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SENSITIVITY ANALYSIS IN GEOMETRIC PROGRAMMING: THEORY AND COMPUTATIONS

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Abstract

This paper surveys the main developments in the area of sensitivity analysis for geometric programming problems, including both the theoretical and computational aspects. It presents results which characterize solution existence, continuity, and differentiability properties for primal and dual geometric programs as well as the optimal value function differentiability properties for primal and dual programs. It also provides an overview of main computational approaches to sensitivity analysis in geometric programming which attempt to estimate new optimal solutions resulting from perturbations in some problem parameters.

Keywords: Sensitivity analysis, perturbed solution, optimal value function, geometric programming.

1. Introduction

The study of sensitivity analysis in geometric programming was an integral part of the original monograph by Duffin, Peterson, and Zener [16]. At about the same time, sensitivity analysis was independently developed for more general nonlinear programming problems by Fiacco and McCormick [21]. While geometric programs form a special class of nonlinear programs, the sensitivity analysis results obtained in nonlinear programming were not applied to geometric programming until the appearance of the article by Dembo [5]. Until recently, almost all sensitivity results in geometric programming were obtained for the dual program [5–7,16,26].

This paper surveys the main developments in the area of sensitivity analysis for geometric programming, including both the theoretical and computational aspects. A unified and comprehensive theoretical framework for sensitivity analysis for general parametric posynomial geometric programs was recently developed in Kyparisis [24]. That paper specialized the sensitivity analysis results in nonlinear programming, obtained by Fiacco and McCormick [21], Fiacco [19,20] and Jittorntrum [22], to posynomial geometric programs, both primal and dual, by exploiting the special structure of these programs. The obtained results characterized solution existence, continuity and differentiability properties for primal and dual geometric programs as well as the optimal value function differentiability properties for primal and dual programs. These results considerably extended and corrected the earlier results of Dembo [5].

A systematic development of computational approaches to sensitivity analysis in geometric programming is due mostly to Dinkel and Kochenberger [6,7,9] and Dinkel, Kochenberger and Wong [11,12], and also Dembo [5]. These papers estimate the new optimal solution resulting from perturbations in some problem parameters (either coefficients c_i or exponents a_{ij}). They also attempt to determine the range of parameter values for which feasible solutions exist and thus perturbed optimal solutions can be computed. The approach of Dinkel et al. [6,7,11] is based on the sensitivity analysis theorem of Duffin et al. [16] for the reduced dual geometric program and an incremental procedure for updating the inverse of a certain Jacobian matrix. This results in a computational method for calculating new optimal solutions which depend on the perturbed problem coefficients. In addition, Dinkel et al. [9,12] have also utilized the sensitivity analysis results of Bigelôw and Shapiro [2], obtained for nonlinear programming problems, to propose an incremental method for computing new optimal solutions depending on the perturbed problem exponents. Furthermore, both Dinkel and Kochenberger [7] and Dembo [5] derive approximate formulas for the feasibility ranges of equations for sensitivity calculations.

There are a number of additional results, of theoretical, computational, or applied nature, that are not surveyed here. Among theoretical results, one should mention the sensitivity analysis results in the monograph by Beightler and Phillips [1] obtained for posynomial primal geometric programs using the "constrained derivatives" approach. Another important paper is the comprehensive survey of geometric programming theory by Peterson [25] which discusses a number of issues pertaining to parametric generalized geometric programs. Peterson's results include characterizations of convexity, continuity and directional differentiability properties of the optimal value function, and the relationship between the absence of the duality gap and certain properties of the optimal value function. Some results on the convexity or concavity properties of the optimal value function and on the optimal value bounds were also obtained by Dembo [5] and Kyparisis [23]. In another development, Fang and Rajasekera [18] used a perturbation approach to prove the main duality theorem for quadratic geometric programs.

Among computational approaches not surveyed here is the method developed by Dinkel et al. [13] which uses sensitivity analysis procedures to accelerate convergence of the harmonic mean algorithm for polynomial geometric programs. Another paper by Dinkel and Tretter [14] uses an interval arithmetic approach to sensitivity analysis in posynomial geometric programming which results in generation of an interval of solution values associated with an interval of coefficient parameter values. The paper by Fang et al. [17] proposes a well-controlled dual perturbation method for posynomial geometric programs which guarantees an almost-optimal primal–dual solution pair. This approach is designed to overcome difficulties of dual solution methods due to the non-differentiability of the dual objective function.

Several authors have also applied sensitivity analysis results obtained for geometric programming problems to some special classes of problems found in applications. These papers include applications to optimal engineering design by Dembo [4] and Dinkel and Kochenberger [8], to constrained entropy maximization models by Dinkel and Kochenberger [9] and Dinkel et al. [10], and to optimization of petroleum drilling operations by Dinkel and Wong [15].

The organization of the papers is as follows. Section 2 presents formulations of the parametric primal and dual geometric programs. Sections 3, 4, and 5 are based on the theoretical sensitivity analysis results in Dembo [5] and Kyparisis [24]. Section 3 presents results on solution differentiability properties for primal geometric programs. Section 4 discusses results on solution differentiability for dual geometric programs and it also relates the results of sections 3 and 4 via duality theory of geometric programming. Section 5 provides an overview of the optimal value differentiability properties for primal and dual programs. Section 6 presents an overview of the computational approaches to sensitivity analysis developed by Dinkel and Kochenberger [6,7,9] and Dinkel et al. [11,12]. Finally, section 7 includes the concluding remarks and directions of future research.

2. Formulation of parametric primal and dual geometric programs

In this section, we formulate the primal and dual geometric programs and their parametric versions. We study the properties of these programs in the subsequent sections of the paper. The material of this section is based on [5,11,16,24].

The *primal posynomial geometric program* has the form (Duffin et al. [16])

$$\begin{aligned} &\text{minimize}_t && g_0(t) && && \textbf{GP} \\ &\text{subject to} && g_k(t) \leqslant 1, && k = 1, \ldots, p, && (2.1) \\ & && t_j > 0, && j = 1, \ldots, m, \end{aligned}$$

where

$$g_i(t) = \sum_{i \in J_k} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad k = 0, 1, \ldots, p,$$

the index sets J_k are defined by $J_k = \{m_k, m_k + 1, \ldots, n_k\}$, $k = 0, 1, \ldots, p$, $m_0 = 1$, $m_1 = n_0 + 1$, $m_2 = n_1 + 1, \ldots, m_p = n_{p-1} + 1$, $n_p = n$, the exponents a_{ij} are arbitrary real numbers and the coefficients c_i are *positive*.

The *dual posynomial geometric program*, associated with the primal geometric program **GP**, has the form (Duffin et al. [16])

$$\text{maximize}_{(\delta,\lambda)} \quad v(\delta, \lambda) = \prod_{i=1}^{n} (c_i/\delta_i)^{\delta_i} \prod_{k=1}^{p} \lambda_k^{\lambda_k} \tag{GD}$$

$$\text{subject to} \quad \sum_{i=1}^{n_0} \delta_i = 1, \tag{2.2}$$

$$\sum_{i=1}^{n} a_{ij}\delta_i = 0, \quad j = 1, \ldots, m, \tag{2.3}$$

$$\sum_{i \in J_k} \delta_i = \lambda_k, \quad k = 1, \ldots, p, \tag{2.4}$$

$$\delta_i \geqslant 0, \quad i = 1, \ldots, n, \tag{2.5}$$

$$\lambda_k \geqslant 0, \quad k = 1, \ldots, p, \tag{2.6}$$

where the sets J_k are defined in **GP**. Under very general assumptions, optimal solutions of the primal program **GP** and the dual program **GD** have been shown by Duffin et al. [16] to satisfy certain duality relationships.

When the constraints (2.2)–(2.6) of the dual geometric program **GD** are solved in terms of a set of basis vector $b^{(j)}$ for $j = 0, 1, \ldots, d$, the following *reduced dual geometric program* is obtained (Duffin et al. [16])

$$\text{maximize}_r \quad v(r) = K_0 \prod_{j=1}^{d} K_j^{r_j} \prod_{i=1}^{n} \delta_i(r)^{-\delta_i(r)} \prod_{k=1}^{p} \lambda_k(r)^{\lambda_k(r)} \tag{$\mathbf{GD_R}$}$$

$$\text{subject to} \quad \delta_i(r) = b_i^{(0)} + \sum_{j=1}^{d} r_j b_i^{(j)} \geqslant 0, \quad i = 1, \ldots, n,$$

where

$$\lambda_k(r) = \lambda_k^{(0)} + \sum_{j=1}^{d} r_j \lambda_k^{(j)}, \quad k = 1, \ldots, p,$$

$$\lambda_k^{(j)} = \sum_{i \in J_k} b_i^{(j)}, \quad j = 0, 1, \ldots, d, \; k = 1, \ldots, p,$$

$$K_j = \prod_{i=1}^{n} c_i^{b_i^{(j)}}, \quad j = 0, 1, \ldots, d,$$

r_j, $j = 1, \ldots, d$ are the new independent variables, the constant vectors $b^{(j)}$ form a basis for the column space of matrix A, composed of coefficients a_{ij} of dual constraints, and $d = n - m - 1$ is called the degree of difficulty. The advantages of considering the reduced dual program $\mathbf{GD_R}$ were discussed in Duffin et al. [16].

The earliest sensitivity analysis results for geometric programs, due to Duffin et al. [16], were based on the dual formulation $\mathbf{GD_R}$ and involved changes in the optimal dual (and primal) solution resulting from perturbations of coefficients c_i

treated as parameters. The exponents a_{ij} were assumed constant. Moreover, the early computational approaches to sensitivity analysis in geometric programming developed by Dinkel et al. [6,7,11] have also been based on the formulation $\mathbf{GD_R}$ and involved perturbations of parameters c_i.

