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Semantic Realism Versus EPR-Like Paradoxes: The Furry, Bohm–Aharonov, and Bell Paradoxes

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We prove that the general scheme for physical theories that we have called semantic realism *(SR) in some previous papers copes successfully with a number of EPR-like paradoxes when applied to quantum physics (QP). In particular, we consider the old arguments by Furry and Bohm–Aharonov and show that they are not valid within a SR framework. Moreover, we consider the Bell–Kochen–Specker and the Bell theorems that should prove that QP is inherently contextual and nonlocal, respectively, and show that they can be invalidated in the SR approach. This removes the seeming contradiction between the basic assumptions of SR and QP, and proves that some problematic features that are usually attributed to QP, as contextuality and nonlocality, occur because of the adoption of a verificationist position, from one side, and from an insufficient adherence to the operational principles that have inspired QP itself, from the other side.*

1. INTRODUCTION

It is well known that the EPR (Einstein–Podolski–Rosen) thought experiment was originally intended to prove the *ontological* incompleteness of quantum physics (QP). Indeed, EPR invented it in such a way that the position and momentum of one of the two elementary particles that appear in it could be reasonably classified as "elements of reality," and deduced the ontological incompleteness from the fact that position and momentum cannot be simultaneously known in QP.

EPR did not use their argument in order to assert that QP leads to paradoxes.(1) But a number of authors worked out and modified the EPR

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argument, deducing paradoxes from it (EPR-like paradoxes in the following; we include the Bell and the Bell–Kochen–Specker, or Bell–KS, theorems in the set of EPR-like paradoxes because of the puzzling consequences of these theorems). Moreover, it has been asserted that these paradoxes can be avoided only accepting one or some of the following disconcerting alternatives.[2]

(i) Existence of superluminal connections
(ii) Retroactions in time
(iii) Variable detection probability
(iv) Breakdown of the "ergodic hypothesis"
(v) Negative probabilities

Of course, the supporters of the orthodox interpretation of QP, or of some variant of it, have provided confutations of the reasonings that lead to paradoxes; yet, in our opinion these answers are unsatisfactory because of the adoption (sometimes implicit or incomplete) of a verificationist theory of truth and meaning. This point has been discussed by us in a previous paper[3] that will be briefly called GS.96 in the following.

We would like to show in the present article that a more satisfactory way out from EPR-like paradoxes can be found by specializing to QP the general scheme for physical theories (*semantic realism* or, briefly, SR) that has been worked out in GS.96 and in some previous papers.[4–8] This result can be obtained since SR embodies some nonstandard features which allow us to look at the problem from an unusual viewpoint. Indeed, SR is based on an epistemological perspective which implies the adoption of a correspondence theory of truth for the language of physics, so that a sort of *realism of properties* is introduced; but SR also embodies some operational requirements that allow one to distinguish *truth* from *epistemic accessibility* (or *testability*) and produce a number of changes in the conventional conception of compound physical systems, consistency, compatibility, testability. In particular, SR propounds a new way (MGP) of characterizing the *truth mode* of empirical physical laws that takes into account the existence of nontrivial relations of pragmatic compatibility on the set of physical properties in some physical theories, as QP: indeed, MGP restricts the set of laboratories (space-time domains) in which an empirical law can be asserted to be true to all *epistemically accessible* laboratories (note that the law could be false, not meaningless, in a laboratory that does not belong to this set). Because of all these features, SR, which reproduces the standard viewpoint when applied to classical physics (CP), introduces some relevant modifications of the standard way of dealing with states, properties, measurements, physical laws, and completeness whenever it is applied to QP (yet it preserves both the mathematical apparatus and the observative consequences of this theory). These changes constitute the background for working out the SR answers to EPR-like paradoxes that we provide in Secs. 3–7 after supplying in Sec. 2 a generalized treatment of the EPR experiment and of some paradoxes in the literature (a brief discussion on probability in a SR framework is also provided in Sec. 8).

We note that our discussion of the paradoxes pointed out by Furry and by Bohm–Aharonov essentially rests on: (i) the semantic differences between states and properties; (ii) the difference between semantic and pragmatic compatibility of properties. However, it does not require MGP. On the contrary, our invalidation of the Bell and Bell–KS theorems requires full use of MGP. Indeed, these theorems assert that QP conflicts with some intuitive requirements regarding physical theories (see Sec. 2), and we show that in all proofs a classical way of characterizing the truth mode of empirical physical laws (MCP: physical laws are true in every laboratory) is implicitly used, while no inconsistency of QP with the aforesaid requirements occurs if MGP is substituted to MCP.

The use of MGP for invalidating the Bell and the Bell–KS theorems is relevant, since these theorems are deep results that ought to show that nonlocality and contextuality are intrinsic features of QP, which do not depend on the philosophical choices that are made *a priori* in the standard (Copenhagen) interpretation, so that they should be unavoidable in QP. Our invalidation proves that this conclusion fails to be true whenever one accepts SR and, in particular, MGP. But this principle is more consistent than MCP with the operational philosophy of QP. Thus, surprisingly, we recover some features of CP in QP (as locality, noncontextuality, and compatibility with some forms of realism) by applying more rigorously one of the basic epistemological choices of QP.

We also observe that our treatment offers an alternative not only to the aforesaid explanations of EPR-like paradoxes in terms of superluminal connections, retroactions in time, etc., but also to the attempts of avoiding nonlocality by limiting the validity of counterfactual reasoning in QP[9, 10] (we have already commented on this latter point in the Introduction to GS.96). Moreover, it suggests new constraints on hidden variables theories for QP that are not consistent with the assumptions that are usually retained to be plausible for this kind of theories.[11, 12] This might explain some difficulties in the existing approaches and opens the way to possibly local and noncontextual theories (see Remark 6.1).

Finally, we notice explicitly that most references in this paper are provided as sample references. The literature on the subject treated here is indeed exceedingly wide, and any attempt of giving a complete account of it in the limited space of an article's bibliography is hopeless.

2. THE EPR-LIKE "PARADOXES"

As we have anticipated in the Introduction, we supply in this section the description of a physical experiment in nonrelativistic QP which includes as particular cases the classical EPR[13] experiment (to be precise, this would actually require a furher generalization of our scheme to continuous observables) and the Bohm[14] reformulation of it.

Let us consider a physical system $\mathfrak{p}$ that is made up of two subsystems, say 1 and 2 (we recall from GS.96, Sec. 2, that the term *physical system* is used here as a synonym of *physical entity*, and that individual samples of physical systems are called *physical objects*, or simply *objects* whenever no confusion is possible) and let us describe it in the framework of the standard Hilbert space quantum theory (HSQT), which will be considered here as a model of the abstract general theory developed in GS.96 (hence, in particular, the restriction of the standard relation of compatibility in HSQT to the set of testable properties is considered as an instance of the relation of pragmatic compatibility introduced in Sec. 6 of GS.96). Let $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$ be the Hilbert spaces of the three systems, respectively, so that $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. Furthermore, let A, B, C be three observables of the whole system, and let A_1, B_1, C_1 be observables of 1, A_2, B_2, C_2 be observables of 2, such that $A = A_1 + A_2$, $B = B_1 + B_2$, $C = C_1 + C_2$ (for the sake of brevity, we identify here every physical observable with the operator that represents it in HSQT). Let us also assume that A_1, B_1, C_1, A_2, B_2, C_2 have discrete nondegenerate spectrum. Then, trivially, the pairs $\{A, A_1\}$, $\{A, A_2\}$, $\{B, B_1\}$, $\{B, B_2\}$, $\{C, C_1\}$, $\{C, C_2\}$ are complete sets of commuting observables in $\mathscr{H}$.

Now, we introduce the following assumptions:

(i) $[A_1, B_1] \neq 0 \neq [A_2, B_2]$; $[A_1, C_1] \neq 0 \neq [A_2, C_2]$; $[B_1, C_1] \neq 0 \neq [B_2, C_2]$;

(ii) three eigenvalues a, b, c exist of A, B, C, respectively, that share a common eigenvector, say $|\kappa\rangle$, which is nonfactorizable, so that it represents a second type pure state S in the standard sense in HSQT (hence in the sense specified in Sec. 3 of GS.96).

By the way, we note that a straightforward generalization of the EPR experiment would have suggested the introduction of two observables A, B only and the assumption $[A, B] = 0$. We shall also see that the observable A would be sufficient in order to reformulate the Furry paradox, the observables A and B would be sufficient in order to reformulate the Bohm–Aharonov paradox, and the three observables A, B, C, are needed in order to reformulate some proofs of the Bell theorem.

