Annals of Operations Research 40 (1992)77–99

LOCATION AND ALLOCATION FOR DISTRIBUTION SYSTEMS WITH TRANSSHIPMENTS AND TRANSPORTION ECONOMIES OF SCALE

James F. CAMPBELL
School of Business Administration, University of Missouri–St Louis, 8001 Natural Bridge Road, St. Louis, MO 63121, USA

Abstract

Locating transshipment facilities and allocating origins and destinations to transshipment facilities are important decisions for many distribution and logistic systems. Models that treat demand as a continuous density over the service region often assume certain facility locations or a certain allocation of demand. It may be assumed that facility locations lie on a rectangular grid or that demand is allocated to the nearest facility or allocated such that each facility serves an equal amount of demand. These assumptions result in suboptimal distribution systems. This paper compares the transportation cost for suboptimal location and allocation schemes to the optimal cost to determine if suboptimal location and allocation schemes can produce nearly optimal transportation costs. Analytical results for distribution to a continuous demand show that nearly optimal costs can be achieved with suboptimal locations. An example of distribution to discrete demand points indicates the difficulties in applying these results to discrete demand problems.

1. Introduction

Transportation economies of scale provide an incentive to consolidate shipments into large vehicle loads. However, large vehicles may be prevented from visiting origins and destinations, due to their size or the lack of infrastructure (e.g. railroads, aircraft). Transshipment facilities provide the capability to transfer shipments between different vehicles or modes of transportation. This paper addresses the inter-dependent problems of locating given numbers of transshipment facilities and allocating demand to these facilities to minimize transportation cost. The cost when suboptimal location and allocation schemes are utilized is compared to the optimal cost to determine if simpler suboptimal location and allocation schemes may result in nearly optimal transportation costs.

Two types of distribution systems are analyzed. One-to-many distribution systems contain a single origin, many destinations and a number of transshipment facilities, which are referred to as terminals in this paper. Shipments travel from the origin to a terminal on one vehicle and then from the terminal to the destination on a second vehicle, or they travel from the origin directly to a destination on a single vehicle without transshipment. One-to-many distribution systems concentrate flows on the origin-to-terminal links. Many-to-many distribution systems contain many origins, many destinations and a number of transshipment facilities, which are referred to as hubs in this paper. Shipments travel from an origin to a destination via either one or two hubs. Many-to-many distribution systems concentrate flows most heavily on the hub-to-hub links.

This paper concentrates primarily on continuous approximation models in which demand is modeled as a continuous density over the service region. In this case, average transportation distances and costs can be expressed as a function of the density of demand (see Eilon et al. [11] and Daganzo [7]).

Continuous approximation models have been used by several authors to analyze one-to-many distribution with one (Geoffrion [12,13], Han [18], Daganzo [9], and Campbell [1,3,4]) or more (Daganzo and Newell [10]) transshipments. A two-step procedure is used in these models to locate terminals and allocate demand. The first step allocates demand by partitioning the service region into non-overlapping subregions. (Throughout this paper, the word subregion refers to the area allocated to a transshipment facility.) The second step locates one transshipment terminal in each subregion. Although terminals are often assumed to be located in the center of each subregion [4,9,10,18], Geoffrion [12,13] showed that the optimal location for each facility is displaced from the center of the subregion toward the origin. However, Geoffrion did not use subregions based on the optimal allocation of demand. Campbell [3] derived the optimal location and allocation schemes for one-to-many distribution to a uniform demand density using the rectilinear metric. Unlike the basic location–allocation problem (Cooper [6]), all destinations should not generally be allocated to the nearest facility.

The many-to-one collection problem in which shipments or people move from many origins to a single destination via transshipment facilities can be handled in a manner similar to that for one-to-many distribution by reversing the direction of flows. Applications of many-to-one collection include passenger transportation (Kanafani [19], and Mirchandani and Odoni [22]) and waste disposal (Marks and Liebman [21], and Wirasinghe and Waters [24]).

Continuous approximation models have also been used to analyze many-to-many distribution problems with uniform or slowly varying densities of demand. Hall [15,16], and Hall and Daganzo [17] focused on how to route shipments via transshipment facilities, i.e. how to allocate origins and destinations to hubs. Daganzo [8] developed a more complete model that includes transportation and inventory costs. All these works employ the rectilinear metric and rely on a two-step procedure for location and allocation. The first step locates facilities by assuming that hubs are arranged in a rectangular grid. The second step allocates demand (i.e. defines subregions) by subdividing the area between the hubs. Campbell [2] derives the optimal location and allocation scheme for many-to-many distribution to a uniform density of demand using the rectilinear metric.

Most previous research using continuous approximation models for one-to-many and many-to-many distribution assumes either the facility locations (e.g. a grid of terminals) or the allocation of demand (e.g. a partition of the service region into non-overlapping subregions). The assumed locations or allocation are generally not optimal. The present paper compares the cost for suboptimal location–allocation schemes to the optimal cost to investigate the effectiveness of some of the assumptions commonly used in continuous models.

Two types of transportation are included in this paper to model transportation economies of scale. For one-to-many distribution, linehaul transportation refers to transportation between the origin and a terminal, and local transportation refers to transportation between a terminal and destination. For many-to-many distribution, linehaul transportation refers to transportation between two hubs and local transportation refers to transportation between an origin or destination and a hub. The transportation rate for linehaul transportation is generally less than the rate for local transportation because of economies of scale in transportation.

Like the basic location–allocation problem, local vehicles are restricted to stop at a single origin or destination. If local vehicles stop at multiple origins or destinations, then the distribution problem is often termed a location-routing problem. Laporte [20] provided a recent summary of research on location-routing problems. Continuous approximation models for location-routing problems have been developed for one-to-many distribution (Campbell [1,4]) and many-to-many distribution (Daganzo [8]) with transshipments.

The remainder of the paper is organized in four sections. Section 2 presents the optimal locations, allocation and transportation cost for one-to-many and many-to-many distribution. Section 3 defines suboptimal location and allocation schemes and compares the cost for combinations of these suboptimal schemes to the optimal cost. Section 4 considers problems with a discrete demand to illustrate the degree to which the results for distribution to a continuous demand apply to distribution to a discrete demand. Section 5 contains conclusions.

2. Optimal locations, allocation and cost

This section presents the optimal locations, allocation and transportation cost for one-to-many and many-to-many distribution to a uniform density of demand. Consider a square service region of side X in which travel follows the rectilinear metric and the directions of travel are parallel to the sides of the region. The two perpendicular components of travel can be treated independently, so the average cost can be viewed as twice the average cost to serve a one-dimensional region of length X. This section presents results for distribution to a one-dimensional region. Results for distribution in two dimensions with the rectilinear metric can be easily derived, as in Campbell [2,3].