More recently, Dembo [5] has considered the following primal posynomial geometric program with variable right hand sides in constraints (2.1) (this problem was also implicitly considered in an earlier paper by Dinkel and Kochenberger [7]):

$$\begin{aligned} &\text{minimize}_t \quad g_0(t) && \mathbf{GP'} \\ &\text{subject to} \quad g_k(t) \leqslant r_k, \quad k = 1, \ldots, p, && (2.7) \\ &\qquad t_j > 0, \quad j = 1, \ldots, m, \end{aligned}$$

where $r_k > 0$, $k = 1, \ldots, p$ and the other quantities are the same as in **GP**. The geometric dual of **GP′** is defined as

$$\text{maximize}_{(\delta,\lambda)} \quad v(\delta, \lambda) = \prod_{i=1}^{n} (c_i/\delta_i)^{\delta_i} \prod_{k=1}^{p} (\lambda_k/r_k)^{\lambda_k} \qquad \mathbf{GD'}$$

$$\text{subject to} \quad (2.2)\text{–}(2.6).$$

Dembo [5] conducted sensitivity analysis for the geometric programs **GP′** and **GD′** where perturbations were allowed not only in the coefficients c_i but also in the coefficients a_{ij} and r_k. The results of Dembo [5] were considerably extended and corrected in a recent paper of Kyparisis [24] who developed a comprehensive framework for sensitivity analysis for general parametric versions of primal and dual geometric programs. These parametric geometric programs are formulated in the second part of this section.

The parametric version of **GP**, denoted by **GP(ϵ)** [24], is obtained when the exponents a_{ij} and the coefficients c_i in **GP** are replaced by $a_{ij}(\epsilon)$ and $c_i(\epsilon)$, where $\epsilon \in \mathbb{R}^r$ is the parameter vector. It is assumed here that a_{ij} and c_i are twice continuously differentiable functions of ϵ. Note that the formulation **GP(ϵ)** allows for arbitrary perturbations of coefficients c_i and exponents a_{ij}. In addition, **GP(ϵ)** also generalizes the formulation **GP′** since the (positive) right hand sides r_k in constraints (2.7) can be accommodated by dividing both sides of inequality constraints (2.7) by r_k and defining the new coefficients $c_i' = c_i/r_k$. Perturbations of r_k can thus be translated into corresponding perturbations of the new coefficients c_i' (see [7]).

In this paper, the following *equivalent parametric convex program*, obtained by using the transformation $t_j = \exp(x_j)$ (see Duffin et al. [16]), is analyzed [24]:

$$\begin{aligned} &\text{minimize}_x \quad f_0(x, \epsilon) && \mathbf{P(\epsilon)} \\ &\text{subject to} \quad f_k(x, \epsilon) \leqslant 1, \quad k = 1, \ldots, p, \end{aligned}$$

where

$$f_k(x, \epsilon) = \sum_{i \in J_k} c_i(\epsilon) \exp\left(\sum_{j=1}^{m} a_{ij}(\epsilon) x_j \right), \quad k = 0, 1, \ldots, p,$$

and the sets J_k are defined in **GP**. The convex equivalent of a simpler unconstrained parametric primal posynomial geometric program, i.e. the unconstrained version of **P**(ϵ), is also separately considered:

$$\text{minimize}_x \quad f_0(x, \epsilon). \qquad \mathbf{P_0}(\epsilon)$$

Similarly to **GP**(ϵ), one can formulate the parametric version of **GD**, denoted by **GD**(ϵ) [24], where a_{ij} and c_i are replaced by $a_{ij}(\epsilon)$ and $c_i(\epsilon)$. In this paper, the following *equivalent parametric concave program*, which is derived by substituting $V(\delta, \lambda, \epsilon) = \log(v(\delta, \lambda, \epsilon))$ for $v(\delta, \lambda, \epsilon)$ in **GD**(ϵ) (see [16]), is analyzed [24]:

$$\begin{aligned} &\text{maximize}_{(\delta,\lambda)} \quad V(\delta, \lambda, \epsilon) = \sum_{i=1}^{n} \delta_i \log(c_i(\epsilon)/\delta_i) + \sum_{k=1}^{p} \lambda_k \log \lambda_k \qquad \mathbf{D}(\epsilon) \\ &\text{subject to} \quad A^{\mathrm{T}}(\epsilon)\delta = e_0, \\ &\qquad B\delta = \lambda, \\ &\qquad \delta \geqslant 0, \quad \lambda \geqslant 0, \end{aligned}$$

where $\log \equiv \log_e$, $A(\epsilon) = [\bar{e} \mid \bar{A}(\epsilon)]$, $\bar{A}(\epsilon) = [a_{ij}(\epsilon)]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $\bar{e}$ corresponds to the constraint (2.2) in **GD**, $e_0 = (1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{m+1}$, and the constant $p \times n$ matrix B corresponds to the constraints (2.4) in **GD**. The concave equivalent of the parametric dual geometric program corresponding to the unconstrained parametric primal geometric program $\mathbf{P_0}(\epsilon)$ has the form [24]

$$\begin{aligned} &\text{maximize}_\delta \quad V_0(\delta, \epsilon) = \sum_{i=1}^{n_0} \delta_i \log(c_i(\epsilon)/\delta_i), \qquad \mathbf{D_0}(\epsilon) \\ &\text{subject to} \quad A_0(\epsilon)^{\mathrm{T}}\delta = e_0, \quad \delta \geqslant 0, \end{aligned}$$

where $A_0(\epsilon) = [\bar{e}_0 \mid \bar{A}_0(\epsilon)]$, $\bar{A}_0(\epsilon) = [a_{ij}(\epsilon)]$, $i = 1, \ldots, n_0$, $j = 1, \ldots, m$, $\bar{e}_0 = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^{n_0}$.

3. Solution differentiability properties for the primal geometric program

This section presents results on the differentiability properties of the perturbed optimal solution (and the associated optimal Lagrange multipliers) of the parametric primal geometric program **P**(ϵ). It is based on the recent paper by Kyparisis [24].

Since **P**(ϵ) is a nonlinear programming problem, the Lagrangian for **P**(ϵ) has the form

$$L_{\mathbf{P}}(x, \mu, \epsilon) = f_0(x, \epsilon) + \sum_{k=1}^{p} \mu_k (f_k(x, \epsilon) - 1),$$

where $\mu = (\mu_1, \ldots, \mu_p)^{\mathrm{T}}$ is the Lagrange multiplier vector. Suppose that x^* is a solution of the unperturbed problem **P**(ϵ^*). We shall assume in the sequel, without any loss of generality, that all the constraints of $\mathbf{P}(\epsilon^*)$ are binding at x^*, i.e. that $f_k(x^*, \epsilon^*) = 1$, $k = 1, \ldots, p$ (if this is not the case, then the nonbinding constraints can be deleted from $\mathbf{P}(\epsilon)$ without affecting the local analysis of solutions of $\mathbf{P}(\epsilon)$ in a neighborhood of x^*). It is well known [21] that, if an appropriate constraint qualification holds at x^*, then the Karush–Kuhn–Tucker (KKT) conditions hold at x^* for $\mathbf{P}(\epsilon^*)$, i.e. there exists $\mu_k^* \geqslant 0$, $k = 1, \ldots, p$, such that

$$\nabla_x L_{\mathbf{P}}(x^*, \mu^*, \epsilon^*)$$
$$= \left[\nabla_x \tilde{f}_0(\bar{A}(\epsilon^*)x^*, \epsilon^*) + \sum_{k=1}^{p} \mu_k^* \nabla_z \tilde{f}_k(\bar{A}(\epsilon^*)x^*, \epsilon^*)\right] \bar{A}(\epsilon^*) = 0, \tag{3.1}$$

$$\tilde{f}_k(\bar{A}(\epsilon^*)x^*, \epsilon^*) - 1 = 0, \quad k = 1, \ldots, p, \tag{3.2}$$

where we have introduced the notation

$$\tilde{f}_k(z, \epsilon) = \sum_{i \in J_k} c_i(\epsilon) \exp(z_i), \quad k = 0, 1, \ldots, p, \; z \in \mathbb{R}^n,$$

so that

$$f_k(x, \epsilon) = \tilde{f}_k(\bar{A}(\epsilon)x, \epsilon), \quad k = 0, 1, \ldots, p.$$

Denote by $M_{\mathbf{P}}^*$ the Jacobian of the KKT system of equations (3.1)–(3.2) with respect to (x, μ), evaluated at (x^*, μ^*, ϵ^*). It can be shown [24] that $M_{\mathbf{P}}^*$ is of the form

$$M_{\mathbf{P}}^* = \begin{bmatrix} \bar{A}^{*\mathrm{T}} H \bar{A}^* & (C\bar{A}^*)^{\mathrm{T}} \\ C\bar{A}^* & 0 \end{bmatrix}, \tag{3.3}$$

where

$$H = \operatorname{diag}\{\ldots, c_i(\epsilon^*) \exp(z_i^*), \; i \in J_0, \ldots, \mu_k^* c_i(\epsilon^*) \exp(z_i^*), \; i \in J_k, \ldots\},$$
$$k \in 1, \ldots, p,$$
$$z^* = \bar{A}^* x^*, \quad \bar{A}^* = \bar{A}(\epsilon^*),$$
$$C = \left[\nabla_z \tilde{f}(\bar{A}^* x^*, \epsilon^*)\right], \quad \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_p)^{\mathrm{T}}.$$

The following theorem relates the second order and regularity conditions for nonlinear programs, which are known to be necessary and sufficient for the nonsingularity of the Jacobian $M_{\mathbf{P}}^*$ [21], to certain conditions on the matrices C and $\bar{A}(\epsilon^*)$ and the Lagrange multiplier vector μ^*.

THEOREM 3.1 [24]

Suppose that the following assumptions hold:

(A1) the KKT conditions hold at x^* for $\mathbf{P}(\epsilon^*)$ with some μ^*,

(A2) $f_k(x^*, \epsilon^*) = 1$, $k = 1, \ldots, p$.

Then, the matrix $M_{\mathbf{P}}^*$ is invertible if and only if the standard second order sufficient condition, the linear independence condition, and the strict complementarity slackness condition [21] hold for $\mathbf{P}(\epsilon^*)$ at x^* with μ^*; i.e., if and only if the following conditions are satisfied:

(A3) rank $(C\bar{A}^*) = p$,

(A4) rank $\bar{A}^* = m$,

(A5) $\mu_k^* > 0$, $k = 1, \ldots, p$.