By using (ii), we get

$$|\kappa\rangle = \sum_h \lambda_h \, |u_{1j(h)} u_{2k(h)}\rangle = \sum_l \mu_l \, |v_{1m(l)} v_{2n(l)}\rangle = \sum_p \nu_p \, |w_{1q(p)} w_{2r(p)}\rangle$$

In the above equations $|u_{1j(h)}\rangle$, $|u_{2k(h)}\rangle$, $|v_{1m(l)}\rangle$, $|v_{2n(l)}\rangle$, $|w_{1q(p)}\rangle$, $|w_{2r(p)}\rangle$ are normalized eigenvectors corresponding to the eigenvalues $a_{1j(h)}$ of A_1, $a_{2k(h)}$ of A_2, $b_{1m(l)}$ of B_1, $b_{2n(l)}$ of B_2, $c_{1q(p)}$ of C_1, $c_{2r(p)}$ of C_2, respectively; the first sum ranges on all indices h such that $a_{1j(h)} + a_{2k(h)} = a$, the second on all indices l such that $b_{1m(l)} + b_{2n(l)} = b$, the third on all indices p such that $c_{1q(p)} + c_{2r(p)} = c$; λ_h, μ_l, ν_p are complex nonzero coefficients such that

$$\sum_h |\lambda_h|^2 = \sum_l |\mu_l|^2 = \sum_p |\nu_p|^2 = 1$$

For the sake of simplicity, we imply the dependence of j, k on h, of m, n on l, of q, r, on p whenever possible in the following. Therefore, we denote by S_{jk}, S_{mn}, S_{qr}, the first type pure states represented by $|u_{1j} u_{2k}\rangle$, $|v_{1m} v_{2n}\rangle$, $|w_{1q} w_{2r}\rangle$, respectively. Furthermore, we denote by S_{1j}, S_{2k}, S_{1m}, S_{2n}, S_{1q}, S_{2r}, the first type pure states represented by $|u_{1j}\rangle$, $|u_{2k}\rangle$, $|v_{1m}\rangle$, $|v_{2n}\rangle$, $|w_{1q}\rangle$, $|w_{2r}\rangle$, respectively, and by A_{1j}, A_{2k}, B_{1m}, B_{2n}, C_{1q}, C_{2r}, respectively, their supports (see Sec. 3 of GS.96). Hence, these supports denote the following testable physical properties.

A_{1j}: "the observable A_1 of subsystem 1 has value a_{1j}";

A_{2k}: "the observable A_2 of subsystem 2 has value $a_{2k} = a - a_{1j}$";

B_{1m}: "the observable B_1 of subsystem 1 has value b_{1m}";

B_{2n}: "the observable B_2 of subsystem 2 has value $b_{2n} = b - b_{1m}$";

C_{1q}: "the observable C_1 of subsystem 1 has value c_{1q}";

C_{2r}: "the observable C_2 of subsystem 2 has value $c_{2r} = c - c_{1q}$."

As in GS.96 (Sec. 3), we denote here by $\perp$, $\bigcap$, $\bigcup$ the lattice operations in the lattice ($\mathscr{E}_e$, $<$) of all properties of p; hence, in particular, $A_{1j}^{\perp}$ and $A_{2k}^{\perp}$ denote the testable properties "the observable A_1 of subsystem 1 has a value different from a_{1j}" and "the observable A_2 of subsystem 2 has a value different from $a_{2k} = a - a_{1j}$," respectively. Analogous interpretations hold for the symbols $B_{1m}^{\perp}$, $B_{2n}^{\perp}$, $C_{1q}^{\perp}$, $C_{2r}^{\perp}$.

By considering the projections that describe A_{1j}, A_{2k}, B_{1m}, B_{2n}, C_{1q}, C_{2r} in HSQT, we see that each property referring to subsystem 1 is (pragmatically) compatible with each property referring to subsystem 2 (see Remark 6.1 of GS.96). In addition, we assume from now on that a_{1j}, b_{1m}, c_{1q} can be chosen in such a way that a property is compatible with another property referring to the same subsystem only if the former is the orthocomplement of the latter (this is generally possible since the observables A_1, B_1, C_1 are not pairwise compatible).

Finally, let us convene to consider only ideal measurements of the kind discussed in GS.96 (Sec. 10), hence measurements that obey the projection postulate. Then, let us consider an ensemble of samples of the physical system $\mathfrak{p}$ in a laboratory i (the reference to which is usually understood in the following) and let us assume that all samples are in the second type pure state S: by using the symbols introduced in GS.96, Sec. 2, this ensemble will be denoted by $\rho_i(S)$. Whenever an ideal measurement of the aforesaid kind is made of A_1 on the physical object 1 in a given sample x, which yields the result a_{1j} (hence A_{1j} is true for x), we conclude that A_2 has value $a_{2k} = a - a_{1j}$ (hence A_{2k} is true for x); moreover, by using the projection postulate, we can say that the state of x after the measurement is S_{jk} (note that, standing on Remark 2.1 of GS.96, we associate here a state to every individual sample of the physical system: one could avoid this procedure by saying that S describes the set $\rho_i(S)$ in i and then considering the subensemble of all samples in $\rho_i(S)$ that yield the result a_{1j} whenever A_1 is measured on 1[(15)]). Analogous conclusions hold whenever ideal measurements of B_1 or C_1 on 1 are considered.

We can now reformulate some arguments that ought to prove that paradoxes follow from the quantum treatment of the physical situation considered above.

The Furry Paradox.[(14, 16, 17)] After the measurement of A_1 that yields the result a_{1j} we know that the physical object 2 in the sample x is in the state S_{2k}. This knowledge is attained without any interaction with 2, so that the state of this object should have been S_{2k} even before the measurement of A_1. Therefore, the initial state of x should have been S_{jk}, hence the ensemble $\rho_i(S)$ coincides with the extension in i of (equivalently, it should be described in i by) a mixture, say T, of the states S_{jk}. This is a paradox. since S and T are empirically distinguishable from a statistical viewpoint[18] (when representing states by means of density matrices, T is represented by the trace class operator

$$\sum_h |\lambda_h|^2 \, |u_{1j} u_{2k}\rangle\langle u_{1j} u_{2k}|$$

which does not coincide with the projection $|\kappa\rangle\langle\kappa|$ that represents S).

Let us add some remarks regarding the above paradox.

First, we observe that Furry's paradox relies on the assumption that the state of the object 2 in the sample should have been S_{2k} even before the measurement of A_1. This assumption is made since it seems unacceptable that an abrupt change in the description of 2 can be induced by a measurement of A_1, performed on 1 without any interaction with 2. One may wonder about the underlying reasons of this unacceptance. It is then apparent that this change is considered a modification of the physical situation at 2 (to be precise, a modification of the physical properties of 2), which is obviously counterintuitive (even if one gives up classical prejudices) in the absence of any interaction with 2, and can be explained only resorting to one of the exotic conjectures quoted in the Introduction.

Second, we observe that, according to the standard quantum theory of ideal measurements, the pure state S and the mixture T yield the same probability for every property of subsystem 2, hence they cannot be statistically distinguished by an observer at 2.[15] This canonical result avoids any further statistical paradox that might occur at 2 because of the fact that the ensemble is described by different states before and after a measurement that does not act on 2.

The Bohm–Aharonov paradox.[19] Let us consider again a sample x of the physical system $\mathfrak{p}$ in the state S. Should a measurement of A_1 be performed, it would yield a result, say a_{1j}, and the state of the physical object 2 in the sample x after the measurement would be S_{2k}. Should a measurement of B_1 be performed, it would yield a result, say b_{1m}, and the state of the physical object 2 in the sample x after the measurement would be S_{2n}. Of course, S_{2k} and S_{2n} are different, hence the object 2 has different and noncompatible properties in the two cases. This entails that the properties of 2 are determined by the measurement that one decides to perform on 1, even if the measurements of A_1 and B_1 do not act directly on 2. Therefore we are compelled either to introduce subjectivity in physics (throught the arbitrary choices of the observer in 1) or, again, to introduce the exotic explanations quoted in the Introduction.

The Contextuality and Nonlocality Paradoxes.[2, 11, 12, 20–25] We call Bell–KS theorem and Bell theorem the proofs that QP is a contextual theory and that contextuality also occurs whenever far apart physical systems are considered, respectively, and classify these results as paradoxes since they imply that QP violates some reasonable epistemological requirements regarding physical theories (this point has been recently stressed by Mermin;[12] see also GS.96, Introduction). In particular, QP should prohibit us to trust even in that minimal form of realism (called *realism of properties* in the Introduction) which consists in maintaining that all testable properties of a given physical system are prefixed, independently of any act of measurement.

The original proofs of the Bell–KS theorem[11,21] were rather complicated but there are some recent proofs[12] that are quite simple and immediate. We mainly refer to these proofs in the following and limit ourselves here to noting that they do not make direct use of the EPR situation described in this section. The original proof of the Bell theorem[20] is based, on the contrary, on an EPR situation, in which seemingly reasonable premises lead to an inequality that is violated in QP; furthermore, most generalizations and variants of the Bell theorem that have been successively proposed are proved following the same procedure (hence, it is usual to call Bell inequality, or briefly BI, every inequality appearing in a proof of this kind). More recently, Greenberger *et al.*[24] provided a proof that gives up with inequalities, taking into account four-particle situations and considering only perfect instead of statistical correlations. Similarly, Mermin[12] obtained a proof that does not resort to inequalities by suitably modifying an eight-dimensional version of the Bell–KS theorem provided by himself. As in the case of the Bell–KS theorem, we refer to all these proofs in the following without giving their details here.

From a historical viewpoint, the Bell theorem is probably more important than the Bell–KS theorem. From a conceptual viewpoint, physicists usually retain that the consequences of the Bell–KS theorem (mainly, contextuality of QP) are less paradoxical than the main consequence of the Bell theorem (nonlocality of QP). Therefore, let us dedicate to the latter theorem some further comments.

