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Evaluating the Validity of Parametrized Relativistic Wave Equations

John R. Fanchi[1]

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We wish to determine the correct partial differential equation(s) for describing a relativistic particle. A physical foundation is presented for using a parametrized wave equation with the general form

$$i\frac{\partial\psi}{\partial s}=\mathbf{K}\psi$$

where s is the invariant evolution parameter. Several forms have been proposed for the generator $\mathbf{K}$ *of evolution parameter translations. Of the proposed generators, only the generator* $\mathbf{K}_2$ *which is proportional to the inner product* $\mathbf{P}_\mu\mathbf{P}^\mu$ *of four-momentum operators can be derived from first principles, notably a probabilistic basis. Although experimental tests must be made to establish the validity of* $\mathbf{K}_2$*, we conclude that* $\mathbf{K}_2$ *is the leading theoretical candidate for the form of a generator of evolution parameter translations.*

1. INTRODUCTION

Several of Professor Rohrlich's publications[1-3] have been concerned with determining the correct Hamiltonian form of a relativistic quantum theory. One aspect of this effort must invariably focus on determining the correct partial differential equation(s) for describing a relativistic particle. This paper assesses the validity of including an invariant evolution parameter in either a Dirac-like equation or a Schrödinger-like equation. Much of the necessary information for choosing between alternative wave equations was brought to light in a recent review.[4]

We present a physical foundation for parametrized relativistic

[1] 6598 S. Cook Way, Littleton, Colorado 80121.

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quantum theories in Sec. 2 and obtain a parametrized wave equation with the general form

$$i\frac{\partial\psi}{\partial s} = \mathbf{K}\psi$$

where s is the invariant evolution parameter, and natural units ($\hbar = c = 1$) are adopted. Section 3 contains several forms that have been proposed for the generator $\mathbf{K}$ of evolution parameter translations. The relative merits of the proposed generators are discussed in Secs. 3 and 4. Of the proposed generators, only the generator $\mathbf{K}_2$ which is proportional to the inner product $\mathbf{P}_\mu\mathbf{P}^\mu$ of four-momentum operators can be derived from first principles, notably the probabilistic basis outlined in Sec. 4. Although experimental tests must be made to establish the validity of $\mathbf{K}_2$, we conclude that $\mathbf{K}_2$ is the leading theoretical candidate for the form of a generator of evolution parameter translations. The role of the Dirac equation is discussed in Sec. 5.

2. SYMMETRY PRINCIPLES

Scientific theories consist of a mathematical formalism with a set of rules for relating mathematical quantities and operations to physical quantities and measurements. Relationships between data measurements are described by equations. One approach to constructing the mathematical formalism is to specify how physical variables should behave when observed from different reference frames. We adopt the fundamental physical assumption —the principle of relativity—that says relationships between physically significant quantities should remain invariant with respect to transformations between different reference frames.

One expression of the principle of relativity is to assert that two references frames are equivalent only if they are at rest with respect to each other. Such transformations include three-dimensional rotations and translations in Euclidean space and are described by the Euclidean group. The Euclidean group requires that the equations of physics be the same for two observers at rest with respect to each other, but the equations may change if the observers are moving relative to one another. If we want the equations of physics to be valid whether we are standing on land, traveling in sea, or moving through space, then the Euclidean group is too restrictive.

We extend the principle of relativity by requiring that the equations of physics be invariant for observers in relative motion. Suppose reference frame F2 is moving relative to frame F1. If we are at rest in frame F1, we

can measure the motion of frame F2 as the change in the spatial location of its origin divided by the change in a monotonically increasing evolution parameter. We require that the evolution parameter increase at the same rate in both reference frames, i.e., it is independent of the frame in which it is measured. We say the two reference frames are in uniform motion relative to one another, and we obtain the Galilei group associated with Newtonian mechanics. The evolution parameter is just the absolute time of Galileo and Newton.

The equations of Newtonian mechanics are invariant with respect to transformations associated with the Galilei group, but Maxwell's equations are not. The Michelson–Morley experiment showed that the invariance of Maxwell's equations should be preserved, even at the expense of Newtonian symmetry. New transformation requirements were needed. They were provided by Einstein's special relativity.

Special relativity requires that inertial (nonaccelerating) observers in equivalent reference frames measure the same speed for the propagation of light in vacuum. The invariant evolution parameter of Newtonian mechanics is replaced by an additional coordinate on a four-dimensional Minkowski space. The symmetry requirements of special relativity are embodied in the inhomogeneous Lorentz group, which is also known as the Poincaré group.

The Poincaré group presumes that the mass of a particle is a constant parameter. Mass is not calculated by the theory, but is put into the theory. One of the outstanding experimental findings of the past half century is the proliferation of particles of varying mass. The concepts of mass state and transitions between mass states are precluded by the assumption of Poincaré invariance. If we hope to improve our understanding of mass states, our invariance requirements must go beyond the transformations associated with the Poincaré group.

2.1. Lie Algebra

We can quantify our discussion of invariance requirements by recognizing that invariance requirements may be represented mathematically as transformations between coordinate systems. Coordinate transformations are represented by continuous symmetry groups, notably Lie groups. An r-parameter Lie group is a set of transformations with the properties of a group and labeled by r continuously varying parameters. Infinitesimal transformations must exist arbitrarily close to the identity element of the group. Infinitesimal generators may be constructed from the infinitesimal transformations to linearize the law of group multiplication. The generators form a vector space called the Lie algebra.