The next theorem is the key sensitivity analysis result for the parametric primal geometric program $\mathbf{P}(\epsilon)$. It states sufficient conditions for the existence, continuity and differentiability of the perturbed optimal solution $x(\epsilon)$ and the associated Lagrange multiplier vector $\mu(\epsilon)$ for $\mathbf{P}(\epsilon)$ in the nondegenerate case, i.e. when the strict complementarity slackness condition (A5) holds. We also note that when some of the constraints of $\mathbf{P}(\epsilon^*)$ are not binding at x^*, this theorem can be stated in an appropriately modified form ([24], proposition 5.2).

THEOREM 3.2 [24]

Suppose that the assumptions (A1)–(A5) hold. Then, for ϵ is some neighborhood of ϵ^*, there exist unique once continuously differentiable vector functions $x(\epsilon)$ and $\mu(\epsilon)$ such that the conditions (A1)–(A5) hold at $x(\epsilon)$ with $\mu(\epsilon)$ for the problem $\mathbf{P}(\epsilon)$ (and thus $x(\epsilon)$ is a unique global minimum of $\mathbf{P}(\epsilon)$).

Moreover, the derivatives $\nabla_\epsilon x(\epsilon^*)$ and $\nabla_\epsilon \mu(\epsilon^*)$ are uniquely determined by the system

$$\begin{bmatrix} \nabla_\epsilon x(\epsilon^*) \\ \nabla_\epsilon \mu(\epsilon^*) \end{bmatrix} = -(M_{\mathbf{P}}^*)^{-1} \begin{bmatrix} \nabla_{\epsilon x}^2 L_{\mathbf{P}}(x^*, \mu^*, \epsilon^*) \\ \nabla_\epsilon \tilde{f}(\bar{A}^* x^*, \epsilon^*) \end{bmatrix} \tag{3.4}$$

and the inverse of matrix $M_{\mathbf{P}}^*$ can be computed using the formulas

$$(M_{\mathbf{P}}^*)^{-1} = \begin{bmatrix} H_{\mathbf{P}} & (C\bar{A}^*)^{\mathrm{T}} \\ C\bar{A}^* & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{\mathbf{P}} & A_{12}^{\mathbf{P}} \\ A_{21}^{\mathbf{P}} & A_{22}^{\mathbf{P}} \end{bmatrix},$$

$$A_{11}^{\mathbf{P}} = H_{\mathbf{P}}^{-1}\left\{ I - (C\bar{A}^*)^{\mathrm{T}}\left[(C\bar{A}^*)H_{\mathbf{P}}^{-1}(C\bar{A}^*)^{\mathrm{T}}\right]^{-1}(C\bar{A}^*)^{\mathrm{T}} H_{\mathbf{P}}^{-1}\right\},$$

$$A_{12}^{\mathbf{P}} = (A_{21}^{\mathbf{P}})^{\mathrm{T}} = H_{\mathbf{P}}^{-1}(C\bar{A}^*)^{\mathrm{T}}\left[(C\bar{A}^*)H_{\mathbf{P}}^{-1}(C\bar{A}^*)^{\mathrm{T}}\right]^{-1},$$

$$A_{22}^{\mathbf{P}} = -\left[(C\bar{A}^*)H_{\mathbf{P}}^{-1}(C\bar{A}^*)^{\mathrm{T}}\right]^{-1}, \quad H_{\mathbf{P}} = \bar{A}^{*\mathrm{T}} H \bar{A}^*,$$

where the matrix $H_{\mathbf{P}}$ is invertible.

We now consider separately the parametric unconstrained geometric program $\mathbf{P}_0(\epsilon)$ and state the result analogous to theorem 3.2. The KKT conditions at x^* for $\mathbf{P}_0(\epsilon^*)$ have the form

$$\nabla_{z_0} \tilde{f}_0(\bar{A}_0(\epsilon^*)x^*, \epsilon^*)\bar{A}_0(\epsilon^*) = 0, \tag{3.5}$$

where

$$\bar{A}_0(\epsilon) = \left[a_{ij}(\epsilon)\right], \quad i = 1, \ldots, n_0, \ j = 1, \ldots, m, \ z_0 \in \mathbb{R}^{n_0}.$$

PROPOSITION 3.1 [24]

Suppose that the following assumptions hold:

(A1′) the KKT conditions hold at x^* for $\mathbf{P}_0(\epsilon^*)$,

(A6) rank $\bar{A}_0^* = m$, where $\bar{A}_0^* = \bar{A}_0(\epsilon^*)$.

Then, for ϵ in some neighborhood of ϵ^*, there exists a unique once continuously differentiable vector function $x(\epsilon)$ such that the conditions (A1′) and (A6) hold at $x(\epsilon)$ for the problem $\mathbf{P}_0(\epsilon)$ (and thus $x(\epsilon)$ is a unique global minimum of $\mathbf{P}_0(\epsilon)$).

Moreover, the derivative $\nabla_\epsilon x(\epsilon^*)$ is uniquely determined by the system

$$\nabla_\epsilon x(\epsilon^*) = -\left(M_{\mathbf{P}_0}^*\right)^{-1} \nabla_{\epsilon x}^2 f_0(x^*, \epsilon^*),$$

where $M_{\mathbf{P}_0}^*$ is the Jacobian of the KKT system of equations (3.5) with respect to x, evaluated at (x^*, ϵ^*), given by

$$M_{\mathbf{P}_0}^* = \bar{A}_0^{*\mathrm{T}} H_0 \bar{A}_0^*,$$

$$H_0 = \operatorname{diag}\left\{ c_1(\epsilon^*) \exp(z_1^*), \ldots, c_{n_0}(\epsilon^*) \exp(z_{n_0}^*) \right\},$$

$$z_0^* = \bar{A}_0^* x^*.$$

The following theorem extends theorem 3.2 to the degenerate case when the strict complementarity slackness assumption (condition (A5)) does not hold for the problem $\mathbf{P}(\epsilon)$. In that case, the assumption concerning the rank of $\bar{A}^*$ must be strengthened.

THEOREM 3.3 [24]

Suppose that, in addition to assumptions (A1)–(A3), the following assumptions hold:

(A7) $\mu_k^* > 0$, $k \in I$, $\mu_k^* = 0$, $k \in K$, where $I = \{1, \ldots, p_I\}$, $K = \{p_{I+1}, \ldots, p\}$, $p_I < p$,

(A8) rank $\bar{A}_I^* = m$, where we denote $\bar{A}_S^* = \bar{A}_S(\epsilon^*)$ and $\bar{A}_S(\epsilon) = [a_{ij}(\epsilon)]$, $i \in J_k$, $k \in \{0\} \cup S$, $j = 1, \ldots, m$, for any index set S.

Then, for ϵ in some neighborhood of ϵ^*, there exist unique vector functions $x(\epsilon)$ and $\mu(\epsilon)$, continuous and such that the conditions (A1)–(A3), (A7)–(A8) hold at $x(\epsilon)$ with $\mu(\epsilon)$ for the problem $\mathbf{P}(\epsilon)$ (and thus $x(\epsilon)$ is a unique global minimum of $\mathbf{P}(\epsilon)$).

Moreover, the directional derivatives of $x(\epsilon)$ and $\mu(\epsilon)$ at ϵ^* in any direction v, denoted by $D_v x(\epsilon^*)$ and $D_v \mu(\epsilon^*)$, respectively, exist and are uniquely determined by the system

$$\left(\bar{A}_I^{*\mathrm{T}} H_I \bar{A}_I^*\right)\dot{x} + \left(C\bar{A}^*\right)^{\mathrm{T}} \dot{\mu} = -\nabla_{\epsilon x}^2 L_{\mathbf{P}}(x^*, \mu^*, \epsilon^*) v, \tag{3.6}$$

$$(C_I \bar{A}_I^*)\dot{x} = \left[\ldots, -\left(\nabla_\epsilon \tilde{f}_k(\bar{A}^* x^*, \epsilon^*) v\right)^{\mathrm{T}}, k \in I, \ldots\right]^{\mathrm{T}}, \tag{3.7}$$

$$(C_K \bar{A}_K^*)\dot{x} \geqslant \left[\ldots, -\left(\nabla_\epsilon \tilde{f}_k(\bar{A}^* x^*, \epsilon^*) v\right)^{\mathrm{T}}, k \in K, \ldots\right]^{\mathrm{T}}, \tag{3.8}$$

$$\dot{\mu}_k \geqslant 0,\ k \in K,\ \dot{\mu}_k\left[\left(\nabla_z \tilde{f}_k(\bar{A}^* x^*, \epsilon^*)\bar{A}^*\right)\dot{x} + \nabla_\epsilon \tilde{f}_k(\bar{A}^* x^*, \epsilon^*) v\right] = 0,\ k \in K, \tag{3.9}$$

where we define $\dot{x} = D_v x(\epsilon^*)$, $\dot{\mu} = D_v \mu(\epsilon^*)$, $\dot{\mu}_k = D_v \mu_k(\epsilon^*)$ and

$H_I = \operatorname{diag}\{\ldots, c_i(\epsilon^*)\exp(z_i^*), i \in J_0, \ldots, \mu_k^* c_i(\epsilon^*)\exp(z_i^*), i \in J_k, \ldots\}$,
 $k \in I$,

$C_S = \left[\ldots, \nabla_{z_S} \tilde{f}_k(\bar{A}^* x^*, \epsilon^*), k \in S, \ldots\right]^{\mathrm{T}}$,

$z_S = (\ldots, z_i, i \in J_k, \ldots)^{\mathrm{T}}, \quad k \in \{0\} \cup S$,

for any index set S.

4. Solution differentiability properties for the dual geometric program

This section presents results on the differentiability properties of the perturbed optimal solution (and the associated optimal Lagrange multipliers) of the parametric dual geometric program $\mathbf{D}(\epsilon)$. The main result of this section, theorem 4.2, was originally proved in Dembo [5]. It extends the earlier result in Duffin et al. [16], stated for perturbations in coefficients c_i only, which is given as theorem 6.1 in section 6. Most of the remaining results in this section were obtained in Kyparisis [24].