We begin by observing that the reasonable premises whose consequences are shown to conflict with QP in the proofs of this theorem vary from proof to proof and are not always explicit. For instance, Wigner[22] emphasizes that his inequality stands (i) on the hypothesis that hidden variables exist which determine the quantum properties of any physical system and (ii) on a locality assumption. Greenberger *et al.*[24] assume the canonical EPR premises of perfect correlation, locality, reality, and completeness of QP. Selleri[23] reports a proof that shows that QP conflicts with the following premises:

R (*reality*): *the results of all conceivable measurements are simultaneously prefixed* (*even in the case of incompatible observables*);

LOC (*locality*): *whenever subsystems* 1 *and* 2 *are sufficiently far apart, a measurement on subsystem* 1 (*or* 2) *does not modify the pre-fixed values of the observables of subsystem* 2 (*or* 1).

Finally, Mermin[12] explicitly states in his proof only a "straightforward assumption of locality," so that locality is directly proved to be inconsistent with QP.

Because of the above differences in the assumptions, the existing interpretations of the Bell theorem are slightly different, but in any case a number of authors agree that this theorem proves the nonlocality of QP. In particular, the proof reported by Selleri shows that QP conflicts with R and/or LOC, while that provided by Mermin shows that QP directly conflicts with LOC. The latter result is consistent with the former: indeed, following Redhead,[25] one can argue that, if QP is accepted and it conflicts with R, then it also conflicts with LOC. Hence, in order to accept QP, one must renounce directly LOC or, more generally, R, which, however, implies renouncing LOC. Therefore, let us comment briefly on these assumptions.

Assumption R is retained by some authors to formalize an epistemological position of ontological realism, while it essentially expresses realism of properties, which is formalized by SR, does not imply any ontological commitment, and it is compatible with some different epistemological positions (see GS.96, Introduction). Assumption LOC expresses the intuitive requirement that the properties of a physical system should not depend on observations performed at distance. Renouncing R implies accepting contextuality and some forms of subjectivism (for instance, embracing Heisenberg's viewpoint,[26] according to which the observer's choice of performing a given measurement on an individual physical system x "actualizes" some properties of the system that were only "potential" before the measurement). Renouncing LOC implies renouncing the principle of separability for far apart systems in QP[15] and/or accepting some of the alternatives listed in the Introduction, which raises serious epistemological problems. The contradiction between R and LOC, from one side, and QP, from the other side, can thus be considered a paradox in the sense specified in GS.96.

3. THE FURRY PARADOX

The answer to the classical argument by Furry and Bohm (Sec. 2) is immediate if our theoretical apparatus in GS.96 is accepted. Indeed, the assumption that, whenever the result a_{1j} is obtained in the laboratory i, the state of the physical object 2 in the sample x of the physical system should have been S_{2k} even before the measurement of A_1, is wrong in our context. For, result a_{1j} allows us to affirm that the physical object 2 has the testable physical property A_{2k}, which is the support of S_{2k} both before and after the measurement (see Sec. 10 of GS.96), but it does not imply that it is in the state S_{2k} (or, equivalently, that the sample of the whole system to which the physical object 2 belongs is in the state S_{jk}) when the measurement is made, because of the difference between the extension of a state and the extension of its support (see in particular Remark 5.1 of GS.96). We can say that the physical object 2 is in the state S_{2k} only after the measurement, as a consequence of the state change from S to S_{jk} of the whole physical object. Thus, one cannot say that the ensemble $\rho_i(S)$ should be described by the mixed state T before the measurement in i, and the paradox considered here is removed.

The above solution requires some further comments. In fact, the attribution of the state S_{2k} to the physical object 2 only after a measurement has been performed on the physical object 1 may still appear intuitively paradoxical (see our first remark after the description of the paradox in Sec. 2). In order to overcome this objection, let us prove that the aforesaid attribution does not imply a change of the testable physical properties of the physical object 2. To this end, let us call $\mathscr{E}_{2S}$ (respectively, $\mathscr{E}^{\perp}_{2S}$) the set of all properties of subsystem 2 that are certainly true (respectively, certainly false) in the state S, i.e., the set of all properties of subsystem 2 that are true (respectively, false) in every laboratory i for every sample of the whole system which is in the state S. Furthermore, let us briefly denote by $\mathscr{E}_{2k}$ the certainly true domain of the state S_{2k}, and let us recall that we have made the convention of implying the dependence of k on the index h in Sec. 2. Finally, let us note that every property E of 2 is represented in HSQT by the prolongation $\hat{P}_E$ to $\mathscr{H}$ of a suitable projection operator P_E defined on $\mathscr{H}_2$. Then, we get:

$E \in \mathscr{E}_{2S}$ iff $\langle \kappa | \hat{P}_E | \kappa \rangle = 1$ iff $\sum_h |\lambda_h|^2 \langle u_{2k} | P_E | u_{2k} \rangle = 1$ iff for every index h, $\langle u_{2k} | P_E | u_{2k} \rangle = 1$ iff for every index h, $E \in \mathscr{E}_{2k}$.

Analogously, we get:
$E \in \mathscr{E}^{\perp}_{2S}$ iff for every index h, $E \in \mathscr{E}^{\perp}_{2k}$.

Thus, for every index h and $k = k(h)$, $\mathscr{E}_{2S} \subseteq \mathscr{E}_{2k}$ and $\mathscr{E}^{\perp}_{2S} \subseteq \mathscr{E}^{\perp}_{2k}$. It follows that the change of state induced by an idealized measurement of A_1 of the kind studied in Sec. 10 of GS.96, which yields the result a_{1j}, just enlarges the set of all properties that are known to be true (or false) for the physical object 2. This means that every physical property of the physical object 2 which is certainly true (or false) in S remains true (or false) with certainty after the measurement of A_1, consistently with the absence of any interaction with this object during the measurement. Furthermore, the measurement itself shows that the property A_{2k}, hence all properties in $\mathscr{E}_{2k}$ must be attributed to the physical object 2 immediately before the measurement (because of the outcome of the measurement) and immediately after it (since the state of this object is S_{2k} after the measurement). More generally, by using arguments analogous to the ones offered in our discussion in Sec. 10 of GS.96, the sets $\mathscr{E}_{i2}^{T}$ and $\mathscr{E}_{i2}^{F}$ of all true and false properties of the physical object 2, respectively, in the laboratory i, can be assumed to remain unaltered during the measurement of A_1. Therefore, we conclude that no change in the physical properties attributed to the physical object 2 must necessarily occur because of the change of state induced by the measurement of A_1, consistently with the absence of any interaction of the measuring apparatus with the physical object 2, and the change of state can be interpreted as a change of the information (see Sec. 5 of GS.96) that is available on the physical objects 1 and 2 (we note that our analysis of ideal measurements in Sec. 10 of GS.96 shows that the sets $\mathscr{E}_{ix}^{T}$ and $\mathscr{E}_{ix}^{F}$ of all true and false properties, respectively of a sample x of the whole system in the laboratory i, may change after a measurement on the physical object 1, consistently with the existence of an interaction of the measuring apparatus with 1, hence with x; of course, also $\mathscr{E}_{i2}^{T}$ and $\mathscr{E}_{i2}^{F}$ may change when considering different samples of system 2, though $\mathscr{E}_{2S}$ and $\mathscr{E}_{2S}^{\perp}$ remain unchanged).

4. THE BOHM–AHARONOV PARADOX

Let us refer to the definitions and symbols introduced in Sec. 2. We preliminarily prove the following statement:

$$\langle u_{1j} | v_{1m} \rangle \neq 0 \text{ iff } \langle u_{2k} | v_{2n} \rangle \neq 0$$

Let us assume that $\langle u_{1j} | v_{1m} \rangle = 0$. Then we get, putting $j' = j(h')$, $k' = k(h')$, $m' = m(l')$, $n' = n(l')$:

$$\begin{aligned} \langle u_{2k} v_{1m} | \kappa \rangle &= \langle u_{2k} | \sum_{h'} \lambda_{h'} \langle v_{1m} | u_{1j'} \rangle \, | u_{2k'} \rangle \\ &= \sum_{h' \neq h} \lambda_{h'} \langle v_{1m} | u_{1j'} \rangle \, \delta_{hh'} = 0 = \langle u_{2k} | \sum_{l'} \mu_{l'} \langle v_{1m} | v_{1m'} \rangle \, | v_{2n'} \rangle \\ &= \sum_{l'} \mu_{l'} \, \delta_{l'} \langle u_{2k} | v_{2n'} \rangle = \mu_l \langle u_{2k} | v_{2n} \rangle \end{aligned}$$

which shows that $\langle u_{2k} | v_{2n} \rangle = 0$, since $\mu_l \neq 0$ (Sec. 2). Hence $\langle u_{2k} | v_{2n} \rangle \neq 0$ implies $\langle u_{1j} | v_{1m} \rangle \neq 0$. By reversing the roles of systems 1 and 2 we get that the converse implication also holds true. This completes our proof.

