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Non-Heisenberg States of the Harmonic Oscillator

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The effects of the vacuum electromagnetic fluctuations and the radiation reaction fields on the time development of a simple microscopic system are identified using a new mathematical method. This is done by studying a charged mechanical oscillator (frequency ω_0) within the realm of stochastic electrodynamics, where the vacuum plays the role of an energy reservoir. According to our approach, which may be regarded as a simple mathematical exercise, we show how the oscillator Liouville equation is transformed into a Schrödinger-like stochastic *equation with a* free *parameter h' with dimensions of action. The role of the physical Planck's constant h is introduced* only *through the zero-point vacuum electromagnetic fields. The perturbative and the exact solutions of the stochastic Schrödinger-like equation are presented for $h' > 0$. The exact solutions for which $h' < h$ are called sub-Heisenberg states. These nonperturbative solutions appear in the form of Gaussian, non-Heisenberg states for which the* initial classical uncertainty relation *takes the form $\langle(\Delta x^2)\rangle\langle(\Delta p)^2\rangle = (h'/2)^2$, which includes the limit of zero indeterminacy ($h \to 0$). We show how the radiation reaction and the vacuum fields govern the evolution of these non-Heisenberg states in phase space, guaranteeing their decay to the stationary state with average energy $\hbar\omega_0/2$ and $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$ at zero temperature. Environmental and thermal effects are briefly discussed and the connection with similar works within the realm of quantum electrodynamics is also presented. We suggest some other applications of the classical non-Heisenberg states introduced in this paper and we also indicate experiments which might give concrete evidence of these states.*

1. INTRODUCTION

Even in its ground state, a microscopic system possesses fluctuations which are associated to the zero-point (or zero temperature) energy which exists

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in ordinary space. The most striking examples are the electric and magnetic vacuum fields which can be indirectly observed.[1,2] The spectral distribution $\rho_0(\omega)$ of these electromagnetic fields is well known and it is related to the correlation function of these fluctuating fields through the ensemble average

$$\langle E_{VF}(t+\tau)\, E_{VF}(t)\rangle = \frac{2\pi}{3}\int_{-\infty}^{\infty} d\omega\, \rho_0(\omega)\, e^{i\omega\tau} \tag{1}$$

Here $E_{VF}(t)$ is the x component of the vacuum electric field at time t and at the origin of the coordinate system.

Within the realm of stochastic electrodynamics (SED) these E_{VF} fields are classical random fields with zero mean.[1,2] However, within quantum electrodynamics (QED) these electromagnetic fields are considered quantized, that is, they have a dual (wave-particle) nature. Nevertheless in both theories (SED and QED) the spectral distribution is such that[1-3]

$$\rho_0(\omega) = h\,|\omega|^3/2\pi^2c^3 \tag{2}$$

where h is the Planck's constant and c the velocity of light.

In this paper we shall study a charged oscillator by comparing some features of the QED and the SED approaches. Our motivations are inspired in the early attempts of Planck,[3] Einstein and Stern,[4] and Nernst[5] to clarify the role of the zero-point energy. Another motivation is the recent tendency to bring classical and quantum theories to a closer (and maybe nonconflicting) relation.[1-4,6]

Within the QED approach, the Heisenberg equation for the one-dimensional motion of a charged oscillator (charge e and mass m) is given by

$$m\ddot{x} = -V'(x) + e[E_{VF}(t) + E_{RR}(t)] \tag{3}$$

where $V(x) = m\omega_0^2 x^2/2$ is the harmonic potential, ω_0 is the oscillator frequency, $x(t)$ is the position operator, and $eE_{RR}(t) \simeq (2/3)(e^2/c^3)\dddot{x}$ is the radiation reaction force.[7] The total quantized electric field acting on the particle, that is, $E_x(t) \equiv E_{VF} + E_{RR}$, is also an operator, and will be considered only a function of time within the nonrelativistic approximation (the vector potential will be denoted $A_x \equiv A_{VF} + A_{RR}$). An important point to notice is that the quantum equation (3) is identical to the corresponding classical equation of motion (see Milonni[2] and the second article by Boyer in Ref. 1). Moreover, by Ehrenfest's theorem, the expectation values of any physical quantity that has linear Heisenberg equations of motion will be identical to the corresponding classical one and, therefore, quite easy to interpret.

Several authors[1-8] consider that the quantum fluctuations associated to the electromagnetic fields, namely E_{VF} in (3), are the source of the quantum fluctuations on the position x of the oscillating charge because *only* E_{VF} depends on h. In fact, it is not difficult to derive the quantum commutation relation between the position and the canonical momentum of the oscillator. From the *stationary solution* of (3) one can show that[8]

$$\left[x, m\dot{x} + \frac{e}{c} A_{VF} - \frac{2e^2}{3c^3}\ddot{x}\right] = [x, m\dot{x}]$$

$$= i\frac{8\pi e^2}{3m}\int_0^\infty d\omega \frac{\omega \rho_0(\omega)}{(\omega^2 - \omega_0^2)^2 + (2e^2\omega^3/3mc^3)^2} = ih \tag{4}$$

follows from the commutation relations associated to the zero-point electromagnetic fields. We observe that the last equality in (4) *is valid only if the radiation reaction force is precisely* $2e^2\ddot{x}/3c^3$. For a more detailed discussion of these matters we refer to the papers quoted in Ref. 8. The validity of (4) within the nonstationary (or transient) regimen, and under special environmental conditions, will be discussed in Sec. 6.

Therefore, we can easily recognize that, as far as the harmonic oscillator is concerned, the QED and the SED descriptions of this system are very similar. As a matter of fact, the Planck's constant h enters in both descriptions *only* through the zero-point fluctuations of the electromagnetic fields [see (2)]. This is the cornerstone of our paper.

Keeping in mind these preliminary considerations, we will show in Sec. 2 how to connect the SED approach with the Schrödinger picture. In order to achieve this goal we shall assume that a phase space probability distribution for the oscillator exists in both classical and quantum approaches.[1] The Ehrenfest theorem, the correspondence principle, and the Heisenberg equation (3) give support to this working hypothesis. Moreover, the classical description is expected to be valid in the $h \to 0$ limit.