Since $\mathbf{D}(\epsilon)$ is a nonlinear programming problem, the Lagrangian for $\mathbf{D}(\epsilon)$ has the form

$$L_{\mathbf{D}}(\delta, \lambda, w, u, \epsilon) = V(\delta, \lambda, \epsilon) + w^{\mathrm{T}}\left(A(\epsilon)^{\mathrm{T}}\delta - e_0\right) + u^{\mathrm{T}}(Bd - \lambda),$$

where $w = (w_0, \ldots, w_m)^{\mathrm{T}}$, $u = (u_1, \ldots, u_p)^{\mathrm{T}}$ are the Lagrange multiplier vectors. Suppose that (δ^*, λ^*) is a solution of the unperturbed problem $\mathbf{D}(\epsilon^*)$. We shall assume in the sequel, unless stated otherwise, that the dual variables are positive, i.e. that $\delta_i^* > 0$, $i = 1, \ldots, n$ and thus also that $\lambda_k^* > 0$, $k = 1, \ldots, m$. This implies that in a neighborhood of (δ^*, λ^*) the constraints $\delta \geqslant 0$ and $\lambda \geqslant 0$ are not binding and can be disregarded.

The KKT conditions at (δ^*, λ^*) for $\mathbf{D}(\epsilon^*)$, which hold under an appropriate constraint qualification, are: there exist w^*, u^* such that

$$\nabla_\delta L_{\mathbf{D}}(\delta^*, \lambda^*, w^*, u^*, \epsilon^*) = \nabla_\delta V(\delta^*, \lambda^*, \epsilon^*) + w^{*\mathrm{T}} A(\epsilon^*)^{\mathrm{T}} + u^{*\mathrm{T}} B = 0, \tag{4.1}$$

$$\nabla_\lambda L_{\mathbf{D}}(\delta^*, \lambda^*, w^*, u^*, \epsilon^*) = \nabla_\lambda V(\delta^*, \lambda^*, \epsilon^*) = 0, \tag{4.2}$$

$$A(\epsilon^*)^{\mathrm{T}}\delta^* - e_0 = 0, \tag{4.3}$$

$$B\delta^* - \lambda^* = 0. \tag{4.4}$$

The following proposition summarizes the well-known facts pertaining to the dual geometric program.

PROPOSITION 4.1 [16]

Suppose that (δ, λ) is a feasible point of $\mathbf{D}(\epsilon)$, i.e. $A(\epsilon)^{\mathrm{T}}\delta = e_0$, $B\delta = \lambda$, and that $\delta_i > 0$, $i = 1, \ldots, n$. Then the following results hold:

(i) the matrix $H_\delta = \nabla_\delta^2 V(\delta, \lambda, \epsilon) = \operatorname{diag}\{-1/\delta_1, \ldots, -1/\delta_n\}$ is negative definite and the matrix $H_\lambda = \nabla_\lambda^2 V(\delta, \lambda, \epsilon) = \operatorname{diag}\{1/\lambda_1, \ldots, 1/\lambda_p\}$ is positive definite;

(ii) the matrix $H_R = H_\delta + B^{\mathrm{T}} H_\lambda B$ is negative semidefinite and it can be expressed as $H_R = H_\delta - B^{\mathrm{T}}(BH_\delta^{-1}B^{\mathrm{T}})^{-1}B$.

Denote by $M_{\mathbf{D}}^*$ the Jacobian of the KKT system of equations (4.1)–(4.4) with respect to (δ, λ, w, u), evaluated at $(\delta^*, \lambda^*, w^*, u^*, \epsilon^*)$. It can be shown [5,24] that $M_{\mathbf{D}}^*$ is of the form (where we denote $A^* = A(\epsilon^*)$)

$$M_{\mathbf{D}}^* = \begin{bmatrix} H_{\delta^*} & 0 & A^* & B^{\mathrm{T}} \\ 0 & H_{\lambda^*} & 0 & -I \\ A^{*\mathrm{T}} & 0 & 0 & 0 \\ B & -I & 0 & 0 \end{bmatrix}. \tag{4.5}$$

The following theorem relates the second order and regularity conditions for nonlinear programs, which are known to be necessary and sufficient for the nonsingularity of the Jacobian $M_{\mathbf{D}}^*$ [21], to certain conditions on the matrices B, H_{δ^*}, and A^*. The sufficiency of these conditions to guarantee the nonsingularity of $M_{\mathbf{D}}^*$ was proved by Dembo [5] and their necessity by Kyparisis [24].

THEOREM 4.1 [5,24]

Suppose that the following assumptions hold:

(B1) the KKT conditions hold at (δ^*, λ^*) for $\mathbf{D}(\epsilon^*)$ with some w^* and u^*,

(B2) $\delta_i^* > 0$, $i = 1, \ldots, n$.

Then, the matrix $M_{\mathbf{D}}^*$ is invertible if and only if the standard second order sufficient conditions and the linear independence condition [21] hold for $\mathbf{D}(\epsilon^*)$ at (δ^*, λ^*) with w^* and u^*; i.e., if and only if the following conditions are satisfied:

(B3) rank $(BH_{\delta^*}^{-1}A^*) = p$,

(B4) rank $A^* = m + 1$.

The next theorem is the main sensitivity analysis result for the parametric dual geometric program $\mathbf{D}(\epsilon)$. It states sufficient conditions for the existence, continuity and differentiability of the perturbed optimal solution $[\delta(\epsilon), \lambda(\epsilon)]$ and the associated Lagrange multiplier vectors $w(\epsilon)$ and $u(\epsilon)$ for $\mathbf{D}(\epsilon)$ in the nondegenerate case, i.e. when the condition (B2) holds. This theorem extends an earlier sensitivity analysis result in Duffin et al. [16], obtained for perturbations in coefficients c_i only, which is given as theorem 6.1 in section 6. Duffin et al. [16] assumed that conditions (B1), (B2) and (B4) hold and also that the Jacobian $M_{\mathbf{D}}^*$ is nonsingular. They also showed that if, in addition, the submatrix with elements $b_i^{(j)}$, $i = 1,\ldots,n_0$, $j = 1,\ldots,d$ (defined in $\mathbf{GD_R}$) has rank d, then the Jacobian $M_{\mathbf{D}}^*$ is nonsingular. Theorem 4.1 uses an alternative additional condition (B3) to guarantee the nonsingularity of $M_{\mathbf{D}}^*$. Since this condition is directly verifiable given the problem data, theorem 4.2 is a more general and direct sensitivity result than those in [16].

THEOREM 4.2 [5,24]

Suppose that the assumptions (B1)–(B4) hold. Then, for ϵ in some neighborhood of ϵ^*, there exist unique once continuously differentiable vector functions $\delta(\epsilon)$, $\lambda(\epsilon)$, $w(\epsilon)$, and $u(\epsilon)$ such that the conditions (B1)–(B4) hold at $[\delta(\epsilon), \lambda(\epsilon)]$ with $w(\epsilon)$ and $u(\epsilon)$ for the problem $\mathbf{D}(\epsilon)$ (and thus $[\delta(\epsilon), \lambda(\epsilon)]$ is a unique global maximum of $\mathbf{D}(\epsilon)$).

Moreover, the derivatives $\nabla_\epsilon\delta(\epsilon^*)$, $\nabla_\epsilon\lambda(\epsilon^*)$, $\nabla_\epsilon w(\epsilon^*)$, $\nabla_\epsilon u(\epsilon^*)$ are uniquely determined by the system

$$\begin{bmatrix} \nabla_\epsilon\delta(\epsilon^*) \\ \nabla_\epsilon\lambda(\epsilon^*) \\ \nabla_\epsilon w(\epsilon^*) \\ \nabla_\epsilon u(\epsilon^*) \end{bmatrix} = -(M_{\mathbf{D}}^*)^{-1} \begin{bmatrix} \nabla_{\epsilon\delta}^2 V(\delta^*, \lambda^*, \epsilon^*) + \dot{A}(\epsilon^*) w^* \\ \nabla_{\epsilon\lambda}^2 V(\delta^*, \lambda^*, \epsilon^*) \\ \dot{A}(\epsilon^*)^{\mathrm{T}} \delta^* \\ 0 \end{bmatrix}, \tag{4.6}$$

where the non-zero part of the three-dimensional matrix $\dot{A}(\epsilon^*)$ is defined by

$$\dot{A}(\epsilon^*) = \left[\nabla_\epsilon a_{ij}(\epsilon^*)\right], \quad i = 1,\ldots,n,\ j = 1,\ldots,m,$$

$$\dot{A}(\epsilon^*) w^* = \left[\ldots, \sum_{j=1}^{m} \nabla_\epsilon^{\mathrm{T}} a_{ij}(\epsilon^*) w_j^*, \ldots\right]^{\mathrm{T}}, \quad i = 1,\ldots,n,$$

and the inverse of matrix $M_{\mathbf{D}}^*$ can be computed using the formulas

$$(M_{\mathbf{D}}^*)^{-1} = \begin{bmatrix} H_{\mathbf{D}} & P^{*\mathrm{T}} \\ P^* & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{\mathbf{D}} & A_{12}^{\mathbf{D}} \\ A_{21}^{\mathbf{D}} & A_{22}^{\mathbf{D}} \end{bmatrix},$$

$$A_{11}^{\mathbf{D}} = H_{\mathbf{D}}^{-1}\left\{I - P^{*\mathrm{T}}\left[P^* H_{\mathbf{D}}^{-1} P^{*\mathrm{T}}\right]^{-1} P^* H_{\mathbf{D}}^{-1}\right\},$$

$$A_{12}^{\mathbf{D}} = \left(A_{21}^{\mathbf{D}}\right)^{\mathrm{T}} = H_{\mathbf{D}}^{-1} P^{*\mathrm{T}}\left[P^* H_{\mathbf{D}}^{-1} P^{*\mathrm{T}}\right]^{-1},$$

$$A_{22}^{\mathbf{D}} = -\left[P^* H_{\mathbf{D}}^{-1} P^{*\mathrm{T}}\right]^{-1},$$

$$P^* = \begin{bmatrix} A^{*\mathrm{T}} & 0 \\ B & -I \end{bmatrix}, \quad H_{\mathbf{D}} = \begin{bmatrix} H_{\delta^*} & 0 \\ 0 & H_{\lambda^*} \end{bmatrix},$$

where the matrix $H_{\mathbf{D}}$ is invertible.