Let K be the number of transshipment facilities and let β be the local transportation cost per shipment per unit distance. Let α be the ratio of the linehaul to local transportation cost per shipment per unit distance, where $0 \leq \alpha \leq 1$, so that $\alpha\beta$ is the linehaul transportation cost per shipment per unit distance. When $\alpha = 1$, local and linehaul transportation costs are equivalent. As α decreases, linehaul cost decreases and in the limiting case of $\alpha = 0$, linehaul transportation incurs no cost.

2.1. ONE-TO-MANY DISTRIBUTION

Suppose the origin is in the center of a one-dimensional service region of length X and the origin is also a terminal. Destinations are distributed over the service region according to a uniform distribution. The optimal transportation cost, terminal locations and allocation scheme were determined in Campbell [3] and the results are presented here.

The optimal transportation cost C_0 is

$$C_0 = \frac{\beta X}{4} \frac{\alpha K + 1}{\alpha + K}. \tag{1}$$

Let x_i denote the location of terminal i measured from the left end of the service region. The optimal terminal locations are

$$x_i = \frac{X}{2} \frac{2i - 1 + \alpha}{K + \alpha}, \quad i = 1, 2, \ldots, K. \tag{2}$$

2.2. MANY-TO-MANY DISTRIBUTION

Consider again a one-dimensional service region of length X containing origins and destinations distributed according to a uniform distribution. The optimal transportation cost, hub locations and allocation scheme were determined in Campbell [2] and a summary of the results follows.

The optimal transportation cost C_M is

$$C_M = \frac{\alpha \beta X}{6} \left[2 + A^3 + 3 \frac{1-\alpha}{\alpha} A^2 + B_1 (1-A)^3 + 3 \frac{B_2}{\alpha} (1-A)^2 \right], \tag{3}$$

where

$$A = 1 + \frac{1 + B_2 + \gamma \left[(1 + B_2)^2 - \alpha(2 - \alpha)(1 - B_1) \right]^{1/2}}{\alpha (1 - B_1)}, \tag{4}$$

$$\gamma = -1, \quad \text{if } K = 2$$

$$\gamma = +1, \quad \text{if } K > 2,$$

$$B_1 = \frac{1 + \alpha^2}{(K-1)^2},$$

and

$$B_2 = \frac{1 - \alpha^2}{K - 1}.$$

The optimal location of hub i is given by

$$x_i = \frac{X}{2} A + X(1 - A) \frac{i-1}{K-1}, \quad i = 1, 2, \ldots, K. \tag{5}$$

For both one-to-many and many-to-many distribution, the routing of shipments defines the allocation. Consider an interval between adjacent transshipment facilities (i.e. terminals or hubs) of width S. A general rule that defines the optimal allocation and routing for both one-to-many and many-to-many distribution is as follows:

(i) All demand within a distance

$$(1 - \alpha)S/2$$

of a transshipment facility should be allocated to (i.e. routed via) the nearest facility.

(ii) All demand not within a distance

$$(1 - \alpha)S/2$$

of a transshipment facility should be routed to minimize the travel *distance.* This may result in "allocation" to more than one facility.

Part (ii) of the above rule produces a unique allocation of all demand for one-to-many distribution, since there is only one origin. However, all demand is not allocated to the nearest terminal. For many-to-many distribution systems, part (ii) allows demand points farther than $(1 - \alpha)S/2$ from a hub to send and receive shipments via two different hubs. This is a prime difference between the allocation pattern in one-to-many and many-to-many distribution: unique allocation for many-to-many distribution is optimal only if $\alpha = 0$.

When $\alpha = 0$, there is no cost for inter-hub transportation, so each shipment is allocated to the nearest hub to minimize local transportation cost. Thus, when $\alpha = 0$, and only when $\alpha = 0$, the optimal location–allocation scheme is to divide the service region into K subregions and locate a facility in the center of each subregion. When $\alpha = 1$, transshipments are not needed and minimizing the total transportation distance minimizes the total transportation cost.

3. Comparison of location–allocation combinations

The previous section presented the optimal transportation cost, locations and allocation for one-to-many and many-to-many distribution to a uniform density of demand. This section investigates several simpler non-optimal, but commonly used schemes for allocation and location. Three allocation schemes (the optimal scheme and two simpler schemes) and two location schemes (the optimal scheme and one simpler scheme) are combined to form composite suboptimal location–allocation strategies. The cost for these suboptimal strategies is compared to the optimal cost to determine whether simpler location and allocation schemes can produce near-optimal costs.

The three allocation schemes considered are:

OA – Optimal Allocation.
Allocate demand to minimize the transportation cost for a given location scheme.

NA – Nearest Allocation.
Allocate all demand to the nearest facility.

EA – Equal Allocation.
Divide the service region into equal size non-overlapping subregions, one for each facility.

The NA and EA allocation schemes require that each point in the service region be allocated to only one facility, unlike the optimal OA allocation scheme. Note that the allocation for the EA allocation scheme is independent of the facility locations.

The two location schemes considered are:

OL – Optimal Location.
Locate facilities to minimize the transportation cost for a given allocation scheme.

CL – Central Location.
Locate K facilities in the center of regions of size X/K. However, these regions of size X/K do not define the allocation.

The central facility locations for the CL location scheme are independent of the allocation scheme and are given by:

$$x_i = \frac{X}{2K}(2i-1), \quad i = 1, 2, \ldots, K. \tag{6}$$

The two location schemes (OL and CL) and three allocation schemes (OA, NA and EA) can be combined in six ways. The combination OL–OA, which produces the optimal transportation cost, was presented in the previous section. The other combinations introduce suboptimality in the location and/or allocation decisions. Two suboptimal combinations using OL are possible: OL–NA and OL–EA. Three suboptimal combinations using CL are possible: CL–OA, CL–NA and CL–EA. However, CL–EA is identical to CL–NA, since central location and allocation to the nearest facility imply that each subregion is the same size.

This section compares the transportation cost for four suboptimal combinations OL–NA, OL–EA, CL–OA, and CL–NA to the cost for the optimal combination OL–OA. One-to-many distribution is considered first, followed by many-to-many distribution.

3.1. ONE-TO-MANY DISTRIBUTION

The transportation cost and terminal locations for the four suboptimal location–allocation combinations are considered in turn. The transportation cost and terminal locations are indicated using the notation $C_0(y, z)$ and $x_i(y, z)$, respectively, where y indicates the location scheme (O for OL or C for CL) and z indicates the allocation scheme (O for OA, N for NA or E for EA). Let the number of terminals be odd, so that there is an even number of terminals to the left and right of the origin, since the origin is itself one of the terminals. Because the demand is uniformly distributed, the optimal terminal locations are symmetric about the origin.

