = \begin{pmatrix} r & 0 \\ 0 & \frac{s}{r^2} \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0 \\ \frac{v}{r^2} \end{pmatrix},$$

in which $r = (s^2 - \upsilon^2)^{1/2}$. This is indeed positive definite by lemma 2 whenever $\upsilon/s < 1$ or $\upsilon > s$ as already mentioned before.

EXAMPLE 3

Drezner and Wesolowsky [2] define the rectilinear asymmetric distance from point a to point x in the plane by

$$d(a, x) = d_1(a, x) + d_2(a, x),$$

where

$$d_1(a, x) = \begin{cases} E\,|x_1 - a_1| & \text{if} \quad x_1 \geq a_1, \\ W\,|x_1 - a_1| & \text{if} \quad x_1 < a_1; \end{cases}$$

and

$$d_2(a, x) = \begin{cases} N\,|x_2 - a_2| & \text{if} \quad x_2 \geq a_2, \\ S\,|x_2 - a_2| & \text{if} \quad x_2 < a_2. \end{cases}$$

It is easy to check that

$$d_1(0, x) = |\frac{1}{2}(W + E)x_1| - \frac{1}{2}(W - E)x_1,$$

and similarly $d_2(0, x) = |\frac{1}{2}(S + N)x_2| - \frac{1}{2}(S - N)x_2$. It follows that the rectilinear asymmetric distance $d(a, x)$ is derived from the linearly perturbed norm $f(N_T, p)$ where $N = l_2$,

$$T = \begin{pmatrix} \frac{1}{2}(W + E) & 0 \\ 0 & \frac{1}{2}(S + N) \end{pmatrix} \quad \text{and} \quad p = \frac{1}{2}\begin{pmatrix} W - E \\ S - N \end{pmatrix}.$$

According to lemma 2 this is a skewed norm iff

$$\max\left\{ \left| \frac{W - E}{W + E} \right|, \left| \frac{S - N}{S + N} \right| \right\} < 1,$$

which is equivalent with $W, E, S, N > 0$, since W, E, S and N are assumed to be nonnegative.

EXAMPLE 4

In the same paper these authors also define an asymmetric Euclidean distance by

$$F(a, x) = \left[1 + \frac{1}{2}(m - 1)(1 - \cos(\Theta - \alpha)) \right] r.$$

(Beware of the typing error in formula (10) in Drezner and Wesolowsky [2, p. 204].) Here (r, Θ) are polar coordinates of $x - a$ and m, α are parameters determining the shape of the distance measure: the unit ball is an ellipse with main axis the segment joining the points with polar coordinates $(1, \alpha)$ and $(1/m, \pi + \alpha)$.

By the fact that $r = \|x - a\|$, $r\cos\Theta = x_1 - a_1$ and $r\sin\Theta = x_2 - a_2$, one easily derives

$$F(a, x) = \frac{1}{2}(m+1)\|x - a\| - \frac{1}{2}(m-1)[(x_1 - a_1)\cos\alpha + (x_2 - a_2)\sin\alpha],$$

showing that this asymmetric Euclidean distance measure is derived from a skewed norm $f(N, p)$ where $N = \frac{1}{2}(m+1)l_2$ is a rescaled Euclidean norm and $p = \frac{1}{2}(m-1)$ $(\cos\alpha, \sin\alpha)$.
According to lemma 2 this is positive definite whenever $(2/(m+1))\|p\| = |m-1|/(m+1) < 1$, holding whenever $m > 0$. In fact, it is sufficient to consider only $m \geq 1$, as implicitly assumed by Drezner and Wesolowsky [2], since the pairs (α, m) and $(\pi + \alpha, 1/m)$ define the same skewed norm, up to a constant scaling factor.

Note that examples 1, 2 and 4 all correspond to gauges with ellipsoidal unit balls, thus, by corollary 4, they are skewed norms derived from the Euclidean norm.

4. Asymmetric Fermat–Weber problems

Let $A \subset \mathbb{R}^n$ be a finite set of destination points. For each $a \in A$ distance to a is calculated by way of a skewed norm $f(N_a, p_a)$ by

$$\begin{aligned} d_a(x) &= f(N_a, p_a)(x - a) \\ &= N_a(x - a) - \langle p_a, x - a \rangle. \end{aligned} \tag{5}$$

A central facility interacting with each of these destinations is to be located so as to minimize the total cost of all interactions. Interaction cost with destination $a \in A$ is supposed to be a linear function of the distance of the site x up to point a. Hence we obtain the following optimization problem

$$\min_{x \in \mathbb{R}^n} FW(x) = \sum_{a \in A} w_a d_a(x), \tag{6}$$

where the $w_a > 0$ are given interaction costs per distance unit. In case all distance measures are symmetric ($p_a = 0$ for all $a \in A$), we obtain the classical Fermat–Weber problem with mixed norms.