According to the classical view, the probability distribution in phase space x and $p \equiv m\dot{x}$ (kinetical momentum) will be denoted by $W(x, p, t)$, and will evolve in time according to the Liouville equation, namely,

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}(\dot{x}W) + \frac{\partial}{\partial p}(\dot{p}W) = 0 \tag{5}$$

Since $\dot{p}/m = \ddot{x}$ is related to the stochastic "vacuum" field $E_{VF}(t)$ [see (3)], Eq. (5) can be transformed into a Fokker–Planck equation in a standard manner.[2] Here, however, we want to transform the Liouville equation directly into a Schrödinger-like stochastic equation.

We shall present the Schrödinger-like equation in a new form through the introduction of a *free parameter* h' with the dimension of action. This is the main result of Sec. 2. It should be stressed, however, that we are not trying to introduce a different Planck's constant as has been conjectured recently.[9] We are explicitly assuming that h is introduced *only* through (2).

In Sec. 3 we show how to apply the approximate methods of perturbation theory, in order to make simple calculations. As an example we calculate the rate of exchange of energy of an arbitrary excited state of the oscillator.

Section 4 is devoted to the discussion of exact solutions of the stochastic Schrödinger-like equation and to the introduction and interpretation of the non-Heisenberg states. The evolution of these states in phase space is given in Sec. 5. In the final section we discuss the limitations of our approach and we suggest some applications of the non-Heisenberg states in more complicated physical systems. We also indicate experiments which may give physical evidence of the non-Heisenberg states.

2. LIOUVILLE EQUATION IN SCHRÖDINGER'S FORM

The transformation of the classical equation (5) into another equation which looks like the Schrödinger equation has been known since the work by Wigner.[10] However, we shall present Wigner's ideas in a different form which will be convenient for our purpose, that is, to obtain Gaussian non-Heisenberg oscillator states and to discuss the effects of E_{VF} and E_{RR} on the time evolution of these states in phase space.

Using a procedure similar to that introduced by Wigner, we shall *define* a function $\tilde{W}(x, y, t)$ through the Fourier transform:[10]

$$\tilde{W}(x, y, t) \equiv \int_{-\infty}^{\infty} dp\, W(x, p, t) \exp\left(-\frac{2ipy}{h'}\right) \tag{6}$$

where h' is a *free parameter* with dimension of action and y is an auxiliary coordinate.

We shall keep $h' \neq h$ (h is the true Planck's constant) in order to stress that h' *has no dynamical meaning*. In other words, we shall see that h [introduced in (2)] has a dynamical meaning and determines the equilibrium state of the system, whereas h' [introduced in (6)] will be related to the intial *shape* of the non-Heisenberg states.

Having defined $\tilde{W}(x, y, t)$ through the Fourier transform (6) one can ask: What will be the equation obeyed by $\tilde{W}(x, y, t)$ if we impose that

$W(x, p, t)$ obeys the Liouville equation (5)? The answer is very simple. The substitution of (6) into (5) leads to[10]

$$ih' \frac{\partial \tilde{W}}{\partial t} + \frac{(h')^2}{2m} \frac{\partial^2}{\partial x \, \partial y} \tilde{W} + 2y[eE_x(t) - V'(x)] \, \tilde{W} = 0 \tag{7}$$

where we have used (3). It is easy to verify the correspondence among the three term of both Eqs. (5) and (7). The second term in (7), for instance, has its origin in the convective term of (5), namely, $\partial(\dot{x}W)/\partial x$.

In order to put (7) into a form which looks more similar to the Schrödinger equation, it is necessary to introduce an auxiliary complex function $\psi(x, t)$ and an additional restrictive hypothesis. Let us, therefore, consider *only* those functions $\tilde{W}(x, y, t)$ that can be written in the form

$$\tilde{W}(x, y, t) \equiv \psi^*(x + y, t) \, \psi(x - y, t) \tag{8}$$

The above assumption [and also the original equation (5) which requires that $W(x, p, t)$ be always positive] will impose some restrictions concerning the $\psi(x, t)$ which are physically acceptable. We shall see, however, that this will lead us to $\psi(x, t)$ functions which are in the form of displaced Gaussian states of the oscillator, namely, a kind of coherent state [see (30)]. Consequently, there is no loss of generality associated with this approach, because the coherent states form a complete set of states of the harmonic oscillator (see Refs. 11 and 17). Moreover, these displaced Gaussian states always generate positive phase-space distributions. Therefore, the hypothesis (8) is very attractive and reasonable from the classical point of view.

Using (8) into (7) we obtain[10]

$$ih' \frac{\partial \psi}{\partial t} = \left[-\frac{(h')^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega_0^2 x^2 - ex(E_{VF} + E_{RR}) \right] \psi \tag{9}$$

Therefore, according to SED, (9) may be interpreted as *classical stochastic* Schrödinger-like equation for the function $\psi(x, t)$ because E_{VF} is the random electric field. Notice, however, that *the equivalence between* (5) and (9) *is exact only for the harmonic oscillator.*[10]

One relevant remark can be made at this point. Nowhere in the above derivation leading to the Schrödinger-like equation (9) is any consideration given to the spectrum of the random force $eE_{VF}(t)$. Therefore, the result (9) holds for any spectrum, in particular for white noise for example. However, the results of Secs. 3, 4, and 5 will be presented considering only the spectral distribution given by (2).

Since E_{VF} is the vacuum field, the solution of Eq. (9) will have properties that depend on the statistical properties of the electromagnetic vacuum fluctuations. We have denoted the total vector potential $A_x(t) = A_{VF}(t) + A_{RR}(t)$, i.e., $cE_x = -\dot{A}_x$. If we introduce another auxiliary function $\Psi(x, t)$ such that

$$\psi(x, t) \equiv \exp\left[-i\frac{ex}{\hbar'}A_x(t)\right]\Psi(x, t) \tag{10}$$

it is possible to show that the equation for $\Psi(x, t)$ will be[12]

$$i\hbar'\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar'\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 + \frac{1}{2}m\omega_0^2x^2\right]\Psi(x, t) \tag{11}$$

which also has to be considered a stochastic Schrödinger-like equation. Notice, however, that (11) and (9) *are equivalent only in the dipole approximation* $[A_x = A_x(t)]$.