OL–NA: The results for OL–NA were derived in Campbell [3] and are repeated here.

$$C_0(\mathrm{O}, \mathrm{N}) = \frac{\beta X}{4} \frac{(\alpha K + 1)(1 - \alpha) + \alpha^2}{(K - 1)(1 - \alpha) + 1}, \tag{7}$$

$$x_i(\mathrm{O}, \mathrm{N}) = \frac{X}{2} \frac{(2i - 1)(1 - \alpha) + \alpha}{(K - 1)(1 - \alpha) + 1}. \tag{8}$$

OL–EA: For the combination OL–EA, one terminal is located in each subregion of size X/K. Consider the ith terminal at location x_i to the left of the origin. The average transportation cost in subregion i is T_i, where

$$T_i = \alpha\beta(X/2 - x_i) + \frac{\beta K}{2X}\left[\frac{iX}{K} - x_i\right]^2 + \frac{\beta K}{2X}\left[x_i - \frac{(i-1)X}{K}\right]^2. \tag{9}$$

The first term accounts for linehaul transportation to the terminal. The second and third terms are for local transportation within the subregion to the right and left of terminal i, respectively. The optimal terminal locations minimize eq. (9) and are given by

$$x_i(\mathrm{O}, \mathrm{E}) = \frac{X}{2K}(2i - 1 + \alpha), \quad i = 1, 2, \ldots, (K-1)/2. \tag{10}$$

Terminal locations for $i > K/2$ are given by $X - x_{K-i+1}(\mathrm{O}, \mathrm{E})$ because of symmetry. The average transportation cost over the service region is

$$C_0(\mathrm{O}, \mathrm{E}) = \frac{2}{K} \sum_{i=1}^{(K-1)/2} T_i + \frac{\beta X}{4K^2}. \tag{11}$$

The first term in eq. (11) accounts for the transportation cost in the $K-1$ subregions to the left and right of the central subregion, which contains the origin. The last term in eq. (11) accounts for the transportation cost in the central subregion containing the origin. From eqs. (9)–(11),

$$C_0(\mathrm{O},\mathrm{E}) = \frac{\beta X}{4}\left[\frac{\alpha(\alpha-1)}{K^2} + \frac{(1-\alpha^2)}{K} + \alpha\right]. \tag{12}$$

The OL–EA combination corresponds to that employed by Geoffrion [12,13] and Campbell [1], in which terminals are displaced from the center of the subregion toward the origin to reduce transportation costs.

CL–OA: The terminal locations for CL–OA are given by eq. (6) and the transportation cost (from Campbell [3]) is

$$C_0(\mathrm{C},\mathrm{O}) = \frac{\beta X}{4}\left[\frac{\alpha(\alpha-1)}{K^2} + \frac{(1-\alpha^2)}{K} + \alpha\right]. \tag{13}$$

Note that the cost for CL–OA is identical to the cost for OL–EA although the terminal locations and the allocation are different.

CL–NA: The terminal locations for CL–NA are given by eq. (6) and the transportation cost (from Campbell [3]) is

$$C_0(\mathrm{C},\mathrm{N}) = \frac{\beta X}{4}\left[\frac{-\alpha}{K^2} + \frac{1}{K} + \alpha\right]. \tag{14}$$

Figure 1 is a comparison of the transportation cost for each suboptimal combination to the optimal transportation cost for $\alpha = 0.5$. Each curve is the percentage difference between the transportation cost for the optimal combination and the transportation cost for OL–OA as a function of the number of terminals. Combinations OL–EA and CL–OA are nearly optimal. The combinations that allocate to the nearest terminal are not nearly optimal, even for large K. Both combinations with nearest allocation converge to a common value that is larger than that for the other two suboptimal combinations. Table 1 displays the maximum percentage difference between the transportation cost for suboptimal combinations and the transportation cost for OL–OA, and the values of α and K that produce the maximum difference. All the maximum differences occur for small numbers of terminals, but even with large numbers of terminals, OL–EA and CL–OA are clearly superior to combinations that allocate to the nearest terminal.

3.2. MANY-TO-MANY DISTRIBUTION

The transportation costs and hub locations for the suboptimal combinations OL–NA, OL–EA, CL–OA and CL–NA for many-to-many distribution are presented

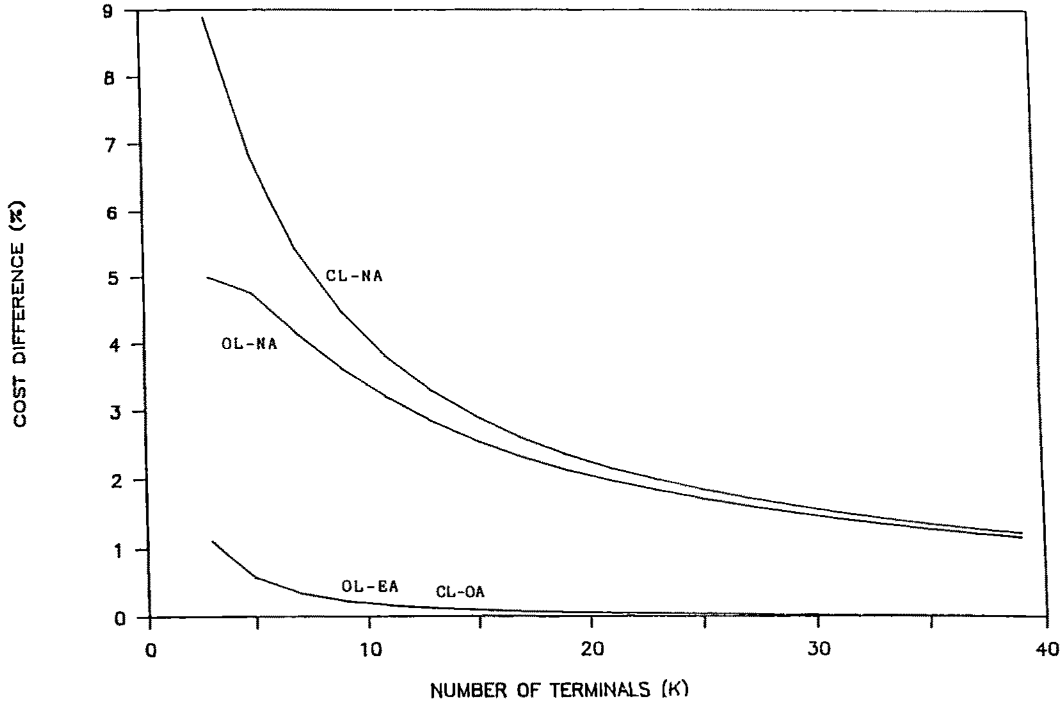

Fig. 1. Transportation cost comparison for one-to-many distribution to a continuous demand with $\alpha = 0.5$.

Table 1

Maximum percentage difference for one-to-many distribution.

Scheme	Maximum % difference	α	K
OL–NA	5.95	0.719	5
OL–EA	1.15	0.577	3
CL–OA	1.15	0.577	3
CL–NA	22.22	1.0	3

in turn. A notation similar to that in the previous section is used (e.g. $C_M(\mathrm{O, N})$ is the transportation cost for many-to-many distribution using optimal hub locations and allocation to the nearest hub).