By recalling our treatment of the consistency relation C in Sec. 5 of GS.96, the above statement can be reformulated and completed as follows:

$$S_{1j} C S_{1m} \text{ iff } S_{2k} C S_{2n} \text{ iff } S_{jk} C S_{mn} \text{ iff } A_{1j} C B_{1m} \text{ iff } A_{2k} C B_{2n}$$

Let us come now to the Bohm–Aharonov paradox (Sec. 2). By using the above results we conclude that two cases may occur.

(i) Let $S_{jk} \subset S_{mn}$. Then, A_{1j} and B_{1m} (equivalently, all properties in the certainly true domains $\mathscr{E}_{1j}$ and $\mathscr{E}_{1m}$ of S_{1j} and S_{1m}, respectively) are consistent, which means that A_{1j} and B_{1m} can be conjointly true (see Sec. 5 of GS.96) for the physical object 1 in a given sample of the physical system. Analogously, the properties A_{2k} and B_{2n} (equivalently, all properties in the certainly true domains $\mathscr{E}_{2k}$ and $\mathscr{E}_{2n}$ of S_{2k} and S_{2n}, respectively) can be conjointly attributed to the corresponding physical object 2. This means that different choices of the measurement to be performed on the physical object 1 do not lead to inconsistent information on the physical object 2, so that no paradox occurs.

(ii) Let $S_{jk} \not\subset S_{mn}$. Then, A_{1j} and B_{1m} are not consistent, that is, they cannot be conjointly true for the physical object 1 (hence, if A_1 is measured and yields the result a_{1j}, we can assert that a measurement of B_1 could not have yielded the result b_{1m} if performed on the same physical object 1). Correspondingly, A_{2k} and B_{2n} cannot be conjointly attributed to the physical object 2. Again, no paradox occurs.

In order to understand the underlying reasons of our solution, it is important to note that in case (i) the properties A_{1j} and B_{1m}, though semantically compatible (consistent), could be not pragmatically compatible (conjointly testable) in the sense specified in Sec. 6 of GS.96, that is, it could occur that $A_{1j} \not\mathscr{K} B_{1m}$. This has no effect on our above arguments, but unveils the deep roots of the Bohm–Aharonov paradox, showing that they lie in the adoption of a verificationist theory of truth and meaning. Indeed, whenever such an adoption is made, A_{1j} and B_{1m}, being not conjointly testable, cannot be conjointly true, hence they are mutually exclusive, which creates the paradox. Our solution is based on the fact that A_{1j} and B_{1m} can be consistent in our approach, though they are not conjointly testable. It is then true that a physical object 2 is in one of the states S_{2k} after a measurement of A_1 on 1, while it is in one of the states S_{2n} after a measurement of B_1 on 1, even if none of these two measurements acts on 2: but this can appear paradoxical only if states are semantically identified with properties (or subsets of properties), which is erroneous in the SR approach to QP (see Sec. 5 of GS.96). If this identification is avoided, one easily deduces, by using our discussion on semantic compatibility and ideal measurements in GS.96 (Secs. 5 and 10), that a change of the state attributed to the physical object 2 may change the set of properties of 2 that are known to be true, but it does not necessarily change the sets $\mathscr{E}^T_{i2}$ and $\mathscr{E}^F_{i2}$ of all properties of 2 that are true, or false (in the laboratory i

where either A_1 or B_1 is measured), respectively. Thus, whenever the possible states after the measurements of A_1 and B_1 are consistent, these measurements provide different but not contradictory information about the physical object 2. Consequently, the opinion that the Bohm–Aharonov paradox exhibits some kind of inherent subjectivity in QP (alternatively, the nonlocality of QP and/or the existence of superluminal connections) is false in our context, which explains it as following from some implicit epistemological preconceptions (in particular, the requirement that states can be physically identifiable with properties in QP, as it occurs in CP[27]).

Finally, let us observe that some physicists could still object to our previous argument that the properties in $\mathscr{E}_{2k}$ could be "contradictory" with respect to the properties in $\mathscr{E}_{2n}$, regardless if S_{2k} is consistent with S_{2n} (such a contradiction could be claimed, for instance, in the Bohm version of the EPR experiment, whenever A_1 is a spin measurement along the z axis and B_1 a spin measurement along a direction which makes an angle ϑ with z, ϑ being such that $0 < \vartheta < \pi/2$). The answer to this objection is rather obvious in our present framework. Indeed we have proved in Sec. 5 of GS.96 that no inconsistency occurs among the properties in $\mathscr{E}_{2k}$ and in $\mathscr{E}_{2n}$ whenever $S_{2k} C S_{2n}$. Seeming contradictions occur in most cases because of the implicit adoption of an unfaithful (usually semiclassical) model for the theory (for instance, a vector model for spin).

5. THE CRITIQUE TO A SAMPLE PROOF OF THE BELL THEOREM

We have seen in Sec. 2 that a number of variants and proofs of the Bell theorem exist. One of us has recently discussed in detail two different ways, both based on MGP, for invalidating a typical sample proof (i.e., the simple and lucid reformulation of Wigner's proof provided by Sakurai[28]) in which an inequality is obtained which is violated by QP.[5-7] We intend to unify these treatments here, to generalize the physical situation that is taken into account and to adapt the arguments to our present context, also aiming to underline some characteristic features of the subject and to supply a pattern for our discussion in Sec. 6 of different cases that occur in the literature.

In order to accomplish our task we must reformulate in our present terms and generalize the Sakurai proof. To this end we need some further definitions and symbols.

Let X, $\mathscr{S}$, $\mathscr{F}_c$ be defined as in GS.96, let $x \in X$, $S \in \mathscr{S}$, $F, F_1, F_2, \ldots, F_n \in \mathscr{F}_c$, and let us put $A(x) = F_1(x) \wedge F_2(x) \wedge \cdots \wedge F(x)$. We denote by $p(F(x)/S(x))$ the conditional probability that a physical object x in the state S has the property F. Furthermore, we recall (see Sec. 2 of GS.96) that, for every finite set Γ, $n(\Gamma)$ denotes the number of elements in Γ, and for every laboratory i, we put:

$$f_i(F(x)/S(x)) = n(\rho_i(F) \cap \rho_i(S))/n(\rho_i(S))$$

$$f_i(A(x)/S(x)) = n(\rho_i(F_1) \cap \rho_i(F_2) \cap \cdots \cap \rho_i(F_n) \cap \rho_i(S))/n(\rho_i(S))$$

By referring to the definitions and interpretation introduced in Sec. 2 of GS.96, one sees that $f_i(F(x)/S(x))$ and $f_i(A(x)/S(x))$ denote relative frequencies and coincide with the values of r that make the statements $(\pi_r x)(F(x)/S(x))$ and $(\pi_r x)(A(x)/S(x))$, respectively, true in i.

Let us add some comments on the functions p and f_i, which are relevant for our treatment.

The quantity $p(F(x)/S(x))$ can always be evaluated in QP by means of the laws of the theory, and for every laboratory $i \in \hat{I}$ one obviously has, in QP, $p(F(x)/S(x)) \cong f_i(F(x)/S(x))$. This approximation can be formalized by using the probability-frequency correlation principle,[27] and it implies that $f_i(F(x)/S(x))$ becomes independent of i whenever the number of elements in D_i that must be taken into account for calculating $f_i(F(x)/S(x))$ is large, as it can occur in every $i \in \hat{I}$ (see Sec. 2(iv) of GS.96). This is consistent with MGP (see Sec. 8 of GS.96), which states that every empirical physical law $V_F = (\pi_r x)(F(x)/S(x))$ is true in every laboratory i where conjointly testable premises are given, the value of r being independent of i.

A different situation occurs regarding $f_i(A(x)/S(x))$. Indeed, proposition P 7.1 of GS.96 states that $A(x)$ (hence $(\pi_r x)(A(x)/S(x))$) is testable whenever $F_1, F_2, \ldots, F_n$ are pairwise pragmatically compatible. In this case a property $F_A \in \mathscr{F}_v$ exists such that $A(x) \equiv F_A(x)$, so that we get

$$f_i(A(x)/S(x)) = f_i(F_A(x)/S(x)) \cong p(F_A(x)/S(x))$$

and the probability $p(F_A(x)/S(x))$, hence $f_i(A(x)/S(x))$, can be evaluated by using the laws of QP. On the contrary, whenever $F_1, F_2, \ldots, F_n$ are not pairwise pragmatically compatible, $A(x)$ is not testable and no prediction nor test can be made in QP of the frequency $f_i(A(x)/S(x))$ (one can also retain that $f_i(A(x)/S(x))$ changes with i in this case, even if $i \in \hat{I}$, so that no probability value $p(A(x)/S(x))$ is reliable if one accepts the aforesaid probability-frequency correlation principle).

Let us come to the physical situation described in Sec. 2. We have already implicitly used in Sec. 3 and 4 the physical law V that rules this situation, for instance when stating that A_{2k} must be true for a given physical object whenever we discover, by means of a measurement of A_1, that

A_{1j} is true for x (Sec. 3). Let L_e, $\mathscr{D}_e$ be defined as in GS.96, and let us write down V by means of L_e. We get

$$V = (\forall x)(S(x) \rightarrow E_S(x))$$

where E_S is the support of S, and $E_S \in \mathscr{D}_e$, since S is a second type state, hence V is a theoretical physical law. In HSQT the theoretical property E_S is represented by the projection on $|\kappa\rangle$, which is one-dimensional.