If we look for solutions of the Schrödinger-like equations (9) or (11), we can use the fact that the Hermite functions, namely

$$\phi_n(x) = \left(\frac{m\omega_0}{\pi\hbar'}\right)^{1/4}\frac{\exp(-(m\omega_0/2\hbar')\,x^2)}{(2^n n!)^{1/2}}H_n\left(x\sqrt{\frac{m\omega_0}{\hbar'}}\right) \tag{12}$$

form an orthogonal and complete set, thus fulfilling the condition

$$\sum_{n=0}^{\infty}\phi_n^*(x)\,\phi_n(y) = \delta(x-y) \tag{13}$$

As a matter of fact, the functions $\psi = \phi_n(x)\exp(-i\varepsilon_n t/\hbar')$ are solutions of (9) and (11) only if $e = 0$. Therefore, they cannot be considered *natural* states of the charged harmonic oscillator. Moreover, the set of "energies" ε_n,

$$\varepsilon_n = \hbar'\omega_0(\tfrac{1}{2} + n) \tag{14}$$

cannot be interpreted as the energy levels of the oscillator because $\hbar'$ is *arbitrary*. The true Planck's constant $\hbar$ only appears through the influence of the vacuum fields $E_{VF}(t)$. Another reason which forbids us to give a classical statistical interpretation to the "states" $(\phi_n(x), \varepsilon_n)$ is that they do not lead to positive-definite probability distribution in phase space,[13] with the exception of the "state" $[\phi_0(x), \varepsilon_0 = \hbar'\omega_0/2]$. However, even in this case, we cannot identify $\phi_0(x)$ with the true equilibrium state of the oscillator because $\hbar'$ is a free parameter. The meaning of $\hbar'$ will be given later on.

Nevertheless, since the set of functions $\phi_n(x)$ is complete, they are useful from the mathematical point of view. We can write the solution of the stochastic Schrödinger-like equation (9) or (11) in the form

$$\psi(x, t) = \sum_{n=0}^{\infty} a_n(t)\, \phi_n(x)\, e^{-i\varepsilon_n t/h'} \tag{15}$$

and find the coefficients $a_n(t)$ by substituting (15) into (9) or (11). We shall discuss exact and approximate solutions of (9) in what follows. Let us first consider the approximate method.

3. PERTURBATIVE ANALYSIS

Let us introduce the *mathematical hypothesis* that for $t = 0$ the ψ function is such that

$$\psi(x, 0) = \phi_l(x) \tag{16}$$

where $\phi_l(x)$ is given by (12) and l is an arbitrary integer. A standard perturbation calculation will give

$$a_n(t) \simeq \delta_{ln} + \frac{ie}{h'} x_{nl} \int_0^t d\tau\, [E_{VF}(\tau) + E_{RR}(\tau)]\, e^{i\omega_{nl}\tau} + \cdots \tag{17}$$

where $h'\omega_{nl} \equiv \varepsilon_n - \varepsilon_l$ and the x_{nl} are defined as

$$x_{nl} \equiv \int_{-\infty}^{\infty} dx\, x\phi_n^*(x)\, \phi_l(x) \simeq -\frac{\ddot{x}_{nl}}{\omega_0^2} \tag{18}$$

We would like to make a few remarks at this point. Firstly, the Planck's constant h contributes to a_n *only* through E_{VF}. Notice, however, that $\langle E_{VF} \rangle = 0$. Secondly, the approximate equality on the right-hand side of (18) is valid if the radiative forces in (3) may be considered less important than the harmonic force $-m\omega_0^2 x$.

The average rate of exchange of energy between the charge and the *total* radiation field is such that

$$\begin{aligned} \dot{\mathscr{E}}(l) &\equiv \left\langle \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\; W(x, p, t)\, e[E_{RR}(t) + E_{VF}(t)]\, \frac{p}{m} \right\rangle \\ &\equiv \dot{\mathscr{E}}_{RR} + \dot{\mathscr{E}}_{VF} \end{aligned} \tag{19}$$

Here it is understood that $W(x, p, t)$ will be calculated by using (6), (8), (15), (16), and (17). The ensemble average indicated by $\langle\ \rangle$ has exactly the same meaning as was indicated in (1). Within this approximate analysis we shall calculate $\dot{\bar{\mathscr{E}}}(l)$ up to the order e^2.

The radiation reaction force will be approximated by

$$eE_{RR}(t) \simeq \frac{2}{3}\frac{e^2}{c^3}\dddot{x} \simeq -\frac{2}{3}\frac{e^2\omega_0^2}{c^3}\dot{x} \equiv -\gamma p \tag{20}$$

where we have introduced the notation $\gamma = 2e^2\omega_0^2/3mc^3$ for the damping constant. The approximation of the radiation reaction force by $-\gamma p$ is consistent with (7), as we shall see more clearly with the nonperturbative calculation that will be presented in Sec. 4.

Therefore, up to order e^2, $\dot{\bar{\mathscr{E}}}_{RR}$ will be given by

$$\dot{\bar{\mathscr{E}}}_{RR} \simeq -\frac{2}{3}\frac{e^2}{c^3}\omega_0^4 \int_{-\infty}^{\infty} dx\, x^2\, |\phi_l(x)|^2 = -\gamma\varepsilon_l \tag{21}$$

where we have used $a_n \simeq \delta_{nl}$ [see (17)] in the first approximation. Such a result, which is essentially the Larmor formula, means that if only the radiation reaction is present, all the oscillator states [see (12)] are *unstable*,[14] because the square of x (or $\ddot{x}/\omega_0^2$) has a nonzero average value in such states.

A convenient form to write this simple result is

$$\dot{\bar{\mathscr{E}}}_{RR} = -\frac{2}{3}\frac{e^2}{c^3}\sum_{n=0}^{\infty} \ddot{x}_{ln}\ddot{x}_{nl} \tag{22}$$

where we have used (13) and (18).

There is another way to derive this result directly from the Larmor formula, that is, avoiding the approximation (20) for the radiation reaction force. Equivalently to (19) we define $\dot{\bar{\mathscr{E}}}_{RR}$ by

$$\begin{aligned}\dot{\bar{\mathscr{E}}}_{RR} &\equiv \left\langle \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, W(x, p, t)\left[-\frac{2}{3}\frac{e^2}{c^3}(\ddot{x})^2\right]\right\rangle \\ &= \left\langle \int_{-\infty}^{\infty} dx\, \psi^*(x, t)\left[-\frac{2}{3}\frac{e^2}{c^3}(\ddot{x})^2\right]\psi(x, t)\right\rangle \end{aligned} \tag{22a}$$

Now we must remember that we want to calculate $\dot{\bar{\mathscr{E}}}_{RR}$ only up to order e^2. Considering (15), (16), and (17) we have $\psi(x, t) \simeq \phi_l(x) + O(e)$. Taking into account the completeness relation (13), it is straightforward to show that (22a) coincides with (22). Therefore, the use of (20) gives a consistent approximation valid up to order e^2.