Under assumptions (B1)–(B4), the derivatives $\nabla_\epsilon \delta(\epsilon^*)$, $\nabla_\epsilon \lambda(\epsilon^*)$, $\nabla_\epsilon w(\epsilon^*)$, $\nabla_\epsilon u(\epsilon^*)$ can also be obtained using an alternative system of equations [5,24]:

$$\begin{bmatrix} \nabla_\epsilon \delta(\epsilon^*) \\ \nabla_\epsilon w(\epsilon^*) \end{bmatrix} = -\begin{bmatrix} H_R & A^* \\ A^{*\mathrm{T}} & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} \nabla^2_{\epsilon\delta} V(\delta^*, \lambda^*, \epsilon^*) + B^{\mathrm{T}} \nabla^2_{\epsilon\lambda} V(\delta^*, \lambda^*, \epsilon^*) + \dot{A}(\epsilon^*) w^* \\ \dot{A}(\epsilon^*)^{\mathrm{T}} \delta^* \end{bmatrix} \tag{4.7}$$

$$\nabla_\epsilon \lambda(\epsilon^*) = B\, \nabla_\epsilon \delta(\epsilon^*),$$

$$\nabla_\epsilon u(\epsilon^*) = H_{\lambda^*} B\, \nabla_\epsilon \delta(\epsilon^*) + \nabla^2_{\epsilon\lambda} V(\delta^*, \lambda^*, \epsilon^*).$$

Theorem 4.2 provides formulas (4.6) (or, alternatively (4.7)) for the first-order derivatives of the optimal solution $[\delta(\epsilon), \lambda(\epsilon)]$ of the perturbed dual program $\mathbf{D}(\epsilon)$. Higher-order sensitivity analysis was also considered in the papers by Dembo [5] and Kyparisis [24]. Dembo [5] derived equations for the approximate second-order derivatives of $[\delta(\epsilon), \lambda(\epsilon)]$ and Kyparisis [24] derived a corrected version of these equations and also obtained equations for the exact second-order derivatives of $[\delta(\epsilon), \lambda(\epsilon)]$.

The next result shows that theorems 3.1, 3.2 and 4.1, 4.2 are closely related. It is based on duality relationships between $\mathbf{P}(\epsilon)$ and $\mathbf{D}(\epsilon)$.

THEOREM 4.3 [24]

Consider the set of sufficient conditions (A1)–(A5) for the differentiability of the solution $x(\epsilon)$ and the Lagrange multiplier vector $\mu(\epsilon)$ for the primal geometric program $\mathbf{P}(\epsilon)$ and the set of sufficient conditions (B1)–(B4) for the differentiability of the solution $[\delta(\epsilon), \lambda(\epsilon)]$ and the Lagrange multiplier vectors $w(\epsilon)$ and $u(\epsilon)$ for the dual geometric program $\mathbf{D}(\epsilon)$.

Then, (A1)–(A5) and (B1)–(B4) are equivalent, i.e., if one set of conditions holds, then the other set of conditions also holds.

We consider separately the parametric dual geometric program $\mathbf{D}_0(\epsilon^*)$ and state the result analogous to theorem 4.2. Assuming that $\delta_i^* > 0$, $i = 1, \ldots, n_0$, the KKT conditions at δ^* for $\mathbf{D}_0(\epsilon^*)$ are: there exists w^* such that

$$\nabla_\delta V_0(\delta^*, \epsilon^*) + A_0(\epsilon^*) w^* = 0, \tag{4.8}$$

$$A_0(\epsilon^*)^{\mathrm{T}} \delta^* - e_0 = 0. \tag{4.9}$$

PROPOSITION 4.2 [5,24] Suppose that the following assumptions hold:

(B1′) the KKT conditions hold at δ^* for $\mathbf{D}_0(\epsilon^*)$,

(B2′) $\delta_i^* > 0$, $i = 1, \ldots, n_0$,

(B5) rank $A_0^* = m + 1$, where $A_0^* = A_0(\epsilon^*)$.

Then, for ϵ in some neighborhood of ϵ^*, there exist unique once continuously differentiable vector functions $\delta(\epsilon)$ and $w(\epsilon)$, such that the conditions (B1′), (B2′) and (B5) hold at $\delta(\epsilon)$ with $w(\epsilon)$ for the problem $\mathbf{D}(\epsilon)$ (and thus $\delta(\epsilon)$ is a unique global maximum of $\mathbf{D}_0(\epsilon)$).

Moreover, the derivatives $\nabla_\epsilon \delta(\epsilon^*)$ and $\nabla_\epsilon w(\epsilon^*)$ are uniquely determined by the system

$$\begin{bmatrix} \nabla_\epsilon \delta(\epsilon^*) \\ \nabla_\epsilon w(\epsilon^*) \end{bmatrix} = -\left(M_{\mathbf{D}_0}^*\right)^{-1} \begin{bmatrix} \nabla_{\epsilon\delta}^2 V_0(\delta^*, \epsilon^*) + \dot{A}_0(\epsilon^*) w^* \\ \dot{A}_0(\epsilon^*)^{\mathrm{T}} \delta^* \end{bmatrix},$$

where $M_{\mathbf{D}_0}^*$ is the Jacobian of the KKT system of equations (4.8), (4.9) with respect to (δ, w), evaluated at $(\delta^*, w^*, \epsilon^*)$, given by

$$M_{\mathbf{D}_0}^* = \begin{bmatrix} H_{\delta^*} & A_0^* \\ A_0^{*\mathrm{T}} & 0 \end{bmatrix},$$

the three-dimensional matrix $\dot{A}_0(\epsilon^*)$ is defined similarly to the matrix $\dot{A}(\epsilon^*)$ used in theorem 4.2, and the inverse of matrix $M_{\mathbf{D}_0}^*$ can be computed using the formulas

$$\left(M_{\mathbf{D}_0}^*\right)^{-1} = \begin{bmatrix} H_{\delta^*} & A_0^* \\ A_0^{*\mathrm{T}} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{\mathbf{D}_0} & A_{12}^{\mathbf{D}_0} \\ A_{21}^{\mathbf{D}_0} & A_{22}^{\mathbf{D}_0} \end{bmatrix},$$

$$A_{11}^{\mathbf{D}_0} = H_{\delta^*}^{-1}\left\{I - A_0^*\left[A_0^{*\mathrm{T}} H_{\delta^*}^{-1} A_0^*\right]^{-1} A_0^{*\mathrm{T}} H_{\delta^*}^{-1}\right\},$$

$$A_{12}^{\mathbf{D}_0} = \left(A_{21}^{\mathbf{D}_0}\right)^{\mathrm{T}} = H_{\delta^*}^{-1} A_0^*\left[A_0^{*\mathrm{T}} H_{\delta^*}^{-1} A_0^*\right]^{-1},$$

$$A_{22}^{\mathbf{D}_0} = -\left[A_0^{*\mathrm{T}} H_{\delta^*}^{-1} A_0^*\right]^{-1}.$$

We note that in the special case when $\dot{A}_0(\epsilon^*) = 0$, i.e. when only the coefficients $c_i(\epsilon)$ are perturbed and the exponents $a_{ij}(\epsilon)$ are constant, the derivatives $\nabla_\epsilon \delta(\epsilon^*)$ and $\nabla_\epsilon w(\epsilon^*)$ can be computed using simpler formulas

$$\nabla_\epsilon \delta(\epsilon^*) = -\left\{H_{\delta^*}^{-1} - H_{\delta^*}^{-1} A_0^*\left[A_0^{*\mathrm{T}} H_{\delta^*}^{-1} A_0^*\right]^{-1} A_0^{*\mathrm{T}} H_{\delta^*}^{-1}\right\} \nabla_{\epsilon\delta}^2 V_0(\delta^*, \epsilon^*),$$

$$\nabla_\epsilon w(\epsilon^*) = -\left[A_0^{*\mathrm{T}} H_{\delta^*}^{-1} A_0^*\right]^{-1} A_0^{*\mathrm{T}} H_{\delta^*}^{-1}\ \nabla_{\epsilon\delta}^2 V_0(\delta^*, \epsilon^*).$$

The above formula for $\nabla_\epsilon \delta(\epsilon^*)$ has been previously obtained in [26] under the assumption that $M_{\mathbf{D}_0}^*$ is invertible.

In section 3 the basic sensitivity analysis result for the primal geometric program $\mathbf{P}(\epsilon)$, stated as theorem 3.2, was extended in theorem 3.3 to the degenerate case. For the dual geometric program $\mathbf{D}(\epsilon)$, it does not seem possible to similarly directly extend theorem 4.2 to the degenerate case when some δ_i^* are zero. The main reason for this is the non-differentiability of the objective function $V(\delta, \lambda, \epsilon)$ at feasible points (δ, λ) when any δ_i or λ_i are zero. The next theorem provides an indirect extension of theorem 4.2, based on theorem 3.3 and duality theory for geometric programs. Note that the conditions (B1) and (B3) assumed in theorem 4.2 cannot be used here since we consider the case where some δ_i^* are zero.

THEOREM 4.4 [24]

Suppose that the assumptions of theorem 3.3 are satisfied. Then, for ϵ in some neighborhood of ϵ^*, there exist unique vector functions $\delta(\epsilon)$ and $\lambda(\epsilon)$, continuous and such that $[\delta(\epsilon), \lambda(\epsilon)]$ is a unique global maximum of $\mathbf{D}(\epsilon)$. If the unique optimal solution $x(\epsilon)$ of the primal geometric program $\mathbf{P}(\epsilon)$ and its associated unique Lagrange multiplier vector $\mu(\epsilon)$ are known (they exist by theorem 3.3), then $[\delta(\epsilon), \lambda(\epsilon)]$ can be computed using formulas

$$\delta_i(\epsilon) = \begin{cases} c_i(\epsilon) \exp\left[\sum_{j=1}^{m} a_{ij}(\epsilon) x_j(\epsilon)\right] / f_0(x(\epsilon), \epsilon), & i \in J_0, \\ \mu_k(\epsilon) c_i(\epsilon) \exp\left[\sum_{j=1}^{m} a_{ij}(\epsilon) x_j(\epsilon)\right] / f_0(x(\epsilon), \epsilon), & i \in J_k,\ k = 1,\dots, p, \end{cases} \tag{4.10}$$

$$\lambda_k(\epsilon) = \sum_{i \in J_k} \delta_i(\epsilon) = \mu_k(\epsilon) / f_0(x(\epsilon), \epsilon), \quad k = 1,\dots, p. \tag{4.11}$$

Moreover, the directional derivatives of $\delta(\epsilon)$ and $\lambda(\epsilon)$ at ϵ^* in any direction v, denoted by $D_v\delta(\epsilon^*)$ and $D_v\lambda(\epsilon^*)$, respectively, exist and can be calculated using formulas (4.10), (4.11) and the directional derivatives $D_v x(\epsilon^*)$ and $D_v\mu(\epsilon^*)$ determined by the systems (3.6)–(3.9) in theorem 3.3.