By taking into account the explicit expression of $|\kappa\rangle$, we get from HSQT:

$$E_S < E_A = (A_{1j} \cap A_{2k}) \cup (A_{1j}^{\perp} \cap A_{2k}^{\perp})$$
$$E_S < E_B = (B_{1m} \cap B_{2n}) \cup (B_{1m}^{\perp} \cap B_{2n}^{\perp})$$
$$E_S < E_C = (C_{1q} \cap C_{2r}) \cup (C_{1q}^{\perp} \cap C_{2r}^{\perp})$$

Since A_{1j} and A_{2k} are pragmatically compatible, $E_A \in \mathscr{F}_e$, and analogously $E_B, E_C \in \mathscr{F}_e$. It follows that the wffs

$$U_A = (\forall x)(S(x) \rightarrow E_A(x))$$
$$U_B = (\forall x)(S(x) \rightarrow E_B(x))$$
$$U_C = (\forall x)(S(x) \rightarrow E_C(x))$$

express empirical physical laws that can be deduced from V. A situation of this kind has been studied in Sec. 8 of GS.96, and it is interesting to note that, since E_A, E_B, E_C are not pairwise pragmatically compatible, the results obtained there prohibit us to assert here that U_A, U_B, U_C are conjointly true in a laboratory i.

In order to find equivalent forms of U_A, U_B, U_C, let us first prove that $E_A(x) \equiv A_{1j}(x) \leftrightarrow A_{2k}(x)$. Indeed, because of standard properties of the classical connective $\leftrightarrow$, the wffs $A_{1j}(x) \leftrightarrow A_{2k}(x)$ and $(A_{1j}(x) \wedge A_{2k}(x)) \vee (\neg A_{1j}(x) \wedge \neg A_{2k}(x))$ are logically equivalent. Since A_{1j} and A_{2k} are compatible, we get (Remark 7.1 of GS.96) $A_{1j}(x) \leftrightarrow A_{2k}(x) \equiv (A_{1j}(x) \cap A_{2k}(x)) \cup (A_{1j}^{\perp}(x) \cap A_{2k}^{\perp}(x)) \equiv (A_{1j} \cap A_{2k}) \cup (A_{1j}^{\perp} \cap A_{2k}^{\perp})(x)$ which proves our statement.

By means of analogous arguments, one can prove that $E_B \equiv (B_{1m}(x) \leftrightarrow B_{2n}(x))$ and that $E_C \equiv (C_{1q}(x) \leftrightarrow C_{2r}(x))$. Hence, we get, because of proposition P 4.1 of GS.96:

$$U_A \equiv (\forall x)(S(x) \rightarrow ((A_{1j}(x) \leftrightarrow A_{2k}(x)))$$
$$U_B \equiv (\forall x)(S(x) \rightarrow ((B_{1m}(x) \leftrightarrow B_{2n}(x)))$$
$$U_C \equiv (\forall x)(S(x) \rightarrow ((C_{1q}(x) \leftrightarrow C_{2r}(x)))$$

We can now proceed with our program of generalizing the Sakurai proof to our present case and then invalidating it by using MGP. To this end, let us firstly accept the basic assumptions R and LOC (Sec. 2), and let us reformulate them referring to the language L defined in GS.96, as follows.

R. *For every laboratory $i \in \hat{I}$ and for every interpretation σ_i of the (individual) variables, every wff of the observative language L has a defined (though possibly unknown) truth value, independently of the act of observation.*

LOC. *Whenever subsystems* 1 *and* 2 *are sufficiently far apart, for every laboratory $i \in \hat{I}$ and for every interpretation σ_i of the (individual) variables, the truth value of a wff of the observative language L regarding subsystem* 1 (*or* 2) *is not influenced by any measurement carried out on subsystem* 2 (*or* 1).

It follows from R and LOC, in every laboratory i:

$$\begin{aligned} &f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x))/S(x)) \\ &\quad = f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x)) \\ &\qquad + f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x) \wedge C_{1q}^{\perp}(x))/S(x)) \end{aligned}$$

(indeed, for every laboratory i, $\rho_i(C_{1q}^{\perp}) = D_i \backslash \rho_i(C_{1q})$, hence the number of physical objects in the state S that have the properties A_{1j} and $B_{2n}^{\perp}$ is equal to the number of objects in S that have the properties A_{1j}, $B_{2n}^{\perp}$ and C_{1q} plus the number of objects in S that have the properties A_{1j}, $B_{2n}^{\perp}$ and $C_{1q}^{\perp}$). Analogously, we get

$$\begin{aligned} &f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x)) \\ &\quad = f_i((A_{1j}(x) \wedge B_{2n}(x) \wedge C_{1q}^{\perp}(x))/S(x)) \\ &\qquad + f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x) \wedge C_{1q}^{\perp}(x))/S(x)) \end{aligned}$$

and

$$\begin{aligned} &f_i((B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x)) \\ &\quad = f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x)) \\ &\qquad + f_i((A_{1j}^{\perp}(x) \wedge B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x)) \end{aligned}$$

By comparing these equations, we obtain the following inequality

$$\begin{aligned} &f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x))/S(x)) \\ &\quad \leqslant f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x)) + f_i((B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x)) \quad (5.1) \end{aligned}$$

Since A_{1j} and $B_{2n}^{\perp}$ are pragmatically compatible, $\rho_i(A_{1j} \cap B_{2n}^{\perp}) = \rho_i(A_{1j}) \cap \rho_i(B_{2n}^{\perp})$. Hence,

$$f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x))/S(x))$$
$$= f_i((A_{1j} \cap B_{2n}^{\perp})(x)/S(x)) \cong p((A_{1j} \cap B_{2n}^{\perp})(x)/S(x))$$

Analogously

$$f_i((B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x))$$
$$= f_i((B_{2n}^{\perp} \cap C_{1q})(x)/S(x)) \cong p((B_{2n}^{\perp} \cap C_{1q})(x)/S(x))$$

Furthermore, let us consider $f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x))$. By using $U_{C'}$, it seems obvious to predict that 1 has the property C_{1q} iff 2 has the property C_{2r}. It follows:

$$f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x)) = f_i((A_{1j}(x) \wedge C_{2r}^{\perp}(x))/S(x)) \qquad (5.2)$$

Hence we get, being A_{1j} and C_{2r} compatible,

$$f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x))$$
$$= f_i((A_{1j} \cap C_{2r}^{\perp})(x))/S(x)) \cong p((A_{1j} \cap C_{2r}^{\perp}(x)/S(x))$$

By substituting in inequality (5.1) we obtain the following *Generalized Bell's inequality*:

GBI. $$p((A_{1j} \cap B_{2n}^{\perp}(x)/S(x))$$
$$\leqslant p((A_{1j} \cap C_{2r}^{\perp})(x)/S(x)) + p((B_{2n}^{\perp} \cap C_{1q})(x)/S(x))$$

GBI obviously reduces to a standard BI whenever suitable specific systems are considered, together with specific observables (e.g., whenever one considers a system of two spin-1/2 particles in the singlet state, GBI reduces to the BI supplied by Sakurai[28]), and it is well known that the probability predicted by QP may violate the BI in these cases, so that GBI is not consistent with QP (it is also well known that a number of empirical tests confirm the quantum predictions[29-31]). Thus, one concludes that either R, or LOC, or both, are not consistent with QP (*Bell's theorem*).

Let us come now to our criticism of this conclusion. Let us analyze the reasoning that leads to (5.2) in detail. First, one considers all samples of the system in any laboratory i such that subsystems 1 and 2 have the properties A_{1j} and $C_{2r}^{\perp}$ (that are pragmatically compatible) respectively, and concludes that in each of them subsystem 1 also has the property $C_{1q}^{\perp}$ because of the empirical physical law $U_{C'}$. Hence he gets the inequality:

$$f_i((A_{1j}(x) \wedge C_{2r}^{\perp}(x))/S(x)) \leqslant f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x)) \qquad (5.3)$$

Second, one considers all samples in i such that subsystem 1 has the properties A_{1j} and $C_{1q}^{\perp}$ (which can occur in our framework but cannot be verified since A_{1j} and $C_{1q}^{\perp}$ are not pragmatically compatible) and concludes that in each of them subsystem 2 has the property $C_{2r}^{\perp}$, again because of the physical law $U_{C'}$. Hence he gets the further inequality:

$$f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x)) \leqslant f_i((A_{1j}(x) \wedge C_{2r}^{\perp}(x))/S(x)) \qquad (5.4)$$

Putting together (5.3) and (5.4), one deduces (5.2).

It is apparent that the above reasoning rests on the assumption that $U_{C'}$ is true in i in every physical situation. But this is incorrect in our context, since we have seen in Sec. 8 of GS.96 that the truth of an empirical physical law can be asserted in a laboratory i only if the premises in i are conjointly testable (criterion MGP). Therefore, let us explore what happens whenever MGP is correctly applied.