In order to obtain a similar expression for $\dot{\mathscr{E}}_{VF}$ a few preliminary steps are necessary for clarity. According to (6) and (8) we can write

$$\int_{-\infty}^{\infty} dp\, W(x, p, t)\, p = \mathrm{Re}\left[\psi^*(x, t)\left(-i\hbar' \frac{\partial}{\partial x}\right)\psi(x, t)\right] \tag{23}$$

and therefore [see (19)]

$$\dot{\mathscr{E}}_{VF} = \left\langle \frac{e}{m} E_{VF}(t)\, \mathrm{Re} \int_{-\infty}^{\infty} dx\, \psi^*(x, t)\left(-i\hbar' \frac{\partial}{\partial x}\right)\psi(x, t)\right\rangle \tag{24}$$

Considering that $\langle E_{VF}\rangle = 0$ and that the correlation function for the vacuum electromagnetic fields is given by (1) and (2), we obtain from (17) and (24) the result

$$\dot{\mathscr{E}}_{VF} \simeq 2\frac{e^2}{\hbar'} \mathrm{Re}\left[\sum_{n=0}^{\infty} \omega_{nl} x_{nl} x_{nl} \int_0^t d\tau\, e^{-\omega_{nl}\tau} \frac{2\pi}{3} \int_{-\omega_{\max}}^{\omega_{\max}} d\omega \frac{\hbar}{2\pi^2 c^3} |\omega|^3 e^{i\omega\tau}\right] \tag{25}$$

where $\omega_{\max}$ is a cutoff frequency consistent with our nonrelativistic approximations (see also Dalibard *et al.* in Ref. 14).

Assuming that $t|\omega - \omega_{nl}| >> 1$, we can make the replacement

$$\int_0^t d\tau\, e^{i\tau(\omega - \omega_{nl})} \to \int_0^{\infty} d\tau\, e^{i\tau(\omega - \omega_{nl})}$$

$$= \frac{i}{\omega - \omega_{nl}} + \pi\delta(\omega - \omega_{nl}) \tag{26}$$

Therefore, we obtain for (25) the following expression:

$$\dot{\mathscr{E}}_{VF} = \frac{2}{3}\frac{e^2}{c^3}\frac{h}{h'}\left[\sum_{n(>l)} \ddot{x}_{ln}\ddot{x}_{nl} - \sum_{n(<l)} \ddot{x}_{ln}\ddot{x}_{nl}\right] \tag{27}$$

provided approximation (18) is valid.

An equivalent result was obtained previously by Dalibard *et al.*[14] within the realm of quantum electrodynamics (QED). These authors, however, used Eq. (11) and $h' = h$.

Combining Eqs. (22) and (27), we can cast Eq. (19) into the interesting form

$$\dot{\mathscr{E}}(l) \simeq -\frac{2}{3}\frac{e^2}{c^3}\left[\left(1 + \frac{h}{h'}\right)\sum_{n(<l)} \ddot{x}_{ln}\ddot{x}_{nl} + \left(1 - \frac{h}{h'}\right)\sum_{n(>l)} \ddot{x}_{ln}\ddot{x}_{nl}\right] \tag{28}$$

which is the main new result of this section and deserves a few comments.

The first term in (28) represents downward transitions [to lower "energy states" $\varepsilon_n < \varepsilon_l = \hbar'\omega_0(1/2 + l)$] while the second term represents upward transitions [see (14)]. Therefore, since the parameter $\hbar'$ is free, both type of transitions are allowed. We also conclude that $\dot{\mathscr{E}}(l)$ is always nonzero in this general case. If, however, we take the particular case in which $\hbar' = \hbar$ we obtain

$$\dot{\mathscr{E}}(l) = -\frac{4}{3}\frac{e^2}{c^3}\sum_{n(<l)} \ddot{x}_{ln}\ddot{x}_{nl} = -\frac{2}{3}\frac{e^2\omega_0^2}{mc^3}(\varepsilon_l - \varepsilon_0) \tag{29}$$

which is the well-known result obtained using QED perturbative calculations.[14] An exact (nonperturbative) calculation should give a different result,[11, 13, 15–19] which may include upward transitions in the general case of nonzero temperatures, for example.[13] These are also expected, for instance, in the (slowly damped) *oscillations* of a squeezed coherent state solution of the harmonic oscillator.[18, 19] In other words, if $\hbar' > \hbar$, for instance, there are upward transitions which are present in the oscillations of the Gaussian (non-Heisenberg) state solutions. These solutions are similar to the Gaussian states first encountered by Schrödinger[15, 19] and will be presented in Sec. 4.

It is also possible to conclude from (29) that the stationary state ["energy" $\varepsilon_0 = \hbar(\omega_0/2)$] cannot be stable in the absence of the vacuum electromagnetic fluctuations which exactly balance the energy loss due to self-reaction.[14] Therefore, a striking conclusion that we have re-encountered in the above analysis is that it *is the vacuum energy that prevents the oscillator (or atomic) collapse* and gives us the criterion to identify the equilibrium state.

4. EXACT SOLUTIONS AND NON-HEISENBERG STATES

Let us see how one can easily find a set of exact (nonperturbative) solutions of the stochastic Schrödinger-like equation (9). We shall also show once more that the free parameter $\hbar'$ has no dynamical role.

Using a well-known procedure (see the paper by Schrödinger[15]), one should look for solutions of (9) or (11) with the form of Gaussian states

$$\psi_c(x, t) = \phi_0(x - x_c(t)) \exp\left[\frac{i}{\hbar'}(xp_c(t) - g(t))\right] \tag{30}$$

Here we are considering $\hbar' \neq \hbar$, and $\phi_0(x)$ is given by (12) with $n = 0$. We shall call these states classical sub-Heisenberg states if $\hbar' < \hbar$. The functions $x_c(t), p_c(t)$, and $g(t)$ are obtained by substituting (30) into (9).