The Lie algebra of parametrized relativistic quantum theory (PRQT) is constructed by assuming that relationships between physically significant quantities are invariant with respect to linear transformations of the form

$$X'_\mu = \Lambda^\nu_\mu x_\nu + a_\mu + b_\mu s \tag{2.1}$$

where $\{\Lambda'_\mu\}$ represent a homogeneous Lorentz transformation, $\{a_\mu\}$ represent translations along the $\{x_\mu\}$ axes, and the four-velocity transformation ("relativistic Galilean boost" of Aghassi *et al.*[5]) is characterized by the four parameters $\{b_\mu\}$. In the parametrized theory with an independent variable s, the theory is assumed to be invariant with respect to the linear transformation

$$s' = s + \Delta s \tag{2.2}$$

where Δs represents translations along the s axis. Except for Greenberger,[6] all parametrized theories assume that the s translation is independent of space-time coordinates. We refer to the set of transformations represented by Eqs. (2.1) and (2.2) as the JARS transformation. The acronym comes from the names of the people who first studied the properties of the set, i.e., Johnson[7] and Aghassi, Roman, and Santilli.[5]

The JARS transformation may be denoted by $\{a, b, \Lambda, \Delta s\}$, which is the group element of a 15-parameter continuous group. The identity element is $\{0, 0, 1, 0\}$ and the inverse element is $\{-\Lambda^{-1}(a - b\Delta s), -\Lambda^{-1}b, \Lambda^{-1}, -\Delta s\}$. The composition law of two-group elements is

$$\begin{aligned}&\{a_2, b_2, \Lambda_2, \Delta s_2\}\{a_1, b_1, \Lambda_1, \Delta s_1\}\\ &\quad = \{a_2 + \Lambda_2 a_1 + (\Delta s_1)b_2, b_2 + \Lambda_2 b_1, \Lambda_2\Lambda_1, \Delta s_1 + \Delta s_2\}\end{aligned} \tag{2.3}$$

which expresses the product of two transformations. We call the group associated with the JARS transformation the JARS group.

Johnson[7] and Aghassi *et al.*[5] presented an algebraic structure with 15 generators. For an infinitesimal transformation the set $\{a_\mu, b_\mu, \Lambda^\nu_\mu, \Delta s\}$ becomes $\{\varepsilon_{p\mu}, \varepsilon_{x\mu}, g^\nu_\mu + \varepsilon^\nu_\mu, \varepsilon\}$, where $(\varepsilon_{p\mu}, \varepsilon_{x\mu}, \varepsilon^\nu_\mu, \varepsilon)$ are first-order infinitesimals. The infinitesimal generator for dynamical development of the system is

$$\mathbf{U} = 1 + i\mathbf{F} \tag{2.4}$$

where

$$\mathbf{F} = \tfrac{1}{2}\mathbf{J}^{\mu\nu}\varepsilon_{\mu\nu} + \mathbf{P}^\mu(\varepsilon_p)_\mu + \mathbf{X}^\mu(\varepsilon_x)_\mu + \mathbf{K}\varepsilon \tag{2.5}$$

Ten generators are the six independent elements of the antisymmetric

angular momentum tensor $\mathbf{J}^{\mu\nu}$ and the four elements of the energy-momentum four-vector $\mathbf{P}^{\mu}$. The generators of the Poincaré group satisfy the 45 usual commutation relations

$$[\mathbf{P}_{\mu}, \mathbf{P}^{\nu}] = 0 \tag{2.6}$$

$$[\mathbf{J}_{\mu\nu}, \mathbf{P}_{\lambda}] = i(\mathbf{P}_{\nu} g_{\mu\lambda} - \mathbf{P}_{\mu} g_{\nu\lambda}) \tag{2.7}$$

$$[\mathbf{J}_{\mu\nu}, \mathbf{J}_{\lambda\sigma}] = i(g_{\mu\sigma}\mathbf{J}_{\nu\lambda} + g_{\mu\lambda}\mathbf{J}_{\nu\sigma} + g_{\nu\sigma}\mathbf{J}_{\mu\lambda} + g_{\lambda\nu}\mathbf{J}_{\sigma\mu}) \tag{2.8}$$

The generator $\mathbf{K}$ generates infinitesimal translations along s. Ten commutators are evaluated by taking the commutator of the generator $\mathbf{K}$ with each of the ten generators of the Poincaré group. The generator $\mathbf{K}$ commutes with all of the generators of the Poincaré group:

$$[\mathbf{J}_{\mu\nu}, \mathbf{K}] = 0 \tag{2.9}$$

$$[\mathbf{P}_{\mu}, \mathbf{K}] = 0 \tag{2.10}$$

The additional four generators are the space-time position operators that satisfy the commutators

$$[\mathbf{X}^{\mu}, \mathbf{X}^{\nu}] = 0 \tag{2.11}$$

$$[\mathbf{P}^{\mu}, \mathbf{X}^{\nu}] = ig^{\mu\nu} \tag{2.12}$$

$$[\mathbf{J}^{\mu\nu}, \mathbf{X}^{\lambda}] = i[g^{\lambda\nu}\mathbf{X}^{\mu} - g^{\lambda\mu}\mathbf{X}^{\nu}] \tag{2.13}$$

In the classical theory, the commutator $[\mathbf{P}^{\mu}, \mathbf{X}^{\mu}]$ vanishes.

The choice of the form of the generator $\mathbf{K}$ of s translations is not unique. Several forms have been proposed[4] in the context of a wave equation. We show that the combination of symmetry principles and canonical transformations leads to a wave equation. We begin by reviewing the concept of canonical transformation and show that the JARS transformation is a canonical transformation. Commutation relations for the operators associated with the JARS transformation are then used to derive a parametrized wave equation.