OL–NA: The results for OL–NA were derived in Campbell [2] and are repeated here.

$$C_M(\mathrm{O,N}) = \frac{\alpha\beta X}{6}\left[2 + E^3 + 3\frac{1-\alpha}{\alpha}E^2 + F_1(1-E)^3 + \frac{3}{\alpha}F_2(1-E)^2\right], \quad (15)$$

where

$$E = (2+\alpha)/4, \qquad \text{if } K = 2,$$

$$E = 1 - \frac{1 + F_2 - [(1+F_2)^2 - \alpha(2-\alpha)(1-F_1)]^{1/2}}{\alpha(1-F_1)}, \qquad \text{if } K > 2,$$

$$F_1 = \frac{1}{(K-1)^2},$$

and

$$F_2 = \frac{1}{K-1}.$$

The optimal location of hub i is given by

$$x_i(\mathrm{O,N}) = \frac{X}{2}E + X(1-E)\frac{i-1}{K-1}, \quad i = 1, 2, \ldots, K. \tag{17}$$

Note the similarity between eqs. (15)–(17) for OL–NA and eqs. (3)–(5) for OL–OA.

OL–EA: The derivation of the transportation cost and hub locations for the combination OL–EA is contained in appendix A. The results are

$$C_M(\mathrm{O,E}) = \frac{\alpha\beta X}{6}\left[2 + \frac{\alpha}{K^3} - \frac{2}{K^2} + \frac{3}{\alpha K} - \frac{\alpha}{K}\right], \tag{18}$$

$$x_i(\mathrm{O},E) = \frac{X}{2K}\left[\alpha + (2i-1)(1-\alpha/K)\right], \quad i = 1, 2, \ldots, K', \tag{19}$$

where

$$K' = \begin{cases} K/2 & \text{if } K \text{ is even,} \\ (K+1)/2 & \text{if } K \text{ is odd.} \end{cases}$$

By symmetry,

$$x_i(\mathrm{O,E}) = X - x_{k-i+1}(\mathrm{O,E}), \quad i = K'+1,\ K'+2, \ldots, K.$$

CL–OA: The hub locations for CL–OA are given by eq. (6). The transportation cost can be derived from eq. (6) and eqs. (1), (4), (19) and (21) in Campbell [2]:

$$C_M(\mathrm{C,O}) = \frac{\alpha\beta X}{6}\left[2 - \frac{\alpha^2}{K^3} + \frac{\alpha^2 - \alpha - 2}{K^2} + 3\,\frac{1-\alpha^2}{\alpha K}\right]. \tag{20}$$

CL–NA: The hub locations for CL–NA are given by eq. (6). The transportation cost can be derived following the procedure in Campbell [2]:

$$C_M(\mathrm{C,N}) = \frac{\alpha\beta X}{6}\left[2 - \frac{2}{K^2} + \frac{3}{\alpha K}\right]. \tag{21}$$

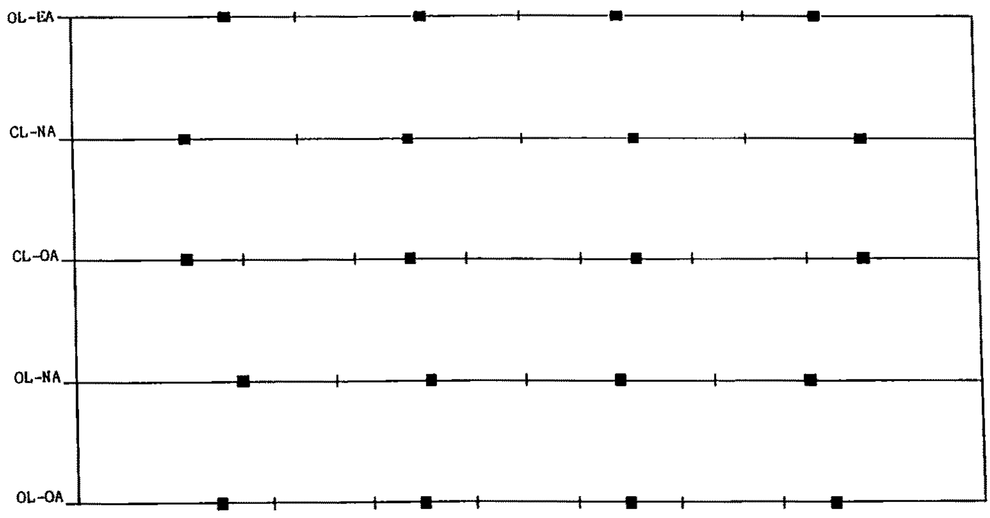

Fig. 2. Location and allocation for many-to-many distribution with $\alpha = 0.5$ and $K = 4$.

Figure 2 shows the hub locations and the allocation for the four suboptimal location–allocation combinations and the optimal combination for $\alpha = 0.5$. For the two combinations with optimal allocation, each interval between two adjacent hubs is divided into three subintervals. Origins and destinations in the central subinterval between each pair of hubs are allocated to both hubs. The other three combinations require allocation to a single hub.

Figure 3 is a comparison of the transportation cost for each suboptimal combination to the optimal transportation cost for $\alpha = 0.5$. Each curve is the percentage difference between the transportation cost for a suboptimal combination and the transportation cost for OL–OA as a function of the number of hubs. The cost resulting from the CL–OA combination is nearly optimal, as was true for one-to-many distribution. However, unlike one-to-many distribution, the OL–EA combination is not nearly optimal and produces a transportation cost closer to that for the combinations with nearest allocation than to that for the CL–OA combination. Both combinations that allocate to the nearest hub perform relatively poorly and the transportation costs for these combinations converge to a common value.

The results for one-to-many and many-to-many distribution with a continuous demand suggest that the combination of central location and optimal allocation is nearly optimal. The results also indicate that allocation to the nearest facility produces relatively poor results, even when optimal locations are utilized. These results seem to suggest that allocating demand optimally may be more important than locating facilities optimally.