It is straightforward to show that $p_c(t) = m\dot{x}_c(t)$ and that $x_c(t)$ must obey the classical equation of motion (3). It is also not difficult to obtain a closed expression for $g(t)$ which will be given by[17]

$$g(t) = \frac{h'\omega_0 t}{2} + \int_0^t dt' \left[\frac{p_c^2(t')}{2m} - \frac{m\omega_0^2 x_c^2(t')}{2} \right] \tag{31}$$

The classical trajectories x_c can be more easily found if we consider the approximation (20) for the radiation reaction force. More rigorous treatments of the radiation reaction can also be used.[7] Here, however, we shall avoid these more elaborated approaches which consider the charge as an extended particle. We shall denote

$$x_c(t) \equiv x_d(t) + x_f(t) \tag{32}$$

where $x_f(t)$ is the fluctuating part. In (32) $x_d(t)$ is the deterministic part, which depends on x_0 (initial position of the Gaussian center), and the initial velocity of the Gaussian center, namely, p_0/m. More explicitly,

$$x_d \equiv \left[x_0 \cos \omega_1 t + \left(\frac{m\gamma x_0 + 2p_0}{2m\omega_1} \right) \sin \omega_1 t \right] e^{-\gamma t/2} \tag{33}$$

where $\omega_1^2 = \omega_0^2 - \gamma^2/4$.

The fluctuating part, x_f, depends on the vacuum field $E_{VF}(t)$ and has the simple form

$$x_f(t) \equiv \frac{e}{m\omega_1} \int_0^t dt'\, E_{VF}(t') \sin[\omega_1(t-t')]\, e\gamma^{(t'-t)/2} \tag{34}$$

With the knowledge of these nonperturbative solutions $\psi_c(x, t)$, one can obtain exact expressions for the coefficients $a_n(t)$ used in the earlier expansion formula (15). We shall not present these expressions here since they have been explicitly written previously[16, 17] (the only difference is that here h' is a free parameter). It is also interesting to recall that for each x_0 and p_0 [see (30)] we have a different state.[16, 17] Therefore, one can obtain various sets of phase-space distributions $W(x, p, t)$ through different ψ_c with different x_0, p_0, and h'. The completeness of these functions was discussed previously (in the particular case $h' = h$) by several authors.[11, 13, 17]

The Wigner function associated to the states $\psi_c(x, t)$ also has a simple form. If we subsitute (30) into (8) and use (6), we get

$$W(x, p, t) = (\pi h')^{-1} \exp\left[-\frac{m\omega_0}{h'}(x - x_c)^2 - \frac{(p - p_c)^2}{h' m\omega_0} \right] \tag{35}$$

which has interesting properties.

Since $\hbar'$ is a free parameter, we can consider, for instance, the particular case

$$\lim_{\hbar' \to 0} W(x, p, t=0) = \delta(x - x_0)\, \delta(p - p_0) \tag{36}$$

which is a typical *deterministic* initial phase-space distribution.

If, for instance, $x_0 = p_0 = 0$ but $\hbar' \neq 0$, we get an initial phase-space distribution

$$W(x, p, t=0) = \frac{1}{\pi\hbar'} \exp\left(- \frac{m\omega_0 x^2}{\hbar'} - \frac{p^2}{\hbar' m\omega_0} \right) \equiv W_{\text{in}}(x, p) \tag{37}$$

which corresponds to an *initial classical uncertainty* relation of the form

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = (\hbar'/2)^2 \tag{38}$$

Therefore, we can analyze the phase-space evolution of our system by using the non-Heisenberg states (30). Our conclusions are not restricted to the perturbative domain [see (28) and (29)], and also include arbitrary initial shape because $\hbar'$ defines the initial Gaussian phase-space distribution [see (37)]. This freedom is possible within SED. Within QED this may be possible provided that we do not postulate the Heisenberg commutation relation for the variables x_0 and p_0 which appear in (33). *The variable x_0 is simply the center of the Gaussian ψ function and not the actual position of the charged particle.*

5. EVOLUTION IN PHASE SPACE

The classical trajectory $x_c(t)$ and momentum $p_c(t) = m\dot{x}_c(t)$, which appear in the Wigner function $W(x, p, t)$ obtained above [see (32)], are *correlated.* The various moments can be obtained from the definitions (33) and (34). These calculations will require the use of the statistical properties of the vacuum electric field [see (1)]. In this section we shall obtain the ensemble-averaged phase-space distribution, namely $\langle W(x, p, t) \rangle$, by using the transition probability[20] in phase space $Q(xpt|x'p')$. In other words, if $W_{\text{in}}(x', p')$ is the initial $(t = 0)$ phase-space distribution, then, at later times, we get

$$\langle W(x, p, t) \rangle = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dp'\, Q(xpt \mid x'p')\, W_{\text{in}}(x', p') \tag{39}$$

The transition probability Q satisfies a Fokker–Planck[20] equation which is the generalization of the Liouville equation (5). Within SED, this equation is well known and can be written as

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p}{m}Q\right)+\frac{\partial}{\partial p}\left[-(\gamma p+m\omega_0^2 x)+D\frac{\partial}{\partial p}\right]Q=0 \tag{40}$$

where D is the diffusion coefficient (see de la Peña and Cetto[2]).

At zero temperature, the diffusion coefficient is given by

$$D=\gamma[\lim_{t\to\infty}\langle p_f^2(t)\rangle]=\frac{\hbar\gamma m\omega_0}{2} \tag{41}$$

where $p_f(t)=m\dot{x}_f(t)$ and $x_f(t)$ is given by (34).

The solution of (40) is also well known and can be written in the form[20,21]

$$2\pi\alpha_1\alpha_2\sqrt{1-\xi^2}\,Q(xpt\mid x'p')$$
$$=\exp\left\{-\left[\frac{(p-p_d)^2}{\alpha_1^2}+\frac{(x-x_d)^2}{\alpha_2^2}-\frac{2\xi}{\alpha_1\alpha_2}(x-x_d)(p-p_d)\right]\Big/2(1-\xi^2)\right\} \tag{42}$$

where $x_d(p_d=m\dot{x}_d)$ is the deterministic trajectory [see (33) and replace $x_0\to x'$ and $p_0\to p'$].