2.2. Canonical Transformations

The canonical transformation of a field $\psi(x, s)$ to $\psi'(x, s)$ by a unitary operator $\mathbf{U}$ is defined by

$$\psi'(x, s) = \mathbf{U}\psi(x, s)\,\mathbf{U}^{-1} \tag{2.14}$$

Since $\mathbf{U}$ is unitary, its adjoint (conjugate transpose) $\mathbf{U}^+$ satisfies the relation

$$\mathbf{U}^+ = \mathbf{U}^{-1} \tag{2.15}$$

so that

$$\mathbf{U}\mathbf{U}^+ = \mathbf{U}^+\mathbf{U} = 1 \tag{2.16}$$

All physical properties of the system described by the original field $\psi(x, s)$ are unchanged in the new system described by the transformed field $\psi'(x, s)$.

An infinitesimal canonical transformation can be obtained by writing $\mathbf{U}$ as

$$\mathbf{U} = 1 + i\mathbf{F} \tag{2.17}$$

where $\mathbf{F}$ is a first-order infinitesimal. If we apply the condition for unitarity in Eq. (2.16) to Eq. (2.17) and neglect second-order infinitesimals, we find

$$\mathbf{U}^+\mathbf{U} = (1 - i\mathbf{F}^+)(1 + i\mathbf{F}) \approx 1 - i(\mathbf{F}^+ - \mathbf{F}) \tag{2.18}$$

The right-hand side of Eq. (2.18) must equal 1 by Eq. (2.16). This is satisfied for all $\mathbf{F}$ only if $\mathbf{F}$ is Hermitian; thus,

$$\mathbf{F}^+ = \mathbf{F} \tag{2.19}$$

The operator $\mathbf{F}$ is the generator of the infinitesimal canonical transformation $\mathbf{U}$.

Substituting Eq. (2.17) into (2.14) gives

$$\psi'(x, s) = (1 + i\mathbf{F})\,\psi(x, s)(1 - i\mathbf{F}) \tag{2.20}$$

or, to first order,

$$\psi'(x, s) = \psi(x, s) + i[\mathbf{F}, \psi(x, s)] \tag{2.21}$$

If the total variation or change of the field $\psi(x, s)$ by an infinitesimal canonical transformation is defined by

$$\delta_0\psi(x, s) = \psi'(x, s) - \psi(x, s) \tag{2.22}$$

then substituting Eq. (2.22) into (2.21) expresses the variation of the field in terms of the commutator of $\mathbf{F}$ with $\psi(x, s)$; thus

$$\delta_0\psi(x, s) = i[\mathbf{F}, \psi(x, s)] \tag{2.23}$$

An infinitesimal JARS transformation can be written as

$$\begin{aligned} \Lambda^{\nu}_{\mu} &= g^{\nu}_{\mu} + \varepsilon^{\nu}_{\mu} \\ a_{\mu} &= \varepsilon_{x\mu} \\ b_{\mu} &= \varepsilon_{p\mu} \\ \delta s &= \varepsilon \end{aligned} \tag{2.24}$$

where $(\varepsilon, \varepsilon_{p\mu}, \varepsilon_{x\mu}, \varepsilon^{\nu}_{\mu})$ are first-order infinitesimals. The infinitesimal generator for dynamical development of the system has the unitary operator representation of Eq. (2.4) with $\mathbf{F}$ given by Eq. (2.5). Since $\mathbf{U}$ is unitary, the JARS transformation is a canonical transformation.

A set of commutation relations of the operators $\mathbf{J}^{\mu\nu}$, $\mathbf{P}^{\mu}$, $\mathbf{X}^{\mu}$, and $\mathbf{K}$ with the field ψ is obtained by developing each side of Eq. (2.23) separately in terms of the infinitesimals $(\varepsilon, \varepsilon_{p\mu}, \varepsilon_{x\mu}, \varepsilon^{\nu}_{\mu})$. The coefficients of the linear terms on both sides are then equated.

The term $\delta_0\psi$ may be expressed in an alternate form by introducing the local variation

$$\delta\psi \equiv \psi'(x', s') - \psi(x, s) \tag{2.25}$$

where x, s and x', s' refer to the same physical point as viewed from canonically transformed reference frames. Performing a Taylor series expansion of $\psi'(x', s')$ and keeping only first-order terms gives

$$\delta\psi = \psi'(x, s) + \partial^{\mu}\psi'(x, s)\,\delta x_{\nu} + \partial_{s}\psi'(x, s)\;\delta s - \psi(x, s) \tag{2.26}$$

where $\partial_s \equiv \partial/\partial s$. Rearranging and using Eq. (2.22) yields

$$\delta_0\psi = \delta\psi - \partial^{\nu}\psi\,\delta x_{\nu} - \partial_{s}\psi\,\delta s \tag{2.27}$$

The increments $\delta\psi$, δx_{ν}, δs are specified in terms of the infinitesimal JARS transformation:

$$\delta x_{\nu} = \varepsilon_{\nu\mu}x^{\mu} + \varepsilon_{x\nu} + \varepsilon_{p\nu} \tag{2.28}$$

$$\delta s = \varepsilon \tag{2.29}$$

$$\delta\psi = \tfrac{1}{2}\Sigma^{\nu\mu}\psi(x, s)\varepsilon_{\nu\mu} \tag{2.30}$$

where $\Sigma^{\nu\mu}$ are the infinitesimal operators of the proper homogeneous Lorentz group.