The next result may be useful when, after solving the dual problem $\mathbf{D}(\epsilon^*)$, one wants to apply theorem 4.4 to that problem.

PROPOSITION 4.3 [24]

Suppose that (δ^*, λ^*) is the optimal (global) solution of $\mathbf{D}(\epsilon^*)$. If the problems $\mathbf{P}(\epsilon^*)$ and $\mathbf{D}(\epsilon^*)$ are canonical [16] and if rank $\bar{A}_I^* = m$, where $I = \{k \mid \lambda_k^* > 0\}$ and $\bar{A}_I^* = \bar{A}_I(\epsilon^*)$, then the unique optimal (global) solution x^* and the (possibly nonunique) optimal Lagrange multiplier vector μ^* for the problem $\mathbf{P}(\epsilon^*)$ exist and are determined from (δ^*, λ^*) by the following formulas:

$$x^* = \left(\bar{A}_I^{*\mathrm{T}} \bar{A}_I^*\right)^{-1} \bar{A}_I^{*\mathrm{T}} z_I^*,$$

$$z_I^* = (\dots, z_i^*, \dots)^{\mathrm{T}}, \quad i \in J_k,\ k \in \{0\} \cup I,$$

$$z_i^* = \begin{cases} \log[\delta_i^* / c_i(\epsilon^*)] + V(\delta^*, \lambda^*, \epsilon^*), & i \in J_0, \\ \log[\delta_i^* / (\lambda_k^* c_i(\epsilon^*))], & i \in J_k,\ k \in I, \end{cases} \tag{4.12}$$

$$\mu_k^* = \lambda_k^* \exp[V(\delta^*, \lambda^*, \epsilon^*)], \quad k = 1,\dots, p,$$

i.e., $\mu_k^* > 0$, if $k \in I$, $\mu_k^* = 0$, if $k \notin I$.

In the nondegenerate case, i.e., when $\delta_i^* > 0$, $i = 1, \dots, n$, the problems $\mathbf{P}(\epsilon^*)$ and $\mathbf{D}(\epsilon^*)$ will be canonical if (δ^*, λ^*) solves $\mathbf{D}(\epsilon^*)$. If, in addition, rank $\bar{A}^* = m$, then

$$x^* = (\bar{A}^{*\mathrm{T}}\bar{A}^*)^{-1}\bar{A}^{*\mathrm{T}}z^*,$$

where z^* is given by (4.12), with $I = \{1, \dots, p\}$.

5. Optimal value differentiability properties for the primal and dual geometric programs

In this section we obtain formulas for the derivatives of the optimal value functions of the primal geometric program $\mathbf{P}(\epsilon^*)$ and the dual geometric program $\mathbf{D}(\epsilon^*)$. These formulas are essentially a consequence of the results in sections 3 and 4 and are based on the paper by Kyparisis [24] as well as on the paper by Dembo [5] where less general versions of these results are given.

Consider the primal geometric program $\mathbf{P}(\epsilon)$. The optimal value function $f^*(\epsilon)$ for $\mathbf{P}(\epsilon)$ is defined by

$$f^*(\epsilon) = \inf_x\{ f_0(x, \epsilon) \mid f_k(x, \epsilon) \leqslant 1,\ k = 1, \dots, p\}.$$

However, because only local behavior of the perturbed solution $x(\epsilon)$ is studied here, the formulas for the first-order and second-order derivatives of $f^*(\epsilon)$ are derived for the "local" optimal value function $f^*(\epsilon)$ defined by

$$f^*(\epsilon) = f_0[x(\epsilon), \epsilon].$$

PROPOSITION 5.1 [24]

Suppose that the assumptions (A1)–(A5) of theorem 3.2 hold. Then, the optimal value function $f^*(\epsilon)$ of $\mathbf{P}(\epsilon)$ is twice continuously differentiable for ϵ in some neighborhood of ϵ^* and

(i) $f^*(\epsilon) = f_0[x(\epsilon), \epsilon] = L_{\mathbf{P}}[x(\epsilon), \mu(\epsilon), \epsilon]$;

(ii)
$$\begin{aligned}
\nabla_\epsilon f^*(\epsilon) &= \nabla_\epsilon L_{\mathbf{P}}[x(\epsilon), \mu(\epsilon), \epsilon] = \nabla_\epsilon f_0[x(\epsilon), \epsilon] + \sum_{k=1}^{p} \mu_k(\epsilon)\, \nabla_\epsilon f_k[x(\epsilon), \epsilon] \\
&= \sum_{i=1}^{n_0}\left[\nabla_\epsilon c_i(\epsilon) + c_i(\epsilon)\left(\sum_{j=1}^{m} \nabla_\epsilon a_{ij}(\epsilon) x_j(\epsilon)\right)\right] \\
&\quad \times \exp\left[\sum_{j=1}^{m} \nabla_\epsilon a_{ij}(\epsilon) x_j(\epsilon)\right] \\
&\quad + \sum_{k=1}^{p} \mu_k(\epsilon)\left\{\sum_{i\in J_k}\left[\nabla_\epsilon c_i(\epsilon) + c_i(\epsilon)\left(\sum_{j=1}^{m} \nabla_\epsilon a_{ij}(\epsilon) x_j(\epsilon)\right)\right]\right. \\
&\quad \left. \times \exp\left[\sum_{j=1}^{m} \nabla_\epsilon a_{ij}(\epsilon) x_j(\epsilon)\right]\right\};
\end{aligned}$$

(iii) $\nabla_\epsilon^2 f^*(\epsilon) = \nabla_\epsilon\left[\nabla_\epsilon L_P[x(\epsilon), \mu(\epsilon), \epsilon]^T\right] = \nabla_\epsilon^2 L_P[x(\epsilon), \mu(\epsilon), \epsilon]$

$$+ \nabla_{\epsilon x}^2 L_P[x(\epsilon), \mu(\epsilon), \epsilon]^T \nabla_\epsilon x(\epsilon) + \sum_{k=1}^{p} \nabla_\epsilon f_k[x(\epsilon), \epsilon] \nabla_\epsilon \mu(\epsilon)$$

$$= \nabla_\epsilon^2 L_P[x(\epsilon), \mu(\epsilon), \epsilon] - N_P^T(\epsilon)(M_P(\epsilon))^{-1} N_P(\epsilon),$$

where $M_P(\epsilon)$ is the Jacobian of the KKT system of equations (3.1), (3.2) with respect to (x, μ) and $N_P(\epsilon)$ is the Jacobian of the KKT system (3.1), (3.2) with respect to ϵ, both evaluated at $[x(\epsilon), \mu(\epsilon), \epsilon]$ (see the analogous formulas for M_P^* given in (3.3) and for N_P^* given as the matrix on the right-hand-side of (3.4)).

In the degenerate case, when the strict complementarity condition (assumption (A5)) does not hold, the following result extends the applicability of proposition 5.1.

PROPOSITION 5.2 [24]

Suppose that the assumptions (A1)–(A3) and (A7), (A8) of theorem 3.3 hold. Then, the optimal value function $f^*(\epsilon)$ of $\mathbf{P}(\epsilon)$ is once continuously differentiable for ϵ in some neighborhood of ϵ^* and

(i) $f^*(\epsilon) = f_0[x(\epsilon), \epsilon] = L_P[x(\epsilon), \mu(\epsilon), \epsilon]$;

(ii) $\nabla_\epsilon f^*(\epsilon) = \nabla_\epsilon L_P[x(\epsilon), \mu(\epsilon), \epsilon] = \nabla_\epsilon f_0[x(\epsilon), \epsilon] + \sum_{k=1}^{p} \mu_k(\epsilon) \nabla_\epsilon f_k[x(\epsilon), \epsilon]$.

For the unconstrained primal geometric program $\mathbf{P}_0(\epsilon)$ the following result summarizes differentiability properties of the optimal value function $f_0^*(\epsilon)$ of $\mathbf{P}_0(\epsilon)$.