These results have implications for modeling distribution systems. If facilities are assumed to be centrally located, then a nearly optimal transportation cost can be achieved by optimally allocating the demand. Thus, assuming a rectangular grid

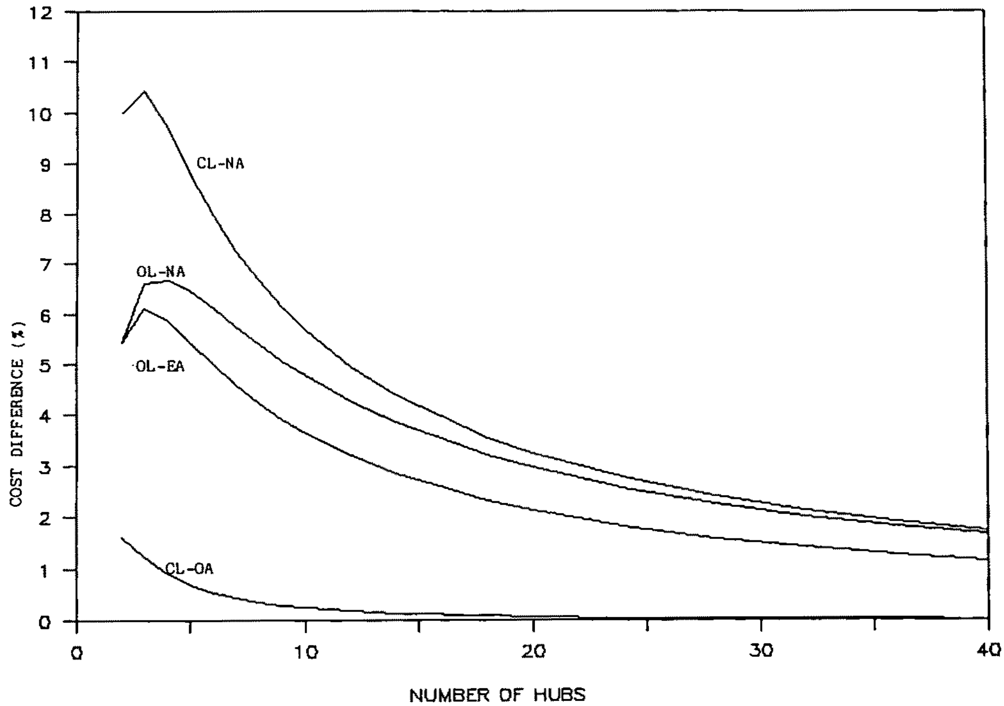

Fig. 3. Transportation cost comparison for many-to-many distribution to a continuous demand with $\alpha = 0.5$.

of facilities is not unreasonable as long as demand is allocated optimally. For one-to-many distribution, if the allocation of demand is assumed by dividing the service region into equal subregions, then a nearly optimal transportation cost can be achieved by optimally locating the terminals.

Allocation to the nearest facility produces poor results for both one-to-many and many-to-many distribution, even when facilities are optimally located. The simple assumption that facilities are located in the center of regions of equal size may be far from optimal, unless local transportation is very expensive relative to linehaul transportation (i.e. α is close to 1).

4. Discrete demand

One important assumption in the preceding analysis is that demand is continuously distributed according to a uniform distribution over the service region. This section compares the transportation cost for the suboptimal location–allocation combinations to the optimal transportation cost for discrete demand problems, to see how well the conclusions from the continuous demand analysis apply to problems with discrete demand. Origins and destinations are specified sites (i.e. points) in the service region and transshipment facilities are restricted to be located at these sites.

This gives rise to a network formulation of the one-to-many and many-to-many distribution problems.

As in the previous section, the optimal location (OL) scheme and the optimal allocation (OA) scheme minimize the transportation cost and the nearest allocation (NA) scheme allocates all demand to the nearest facility. However, the equal allocation (EA) and central location (CL) schemes are not as straightforward for discrete demand points as for a continuous uniform demand density.

For a continuous uniform demand density, division of the service region into K equal area subregions also results in K equal demand subregions. This is not true in general for non-uniform or discrete demands. Thus, the EA scheme is based on dividing the service region into K contiguous non-overlapping subregions which contain approximately equal amounts of demand. These subregions should be reasonably compact (i.e. not too elongated) to minimize local transportation cost. There may be a very large number of ways to divide a set of demand points into K spatially contiguous subsets of approximately equal demand. Selecting different subsets of demand will produce different allocations and different results. Only one division of the demand into subsets for each value of K is explored in this paper.

For distribution to a continuous uniform demand density, the CL scheme produces locations that are central with respect to both the area of the subregion and the demand in the subregion. For discrete demand points, the locations for the CL schemes are determined by first dividing the demand into K compact subsets of approximately equal demand, and then locating a facility at the demand point closest to the center of gravity of each subset of demand. A more exact method would locate the facility by solving a 1-median problem for each subset of demand. However, the center of gravity method is simpler and, in many cases, the demand point closest to the center of gravity is also the median.

4.1. ONE-TO-MANY DISTRIBUTION

The location–allocation problem for one-to-many distribution with discrete destinations can be formulated as a p-median problem (Hakimi [14]) in which the cost to allocate a destination at site i to a terminal at site j is

$$C_{ij} = \beta w_i (d_{ji} + \alpha d_{0j}),$$

where

d_{ji} is the distance from site j to site i,

d_{0j} is the distance from the origin to site j, and

w_i is the demand at site i.

This is a special case of the p-median problem in which one site (the origin) is allocated to all the terminals. Each terminal location is influenced by the ratio of the local and linehaul transportation rates and by the weight of the origin allocated to the terminal. The allocation of destinations to terminals determines the amount of demand served by each terminal and thus, the weight of the origin allocated to each terminal. Unlike the basic p-median problem, all destinations should not be allocated to the nearest terminal.

The location, allocation and transportation cost were found for the optimal and suboptimal combinations by enumerating the location and allocation patterns for an example consisting of 50 sites (destinations) in a square service region (from p. 57 of Eilon et al. [11]). All destinations were assumed to have the same demand and Euclidean distances were used. The origin was assumed to be located at the lower left-hand corner of the square service region, instead of in the center of the service region. Thus, the square service region may be viewed as one-quarter of a larger service region that contains a centrally located origin.

Results for one-to-many distribution via 2, 3 and 4 terminals are contained in table 2. This table shows the percentage cost difference between the cost for each suboptimal combination and the optimal cost for two values of α. The results for

Table 2

Transportation cost comparison for one-to-many distribution to a discrete demand.

		Percentage cost differences		
α	Combination	$K=2$	$K=3$	$K=4$
0.5	OL–EA	1.04	2.31	3.53
	OL–NA	0.99	3.05	2.10
	CL–OA	4.90	2.98	4.24
	CL–NA	8.73	7.96	11.63
0.83	OL–EA	3.37	4.80	2.95
	OL–NA	3.85	3.63	2.69
	CL–OA	3.14	2.50	2.27
	CL–NA	18.62	14.74	18.52

$\alpha = 0.5$ differ from the continuous demand results (shown in fig. 1) in that the CL–OA and OL–EA combinations are not nearly optimal and OL–NA performs about as well or better than CL–OA and OL–EA. However, the poor results for CL–NA are similar to the continuous results.

For discrete demand problems, a combination with nearest allocation may be better than either CL–OA or OL–EA, especially if α is small, since small values of α favor schemes with allocation to the nearest facility. (For $\alpha = 0$, allocation to the nearest facility is optimal.) The CL–OA combination tends to perform better, relative to the other suboptimal combinations, for large values of α, since large values of α place more emphasis on the allocation of destinations to reduce linehaul transportation cost. In the lower part of table 2 ($\alpha = 0.83$), the CL–OA combination produces lower costs than the other suboptimal combinations, but only by a small amount.