Combining Eqs. (2.27)–(2.30) gives

$$\begin{aligned} \delta_0\psi = {} & \tfrac{1}{2}[(\Sigma^{\nu\mu}\psi + (x^{\nu}\partial^{\mu} - x^{\mu}\partial^{\nu})\psi]\varepsilon_{\nu\mu} \\ & - (\partial^{\nu}\psi)\varepsilon_{x\nu} - (\partial^{\nu}\psi)\varepsilon_{p\nu}s - (\partial_{s}\psi)\varepsilon \end{aligned} \tag{2.31}$$

where the $\varepsilon_{\nu\mu}$ terms has been rewritten in the form

$$(\partial^\nu \psi)\varepsilon_{\nu\mu} x^\mu = \tfrac{1}{2}[(x^\nu \partial^\mu - x^\mu \partial^\nu)\psi]\varepsilon_{\nu\mu} \tag{2.32}$$

and the antisymmetry relation $\varepsilon_{\nu\mu} = -\varepsilon_{\mu\nu}$ has been used. The commutator $i[\mathbf{F}, \psi]$ is obtained by substituting Eq. (2.5) into Eq. (2.23); thus

$$\begin{aligned} i[\mathbf{F}, \psi] = {} & \tfrac{1}{2} i[\mathbf{J}^{\nu\mu}, \psi]\varepsilon_{\nu\mu} + i[\mathbf{P}^\nu, \psi]\varepsilon_{p\nu} \\ & + i[\mathbf{X}^\nu, \psi]\varepsilon_{x\nu} + i[\mathbf{K}, \psi]\varepsilon \end{aligned} \tag{2.33}$$

Comparing Eqs. (2.31) with (2.33) yields the commutation relations

$$i[\mathbf{J}^{\nu\mu}, \psi] = \Sigma^{\nu\mu}\psi + (x^\nu \partial^\mu - x^\mu \partial^\mu)\psi \tag{2.34}$$

$$i[\mathbf{P}^\nu, \psi] = -\partial^\nu \psi \tag{2.35}$$

$$i[\mathbf{X}^\nu, \psi] = -s(\partial^\nu \psi) \tag{2.36}$$

and

$$i[\mathbf{K}, \psi] = -\partial_s \psi \tag{2.37}$$

Equations (2.34)–(2.37) can be considered the defining relations for the operators $\mathbf{J}^{\nu\mu}$, $\mathbf{P}^\nu$, $\mathbf{X}^\nu$, $\mathbf{K}$.

2.3. Wave Equation

We derive a wave equation from the results presented above by rearranging Eq. (2.37) so that it appears as an operator acting on an arbitrary field ϕ:

$$\{i(\partial_s \psi) - [\mathbf{K}, \psi]\}\phi = 0 \tag{2.38}$$

Expanding the commutator bracket gives

$$\begin{aligned} [\mathbf{K}, \psi]\phi &= [\mathbf{K}\psi - \psi\mathbf{K}]\phi \\ &= \psi(\mathbf{K}\phi) + (\mathbf{K}\psi)\phi - \psi(\mathbf{K}\phi) \\ &= (\mathbf{K}\psi)\phi \end{aligned} \tag{2.39}$$

Combining Eq. (2.39) with Eq. (2.38) yields

$$\{i(\partial_s \psi) - (\mathbf{K}\psi)\}\phi = 0 \tag{2.40}$$

A wave equation is obtained by observing that Eq. (2.40) is satisfied for any field ϕ when the bracketed term vanishes; thus,

$$i\frac{\partial \psi}{\partial s} = \mathbf{K}\psi \tag{2.41}$$

3. GENERATOR OF EVOLUTION PARAMETER TRANSLATIONS

The choice of the form of the generator $\mathbf{K}$ of s translations is not unique. Several forms have been proposed. The most common forms are the Dirac-like form

$$\mathbf{K}_1 = \alpha_1 \gamma_\mu \mathbf{P}^\mu \tag{3.1}$$

where γ_μ are Dirac gamma matrices, and the Schrödinger-like form

$$\mathbf{K}_2 = \alpha_2 \mathbf{P}_\mu \mathbf{P}^\mu \tag{3.2}$$

A third form was suggested by Johnson[7]:

$$\mathbf{K}_3 = \alpha_3 \sqrt{\mathbf{P}_\mu \mathbf{P}^\mu} \tag{3.3}$$

Each generator applies to the dynamics of a free particle. The quantities α_j are constants with respect to the operators. For example, Johnson[7] defined $\alpha_1 = i/m$ and $\alpha_3 = 1$; Aghassi *et al.*[5] chose $\alpha_2 = 1$, while Pearle[8] and Johnson[7] considered $\alpha_2 = 1/2m$, where m is the mass of the particle.

Our focus is on the order of the $\mathbf{P}^\mu$ operators relative to the order of s. We would like to know which form is the correct form to use. Each of the forms can be constructed from more basic principles, but in every formulation except one the order of $\mathbf{P}^\mu$ relative to the order of s is determined by a model of the system—such as an assumed Lagrangian—rather than fundamental assumptions such as probability conservation. Only the probabilistic formulation results in a relationship between the order of $\mathbf{P}^\mu$ and the order of s that is established from first principles. Before discussing the probabilistic formulation, we review alternative constructions of parametrized wave equations. This review should be viewed as representative of the constructions that have been made in the past. A more comprehensive review is given in Fanchi.[4]

3.1. Evaluation of $\mathbf{K}_3$

Johnson argued that Eq. (3.3) could be used to eliminate certain unphysical representations by requiring that the Hamiltonian be Hermitian

and positive-definite. These "unphysical" representations correspond to spacelike (tachyonic) states that may be physical, but their existence has not been established,[9] although experimentalists have measured spacelike masses for neutrinos.[10,11] By presenting a physical rationale for eliminating these mathematically admissible states, Johnson had to explain why exact localization was not possible for a physical state. Furthermore, the positive-definite requirement applied to the Hamiltonian conflicts with the allowed existence of negative energy states in the rest frame of a particle.