COROLLARY 5.1 [24]

Consider the problem $\mathbf{P}_0(\epsilon)$ and suppose that the assumptions (A1′) and (A6) of proposition 3.1 hold. Then, the optimal value function $f_0^*(\epsilon)$ of $\mathbf{P}_0(\epsilon)$ is twice continuously differentiable for ϵ in some neighborhood of ϵ^* and

(i) $f_0^*(\epsilon) = f_0[x(\epsilon), \epsilon]$;

(ii) $\nabla_\epsilon f_0^*(\epsilon) = \nabla_\epsilon f_0[x(\epsilon), \epsilon]$

$$= \sum_{i=1}^{n_0}\left[\nabla_\epsilon c_i(\epsilon) + c_i(\epsilon)\left(\sum_{j=1}^{m} \nabla_\epsilon a_{ij}(\epsilon) x_j(\epsilon)\right)\right]$$

$$\times \exp\left[\sum_{j=1}^{m} \nabla_\epsilon a_{ij}(\epsilon) x_j(\epsilon)\right];$$

(iii) $\nabla_\epsilon^2 f_0^*(\epsilon) = \nabla_\epsilon\left[\nabla_\epsilon f_0[x(\epsilon), \epsilon]^T\right] = \nabla_\epsilon^2 f_0[x(\epsilon), \epsilon]$

$$- \nabla_{\epsilon x}^2 f_0[x(\epsilon), \epsilon]^T \left(\bar{A}_0^T(\epsilon) H_0 \bar{A}_0(\epsilon)\right)^{-1} \nabla_{\epsilon x}^2 f_0[x(\epsilon), \epsilon].$$

In the remainder of this section we consider the dual geometric programs $\mathbf{D}(\epsilon)$ and $\mathbf{D}_0(\epsilon)$. The optimal value function $V^*(\epsilon)$ for $\mathbf{D}(\epsilon)$ is defined by

$$V^*(\epsilon) = \sup_{(\delta,\lambda)}\left\{V(\delta, \lambda, \epsilon) \mid A(\epsilon)^{\mathrm{T}}\delta = e_0,\ B\delta = \lambda,\ \delta \geqslant 0,\ \lambda \geqslant 0\right\}.$$

However, since only local behavior of the perturbed solution $[\delta(\epsilon), \lambda(\epsilon)]$ is studied here, the formulas for the first-order and second-order derivatives of $V^*(\epsilon)$ are derived for the "local" optimal value function $V^*(\epsilon)$ defined by

$$V^*(\epsilon) = V[\delta(\epsilon), \lambda(\epsilon), \epsilon].$$

A special case of the formula for the first-order derivative of $V^*(\epsilon)$ given in proposition 5.3, part (ii), was obtained by Duffin et al. [16] for the perturbations in coefficients c_i only (compare formula (6.1) in section 6).

PROPOSITION 5.3 [24]

Suppose that the assumptions (B1)–(B4) of theorem 4.2 hold. Then, the optimal value function $V^*(\epsilon)$ of $\mathbf{D}(\epsilon)$ is twice continuously differentiable for ϵ in some neighborhood of ϵ^* and

(i) $V^*(\epsilon) = V[\delta(\epsilon), \lambda(\epsilon), \epsilon] = L_{\mathrm{D}}[y(\epsilon), \epsilon]$;

(ii)
$$\begin{aligned}\nabla_\epsilon V^*(\epsilon) &= \nabla_\epsilon L_{\mathrm{D}}[y(\epsilon), \epsilon] = \nabla_\epsilon V[\delta(\epsilon), \lambda(\epsilon), \epsilon] + w(\epsilon)^{\mathrm{T}}\dot{A}^{\mathrm{T}}(\epsilon)\delta(\epsilon)\\ &= \sum_{i=1}^{n}[\delta_i(\epsilon)/c_i(\epsilon)]\ \nabla_\epsilon c_i(\epsilon) + w(\epsilon)^{\mathrm{T}}\dot{A}^{\mathrm{T}}(\epsilon)\delta(\epsilon);\end{aligned}$$

(iii)
$$\begin{aligned}\nabla_\epsilon^2 V^*(\epsilon) &= \nabla_\epsilon^2 L_{\mathrm{D}}[y(\epsilon), \epsilon] + \nabla_{\epsilon\delta}^2 L_{\mathrm{D}}[y(\epsilon), \epsilon]^{\mathrm{T}}\ \nabla_\epsilon\delta(\epsilon)\\ &\quad + [\dot{A}^{\mathrm{T}}(\epsilon)\delta(\epsilon)]^{\mathrm{T}}\ \nabla_\epsilon w(\epsilon)\\ &= \nabla_\epsilon^2 L_{\mathrm{D}}[y(\epsilon), \epsilon] - N_{\mathrm{D}}^{\mathrm{T}}(\epsilon)(M_{\mathrm{D}}(\epsilon))^{-1}N_{\mathrm{D}}(\epsilon),\end{aligned}$$

where $y(\epsilon) = [\delta(\epsilon), \lambda(\epsilon), w(\epsilon), u(\epsilon)]^{\mathrm{T}}$, $\dot{A}(\epsilon)$ is defined in the same manner as $\dot{A}(\epsilon^*)$, $M_{\mathrm{D}}(\epsilon)$ is the Jacobian of the KKT system of equations (4.1)–(4.4) with respect to (δ, λ, w, u) and $N_{\mathrm{D}}(\epsilon)$ is the Jacobian of the KKT system (4.1)–(4.4) with respect to ϵ, both evaluated at $[\delta(\epsilon), \lambda(\epsilon), w(\epsilon), u(\epsilon)]$ (see the analogous formulas for M_{D}^* given in (4.5) and for N_{D}^* given as the matrix on the right-hand-side of (4.6)).

In the degenerate case, when some δ_i^* are zero and thus assumption (B2) does not hold, the following result indirectly extends the applicability of proposition 5.2. Its proof is based on proposition 5.2, the result of Dembo [3], and the following relationship between the optimal value functions $f^*(\epsilon)$ of $\mathbf{P}(\epsilon)$ and $V^*(\epsilon)$ of $\mathbf{D}(\epsilon)$

$$V^*(\epsilon) = \log f^*(\epsilon).$$

This relationship can also be used to derive alternative formulas for the derivatives of $V^*(\epsilon)$ under the assumptions of proposition 5.1.

PROPOSITION 5.4 [5,24]

Suppose that the assumptions (A1)–(A3) and (A7), (A8) of theorem 3.3 hold. Then, the optimal value function $V^*(\epsilon)$ of $\mathbf{D}(\epsilon)$ is once continuously differentiable for ϵ in some neighborhood of ϵ^* and

(i) $V^*(\epsilon) = V[\delta(\epsilon), \lambda(\epsilon), \epsilon]$;

(ii) $\nabla_\epsilon V^*(\epsilon) = \nabla_\epsilon V[\delta(\epsilon), \lambda(\epsilon), \epsilon] + x(\epsilon)^{\mathrm{T}} \dot{A}(\epsilon)^{\mathrm{T}} \delta(\epsilon)$,

where $x(\epsilon)$ is the optimal solution of $\mathbf{P}(\epsilon)$.

For the dual geometric program $\mathbf{D}_0(\epsilon)$, the following result summarizes differentiability properties of the optimal value function $V_0^*(\epsilon)$ of $\mathbf{D}_0(\epsilon)$.

COROLLARY 5.2 [24]

Consider the problem $\mathbf{D}_0(\epsilon)$ and suppose that the assumptions (B1′), (B2′) and (B5) of proposition 4.2 hold. Then, the optimal value function $V_0^*(\epsilon)$ of $\mathbf{D}_0(\epsilon)$ is twice continuously differentiable for ϵ in some neighborhood of ϵ^* and

(i) $V_0^*(\epsilon) = V_0[\delta(\epsilon), \epsilon]$;

(ii)
$$\begin{aligned}\nabla_\epsilon V_0^*(\epsilon) &= \nabla_\epsilon V_0[\delta(\epsilon), \epsilon] + w(\epsilon)^{\mathrm{T}} \dot{A}_0(\epsilon)^{\mathrm{T}} \delta(\epsilon) \\ &= \sum_{i=1}^{n} [\delta_i(\epsilon)/c_i(\epsilon)]\ \nabla_\epsilon c_i(\epsilon) + w(\epsilon)^{\mathrm{T}} \dot{A}_0(\epsilon)^{\mathrm{T}} \delta(\epsilon);\end{aligned}$$

(iii)
$$\begin{aligned}\nabla_\epsilon^2 V_0^*(\epsilon) &= \left[\nabla_{\epsilon\delta}^2 V_0[\delta(\epsilon), \epsilon] + \dot{A}_0(\epsilon) w(\epsilon)\right]^{\mathrm{T}} \nabla_\epsilon \delta(\epsilon) \\ &\quad + \left[\dot{A}_0(\epsilon)^{\mathrm{T}} \delta(\epsilon)\right]^{\mathrm{T}} \nabla_\epsilon w(\epsilon) + \nabla_\epsilon^2 V_0[\delta(\epsilon), \epsilon] + w(\epsilon)^{\mathrm{T}} \ddot{A}_0(\epsilon)^{\mathrm{T}} \delta(\epsilon),\end{aligned}$$

where the non-zero part of the four-dimensional matrix $\ddot{A}_0(\epsilon)$ is given by

$\ddot{A}_0(\epsilon) = \left[\nabla_\epsilon^2 a_{ij}(\epsilon)\right], \quad i = 1, \ldots, n_0, \ j = 1, \ldots, m.$

6. Computational approaches to sensitivity analysis for geometric programs

In this section we review computational approaches to sensitivity analysis for geometric programs developed by Dinkel et al. [6,7,9,11,12]. We consider primarily the reduced dual geometric program $\mathbf{GD_R}$, in which the coefficients c_i, $i = 1, \ldots, n$ serve as perturbation parameters and the exponents a_{ij} are assumed to be constant, and present the incremental procedures for estimating the new optimal solution of $\mathbf{GD_R}$ resulting from changes in coefficients c_i. We also briefly mention the extension of this approach to the situation with perturbed exponents a_{ij} and present a result on the range of coefficients c_i values for which feasible solutions exist.

The following sensitivity analysis theorem, which relates changes in the primal coefficients c_i to changes in the optimal dual solution of the program $\mathbf{GD_R}(c)$, was proved by Duffin et al. ([16], appendix B). This theorem forms the basis for the computational approach to sensitivity analysis developed by Dinkel et al. [6,7,9,11,12].