The function $\alpha_1(t)$ is given by[21]

$$\alpha_1^2(t)=\frac{D}{\gamma}\left\{1-\left[1+\frac{\gamma^2}{2\omega_1^2}\sin^2(\omega_1 t)-\frac{\gamma}{2\omega_1}\sin(2\omega_1 t)\right]e^{-\gamma t}\right\} \tag{43}$$

whereas $\alpha_2(t)$ is such that

$$\alpha_2^2(t)=\frac{D}{\gamma m^2\omega_0^2}\left\{1-\left[1+\frac{\gamma^2}{2\omega_1^2}\sin^2(\omega_1 t)+\frac{\gamma}{2\omega_1}\sin(2\omega_1 t)\right]e^{-\gamma t}\right\} \tag{44}$$

The correlation ξ is given by

$$\xi\alpha_1\alpha_2=\frac{D}{m\omega_1^2}\sin^2(\omega_1 t)\,e^{-\gamma t} \tag{45}$$

and, since $Q(xpt|x't')$ is a *classical* transition probability within SED, we also have

$$\lim_{t\to 0}Q(xpt\mid x'p')=\delta(x-x')\,\delta(p-p') \tag{46}$$

Let us assume that $W_{in}(x'|p')$ is given by (37), which characterizes a simple initial non-Heisenberg state in phase space. We now want to calculate the average energy $\mathscr{E}(t)$ as a function of time. The explicit definition is

$$\mathscr{E}(t) \equiv \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \left(\frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 \right) \langle W(x, p, t) \rangle$$

$$\equiv \frac{\overline{p^2}}{2m} + \frac{1}{2} m\omega_0^2 \overline{x^2} \tag{47}$$

Using (39) and also (42) we obtain for $\overline{x^2}$ the following result:

$$\overline{x^2} = \frac{h'}{2m\omega_0} \left[\cos^2(\omega_1 t) + \left(1 + \frac{\gamma^2}{4\omega_1^2} \right) \sin^2(\omega_1 t) + \frac{\gamma}{2\omega_1} \sin(2\omega_1 t) \right] e^{-\gamma t}$$

$$+ \frac{h}{2m\omega_0} \left\{ 1 - \left[1 + \frac{\gamma^2}{2\omega_1^2} \sin^2(\omega_1 t) + \frac{\gamma}{2\omega_1} \sin(2\omega_1 t) \right] e^{-\gamma t} \right\} \tag{48}$$

and a similar expression for $\overline{p^2}$.

It is interesting to observe that $\overline{x^2} = h'/2m\omega_0$ for $t = 0$. However, for $\gamma t \gg 1$ we get $\overline{x^2} = h/2m\omega_0$ as expected. Notice that here h is the Planck's constant whose origin is the vacuum field which was introduced through (34), (1), and (2).

The final result to the average energy is

$$\mathscr{E}(t) = \frac{h\omega_0}{2} + \frac{(h' - h)\,\omega_0}{2} e^{-\gamma t} \left[1 + \frac{\gamma^2}{2\omega_0^2} \sin^2(\omega_0 t) \right] \tag{49}$$

where we have neglected γ^4/ω_0^4 as compared to 1. It is easy to see that $\mathscr{E}(t)$ varies from $\mathscr{E}(0) = h'\omega_0/2$ to $\mathscr{E}(\infty) = h\omega_0/2$, which is the average energy of the mechanical oscillator in the stationary regime. The above result (49) is more general than our previous formula for $\dot{\mathscr{E}}(l)$, presented in Sec. 3 [see (28) and (29)]. The reason is that the initial state ψ_c is an exact solution of (9), that is ψ_c has contributions from all states $\phi_l(x)$ introduced in (12). Since h' is arbitrary, the states ψ_c are associated [see (30) and (35)] to a large class of phase-space distributions. However, if $h' < h$, these Gaussian phase-space distributions violate the Heisenberg principle and, therefore, their real existence remains dubious in this special case. Nevertheless, examples of thermal phase-space distributions [see (37)] with $h' = h\coth(h\omega_0/2kT) > h$ were discussed previously using the familiar quantum mechanical approach[22] (without the zero-point field). This will be commented upon in Sec. 6.

6. CONCLUSION AND DISCUSSION

Let us briefly discuss the effects of nonzero temperatures and also other interesting environmental effects on the evolution of the non-Heisenberg states analyzed in this paper.

If the oscillator is inside a cavity with temperature T, then the spectral distribution of the vaccum electromagnetic fluctuations is given by[1,2]

$$\rho_T(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\left[\frac{1}{2} + \frac{1}{\exp(\hbar\omega/kT) - 1}\right] \equiv \frac{\hbar\omega^3}{2\pi^2 c^3}\coth\left(\frac{\hbar\omega}{2kT}\right) \tag{50}$$

instead of (2). Therefore, a simple rule, to generalize the result of our previous sections to finite temperatures, is to replace $\hbar$ by $\hbar\coth(\hbar\omega_0/2kT)$ in every place in which Planck's constant appears, for instance, in the diffusion coefficienty D given in (41) and also in Eq. (28), which is the main result of Sec. 3. If we want, for instance, to describe finite temperature effects (and dissipation by radiation reaction) on the oscillator, within the usual quantum formalism, we should use (9) with $\hbar' = \hbar$. The thermal radiation effects will appear through E_{VF}, which now has the spectral distribution (50). However, the oscillator equilibrium distribution in configuration space, that is, the "ground" state probability distribution at finite temperature, can be obtained directly from (12) by using a simple algorithm, namely, we take $|\phi_0(x)|^2$ with $\hbar' = \hbar\coth(\hbar\omega_0/2kT)$. The detailed justification of this result (within quantum mechanics) is not trivial.[22]

We also want to make some comments concerning the limitations of the application of the stochastic Schrödinger-like equations (9) or (11). As we have explained in Sec. 2, Eqs. (9) and (11) are equivalent to the Liouville equation (5) *only in the particular case in which the potential* $V(x, t)$ *can be written as*

$$V(x, t) = a(t)\,x^2 + b(t)\,x + c(t) \tag{51}$$

where $a(t)$, $b(t)$, and $c(t)$, are arbitrary functions of the time (see G. Manfredi *et al.* in Ref. 10). In other words, the time-dependent potential must be a quadratic polynomal in x. Despite this limitation, Eqs. (9) and (11) are useful because *many* physical systems can be modeled with such a potential.[23] Moreover, exact solutions of the Schrödinger equation with the potential (51) are well known (see Kleber,[23] for instance) and can be applied to (9).

On the other hand, *our previous equation* (7) *does not have the above-mentioned limitation and can be applied to a potential* $V(x)$ *with a more general* x *dependence.* Therefore, we also want to say a few words as far as

the computional use of Eq. (7) is concerned. Firstly, an expansion of the type

$$\tilde{W}(x, y, t) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} C_{ln}(t)\, \phi_l^*(x+y)\, \phi_n(x-y) \tag{52}$$

where $C_{ln}(t)$ are unknown time-dependent coefficients, may be useful. The Hermite functions $\phi_n(x)$ [see (12)] form an orthogonal and complete basis [see França and Marshall[13]] which allows us to write (52). The substitution of (52) into (7) will give a linear first-order differential equation for the coefficients $C_{ln}(t)$. This equation may be solved analytically or, at least, numerically. Therefore, we would like to suggest that the application of Eq. (52) and (7) to other physical systems deserves further attention.