Johnson interpreted Eq. (3.3) as applying to massive, free particles. Johnson's attempts to describe massive, spin-1/2 particles led to an ambiquity in the definition of the Hamiltonian. In particular, he found that Eq. (3.1) has the same commutator as Eq. (3.3) when taken with the Poincaré group operators $J^{\mu\nu}$ and P^{μ}. He argued that Eq. (3.3) worked better with other physical observables such as position and spin than did Eq. (3.1), but he was unable to directly connect Eq. (3.3) with the Dirac theory of spin-1/2 particles.

Another problem with Eq. (3.3) arises when interactions are considered. Johnson and Chang[12] used a transformation like the Foldy–Wouthuysen transformation to solve the problem of a Dirac particle in an external electromagnetic field for a Hamiltonian with the form of Eq. (3.1). They noted that Eq. (3.3) was difficult to transform because of the asymmetrical treatment of its zeroth component. These problems discourage us from considering Eq. (3.3) any further.

3.2. Parametrized Hamilton–Jacobi Theory

Fock[13] defined a classical action as an integral of a Lagrangian function with respect to proper time:

$$A_{\text{Foc}} = \int L_{\text{Foc}}\, ds \tag{3.4}$$

Proper time was treated as an independent invariant parameter. The classical Lagrangian for an electron interacting with an electgromagnetic field is given by

$$L_{\text{Foc}} = -m\dot{x}_{\mu}\dot{x}^{\mu} - \frac{m}{2} - e\dot{x}_{\mu}A^{\mu} \tag{3.5}$$

where

$$\dot{x}_{\mu} = \frac{dx_{\mu}}{ds} \tag{3.6}$$

The Hamilton–Jacobi equation corresponding to Fock's Lagrangian is

$$\frac{\partial A_{\text{Foc}}}{\partial s}+\frac{1}{2m}\left[(\nabla A_{\text{Foc}}+e\vec{A})^2-\left(\frac{\partial A_{\text{Foc}}}{\partial t}-e\Phi\right)^2\right]+m^2=0 \tag{3.7}$$

Given a classical basis, Fock presented the quantum analog of the Hamilton–Jacobi Eq. (3.7):

$$i\frac{\partial F}{\partial s}=\frac{1}{2m}\left[(i\partial^\mu-eA^\mu)(i\partial_\nu-eA_\nu)-m^2-\frac{ie}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\right]F \tag{3.8}$$

where $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ is the field strength tensor. The function F satisfies a partial differential equation that is second order in space-time and first order in proper time. Solutions of Eq. (3.8) had to be stationary states with zero eigenvalues in Fock's formulation.

3.3. Parametrized Path Integrals

Feynman[14] outlined a procedure to extend his path integral formulation of nonrelativistic quantum mechanics to quantum electrodynamics (QED) in 1950. Morette[15] succinctly presented the relativistic path integral formalism by writing the probability amplitude K for a particle to go from space-time point x^{A} to space-time point x^{B} as

$$K(x^{\text{B}}, x^{\text{A}})=\int \exp(iS[x])\,d(\text{paths}) \tag{3.9}$$

The integral is over all paths $x(s)$ from x^{A} to x^{B}. The action functional $S[x]$ is given by

$$S[x]=\int_{s^{\text{A}}}^{s} L\{x(s), \dot{x}(s)\}\,ds \tag{3.10}$$

where L is the Lagrangian.

Like the Hamilton–Jacobi and variational approaches, the key to the form of the wave equation was the postulated Lagrangian used in the action. For example, following a suggestion made by Feynman,[16] Morette[15] applied Feyman's path integral approach to a Dirac particle in a constant, external electromagnetic field. She used a Lagrangian of the form

$$L_{\text{Mor}}=\frac{m}{2}\dot{x}_\mu\dot{x}^\mu+\frac{e}{2}(\gamma_\mu\dot{x}^\mu)(\gamma_\nu A^\nu) \tag{3.11}$$

with the corresponding wave equation

$$2im\frac{\partial}{\partial s}\psi_{\text{Mor}}(x,s)=\{\gamma_\mu[\partial^\mu-ieA^\mu]\}^2\psi_{\text{Mor}}(x,s) \tag{3.12}$$

Feynman[17] used a path integral formulation to derive Fock's parametrized spin-1/2 wave equation. His Lagrangian had the form

$$L_{\text{FF}}=-\left[\frac{1}{2}\dot{x}_\mu\dot{x}^\mu+e\dot{x}_\mu A^\mu-\frac{e}{4}\sigma_{\mu\nu}F^{\mu\nu}\right] \tag{3.13}$$

The interaction term is assumed as a result of "minimal coupling." In every case, including more modern path intgegral formulations such as Barut and Duru,[18] a Lagrangian "model" determinies the form of the wave equation.