Since $FW(x)$ is a convex function as a positive linear combination of convex functions, the optimality condition at point x is $0 \in \partial FW(x)$. By the linearity of the subdifferential operator we have $\partial FW(x) = \sum_{a \in A} w_a \partial d_a(x)$, and by (5)

$$\partial d_a(x) = \partial N_a(x - a) - p_a.$$

We thus obtain the following property.

LEMMA 5

x is an optimal solution to the Fermat–Weber problem (6) iff there exists for each $a \in A$ a subgradient $g_a \in \partial N_a(x-a)$ such that

$$\sum_{a \in A} w_a q_a = \sum_{a \in A} w_a p_a . \qquad \square$$

COROLLARY 6

If for all $a \in A$ the norm N_a is differentiable at the point $x-a$, then x is an optimal solution iff

$$\sum_{a \in A} w_a \nabla N_a(x-a) = \sum_{a \in A} w_a p_a . \qquad \square$$

This optimality condition, however, never applies when x coincides with one of the destinations, $b \in A$ say. In the sequel we will suppose all norms N_a to be *smooth*, i.e. differentiable everywhere except at the origin, in order to be able to derive an easy optimality condition in case of such a coincidence.

THEOREM 7

The site $x = b \in A$ is an optimal solution to the Fermat–Weber problem iff

$$N_b^{\mathrm{d}}(G_b) \le w_b,$$

where $G_b = \sum_{a \in A \setminus \{b\}} w_a \nabla N_a(b-a) - \sum_{a \in A} w_a p_a .$

If the inequality is strict, then b is the only optimal solution.

Proof

Applying lemma 5 at point $x = b$ with the remark that for each $a \neq b$ N_a is differentiable at point $b-a$, we obtain the optimality condition:

$$w_b q = -G_b \text{ for some } q \in \partial N_b(0),$$

or, by (3), for some q with $N^{\mathrm{d}}(q) \le 1$. $\square$

Note that when all $p_a = 0$ we recover the fixed-point optimality conditions of Juel and Love [6] – see also Plastria [11].

EXAMPLE 5

The one-centroid problem on an inclined plane was defined by Hodgson et al. [5] to determine the location of an unknown centroid on the inclined plane minimising the total work necessary for sliding blocks of different masses to this centroid. Denoting by $a \in A$ the initial locations of the different blocks, with weights m_a, and using further the notations of example 1, we obtain an asymmetric Fermat–Weber problem (6) with weights $w_a = m_a$, and distance measures $d_a(x) = f(N_a, p_a)$, where $N_a = N_T$, $N = l_2$, $T = g\mu\begin{pmatrix}\cos\Theta & 0\\ 0 & 1\end{pmatrix}$ and $p = (0, -g\tan\Theta)$ for each $a \in A$.

The norm N_T is differentiable (except for 0) with gradient

$$\begin{aligned}\nabla N_T(y) &= \nabla(N \circ T)(y) = T'\nabla N(Ty)\\ &= T'\frac{Ty}{N(Ty)}\\ &= \frac{g\mu}{(y_1^2\cos^2\Theta + y_2^2)^{1/2}}\begin{pmatrix} y_1\cos^2\Theta\\ y_2\end{pmatrix}.\end{aligned}$$

Therefore,

$$G_b = \begin{pmatrix} G_{b1}\\ G_{b2}\end{pmatrix} = \begin{pmatrix} \mu g \sum_{a\in A\setminus\{b\}} \frac{m_a(b_1 - a_1)\cos^2\Theta}{(b_1^2\cos^2\Theta + b_2^2)^{1/2}}\\ \mu g \sum_{a\in A\setminus\{b\}} \frac{m_a(b_2 - a_2)}{(b_1^2\cos^2\Theta + b_2^2)^{1/2}} - g\tan\Theta\sum_{a\in A} m_a\end{pmatrix},$$

which corresponds (after adapting notations) with the expression for G_k in Hodgson et al. [5, p. 224]. According to theorem 7 the optimality condition at $x = b$ is

$$N_b^{\mathrm{d}}(G_b) \le m_b,$$

or, by lemma 1,

$$N^{\mathrm{d}}((T^{-1})'G_b) \le m_b.$$

Since $N^{\mathrm{d}} = N = l_2$ and

$$(T^{-1})' = \frac{1}{\mu g}\begin{pmatrix} \frac{1}{\cos\Theta} & 0\\ 0 & 1\end{pmatrix},$$

this condition becomes:

$$\|(G_{b1}/\cos\Theta, G_{b2})\| \le \mu g m_b,$$

which differs from the condition $\|G_b\| \le \mu g m_b$ erroneously stated by Hodgson et al. [5]. This error was already noted by Chen [1].

The interested reader will easily derive the optimality conditions in a similar way for a Fermat–Weber problem involving flights in the presence of wind, using the development in example 2.