4.2. MANY-TO-MANY DISTRIBUTION

The location–allocation problem for many-to-many distribution with discrete demand (i.e. origins and destinations) can be formulated as an integer programming problem as follows. Assume that every demand site is both an origin and a destination. Define:

n = number of demand sites;

p = number of terminals to be located;

$$X_{ijkm} = \begin{cases} 1 & \text{if the flow from site } i \text{ to site } j \text{ goes via first hub } k \text{ and then hub } m, \\ 0 & \text{otherwise;} \end{cases}$$

$$Y_k = \begin{cases} 1 & \text{if a hub is established at site } j, \\ 0 & \text{otherwise;} \end{cases}$$

C_{ijkm} = transportation cost to deliver demand w_{ij} from site i to site j via hubs at sites k and m (in order i–k–m–j).

The location–allocation problem for many-to-many distribution can then be formulated as follows:

MTM minimize
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n} X_{ijkm}C_{ijkm} \tag{22}$$

subject to
$$\sum_{k=1}^{n}\sum_{m=1}^{n} X_{ijkm} = 1, \qquad \text{for all } i, j, \tag{23}$$

$$\sum_{k=1}^{n} Y_k = p, \tag{24}$$

$$0 \le X_{ijkm} \le Y_k, \qquad \text{for all } i, j, k, m, \tag{25a}$$

$$0 \le X_{ijkm} \le Y_m, \qquad \text{for all } i, j, k, m. \tag{25b}$$

Constraints (23) ensure that every origin–destination pair is routed via hubs. If $k = m$ in constraint (23), then shipments are routed via only one hub. If $k \neq m$, then shipments are routed via two hubs. Constraint (24) establishes p hubs. Constraints (25a) and (25b) prevent the routing of shipments via sites that do not have hubs. These constraints are an interesting feature of the problem, since setting X_{ijkm} equal to one forces two hubs to be established (unless $k = m$).

If the transportation cost is proportional to the distance between two sites and α is the ratio of the linehaul to local transportation rate, then

$$C_{ijkm} = \beta w_{ij}(d_{ik} + d_{mj} + \alpha d_{km}),$$

where

d_{ij} is the travel distance from site i to site j, and
w_{ij} is the demand for origin–destination pair i, j.

In general, $C_{iikm} = 0$, since $w_{ii} = 0$.

Problem **MTM** corresponds to the OL–OA combination. This is a different problem than the hub location problem formulated by O'Kelly [23], because each site is not restricted to be allocated to only one hub. Problem **MTM** can be converted to O'Kelly's hub location problem by adding constraints that restrict each origin and destination to send and receive all shipments via a single hub (see Campbell [5]).

Instead of solving integer programming formulations, the location–allocation problems corresponding to the four suboptimal combinations and to the optimal combination were solved by enumeration for an example that has 20 demand sites in a square service region. (The 20 sites are the first 20 of the 50 from p. 57 of Eilon et al. [11].) Each site is both an origin and a destination and all origin–destination pairs were assumed to have the same demand.

The terminal locations, the allocation and the transportation cost with 2, 3, 4 and 5 hubs and $\alpha = 0.5$ were found by enumeration. Solutions for problems with more demand sites or larger numbers of hubs required excessive amounts of computer time for enumeration. The results, which are shown in table 3, differ from the continuous demand results (shown in fig. 3) in that the CL–OA combination is not clearly superior to the other suboptimal combinations. However, CL–OA is the best combination and CL–NA is the worst combination, as was true for a continuous demand.

Table 3

Transportation cost comparison for many-to-many distribution to a discrete demand with $\alpha = 0.5$.

	Percentage cost difference			
Combination	$K=2$	$K=3$	$K=4$	$K=5$
OL–EA	6.66	6.98	4.22	3.92
OL–NA	4.76	3.96	3.44	4.48
CL–OA	4.36	2.39	3.09	0.15
CL–NA	6.67	9.59	12.18	4.98

Figure 4 shows the optimal hub locations and the optimal allocation for three hubs. This figure does not show the routing of each shipment, but does show which transportation links are used in the optimal distribution system, i.e. for the OL–OA combination. Of the seventeen demand sites not occupied by hubs, two send shipments to all three hubs, six send shipments to two hubs, and nine are allocated to a single hub. Table 4 shows how many sites are allocated to different numbers of hubs for

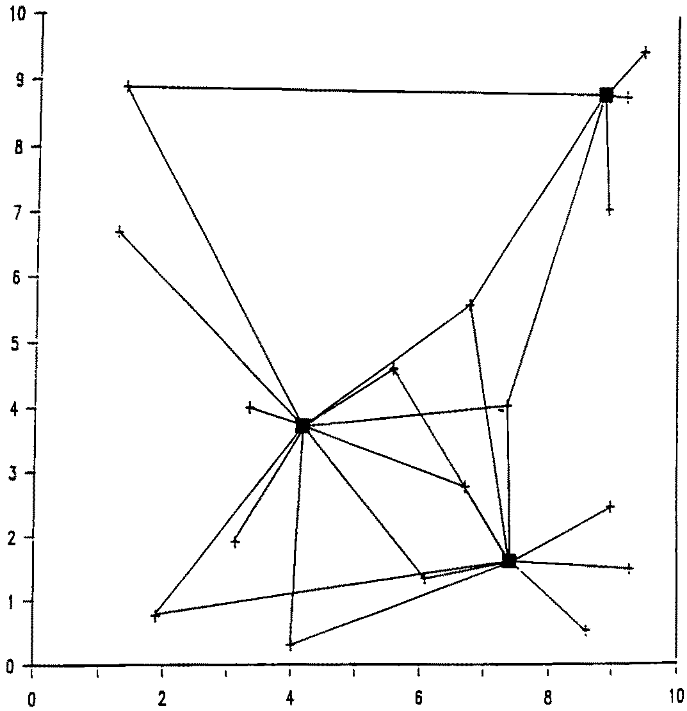

Fig. 4. Location–allocation for $\alpha = 0.5$ and $K = 3$.

Table 4

Allocation of sites for many-to-many distribution.

	Number of sites allocated to:		
K	1 hub	2 hubs	3 hubs
2	13	5	–
3	9	6	2
4	9	5	2
5	7	6	2

the example problem with 2, 3 4 and 5 hubs and $\alpha = 0.5$. For smaller values of α, more sites should be allocated to a single facility, because minimizing local transportation distance is more important than linehaul distance.

5. Conclusions

This paper compared the transportation cost resulting from suboptimal location and allocation schemes to the optimal cost. Results from distribution to a continuous demand indicated that nearly optimal costs can be achieved by centrally locating the transshipment facilities and optimally allocating the demand. For one-to-many distribution, the equal allocation scheme was also nearly optimal if terminals were optimally located. The combination of centrally located facilities and allocation to the nearest facility produced poor results for both one-to-many and many-to-many distribution.