3.4. Miscellaneous Extensions from a Classical Basis

Pavsic presented a phase space form of an action with a Lagrange multiplier.[19] He combined the action with a variational principle to derive parametrized equations of motion without the mass-shell constraint in classical mechanics, and both first and second quantized formulations.[20] Although initially Pavsic worked only with a Dirac-like equation,[21] he eventually included an equation with $\mathbf{K}_2$ in his variational formulation.[20] The form of the resulting wave equation was dependent on the postulated form of the Lagrangian in the action.

Horwitz and Piron[22] introduced a parametrized quantum theory with a generator like $\mathbf{K}_2$ as a quantum mechanical analog to a Hamiltonian formulation of relativistic classical mechanics. More recently, Evans[23] has tried to justify $\mathbf{K}_1$, but his development was based on an extension of a classical Hamiltonian, and his choice of probability density is not positive definite. Parametrized wave equations developed as a quantum mechanical extension of relativistic classical physics invariably depend on an assumed Lagrangian or Hamiltonian model. We are loking for a formulation of a wave equation that does not depend on a particular physical model, but is the result of fundamental physical insight. It is possible to construct a formalism that has the attributes of quantum mechanics from fundamental concepts of probability theory.[4,24,25] This approach is outlined in Sec. 4.

4. PROBABILISTIC BASIS

The Born representation and conservation of probability are at the heart of the derivation of a field equation which satisfies the assumptions

of conventional probability theory. Construction of wave equations that are consistent with probability theory begins with the assumption that a conditional probability density $\rho(y|s)$ exists. The symbol y denotes a set of $4N$ coordinates with N being the number of particles in the hypervolume D^N for which $\rho(y|s)$ has nonzero values. The μth component of the position four-vector of particle a is written as y_a^μ, where $\mu = 0, 1, 2, 3$. Indices 1, 2, 3 signify space components and 0 signifies the geometric time component.

According to probability theory, $\rho(y|s)$ must be positive definite and normalizable; thus,

$$\rho(y|s) \geqslant 0 \tag{4.1}$$

and

$$\int_{D^N} \rho(y|s)\, dy = 1 \tag{4.2}$$

where

$$dy = \prod_{a=1}^{N} d^4 y_a, \qquad d^4 y_a = dy_a^0\, dy_a^1\, dy_a^2\, dy_a^3 \tag{4.3}$$

Conservation of probability implies

$$\frac{\partial \rho}{\partial s} + \sum_{a=1}^{N} \frac{\partial \rho\, V_a^\mu}{\partial y_a^\mu} = 0 \tag{4.4}$$

where Einstein's summation convention for Greek indices is assumed and the term ρV_a^μ represents the μth component of probability flux of particle a.

Suppose ρ depends on L internal variables having discrete parameter values $\{l_i\}$ such that

$$\rho(y|s)\, dy = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_L} \rho(y, l_1, l_2, \ldots, l_L|s)\, dy \tag{4.5}$$

where the ith sum extends over the entire range of allowed values of the ith parameter l_i. The term $\rho(y, l_1, \ldots, l_L|s)\, dy$ is the probability of observing a system having parameter values $l_1, \ldots, l_L$ in the configuration y within an infinitesimal $4N$-volume dy at the historical time s.

Denote the allowed values of the ith discrete parameter l_i as L_i. Then the number of terms Λ in the L sums of Eq. (4.5) is just the product of L_i, namely

$$\Lambda = \prod_{i=1}^{L} L_i \tag{4.6}$$

The L sums can be replaced by a single sum over the range $1 \leqslant \lambda \leqslant \Lambda$. This requires assigning a one-to-one correspondence between λ and each term of the L sums. Doing so lets us write Eq. (4.5) as

$$\rho(y|s)\,dy = \sum_{\lambda=1}^{\Lambda} \rho_\lambda(y|s)\,dy$$
$$\rho_\lambda(y|s) \equiv \rho(y, l_1, \ldots, l_L|s) \tag{4.7}$$

where the subscript λ of $\rho_\lambda(y|s)$ represents a set of values of the discrete parameters $\{l_i\}$.

For ρ to be differentiable and non-negative,[24] its derivative must satisfy $\partial\rho/\partial s = 0$ if $\rho = 0$. The positive-definite requirement for $\rho_\lambda(y|s)$ is satisfied by writing $\rho_\lambda(y|s)$ in the Born representation:

$$\rho_\lambda(y|s) = \psi_\lambda^*(y, s)\,\psi_\lambda(y, s) \tag{4.8}$$

The scalar eigenfunctions ψ_λ are specified only to within a gauge transformation of the first kind. Consequently ψ_λ can be written as

$$\psi_\lambda(y, s) = [\rho_\lambda(y|s)]^{1/2} \exp[i\xi(y, s)] \tag{4.9}$$

where the real scalar function $\xi(y, s)$ is assumed independent of the discrete parameters. This restriction is realistic if the probability density of the physical system can be decomposed such that the parameters $\{l_i\}$ characterize mutually independent events. Systems characterized by an eigenfunction with the product form $\psi(\text{space-time}) \otimes \psi(\text{spin}) \otimes \psi(\text{isospin}) \ldots$ fulfill the independence requirement.