EXAMPLE 6

Consider the asymmetric Fermat–Weber problem as proposed by Drezner and Wesolowsky [2]: the objective to be minimised is

$$FW(x) = \sum_{a \in A} w_a d_a(x),$$

where $d_a(x) = f(N_a, p_a)(x - a)$ with

$$N_a = \tfrac{1}{2}(m_a + 1)l_2 \quad \text{and} \quad p_a = \tfrac{1}{2}(m_a - 1)(\cos\alpha_a, \sin\alpha_a)$$

(see example 4). We easily find from theorem 7 that $b \in A$ is the optimal solution iff $\|G_b\| \le 2w_b/(m_b + 1)$, where

$$G_b = \sum_{a \in A \setminus \{b\}} \frac{w_a(m_a + 1)}{2\|b - a\|}(b - a) - \sum_{a \in A} w_a p_a .$$

The fact that some destination may be an optimal solution is totally ignored by these authors: the optimality conditions ([2] formulae (12), p. 205), which correspond to those of our corollary 6, indeed only make sense at points x different from any destination. Their proof of convergence of their adapted Weiszfeld algorithm is then only valid in such cases. This algorithm may undoubtedly be adapted in a way similar to Kuhn and Kuenne [7], as was done by Hodgson et al. [5] for the inclined plane case, and its convergence proven. This, however, falls outside the scope of this paper.

5. A dominance theorem for asymmetric distance

A well known theorem of Witzgall [16] states that if, in a Fermat–Weber problem where all distances are measured by the same metric, some destination is dominant, i.e. outweighs all other destinations taken together, then this destination is the optimal site for the central facility.

The easy proof of this theorem explicitly relies on the symmetry of the distance measure, and fails without this property. It is easy to construct one-dimensional examples in which the theorem does not hold.

In this section we derive a generalisation of this dominance theorem for Fermat–Weber problems with a skewed norm, which reduces to Witzgall's result in the case of symmetry. Note that Goldman, in his preface to Witzgall's seminal report [16], recommends research into this kind of generalisation.

Let d be an asymmetric distance measure on $\mathbb{R}^n$. We define its *skewness* $s(d)$ by

$$s(d) = \sup\left\{ \frac{d(x, y)}{d(y, x)} \middle| x \neq y \in \mathbb{R}^n \right\}.$$

This skewness may be infinite for general distance measures, but when d is derived from a gauge it is always finite and ≥ 1. Note that $s(d) = 1$ expresses d is symmetric.

LEMMA 8

If d is derived from the skewed norm $f(N, p)$, then

$$s(d) = \frac{1 + N^{\mathrm{d}}(p)}{1 - N^{\mathrm{d}}(p)}.$$

Proof

For any $x \neq y \in \mathbb{R}^n$ we have:

$$\frac{d(x,y)}{d(y,x)} = \frac{N(y-x) - \langle p, y-x\rangle}{N(y-x) + \langle p, y-x\rangle}$$

$$= 1 - \frac{2}{\dfrac{N(y-x)}{\langle p, y-x\rangle} + 1}.$$

This expression is maximised when $\langle p, y-x\rangle/(N(y-x))$ is minimal and negative. By the Cauchy–Schwartz inequality (2) this latter minimum is equal to $-N^{\mathrm{d}}(p)$. Hence,

$$s(d) = 1 - \frac{2}{\dfrac{-1}{N^{\mathrm{d}}(p)} + 1} = \frac{1 + N^{\mathrm{d}}(p)}{1 - N^{\mathrm{d}}(p)}. \qquad \square$$

Consider now a Fermat–Weber problem

$$\min_{x \in \mathbb{R}^n} FW(x) = \sum_{a \in A} w_a d(a, x), \tag{7}$$

in which all distances are calculated by way of a fixed smooth skewed norm $f(N, p)$.

THEOREM 9

If for $b \in B$ we have

$$s(d) \sum_{a \in A\setminus\{b\}} w_a \leq w_b,$$

then $x = b$ is an optimal solution to (7). If the inequality is strict, then it is the unique optimal solution.

Proof

Let us check the destination optimality criterion of theorem 7.

$$N^{\mathrm{d}}(G_b) \leq \sum_{a \in A\setminus\{b\}} w_a N^{\mathrm{d}}(\nabla N(b-a)) + \sum_{a \in A} w_a N^{\mathrm{d}}(p)$$

$$= (1 + N^{d}(p)) \sum_{a \in A \setminus \{b\}} w_a + w_b N^{d}(p)$$

since $N^{d}(\nabla N(x)) = 1$ for any x

$$\leq (1 + N^{d}(p)) w_b / s(d) + w_b N^{d}(p)$$

$$= w_b.$$

When the inequality among weights is strict, the last inequality above is also strict, showing that $N^{d}(G_b) < w_b$, whence that $x = b$ is the unique optimal solution. □

Note that in the case of a symmetric distance this reduces to Witzgall's theorem since $s(d) = 1$.

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