In our opinion, the application of (7) to the three-dimensional motion of the electron in the Coulomb field of a proton, for instance, may reveal the possible existence of sub-Heisenberg states in the hydrogen atom. This is expected within the classical SED approach. In this case, Eq. (7) can be easily generalized to the three spatial dimensions with $V(r) = -e^2/r$, where r is the distance between the electron and the proton, namely

$$\left\{ ih' \frac{\partial}{\partial t} + \frac{(h')^2}{2m} \frac{\partial^2}{\partial \mathbf{r} \cdot \partial \boldsymbol{\xi}} + 2\boldsymbol{\xi} \cdot \left[-\frac{e^2 \mathbf{r}}{r^3} + e(\mathbf{E}_{VF} + \mathbf{E}_{RR}) \right] \right\} \tilde{W}(\mathbf{r}, \boldsymbol{\xi}, t) = 0 \tag{53}$$

Here $\tilde{W}(\mathbf{r}, \boldsymbol{\xi}, t)$ is the three-dimensional generalization of (6) and $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z)$ is the vector associated to the auxiliary coordinates. If a solution of Eq. (53) is constructed, then the probability distribution in configuration space will be given by the ensemble average $\langle \tilde{W}(\mathbf{r}, 0, t) \rangle$. At this point it is worthwhile to stress that we are unable to make any comment, concerning the success or failure of (53) for a *full* modeling of the hydrogen atom, without doing explicit calculation. In this regard, the mathematical similarity between the Coulomb and the harmonic oscillator problems has been stressed by several authors[24] and may or may not be useful. Nevertheless, for almost circular classical orbits [$r(t) \simeq$ const], Eq. (53) can be transformed into a stochastic Schrödinger-like equation for a frequency-modulated harmonic oscillator. One can show that this equation has exact solutions in the form of squeezed coherent states[23] which may be observed, associated to quasi-circular Rydberg atomic states. Therefore, at least these particular cases may be treated within SED using the proposal developed in our paper (see Suárez Barnes *et al.*[23]).

Classical stochastic electrodynamics allows a simple mathematical introduction of sub-Heisenberg states by using the free parameter h'. These states on their turn, are associated to trajectories and, consequently, may

provide a *natural*[25] connection between the "quantum regimen" and the classical chaotic dynamics. This connection is a difficult problem because at the microscopic level, the Heisenberg uncertainty principle precludes the notion of classical orbits since position and momentum cannot be measured with infinite precision at each instant. "This implies that most definitions of classical chaos, which require the existence of orbits, lose validity."[25] At the same time, $h \neq 0$ within the SED scheme. Therefore, we suggest that the use of the SED approach explained here may also be useful to the study of chaotic phenomena (classical or quantum).[25-27] For instance, a remarkably simple example of classical chaotic behavior with a bifurcation diagram is given for an extremely simple nonautonomous circuit.[28] It is a series connection of a linear resistor, a linear inductor, and a two-segment, piecewise-linear capacitor driven by a sinusoidal voltage source. The equation of motion for the charge in the capacitor is similar to (3) because the capacitor is piecewise linear.[28] We suggest that our method (see Sec. 4) may be useful to study the connection between the classical and the "quantum" behaviour of this system.

Let us consider another interesting simple system. This occurs when a charged oscillator is inside resonant cavities[29] or between two perfect plane mirrors.[30, 31] In the latter case, the emission and the absorption of the oscillator depend on the position and the orientation which the oscillator has with respect to the mirror plates. If the oscillator is oriented *parallel* to the plates, the damping constant γ [see (20)] is modified to another one [$\gamma_{\parallel}(\omega_0)$] which is a more complicated function of ω_0, namely,[30, 31]

$$\gamma_{\parallel}(\omega_0) = \gamma \frac{3\pi c}{2\omega_0 a} \sum_{s=0}^{[\omega_0 a/\pi c]} \left[1 + \left(\frac{\pi c s}{\omega_0 a}\right)^2\right] \sin^2\left(\frac{\pi b s}{a}\right) \tag{54}$$

Here, a is the distance between the mirror plates and b ($b < a$) is the distance from the oscillator and one of the plates. An interesting remark is that $\gamma_{\parallel}(\omega_0) = 0$ if $\omega_0 < \pi c/a$. In this case the oscillator cannot lose energy and *all states are stable*, including the *nonphysical states* (12), which we have introduced in Sec. 2, using the free parameter $h' \neq h$. In other words, the charged oscillator behaves as an uncharged *Newtonian* oscillator. Even the true Planck's constant h has no role in this particular case. The reason is that the spectral distribution is also modified by the perfect mirrors, that is, $\rho_0(\omega)$ is replaced by $\rho_{\parallel}(\omega)$ such that[30, 31]

$$\rho_{\parallel}(\omega) = \frac{\gamma_{\parallel}(\omega)}{\gamma} \frac{\hbar\omega^3}{2\pi^2 c^3}, \qquad \omega \geq 0 \tag{55}$$

Therefore, $\rho_{\parallel}(\omega) = 0$ for frequencies $0 \leq \omega \leq \pi c/a$. If the oscillator frequency ω_0 is this interval, then the "vacuum" field E_{VF} is unable to excite the

oscillator which is oriented *parallel* to the perfect mirror plates (the situation is completely different if the oscillator is oriented perpendicularly to the plates). Therefore, the equation of motion (3) becomes simply

$$\ddot{x}(t) = -\omega_0^2 x(t) \tag{56}$$

which is the Newton equation of an *uncharged* oscillator.