4.1. The Stückelberg Equation

A field equation is derived by expressing the four-velocity of the ath particle V_a^μ as

$$V_a^\mu(y, s) = \varepsilon_a \left[\varepsilon' \frac{\partial \xi(y, s)}{\partial y_{a\mu}} + \varepsilon'' A_a^\mu(y, s) \right] \tag{4.10}$$

where ε', ε'', and ε_a are real c-numbers whose values are not yet specified. We assume that A_a^μ and, consequently, V_a^μ are independent of λ and are real, scalar functions. Using these assumptions with the procedure in Fanchi and Wilson,[26] we derive the field equation

$$i\varepsilon' \frac{\partial \Psi}{\partial s} = \sum_{a=1}^{N} \frac{\varepsilon_a}{2} \pi_a^\mu \pi_{a\mu} \Psi + \mathbf{V}\Psi \tag{4.11}$$

where $\mathbf{V}$ is a $\Lambda \times \Lambda$ Hermitian matrix. The operators π_a^μ and p_a^μ are defined by

$$\pi_a^\mu \equiv \mathbf{p}_a^\mu + \varepsilon'' A_a^\mu \tag{4.12}$$

and

$$\mathbf{p}_a^\mu \equiv \frac{\varepsilon'}{i} \frac{\partial}{\partial y_{a\mu}} \tag{4.13}$$

The single-particle form of Eq. (4.11) resembles an equation first studied by Stückelberg.[27] For ease of reference, we refer to Eq. (4.11) as the Stückelberg equation.

Equation (4.11) may be written in the form

$$i\varepsilon' \frac{\partial \Psi}{\partial s} = \mathbf{K}\Psi \tag{4.14}$$

where

$$\mathbf{K} = \sum_{a=1}^{N} \frac{\varepsilon_a}{2} \pi_a^\mu \pi_{a\mu} \mathbf{I} + \mathbf{V} \tag{4.15}$$

The operator $\mathbf{K}$ is Hermitian since it is the sum of two Hermitian operators. In this formulation, the definition of the expectation value of an observable Ω is

$$\langle \Omega \rangle \equiv \int_{D^N} \Psi^+ \Omega \Psi \, dy \tag{4.16}$$

Field equations for a many-body system which explicitly exhibits the "minimal coupling" electromagnetic interaction are obtained by making the identifications

$$\begin{aligned} \varepsilon' &= \hbar \\ \varepsilon'' &= -e/c \\ e_a &= 1/m_a \end{aligned} \tag{4.17}$$

where m_a is a c-number with mass units. We identify the four-vector A_a^μ as the external four-vector potential acting on the ath particle. Additional interactions may be incorporated via the presently unspecified Hermitian matrix $\mathbf{V}$ of Eq. (4.11). With these identifications, π_a^μ becomes the

four-momentum operator with "minimal coupling" of the ath particle. Equation (4.11) may be written in the form

$$i\hbar \frac{\partial \Psi}{\partial s} = \mathbf{K}\Psi \tag{4.18}$$

where $\mathbf{K}$ is the generator of s translations with the explicit form

$$\mathbf{K} \equiv \sum_{a=1}^{N} \left(\frac{\pi_a^{\pi} \pi_{a\mu}}{2m_a} \right) \mathbf{I} + \mathbf{V} \tag{4.19}$$

Equations (4.18) and (4.19) support the use of an operator of the form of $\mathbf{K}_2$ in the wave equation.

Finally we note that Lopez and Perez[28] used some of the ideas presented above to derive a wave equation with the generator $\mathbf{K}_1$. They found that they could not introduce interactions without returning to a generator of the form of $\mathbf{K}_2$.

5. DIRAC-LIKE 4×4 REPRESENTATION

If $\mathbf{K}_2$ is the correct form of the generator of s translations, an obvious question arises: why has the Dirac equation been so successful? We begin to answer this question by considering the two-component Stückelberg equation for a single particle:

$$i \frac{\partial}{\partial s} \mathsf{u} = \mathbf{K}\mathsf{u} = \left[\frac{\pi_\mu \pi^\mu}{2m} \mathbf{I} + \mathbf{V} \right] \mathsf{u} \tag{5.1}$$

The matrices $\mathbf{I}$, $\mathbf{K}$, $\mathbf{V}$ are 2×2 matrices and u is a 2×1 column vector. Equation (5.1) is the 2×2 representation of the spin-1/2 Stückelberg equation. Conventional treatments of the spin-1/2 particle based on the Dirac equation use a 4×4 representation. Our interest here is to construct a 4×4 representation of the spin-1/2 Stückelberg equation.

5.1. 4×4 Representation of the Spin-1/2 Stückelberg Equation

Equation (5.1) corresponds to two equations containing second-order space-time derivatives. Following a procedure first introduced by Davidon,[29] we write Eq. (5.1) as four equations containing first-order space-time derivatives:

$$i\pi_\mu \mathbf{A}^\mu \mathsf{u} = \mathsf{v} \tag{5.2}$$

and

$$-i\,\pi_\nu \mathbf{B}^\nu \mathsf{v} = 2mi\frac{\partial}{\partial s}\mathsf{v} \tag{5.3}$$

The 2×2 matrices $\{\mathbf{A}^\mu, \mathbf{B}^\mu\}$ are as yet unspecified, and v is defined by Eq. (5.2). Substituting Eq. (5.2) into (5.3) gives

$$\pi_\nu \pi_\mu \mathbf{B}^\nu \mathbf{A}^\mu \mathsf{u} = 2mi\frac{\partial}{\partial s}\mathsf{u} \tag{5.4}$$

Comparing Eq. (5.4) with (5.1) yields

$$\pi_\nu \pi_\mu \mathbf{B}^\nu \mathbf{A}^\mu = 2m\mathbf{K} = \pi_\mu \pi^\mu \mathbf{I} + 2m\mathbf{V} \tag{5.5}$$

Equation (5.5) expresses $\{\mathbf{A}^\mu, \mathbf{B}^\mu\}$ in terms of the Hamiltonian $\mathbf{K}$.