The discrete demand results differed from the continuous demand results in important ways. The combination of central location and optimal allocation was not nearly optimal, nor was it significantly better than the combinations that used optimal locations (i.e. optimal location with nearest allocation and optimal location with equal allocation). Another difference was that allocation to the nearest facility may produce good results (as good as the other suboptimal combinations considered) for discrete problems. Thus, the suggestion from the continuous results that allocation to the nearest facility is undesirable, is not supported by the discrete results. However, one similarity in the discrete and continuous results was the poor performance of the combination of central location and allocation to the nearest facility.

These results indicate that caution must be exercised when applying results based on a continuous uniform density of demand to problems with discrete demand points. The continuous demand results indicate the limiting case when the number of demand sites is very large and the demand sites are randomly scattered over the service region. If the number of demand sites is small or the demand is clustered, then the continuous results would not apply. When the number of discrete demand points is small relative to the number of facilities, then the cost to serve a discrete demand is likely to be less than the cost to serve a continuous demand, especially for small α. Thus, the difference between the continuous and discrete results may not be surprising in view of the non-uniform nature of the discrete demand.

For many-to-many distribution, allocation to a single hub is generally not optimal. The combination of optimal location and optimal allocation for many-to-many distribution provides a lower bound on the optimal cost for the hub location problem that requires each demand point to be allocated to a single hub. This lower bound is likely to be better for smaller values of α, since the continuous results indicate that the area in which demand should be allocated to more than one hub decreases as α increases. Thus, the optimal locations for the many-to-many distribution problem may provide a starting point for solving the hub location problem with unique allocation. The discrete form of the many-to-many distribution problem is easier to solve than the discrete form of the hub location problem with unique allocation. However, converting the solution from the many-to-many distribution problem to solve the problem with unique allocation requires overcoming the combinatorics of uniquely allocating all the demand points to a single hub.

Appendix

This appendix formulates the transportation cost for the combination of optimal location and equal allocation (OL–EA) for many-to-many distribution. If origins and destinations are distributed according to a uniform distribution over a line segment of length X, then the average distance between an origin and destination is $X/3$. If hubs were located at all origins and destinations, then all travel would be on linehaul vehicles and the average transportation cost would be $\alpha\beta X/3$. However, there is generally not a hub at every origin and destination, so the average transportation cost is larger than $\alpha\beta X/3$.

For an origin–destination pair separated by a distance d, define the circuity cost to be the transportation cost in excess of $\alpha\beta d$. The average circuity cost is the amount by which the average transportation cost exceeds $\alpha\beta X/3$. The circuity cost accounts for the local transportation and for the additional cost due to backtracking to a hub. The average transportation cost can be written as

$$C_M(\mathrm{O}, \mathrm{E}) = \alpha\beta X/3 + Q, \tag{A1}$$

where Q is the average circuity cost.

The EA allocation scheme divides the one-dimensional service region of length X into K subregions of length X/K. The average circuity cost can be written as the average of the average circuity cost incurred in each subregion of size X/K:

$$Q = \sum_{i=1}^{K} q_i/K, \tag{A2}$$

where q_i is the average circuity cost in subregion i.

Circuity cost is incurred in a subregion for travel between the hub and all the origins and destinations located in that subregion. There are two cases to consider to determine the circuity cost in a subregion. The probability of each case and the circuity cost for each case are derived for subregion i.

Case 1: Either the origin or destination is in the subregion.

The probability that an origin is in subregion i of length X/K is $1/K$. The probability that a destination is outside of subregion i is $(K-1)/K$. Let P_1 be the probability of case 1.

$$P_1 = 2(K-1)/K^2. \tag{A3}$$

The factor 2 is required because either the origin or the destination is in the subregion for case 1. Without loss of generality, the circuity cost will be formulated for an origin in the subregion and a destination outside the subregion. By symmetry, the results are identical when the origin and destination are reversed.

All shipments from the origin in the subregion travel first to the hub. If the shipment backtracks to reach the hub, then circuity cost is incurred at a rate of $\beta(1 + \alpha)$ per unit distance to the hub. This accounts for local transportation to the hub and linehaul transportation from the hub back to the origin. If the shipment does not backtrack to the hub, then circuity cost is incurred at a rate of

$\beta(1-\alpha)$ per unit distance to the hub. This accounts for local instead of linehaul transportation to the hub.

Let the hub be located a distance y from the left edge of the subregion and let w be the distance from the origin to the hub. For $w < y$, the shipment backtracks when the destination is to the left of the subregion, so the circuity cost can be written as

$$g_1(w) = \beta(1+\alpha)w\,\frac{i-1}{K-1} + \beta(1-\alpha)w\,\frac{K-i}{K-1}. \tag{A4}$$

The factor $(i-1)/(K-1)$ in the first term is the probability that the destination is located to the left of subregion i for case 1 and the factor $(K-i)/(K-1)$ in the second term is the probability that the destination is to the right of subregion i for case 1. For $w > y$, the circuity cost can be formulated similarly as

$$g_r(w) = \beta(1-\alpha)w\,\frac{i-1}{K-1} + \beta(1+\alpha)w\,\frac{K-i}{K-1}. \tag{A5}$$

The average circuity cost for case 1 is then

$$G_1 = \frac{K}{X}\int_0^y g_1(w)\mathrm{d}w + \frac{K}{X}\int_0^t g_r(w)\mathrm{d}w, \tag{A6}$$

where

$$t = X/K - y$$

is the distance from the hub to the right boundary of the subregion. Performing the integration produces the average circuity cost of case 1:

$$G_1 = \beta y^2 - \beta\left[1 + \alpha\,\frac{K-2i+1}{K-1}\right]\left[y\,\frac{X}{K} + \frac{X^2}{2K^2}\right]. \tag{A7}$$

Case 2: Origin and destination are both in the subregion.

The probabiity that an origin and destination are both in subregion i is

$$P_2 = \frac{1}{K^2}. \tag{A8}$$

Because all shipments are assumed to visit at least one hub, the shipment is sent from the origin to the hub and then to the destination. There is no linehaul transportation for case 2. The circuity cost for case 2 is determined by analyzing three subcases. The first and second are for the situation in which the origin and destination are on the same side of the hub. The third is for the situation in which the origin and destination are on opposite sides of the hub.

Case 2a: Origin and destination are both to the left of the hub.

The probability of case 2a is

$$P_{2a} = \frac{y^2}{(X/K)^2}. \tag{A9}$$

Let w be the distance from the hub to the nearer of the origin and destination and let υ be the distance from the hub to the farther of the origin and destination. The circuity cost is the cost in excess of the cost for linehaul transportation from the origin to the destination. Thus, the average circuity cost for case 2a is

$$G_{2a} = \frac{\beta \int_0^y \int_w^y w + \upsilon - \alpha(\upsilon - w) \mathrm{d}\upsilon \, \mathrm{d}w}{\int_0^y \int_w^y \mathrm{d}\upsilon \, \mathrm{d}w}$$

$$= \beta y(1 - \alpha/3). \tag{A10}$$

Case 2b: Origin and destination are both to the right of the hub.