It is interesting to see what happens, from the QED point of view, with the "Newtonian" oscillator if we remove the perfect mirrors at $t=0$, allowing radiation of all frequencies to excite the oscillator for times $t>0$. Within the QED approach the electromagnetic fields E_{VF} and E_{RR} are operators. Therefore, we can assume, without proof, the validity of Eq. (3) as the Heisenberg equation for the position operator $x(t)$ as we said in the introduction. The solution of the Heisenberg equation can be written as before [see (32) to (34)]. Due to the effect of the perfect mirror plates, the oscillator is initially Newtonian and we can consider x_0 and p_0 [see (33)] as *commuting variables*. Therefore, after removing the perfect mirror plates at $t=0$ it is possible to show that the commutation relation at later times will be (see Ref. 3 and 32, and also Eckardt in Ref. 8)

$$[x, p] = [x_0, p_0]\, e^{-\gamma t} + ih(1 - e^{-\gamma t}) = ih(1 - e^{-\gamma t}) \tag{57}$$

This qualitative analysis shows that, in principle, sub-Heisenberg states are possible within the realm of QED because $[x, p] \neq ih$ for small times ($\gamma t \ll 1$). However, when the conductivity of the plates is finite, the discontinuities of (54) become smooth because there is always some residual noise at all frequencies,[33] that is, $\rho_{\parallel}(\omega) \neq 0$ for $0 \leqslant \omega \leqslant \pi c/a$. This residual noise is very small[33] but may forbid practical observations of sub-Heisenberg states (or trajectories) by this method. However, other related interesting phenomena (as the superpression of spontaneous emission) have been observed experimentally[34] and explained theoretically within the realm of SED.[6] To our knowledge, the QED calculation was not published (see Ref. 34 and J. P. Dowling in Ref. 31) and within the SED calculation explicit use was made of the concept of trajectory. In this regard, the (indirect) observation of the *particle trajectory* seems to be possible despite the position-momentum uncertainty relation (see Refs. 6 and 35).

We believe that these conclusions may be extended to other dynamical variables, such as the spin and the magnetic dipole. The reason is that every physical system is exposed to fluctuating forces and, consequently, will exchange *angular momentum* with its surroundings. Schiller and Tesser,[8] Boyer,[36] and Barranco, Brunini, and França[37] have discussed

this point within QED and SED. It is shown clearly that the concept of "spin vector trajectory" is useful and is also in good agreement with the experimental observations.[36,37] Therefore, in our opinion, the paramagnetic behavior may be interpreted as experimental evidence of the spin vector trajectory.[37] Moreover, quite recently, it has been suggested that new detectors constructed with the aid of modern quantum optics may provide a way to understand better the relation between path detection and the uncertainty principle.[38]

Here, by using a simple mathematical approach, we have shown that the harmonic oscillator is an essentially *classical* system. This has been known since the pioneering works by Wigner (1932), within conventional quantum mechanics,[10] and by Marshall (1963), within the realm of random (or stochastic) electrodynamics.[1] Within our approach, *which may be considered only a simple mathematical exercise*, we have shown how to connect the oscillator Liouville equation with a stochastic Schrödinger-like equation, using a convenient *arbitrary* constant h'. This *freedom* allow us to see very clearly, in Eqs. (12) to (15), that the energy levels and eigenfunctions of the harmonic oscillator are only a useful *mathematical* basis of functions: "No excited state of the harmonic oscillator is a real physical state."[13] Therefore, the usual "quantization" of the free electromagnetic field, as an infinite set of uncoupled *harmonic oscillators*, may be considered as a *formal* mathematical treatment of a *classical* random field.[1-3]

In this regard, we would like to recall that Born, Heisenberg, and Jordan (1926) and Dirac (1927), showed how to systematically "quantize" free electromagnetic field by exploiting the representation of each field mode as a *harmonic oscillator.*[39] They, therefore, *rediscovered the classical zero-point electromagnetic fluctuations* used by Planck, Einstein, Stern, Nernst, Debye and others.[1-5] In fact, "zero-point energy was a *hot topic* during the decade preceding Heisenberg's matrix mechanics and Schrödinger's wave mechanics."[40] We have to recognize that, at present, there is also an increasing theoretical and experimental interest in many other manifestations of the zero-point electromagnetic fluctuations.[41-48] Therefore, we think that our paper may be useful within this wide context.

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45. Claudia Eberlein, in "Sonoluminescence as Quantum Vacuum Radiation," University of Illinois, Urbana, Illinois 61801–3080, USA, preprint (May 1995). See also S. J. Putterman, *Sci. Am.* **272**, February 1995, p. 32.
46. H. M. França and A. Maia, Jr., in "Maxwell electromagnetic theory, Planck's radiation law, and Bose–Einstein statistics," preprint IFUSP (May, 1995), submitted to *Found. Phys.*
47. K. Dechoum, H. M. França, and A. Maia, Jr., in "Some observable effects of the current fluctuations in a long solenoid: the significance of the vector potential," preprint IFUSP (September, 1995), submitted to *Found. Phys.*
48. A. V. Barranco and H. M. França, *Found. Phys. Lett.* **5**, 25 (1992).

Erratum

K. Dechoum and H. M. França

The following misprints appeared in the paper "Non-Heisenberg States of the Harmonic Oscillator," *Found. Phys.* **25**, 1599–1620 (1995).

Page 1599 (line 15, abstract): replace $(\hbar \to 0)$ by $(\hbar' \to 0)$.

Page 1603 [Eq. (7)]: replace $i\hbar' \dfrac{\partial \tilde{W}}{\partial t}$ by $-i\hbar' \dfrac{\partial \tilde{W}}{\partial t}$.

Page 1607 [Eq. (24)]: replace $\hbar'$ by $\hbar'$.

Page 1607 [Eq. (27)]: replace $-l$ by l in the second summation.

Page 1607 (line 6 from below): the correct phrase is "These authors, however, used Eq. (2) and Eq. (3), that is, their approach is based on the Heisenberg picture."

Page 1608 (line 13): replace $\hbar' > \hbar$ by $\hbar > \hbar'$.

Page 1609 [Eq. (34)]: the correct factor is $\exp[\gamma(t' - t)/2]$.

Page 1609 (last line): the complete line is "which has interesting properties in the limit $t \to 0$. It should be remarked that the limit $t \to \infty$ is meaningful only in the case $\hbar' \to 0$."

Page 1610 (line 13): replace $\hbar'$ by $\hbar'(<\hbar)$.

Page 1613 (line 15): replace E_{TF} by E_{TF}^{thermal}.

Page 1613 (line 16): replace (50) by $p_{TF}(\omega) - p_0(\omega)$ [see (50)].

Page 1614 [Eq. (53)]: replace $i\hbar'(\partial/\partial t)$ by $-i\hbar'(\partial/\partial t)$.

Page 1614 (last line): replace "trajectories and" by "trajectories $(\hbar' \to 0)$ and."

Page 1615 (line 16): replace (see Sec. 4) by (see Sec. 5).

Page 1615 (last line): replace "ω_0 is this" by "ω_0 is within this."

Page 1620 (Ref. 46): the correct reference is "H. M. França, A. Maia, Jr., and C. P. Malta, *Found. Phys.* **26**, 1055 (1996)."