Equations (5.2) and (5.3) can be combined to form the matrix equation

$$i\pi_\mu \mathbf{\Gamma}^\mu \Psi = \mathbf{M}\Psi \tag{5.6}$$

where the 4×4 matrices $\mathbf{\Gamma}^\mu$, $\mathbf{M}$ are

$$\mathbf{\Gamma}^\mu = \begin{bmatrix} 0 & \mathbf{A}^\mu \\ -\mathbf{B}^\mu & 0 \end{bmatrix} \tag{5.7}$$

$$\mathbf{M}^\mu = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 2mi(\partial/\partial s)\mathbf{I} \end{bmatrix} \tag{5.8}$$

and the 4×1 column vector Ψ is

$$\Psi = \begin{bmatrix} \mathsf{v} \\ \mathsf{u} \end{bmatrix} \tag{5.9}$$

Equation (5.6) is the 4×4 representation of the spin-1/2 Stückelberg equation.

5.2. Stationary States

A stationary mass state equation may be derived from Eq. (5.6) by performing a nonunitary transformation. In this case Ψ is an eigenfunction of $\mathbf{K}$ such that

$$\Psi(x, s) \sim \Psi(x)\, e^{-ims/2} \tag{5.10}$$

so that

$$\mathbf{M}\psi = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & m^2\mathbf{I} \end{bmatrix} \psi \tag{5.11}$$

Using Davidon's transformation matrix

$$\mathbf{T} \equiv \begin{bmatrix} m^{1/2} & 0 \\ 0 & m^{-1/2} \end{bmatrix} \tag{5.12}$$

we multiply Eq. (5.11) on the right by $\mathbf{T}^* = \mathbf{T}$ and expand as

$$(i\mathbf{T}^*\pi^\mu\Gamma^\mu\mathbf{T} - \mathbf{T}^*\mathbf{M}\mathbf{T})\,\mathbf{T}^{-1}\psi = 0 \tag{5.13}$$

In terms of the transformed variables

$$\begin{aligned} \Gamma'^\mu &= \mathbf{T}^*\Gamma^\mu\mathbf{T} = \Gamma^\mu \\ \mathbf{M}' &= \mathbf{T}^*\mathbf{M}\mathbf{T} = m\mathbf{I} \end{aligned} \tag{5.14}$$

and

$$\psi' = \mathbf{T}^{-1}\psi = \begin{bmatrix} m^{-1/2}\mathsf{v} \\ m^{1/2}\mathsf{u} \end{bmatrix} \tag{5.15}$$

Eq. (5.13) becomes

$$(i\pi^\mu\Gamma - m\mathbf{I})\,\psi' = 0 \tag{5.16}$$

Equation (5.16) is a 4×4 representation of the stationary mass state equation.

It is tempting to identify $\{\Gamma^\mu\}$ as the Dirac gamma matrices by writing

$$\begin{aligned} \mathbf{A}_D^\mu &\equiv [-i\mathbf{I}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z] \\ \mathbf{B}_D^\mu &\equiv -[i\mathbf{I}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z] \end{aligned} \tag{5.17}$$

where $\{\boldsymbol{\sigma}_j,\ j = x, y, z\}$ are the Pauli matrices. The resulting equation would be the Dirac equation.

Davidon[29] pointed out that defining $\{\Gamma^\mu\}$ by Eq. (5.17) is inconsistent with the scalar product we have adopted, namely Eq. (4.16). Preferring to retain the Dirac equation in the form that made it so successful, Davidon introduced an inner product with an indefinite metric. The indefinite metric allows both positive and negative probabilities, which is the same difficulty suffered by conventional treatments based on the Dirac equation.

We are at a dilemma. If we insist on preserving probabilistic concepts, we cannot accept the Dirac equation as part of PRQT. Is this a defect in PRQT, or is it a clue that the Dirac equation, even with its history of success, is not quite right?

Mayants[25] and Wu and Hwang[30] have pointed out flaws in the Dirac equation. Mayants noted that the Dirac equation predicts that the expectation value of the velocity of any free, relativistic spin-1/2 particle is the speed of light. This is a well-known consequence of the Dirac theory. Mayants argues that this result contradicts both the principles of relativity theory and experimental data. He concludes that the Dirac equation must be incorrect. Wu and Hwang[30] note that it is very difficult to apply the single-particle Dirac equation to many particles. They further observe that the Dirac theory—even though it was originally intended to be a single-particle theory—could not be interpreted within the confines of a single-particle theory. Klein's paradox and the Stückelberg–Feynman interpretation are related to this problem.[31]

The difficulties of the Dirac equation suggest that a better approach may exist. We know from the above analysis that we can construct a 4×4 representation of a new spin-1/2 field equation. Furthermore, we have consistently rejected formulations that require unconventional probabilistic concepts such as negative probabilities. Therefore, we must be prepared to follow PRQT to its logical conclusion. In the process we may obtain a new perspective that will help us better understand the standard paradigm. For these reasons we conclude that the usual form of the Dirac equation is inappropriate for PRQT.

Does this mean that PRQT cannot calculate experimental results to the same degree of accuracy as the Dirac equation or its second-order counterpart? It does not. What it does mean is that a new model of a spin-1/2 particle interacting with an electromagnetic field is needed.

A model of the spin-1/2 particle interacting with a four-vector potential has been proposed by Horwitz, Piron, and Reuse.[32-34] The validity of this model depends on its ability to match experimental results. Reuse[35] and Grelland[36] have shown that the model does a statisfactory job of reproducing experimental results for an electron bound by a Coulomb potential, but more applications are needed to further evaluate the model.

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