Obviously, the reasoning that leads to (5.3) holds unaltered, since A_{1j} and $C_{2r}^{\perp}$ are pragmatically compatible, hence $A_{1j}(x)$ and $C_{2r}^{\perp}(x)$ are conjointly testable premises, and MGP assures that $U_{C'}$ must be true in i in this case. On the contrary, the reasoning that leads to (5.4) breaks down. Indeed, MGP does not assure that $U_{C'}$ remains true whenever $A_{1j}(x)$ and $C_{1q}^{\perp}(x)$ are assumed to be true, since A_{1j} and $C_{1q}^{\perp}$ are not pragmatically compatible, hence $A_{1j}(x)$ and $C_{1q}^{\perp}(x)$ are not conjointly testable premises. Thus we cannot obtain equality (5.2), which is basic for deducing GBI, since it allows us to substitute the frequency $f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x))$ with the probability $p((A_{1j}^{\perp} \cap C_{2r}^{\perp})(x)/S(x))$.

This result shows that we have reached our goal. Indeed, it leads us to invalidate GBI because of MGP, hence because of the possible breakdown of an empirical physical law under conditions in which it cannot be verified, without invalidating R and/or LOC. This implies that the violation of GBI that occurs in QP can be explained without renouncing R and/or LOC, which are thus reconciled with QP.

It is interesting to note that, whenever MGP is adopted, $f_i((A_{1j}(x) \wedge B_{2n}^{\perp}(x))/S(x))$ and $f_i((B_{2n}^{\perp}(x) \wedge C_{1q}(x))/S(x))$ can still be replaced by $p((A_{1j} \cap B_{2n}^{\perp})(x)/S(x))$ and $p((B_{2n}^{\perp} \cap C_{1q})(x)/S(x))$, respectively, as above. By substituting in (5.1) we obtain the following *weakened Bell's inequality* in place of GBI:

$$\text{WBI.} \quad p((A_{1j} \cap B_{2n}^{\perp})(x)/S(x)) \leqslant f_i((A_{1j}(x) \wedge C_{1q}^{\perp}(x))/S(x)) + p((B_{2n}^{\perp} \cap C_{1q})(x)/S(x)),$$

which holds in every laboratory i.

It is apparent that WBI, which is not an empirical law in the sense specified in Sec. 8 of GS.96, cannot conflict with the predictions of QP (which regard probability values) nor can be contradicted by any experimental result, since A_{1j} and $C^{\perp}_{1q}$ are not pragmatically compatible, and $f_i((A_{1j}(x) \wedge C^{\perp}_{1q}(x))/S(x))$ (which might take different values in different laboratories) cannot be measured.

6. THE GENERAL CRITIQUE TO THE BELL THEOREM

We have invalidated in Sec. 5 a sample proof of the Bell theorem. But we have seen in Sec. 2 that a number of proofs of this theorem exist. Without giving the details of each proof, we will discuss in this section how our arguments in the sample case can be adapted to some different relevant cases in the literature.

We have divided in Sec. 2 the existing proofs of the Bell theorem into two classes: the class of all proofs that make resort to an inequality (BI) and the class of those that do not. Our sample case in Sec. 5 belongs to the former class, and our discussion constitutes a paradigm for invalidating all proofs of this class, even if our arguments must obviously be suitably modified case by case. Thus, for instance, in the Wigner[22] proof of a BI, which is an ancestor of the Sakurai proof, MGP invalidates Wigner's statement that "$(\sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3) = 0$ unless $\tau_1 = -\sigma_1$, $\tau_2 = -\sigma_2$, $\tau_3 = -\sigma_3$." It must, however, be noted that in some cases MGP does not lead to question a BI directly. This occurs, in particular, in the original treatment by Bell[20] and, more generally, whenever the BI takes the form $\Delta \leqslant 2$, where Δ is the sum of four "correlation functions."[2, 23] Indeed, the proof of this BI does not require in these cases the use of physical laws, hence it is not questioned by MGP (in this sense, $\Delta \leqslant 2$ should be compared to the inequality (5.1), rather than to GBI). But in this subclass of proofs the quantum expressions of every correlation function that appears in Δ is obtained by using physical laws similar to U_A, U_B, U_C, and the conjoint attribution of all values of the correlation functions calculated in this way to a given ensemble $\rho_i(S)$ in a laboratory i is prohibited by MGP. Thus, the quantity Δ_Q obtained by substituting in Δ the quantum expressions of the correlation functions must be considered a mathematical quantity (the different terms in it being testable in different laboratories) that cannot be identified with Δ whenever MGP is accepted (hence, no paradox occurs whenever one obtains that $\Delta_Q > 2$ in special cases). This invalidates the proofs in the subclass that we are considering, since they rest on the identification $\Delta_Q = \Delta$.

Our arguments in Sec. 5 can, however, be also adapted to provide a critique of those proofs of the Bell theorem that do not use inequalities. Let us consider for instance the proof supplied by Greenberger, Horne, and Zeilinger (GHZ proof), as reported in 1990 in the paper by Greenberger *et al.* (GHSZ paper).[24] Limiting ourselves to a nonformalized and intuitive discussion, let us note that in this proof the following general theoretical law is stated:

$$E^{\Psi}(\hat{n}_1, \hat{n}_2, \hat{n}_3, \hat{n}_4) = -\cos(\phi_1 + \phi_2 - \phi_3 - \phi_4) \tag{6.1}$$

[see Eq. (9) in the GHSZ paper]. Then, a set of empirical physical laws is deduced from Eq. (6.1) by assuming locality, reality, and completeness (but the last assumption does not seem strictly needed), as follows.

$$A_\lambda(0)\, B_\lambda(0)\, C_\lambda(0)\, D_\lambda(0) = -1 \tag{6.2}$$

$$A_\lambda(\phi)\, B_\lambda(0)\, C_\lambda(\phi)\, D_\lambda(0) = -1 \tag{6.3}$$

$$A_\lambda(\phi)\, B_\lambda(0)\, C_\lambda(0)\, D_\lambda(\phi) = -1 \tag{6.4}$$

$$A_\lambda(2\phi)\, B_\lambda(0)\, C_\lambda(\phi)\, D_\lambda(\phi) = -1 \tag{6.5}$$

and

$$A_\lambda(\vartheta + \pi)\, B_\lambda(0)\, C_\lambda(\vartheta)\, D_\lambda(0) = 1 \tag{6.6}$$

[See Eqs. (12a), (12b), (12c), (12d), and (17), respectively, in the GHSZ paper]. Finally, a contradiction is proven to occur when simultaneously applying Eqs. (6.2)–(6.6):

$$A_\lambda(\pi) = A_\lambda(0)$$

$$A_\lambda(\pi) = -A_\lambda(0)$$

According to GHSZ, this contradiction should show that the quantum law (6.1) is inconsistent with locality, reality and completeness. More specifically, it proves the inconsistency of these assumptions with the perfect correlations between spin directions that occur because of Eq. (6.1) in special cases. But GHZ assume in their proof that perfect correlation simultaneously holds in different directions, that is, they implicitly assume the classical principle MCP (see GS.96) or, equivalently, the simultaneous unrestricted validity of all empirical laws deduced from the theoretical law (6.1). This is no longer correct if MGP is adopted, as it can easily be proved by using the results obtained in Sec. 8(i) of GS.96 (our argument can be summarized by saying that, whenever one of the empirical physical laws (6.2)–(6.6) is assumed to hold in a laboratory *i*, it constitutes some premises in *i* that are easily seen to be noncompatible with the premises that one introduces further in order to deduce predictions from another law of the set; thus, one cannot use all the aforesaid empirical laws conjointly in the laboratory *i*, which is sufficient to invalidate the GHZ proof, and no inconsistency occurs between QP, here represented by the theoretical law (6.1), and the assumptions of locality and reality).

Let us now compare the above intuitive invalidation of the GHZ proof by means of MGP with the more formal invalidation of the Sakurai proof in Sec. 3. It is then apparent that the reality and locality assumptions here correspond to assumptions R and LOC, respectively, in Sec. 5, while the perfect correlation laws here correspond to the empirical laws U_A, U_B, U_C in Sec. 5. In both cases, the standard viewpoint assumes implicitly the unrestricted validity of these laws in QP (hence MCP is implicitly adopted); it follows that the contradiction between the assumptions, from one side, and the consequences of the aforesaid laws, from the other side, is interpreted as a contradiction of the assumptions themselves with QP. This argument fails whenever MGP is accepted, since MGP restricts the validity of physical laws in QP, so that the aforesaid contradiction does not imply any more a conflict between the assumptions and QP.

Finally, we recall that the GHZ proof has been modified or generalized by several authors.[12, 32, 33] However, the arguments expounded above can be adapted in order to invalidate these new proofs whenever our SR viewpoint is accepted. In particular, our critique of the Mermin version of the GHZ proof provided in Mermin's 1993 paper[12] can be directly obtained from the critique of the Mermin proof of the Bell–KS theorem in the next section.