This case is analogous to case 2a if y is replaced by t. Thus, the probability of case 2b is

$$P_{2b} = \frac{t^2}{(X/K)^2} \tag{A11}$$

and the average circuity cost is

$$G_{2b} = \beta t(1 - \alpha/3). \tag{A12}$$

Case 2c: Origin and destination are on opposite sides of the hub.

The probability of case 2c is

$$P_{2c} = \frac{2yt}{(X/K)^2}. \tag{A13}$$

For case 2c, define w to be the distance from the hub to the demand point (origin or destination) to the left of the hub and υ to be the distance from the hub to the demand point to the right of the hub. The circuity cost is the cost in excess of $\alpha\beta(w + \upsilon)$. The average circuity cost is

$$G_{2c} = \frac{\beta \int_0^y \int_0^t (w + \upsilon)(1 - \alpha) \, \mathrm{d}\upsilon \, \mathrm{d}w}{\int_0^y \int_0^t \mathrm{d}\upsilon \, \mathrm{d}w}$$

$$= \beta(1 - \alpha) \frac{X}{2K}. \tag{A14}$$

The average circuity cost for case 2 is then

$$G_2 = P_{2a}G_{2a} + P_{2b}G_{2b} + P_{2c}G_{2c}$$
$$= \beta[2y^2(X/K)^2 - 2y(X/K) + 1 - \alpha/3]. \quad \text{(A15)}$$

The average circuity cost in a subregion of size X/K is the sum of the products of the probabilities and average circuity costs for cases 1 and 2:

$$q_i = P_1G_1 + P_2G_2. \quad \text{(A16)}$$

The optimal value of y can be found by solving

$$\frac{\mathrm{d}}{\mathrm{d}y} q_i = 0$$

for y. The result is

$$y^* = \frac{X}{2K}\left[1 + \alpha - \alpha\,\frac{2i-1}{K}\right], \quad i = 1, 2, \ldots, K', \quad \text{(A17)}$$

where

$$K' = \begin{cases} K/2 & \text{if } K \text{ is even} \\ (K+1)/2, & \text{if } K \text{ is odd.} \end{cases}$$

Because y^* is the distance from the hub to the left edge of the subregion, each hub is displaced from the center of the subregion toward the center of the service region by an amount

$$y^* - \frac{X}{2K} = \alpha\frac{X}{2K}[1 - (2i-1)/K], \quad i = 1, 2, \ldots, K'. \quad \text{(A18)}$$

Thus, hubs farther from the center of the service region are displaced a greater amount than hubs closer to the center of the service region. The optimal hub locations for the OL–EA combination are then

$$x_i = (i-1)\frac{X}{K} + y^*$$
$$= \frac{X}{2K}[\alpha + (2i-1)(1 - \alpha/K)], \quad i = 1, 2, \ldots, K'. \quad \text{(A19)}$$

The hub locations in the right half of the service region are determined by symmetry. Equations (A3), (A7), (A8), (A15), (A16) and (A17) provide the average circuity cost in subregion i. Equations (A1) and (A2) then provide the average transportation cost for the OL–EA combination:

$$C_M(\mathrm{O}, \mathrm{E}) = \frac{\alpha\beta X}{6}\left[2 + \frac{\alpha}{K^3} - \frac{2}{K^2} + \frac{3}{\alpha K} - \frac{\alpha}{K}\right]. \quad \text{(A20)}$$

References

[1] J.F. Campbell, Designing logistics systems by analyzing transportation, inventory and terminal costs tradeoffs, J. Bus. Logist. 11(1990)159.

[2] J.F. Campbell, Freight consolidation and routing with transportation economies of scale, Transp. Res. 24B(1990)345.

[3] J.F. Campbell, Location–allocation for distribution to a continuous demand with transshipments, Naval Res. Logist. 39(1992)635.

[4] J.F. Campbell, One-to-many distribution with transshipments, Transp. Sci. (to appear).

[5] J.F. Campbell, Integer programming formulation of discrete hub location problems, Eur. J. Oper. Res. (to appear).

[6] L. Cooper, Location–allocation problems, Oper. Res. 11(1963)331.

[7] C.F. Daganzo, The distance traveled to visit N points with a maximum of C stops per tour, Transp. Sci. 18(1984)331.

[8] C.F. Daganzo, The break-bulk role of terminals in many-to-many logistic networks, Oper. Res. 35(1987)543.

[9] C.F. Daganzo, A comparison of in-vehicle and out-of-vehicle freight consolidation strategies, Transp. Res. 22B(1988)173.

[10] C.F. Daganzo and G.F. Newell, Configuration of physical distribution networks, Networks 16(1986)113.

[11] S. Eilon, C.D.T. Watson-Gandy and N. Christofides, *Distribution Management: Mathematical Modelling and Practical Analysis* (Griffin, London 1971).

[12] A.M. Geoffrion, The purpose of mathematical programming is insight, not numbers, Interfaces 7(1976)81.

[13] A.M. Geoffrion, Making better use of optimization capability in distribution system planning, AIIE Trans. 11(1979)96.

[14] S.L. Hakimi, Optimum locations of switching centers and the absolute centers and medians of a graph, Oper. Res. 12(1964)450.

[15] R.W. Hall, Travel distance through transportation terminals on a rectangular grid, J. Oper. Res. Soc. 35(1984)1067.

[16] R.W. Hall, Comparison of strategies for routing shipments through transportation terminals, Trans. Res. 21A(1987)421.

[17] R.W. Hall and C.F. Daganzo, Travel distance through transportation terminals on a rectangular grid: Alternate routing strategies, General Motors Research Laboratories, Research Publication GMR-4719 (1984).

[18] A.F. Han, One-to-many distribution of nonstorable items: Approximate analytic models, Ph.D. Thesis, Department of Civil Engineering, University of California, Berkeley, CA (1984).

[19] A. Kanagfani, Location models for off-airport terminals, Transp. Res. 6(1972)372.

[20] G. Laporte, Locaton-routing problems, in: *Vehicle Routing: Methods and Studies*, ed. B.L. Golden and A.A. Assad (Elsevier Science, 1988).

[21] D.H. Marks and J.C. Liebman, Location model: A solid waste collection example, J. Urban Planning Dev. Div. 97(1971)15 (*Proc. ASCE*).

[22] P.B. Mirchandani and A.R. Odoni, Locating new passenger facilities on a transportation network, Transp. Res. 13B(1979)113.

[23] M.E. O'Kelly, A quadratic integer program for the location of interacting hub facilities, Eur. J. Oper. Res. 32(1987)393.

[24] S.C. Wirasinghe and N.M. Waters, An approximate procedure for determining the number, capacities and locations of solid waste transfer-stations in an urban region, Eur. J. Oper. Res. 12(1983)105.