Remark 6.1. It follows from our discussion on the contextuality and nonlocality paradoxes in Sec. 2 that the invalidation of the Bell theorem in the SR approach to QP prohibits one to assert that QP necessarily conflicts with LOC. But one may then wonder what would happen if he should perform a suitable test of a given Bell inequality. Would the inequality be violated or not? Our treatment allows us to provide a simple but not trivial answer (see also GS.96, Introduction). Indeed, according to the SR approach to QP, a Bell inequality either is incorrect (as the Wigner inequality and GBI in Sec. 5), or is a correct theoretical formula which is not epistemically accessible in QP (as the original Bell inequality and WBI in Sec. 5). In both cases, the inequalities that can be tested in QP cannot be identified with Bell inequalities, so that no contradiction can occur, and any devised physical experiment on Bell inequalities actually tests something else (correlations among properties of physical objects in accessible contexts). Thus, a Bell inequality does not provide a method for testing experimentally whether either QP or LOC is correct, contrary to the belief of many physicists concerned with the foundations of QP.

However, any experimental confirmation of the quantum inequalities is not irrelevant from a SR viewpoint. Indeed, our discussion in this section and in Sec. 5 shows that the contradiction between the Bell inequalities and the inequalities predicted by QP is removed only if the MCP principle fails to be true in the specific case that is considered. Thus, if quantum inequalities turn out to be satisfied, there must be empirical physical laws of QP (as the perfect correlation law quoted above) which fail to be true in physical contexts that are not directly accessible according to QP itself. This suggests, in particular, that a hidden variables theory for QP (see GS.96, Remark 8.2) should violate the Kochen and Specker constraints[11, 12] in laboratories where (pragmatically) noncompatible premises are assumed (of course, these constraints should be satisfied in every laboratory where the assumed premises are pairwise compatible). To the best of our knowledge, an attempt at constructing a hidden variables theory of this kind has never been done, and the above suggestion seems to us an interesting byproduct of the SR approach to QP, since it opens the way to possibly noncontextual and local theories.

7. THE CRITIQUE TO THE BELL–KS THEOREM

Both the formal invalidation of a particular proof of the Bell theorem in Sec. 5 and the informal arguments that we have provided in Sec. 6 in order to extend the invalidation to different proofs of this theorem can be modified and generalized so as to obtain formal or intuitive invalidations of the existing proofs of the Bell–KS theorem. For the sake of brevity, we will limit ourselves here to consider the informal arguments only and to apply them to the simple and immediate version of the Bell–KS theorem in four (or more) dimensions provided by Mermin.[12] It is rather apparent that our arguments apply also to the other proofs in the literature, and we leave this exercise to the reader.

By using Mermin's symbols and procedures, let us introduce a set of nine dichotomic observables (eigenvalues ± 1) for a physical system made up of two subsystems (labeled 1 and 2; the observables are represented in terms of the Pauli matrices for two independent spin-1/2 particles), arranged as in Table I.

Table I

σ_x^1	σ_x^2	$\sigma_x^1\sigma_x^2$
σ_y^2	σ_y^1	$\sigma_y^1\sigma_y^2$
$\sigma_x^1\sigma_y^2$	$\sigma_x^2\sigma_y^1$	$\sigma_z^1\sigma_z^2$

By multiplying the elements in each row and column in Table I and using the commutation rules of spin-like observables, we get

$$(\sigma_x^1)(\sigma_x^2)(\sigma_x^1\sigma_x^2)=I \tag{7.1}$$

$$(\sigma_y^2)(\sigma_y^1)(\sigma_y^1\sigma_y^2)=I \tag{7.2}$$

$$(\sigma_x^1\sigma_y^2)(\sigma_x^2\sigma_y^1)(\sigma_z^1\sigma_z^2)=I \tag{7.3}$$

$$(\sigma_x^1)(\sigma_y^2)(\sigma_x^1\sigma_y^2)=I \tag{7.4}$$

$$(\sigma_x^2)(\sigma_y^1)(\sigma_x^2\sigma_y^1)=I \tag{7.5}$$

$$(\sigma_x^1\sigma_x^2)(\sigma_y^1\sigma_y^2)(\sigma_z^1\sigma_z^2)=-I \tag{7.6}$$

The above equalities express derived general theoretical laws. Indeed, each of them is obtained by defining a new observable (the product on the left side) and by deducing that it must be equal to the identity I because of primitive theoretical laws (the commutation rules of spin-like observables; note that these laws hold for every possible state of the physical system, hence a formal expression of them by means of a general formalized language, as the language L^* mentioned in GS.96, Sec. 8, contains quantification on state variables: this means that they cannot be expressed by means of L_e, which does not contain predicative variables). Each of them contains the product of three pairwise commuting observables (the observables in parentheses, which can be further expressed as products of spin-like observables), but the observables that appear in one of the products do not necessarily commute with the observables that appear in another product. Now, let us select one of the above theoretical laws and let us consider all the properties associated to the three observables in it (we recall that the standard procedure for associating a family of properties to a physical observable Ω consists in considering all pairs (Ω, Δ), with Δ a Borel subset of the real line, and interpreting (Ω, Δ) as the property "the value of the observable Ω lies in the Borel set Δ"; of course, each property in the family is represented by a projection in HSQT). Since the three observables commute, we expect that all these properties are pragmatically compatible, and that all the empirical laws that can be deduced from the selected theoretical law (and that, roughly speaking, take the form of statements attributing a testable property with a certain specified probability to all physical objects in a given state) are simultaneously true in a laboratory $i \in \hat{I}$ [a situation of this kind occurs in the particular case discussed in GS.96, Sec. 8(i)]. But let us consider the empirical laws deduced from two or more of the laws (7.1)–(7.6) such that the products on the left member contain noncommuting observables (as $(\sigma_x^1\sigma_x^2)$ in Eq. (7.1) and

$(\sigma_x^1 \sigma_y^2)$ in Eq. (7.3)). Then, we expect that there are properties associated to these observables that are not pragmatically compatible, so that, taking into account MGP, we cannot assert that all the aforesaid empirical laws can be simultaneously true in i: indeed, if two or more of these laws are true in i, and they lead to predicting properties that are not pragmatically compatible, this prohibits us from assuming that further empirical laws are true in i [even a situation of this kind may occur in the particular case discussed in GS.96, Sec. 8(i)]. This result is sufficient to invalidate Mermin's proof of the Bell–KS theorem if an SR perspective is accepted. Indeed, it means that in every empirical situation the truth values of all statements regarding physical properties of a given physical object (which are defined independently of any observative context) are not bound to respect simultaneously all the conditions imposed by Eqs. (7.1)–(7.6). Thus, the impossibility that the observables appearing in Eqs. (7.1)–(7.6) take values that satisfy all these equations simultaneously does not imply that no value can be attributed to the observables independently of the set of observations that one decides to perform (contextuality), but rather that the values of the observables are not bound to satisfy in every laboratory and in every state Eqs. (7.1)–(7.6) simultaneously (it must, however,be stressed that no direct test on a physical object can reveal whether this actually occurs).

Remark 7.1. We have seen above that an adequate formalization of the theoretical laws (7.1)–(7.6) ought to be done by means of statements of the general theoretical language L^*. We recall from GS.96, Sec. 8, that second-order statements of this kind may have no defined truth value, being considered as purely formal statements from which empirical laws expressed by means of L_e can be deduced, the truth mode of which is established and limited by the MGP principle. Thus, the possibility that Eqs. (7.1)–(7.6) are not simultaneously verified by the values of the observables does not appear a blasphemous violation of some universal truth. On the other side, a violation of this kind cannot appear in any physical situation that is epistemically accessible, hence it does not imply any modification of the observative content of QP.

We also observe that the expressions of the observables in parentheses in Eqs. (7.1)–(7.6) as products of spin-like observables is used only to obtain the commutators of the observables themselves. Should one consider the products in the left members of Eqs. (7.1)–(7.6) directly in terms of spin-like observables, together with all properties associated with these latter observables, it may occur that the empirical laws deduced from one of these laws cannot be asserted to be simultaneously true in a laboratory $i \in \hat{I}$. To be precise, this occurs in the case of Eqs. (7.3) and (7.6), where the products on the left contain noncommuting spin-like observables.

Remark 7.2. The proof of the Bell–KS theorem considered here exhibits an instance of a physical situation characterized by a set of theoretical physical laws

$$V_1 = f_1(R_{11}, R_{12}, ..., R_{1m_1})$$
$$V_2 = f_2(R_{21}, R_{22}, ..., R_{2m_2})$$
$$\dots$$
$$V_n = f_n(R_{n1}, R_{n2}, ..., R_{nm_n})$$

and by the commutation rules

$$[R_{jk}, R_{j'k'}] = 0 \qquad \text{if} \quad j = j',$$
$$[R_{jk}, R_{j'k'}] \neq 0 \qquad \text{for some} \quad k, k' \quad \text{if } j \neq j'$$

Our negative conclusions on the simultaneous truth in a laboratory i of the empirical physical laws deduced from Eqs. (7.1)–(7.6) can then be easily extended to the general case. In addition, the conclusions in the second part of Remark 7.1 can be generalized by assuming that the observables R_{jk} in the above equations can be expressed as functions of further observables,

$$R_{jk} = g_{jk}(S_1^{jk}, S_2^{jk}, ..., S_{p_{jk}}^{jk})$$

fixing j, k, and considering the cases: (i) for every l and l', $[S_l^{jk}, S_{l'}^{jk}] = 0$; (ii) some l and l' exist such that $[S_l^{jk}, S_{l'}^{jk}] \neq 0$.

Remark 7.3. Our arguments in this section prove that contextuality is not an inherent feature of QP. But it must be noted that MGP introduces a new kind of contextuality in QP, which refers to the validity of empirical physical laws, not to the assignability of truth values to observative statements. We will call this kind of contextuality *pragmatic*,

Table II

	Semantic contextuality	Semantic contextuality for separated systems (nonlocality)	Pragmatic contextuality
Canonical viewpoint	Yes (Bell–KS theorem)	Yes (Bell theorem)	No
SR viewpoint	No (SR + MGP)	No (SR + MGP)	Yes

differentiating it from the conventional kind of contextuality, which will be called *semantic* here. Then, Table II synthesizes the differences between the conventional approaches and the SR approach.

8. SOME REMARKS ON PROBABILITY

We have introduced in Sec. 5 a function f_i denoting a (relative) frequency, and a function p denoting a (conditional) probability. While the former has been defined by using statistical statements in the language L_e (see Sec. 4 of GS.96), an analogous procedure cannot be applied to the latter, since no reference to abstract probability is made in L_e.

If one wants to formalize probabilistic statements, L_e must be suitably enlarged by means of a family of probability operators,[27] so as to obtain a broader language L_e^π (the link between probabilistic and statistical statements in L_e^π can then be established by the probability-frequency correlation principle[27]). For the sake of brevity, we will consider here only operators of the form Π_r, with $r \in [0, 1]$. Then, the statement "the probability that the physical object x in the state S has the property E is r" can be formalized in L_e^π by the (open) wff $\Pi_r(E(x)/S(x))$. It is important to note that the truth value of this wff in a laboratory i does not depend only on the interpretation σ_i that is chosen in i, but rather on the whole family $\{\sigma_j\}_{j \in I}$ (where each σ_j is chosen in j so that $\sigma_j(x) \in \rho_j(S)$).

A simple scheme for probabilistic physical laws is provided by the following wff:

$$A_r^\pi = (\forall x)\, \Pi_r(E(x)/S(x))$$

The assertion that A_r^π is true (which entails that $A_{r'}^\pi$, with $r' \neq r$, must be false) is equivalent to the assertion that $\Pi_r(E(x)/S(x))$ is true whatever the family $\{\sigma_j\}_{j \in I}$ may be. In this case, and only in this case, can $p(E(x)/S(x))$ be defined, and it coincides with r. Thus, the function p is defined by using a probabilistic statement in L_e^π.

Let us add some remarks. First, we observe that, whenever A_r^π is true, it associates a probability value r to the pair $(S, E) \in \mathscr{S} \times \mathscr{E}_e$. By using known techniques in logic, the language L_e^π can be further enlarged by introducing, for every $r \in [0, 1]$, a second-order diadic predicate R whose domain is $\mathscr{S} \times \mathscr{E}_e$ and which is such that $R(S, E)$ is true iff A_r^π is true, so that R can be intensionally interpreted as a property of the pair (S, E). In this sense the assignement of a probability value can be regarded as the statement of a property; this property is not a physical property that can be attributed to a physical object x (this is syntactically evident, since the formula $R(x, E)$ is not a wff) but rather a second-order property, which can be attributed to first-order predicates that are interpreted as states or physical properties.

Second, let us note that, whenever A_r^{π} is true, in every laboratory an *individual* probability r can be associated to every physical object x in the state S, which can be interpreted as the probability that x has the property E. It is then interesting to recall that individual probabilities have been considered by Selleri[23] in order to show that R and LOC imply the Bell theorem even if this kind of probabilities is introduced. However, it is apparent that our solution of the paradox applies also in this case. In addition, we stress that the individual probability r can be associated to all physical objects in the state S only conventionally. Indeed, one expects that r may change for a prefixed physical object x whenever a change of the state S of x occurs (that is, of the information regarding x), even if it can happen that there is no change in the testable physical properties of x (see Secs. 9 and 10 of GS.96). This shows again from a different viewpoint that r cannot generally be treated as a physical property of x.

The above conclusion does not hold in CP whenever a change in r occurs due to a shift from a pure state S to a pure state S'; indeed, r changes from 0 to 1, or conversely, in this case, hence a change in the attribution of the testable physical property E to the physical object x occurs, which is associated to the change of r. In this sense r can be considered a physical property of x in CP (but nonpure states must be excluded).

REFERENCES

1. M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).
2. F. Selleri, "History of the Einstein–Podolsky–Rosen paradox," in *Quantum Mechanics Versus Local Realism*, F. Selleri, ed. (Plenum, New York, 1988), p. 1.
3. C. Garola and L. Solombrino, "The theoretical apparatus of semantic realism: a new language for classical and quantum physics," *Found. Phys.* **26**, 1121 (1996).
4. C. Garola, "Semantic incompleteness of quantum physics and EPR-like paradoxes," *Int. J. Theor. Phys.* **32**, 1863 (1993).
5. C. Garola, "Reconciling local realism and quantum physics: a critique to Bell," *Teor. Mat. Fiz.* **99**, 285 (1994).
6. C. Garola, "Criticizing Bell: local realism and quantum physics reconciled," *Int. J. Theor. Phys.* **34**, 269 (1995).
7. C. Garola, "Questioning Nonlocality: an operational critique to Bell's Theorem," in *The Foundations of Quantum Mechanics. Historical Analysis and Open Questions*, C. Garola and A. Rossi, eds. (Kluwer Academic, Dordrecht, 1995), p. 273.
8. C. Garola, "Pragmatic versus semantic contextuality in quantum physics," *Int. J. Theor. Phys.* **34**, 1383 (1995).
9. W. M. De Muynck, W. De Baere, and H. Maartens, "Interpretation of quantum mechanics, joint measurement of incompatible observables, and counterfactual definiteness," *Found. Phys.* **24**, 1589 (1994).

10. H. P. Stapp, "Comments on 'Interpretation of quantum mechanics, joint measurement of incompatible observables, and counterfactual definiteness'," *Found. Phys.* **24**, 1665 (1995).
11. S. Kochen and E. P. Specker, "The problem of hidden variables in quantum mechanics," *J. Math. Mech.* **17**, 59 (1967).
12. N. D. Mermin, "Hidden variables and the two theorems of John Bell," *Rev. Mod. Phys.* **65**, 803 (1993).
13. A. Einstein, B. Podolsky, and N. Rosen, "Can quantum mechanical description of reality be considered complete?," *Phys. Rev.* **47**, 777 (1935).
14. D. Bohm, *Quantum theory* (Prentice-Hall, Englewood Cliffs, New Jersey, 1951).
15. B. D'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, Massachusetts, 1976).
16. W. H. Furry, "Note on the quantum-mechanical theory of measurement," *Phys. Rev.* **49**, 393 (1936).
17. W. H. Furry, "Remarks on measurements in quantum theory," *Phys. Rev.* **49**, 476 (1936).
18. D. Fortunato, "A property of the projection operator associated to a mixture of the second type," *Lett. Nuovo Cimento* **15**, 289 (1976).
19. D. Bohm and Y. Aharonov, "Discussion of experimental proofs for the paradox of Einstein, Rosen, and Podolsky," *Phys. Rev.* **108**, 1070 (1957).
20. J. S. Bell, "On the Einstein–Podolsky–Rosen paradox," *Physics* **1**, 195 (1964).
21. J. S. Bell, "On the problem of hidden variables in quantum mechanics," *Rev. Mod. Phys.* **38**, 447 (1966).
22. E. P. Wigner, "On hidden variables and quantum mechanical probabilities," *Am. J. Phys.* **38**, 1005 (1970).
23. F. Selleri, "Even local probabilities lead to the paradox," in *Quantum Mechanics Versus Local Realism*, F. Selleri, ed. (Plenum, New York, 1988) p. 149.
24. D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities," *Am. J. Phys.* **58**, 1131 (1990).
25. M. Redhead, *Incompleteness, Nonlocality and Realism* (Clarendon, Oxford, 1987).
26. W. Heisenberg, *Physics and Philosophy* (Harper & Row, New York, 1958).
27. C. Garola, "Classical foundations of quantum logic," *Int. J. Theor. Phys.* **30**, 1 (1991).
28. J. J. Sakurai, *Modern Quantum Mechanics* (Benjamin, Reading, Massachusetts, 1985).
29. A. Aspect, P. Grangier, and G. Roger, "Experimental tests of realistic local theories via Bell's theorem," *Phys. Rev. Lett.* **47**, 460 (1981).
30. A. Aspect, P. Grangier, and G. Roger, "Experimental realization of Einstein–Podolsky–Rosen–Bohm gedankenexperiment: a new violation of Bell's inequalities," *Phys. Rev. Lett.* **49**, 91 (1982).
31. A. Aspect, J. Dalibard, and G. Roger, "Experimental tests of Bell's inequalities using time-varying analyzers," *Phys. Rev. Lett.* **49**, 1804 (1982).
32. K. C. Clifton, M. L. G. Redhead, and J. N. Butterfield "Generalization of the Greenberger–Horne–Zeilinger algebraic proof of nonlocality," *Found. Phys.* **21**, 149 (1991).
33. N. D. Mermin, "Quantum mysteries revisited," *Am. J. Phys.* **58**, 731 (1990).