THEOREM 6.1 [16]

Suppose that the primal geometric program **GP** has $d > 0$ and rank $\bar{A} = m$, where $\bar{A} = [a_{ij}]$. If the solution of **GD** is such that $\delta_i^* > 0$, $i = 1,\ldots, n$ and if the matrix $J(\delta)$ with components (compare $\mathbf{GD_R}$)

$$J_{ij}(\delta) = \sum_{l=1}^{n} b_l^{(i)} b_l^{(j)}/\delta_l - \sum_{k=1}^{p} \lambda_k^{(i)} \lambda_k^{(j)}/\lambda_k(\delta), \quad i,\ j = 1,\ldots, d,\ \lambda_k(\delta) = \sum_{i \in J_k} \delta_i,$$

is nonsingular at δ^*, then the functions which give the optimal solution value δ^* and the optimal objective value $v(\delta^*)$ ($:= v(\delta^*,\ \lambda(\delta^*))$) in terms of variable coefficient vector c are differentiable on an open neighborhood of c. These differentials are

$$\mathrm{d}v/v^* = \sum_{i=1}^{n} \delta_i^*\ \mathrm{d}c_i/c_i, \tag{6.1}$$

$$\mathrm{d}\delta_i = \sum_{j=1}^{d} \left\{ b_i^{(j)} \sum_{k=1}^{d} \left[J_{jk}^{-1}(\delta^*) \sum_{l=1}^{n} b_l^{(k)}\ \mathrm{d}c_l/c_l \right] \right\}, \quad i = 1,\ldots, n, \tag{6.2}$$

where $J_{jk}^{-1}(\delta^*)$ represents the components of the inverse of $J(\delta^*)$, $v^* = v(\delta^*)$, and

$$\mathrm{d}\lambda_k = \sum_{i \in J_k} \mathrm{d}\delta_i, \quad k = 1,\ldots, p. \tag{6.3}$$

Theorem 6.1 holds only if there are no nonbinding primal constraints at the optimal solution, since it requires that $\delta_i^* > 0$, $i = 1,\ldots, n$. Thus it is assumed from now on that the problem has been reformulated, if necessary, to satisfy this condition.

The new dual solution and the optimal dual objective value can be estimated for differential changes $\mathrm{d}c_i$, which maintain the positivity conditions on all dual variables, using the formulas

$$\delta_1' = \delta_i^* + \mathrm{d}\delta_i, \quad i = 1,\ldots, n, \tag{6.4}$$

$$v' = v^* + v^* \sum_{i=1}^{n} \delta_i^*\ \mathrm{d}c_i/c_i, \tag{6.5}$$

and $\lambda_k' = \sum_{i \in J_k} \delta_i'$, $k = 1,\ldots, p$, where $\mathrm{d}\delta_i$ is given by (6.2). Once the dual solution is known the estimate of the new primal solution, t', is computed from the duality relationships [16] as:

$$(\log t_j') = [\bar{A}^{\mathrm{T}}\bar{A}]^{-1}\bar{A}^{\mathrm{T}}Z, \tag{6.6}$$

where Z is a vector with elements

$$Z_i = \begin{cases} \log \delta_i' + \log v(\delta') - \log(c_i + \mathrm{d}c_i), & i = 1, \ldots, n_0, \\ \log \delta_i' - \log \lambda_k' - \log(c_i + \mathrm{d}c_i), & i = n_0 + 1, \ldots, n. \end{cases} \tag{6.7}$$

In the practical application of these results, the differential changes $\mathrm{d}c_i/c_i$ of eqs. (6.1) and (6.2) are approximated by the difference form $(c_i' - c_i)/c_i$, where c_i' and c_i are, respectively, the new and old values of the coefficient. It is clear from eqs. (6.1) and (6.2) that the use of the difference form in place of $\mathrm{d}c_i/c_i$ results in the introduction of error into the computations of bounds and new values of the variables. The accuracy of the results when using the difference form can be improved by using an incremental procedure proposed in [7] which allows for the continued updating of J^{-1}. The procedure considers a "small" change about δ^* and computes a new solution δ^1 using the formulas

$$\delta_i^1 = \delta_i^* + \mathrm{d}\delta_i^1,$$

$$\mathrm{d}\delta_i^1 = \sum_{j=1}^{d} \left\{ b_i^{(j)} \sum_{k=1}^{d} \left[J_{jk}^{-1}(\delta^*) \sum_{l=1}^{n} b_l^{(k)}(c_l' - c_l)/c_l \right] \right\}, \qquad i = 1, \ldots, n.$$

This new solution δ_i^1 is used as the point about which a new solution δ^2 is computed for a small change, that is

$$\delta_i^2 = \delta_i^1 + \mathrm{d}\delta_i^2,$$

$$\mathrm{d}\delta_i^2 = \sum_{j=1}^{d} \left\{ b_i^{(j)} \sum_{k=1}^{d} \left[J_{jk}^{-1}(\delta^1) \sum_{l=1}^{n} b_l^{(k)}(c_l'' - c_l')/c_l' \right] \right\}, \qquad i = 1, \ldots, n.$$

This procedure can be continued until some desired level of change is achieved or until termination. The incremental procedure has the effect of allowing for updating the evaluation of J^{-1} according to

$$J_{ij}^{-1}\left(\delta^* + \sum_{t=0}^{T-1} \mathrm{d}\delta^{(t)} \right),$$

where T defines the number of increments and $\mathrm{d}\delta^{(0)} = 0$. Several types of incrementing procedures have been discussed in Dinkel et al. [7,11,12]. These discussions centered around different methods for choosing the increment size. For example, the increment size is defined in [11] as a percentage of the original coefficient so that for $(c_i' - c_i)/c_i = \Delta/c_i$ no incrementing implies all computations are made using $J^{-1}(\delta^*)$, 50 percent increments implies $\Delta/c_i = 0.5$ for each increment or $c_i' = 1.5c_i$, etc. An analysis of the accuracy of the primal and dual solutions computed using the incremental procedure was conducted in Dinkel et al. [11,12]. The error in the solution was computed using the formula

percent error = ((true − predicted)/true) × 100 percent,

where "true" corresponds to the variable value as determined by the optimal solution and "predicted" corresponds to the variable value as determined by the sensitivity analysis procedures. The numerical results in [11,12] show that the primal solution exhibits less error than the dual solution and that both the primal and objective functions display less error than the variables. Overall, these results showed the ability of the incremental procedure to control the error, especially in view of large parameter changes.

In addition to computing a new dual solution via eqs. (6.4) and (6.5) and the new primal solution via eqs. (6.2) and (6.3), Dinkel et al. [7,11] also propose a method for computing approximate bounds on allowable changes in a coefficient. This is done by determining Δ such that

$$0 = \delta_i' = \delta_i^* + \sum_{j=1}^{d} \left\{ b_i^{(j)} \sum_{k=1}^{d} \left[J_{jk}^{-1}(\delta^*) \sum_{l=1}^{n} b_l^{(k)} \Delta / c_l \right] \right\}, \quad i = 1, \ldots, n.$$

For decreases in the coefficients a lower bound of zero is imposed in order to preserve the posynomial nature of the program. The following result in Dinkel and Kochenberger [7] develops explicit formulas for such approximate ranges of allowable changes in the coefficients c_i which can be easily programmed for automatic computation.

THEOREM 6.2 [7]

Suppose that the solution of the dual geometric program **GD** is such that $\delta_i^* > 0$, $i = 1, \ldots, n$ and that $J^{-1}(\delta^*)$ exists. If c_s changes for any $1 \leqslant s \leqslant n$, then the approximate range of values for c_s' for which (6.1)–(6.5) are valid is given by

$$\max_{i: D_{is} > 0} \{ c_s - \delta_i^* c_s D_{is}^{-1}, 0 \} < c_s' < \min_{i: D_{is} < 0} \{ c_s - \delta_i^* c_s D_{is}^{-1} \},$$

where

$$D_{is} = \sum_{j=1}^{d} \left\{ b_i^{(j)} \sum_{k=1}^{d} \left[J_{jk}^{-1}(\delta^*) b_s^{(k)} \right] \right\}.$$

A similar result is given in [7] for allowable changes in the right hand side coefficients r_k and related results can also be found in Dembo [5].

In the preceding discussion it was assumed that the original problem was reformulated to delete any constraints which are nonbinding at the optimal solution. Since changes in the problem coefficients are likely to affect such constraints, Dinkel and Kochenberger [7,11] also propose an analysis of nonbinding constraints which proceeds as follows. Theorem 6.1 [16] precludes the direct inclusion of nonbinding constraints, however, eqs. (6.4), (6.6), and (6.7) provide a means of computing the new primal associated with a change in the coefficients.

Hence one can evaluate any nonbinding constraints as the solution changes, that is, the value of t' from eq. (6.6) is used to evaluate $g_k(t)$ for those constraints for which $g_k(t^*) < 1$ (t^* denotes the optimal solution of **GP**). It follows that as long as such constraints remain nonbinding the analysis remains valid. Termination of the procedure occurs if (i) a binding constraint becomes nonbinding, (ii) a nonbinding constraint becomes binding, or (iii) a maximum limit in the changes is exceeded.

An extension of the computational approaches discussed so far in this section to the case where the exponents a_{ij} are perturbed was also proposed by Dinkel and Kochenberger [9,12]. In that case theorem 6.1 does not apply and the analysis in [9,12] is based on the results on the directional differentiability of the perturbed optimal solution of a parametric nonlinear programming problem, due to Bigelow and Shapiro [2] (these results were subsequently generalized by Jittorntrum [22]). The application of the results in [2] to the optimal dual solution δ of **GD** (which depends on perturbations in exponents a_{ij}) results in formulas analogous to the systems (4.6) or (4.7) in section 4, except that the first-order derivatives are replaced by corresponding directional derivatives. Dinkel and Kochenberger [9,12] propose to replace the differential changes in these formulas by discrete approximation in order to compute approximate dual solutions for perturbed exponents a_{ij}. An incremental procedure which replaces a large change in a_{ij} by a sequence of smaller changes is used in actual computations in [9,12], similarly as in the method discussed in the earlier part of this section. When the approximate dual solution is finally obtained, the corresponding primal solution can be calculated using the duality relationships.

7. Summary and conclusions

This survey demonstrates that sensitivity analysis has been an integral part of the developments in geometric programming theory and applications in the past two decades. Research in this area has initially proceeded separately from similar studies for nonlinear programming problems, which subsume the class of geometric programs. In the last decade, however, the general theory of nonlinear programming sensitivity analysis has been extensively applied to geometric programming, taking into account unique features of geometric programs. This offers the promise of continued development in this area parallel to the developments in sensitivity analysis in nonlinear programming.

The following directions of future research on the topic of sensitivity analysis for geometric programming problems seem to be promising and should lead to new significant advances in geometric programming:

(1) extension of the existing theoretical framework for sensitivity analysis for parametric posynomial geometric programs [5,16,24] to signomial (or polynomial) geometric programs (i.e., programs with arbitrary coefficients c_i);

(2) further development of sensitivity analysis for parametric posynomial geometric programs using the general recent results obtained in nonlinear programming [20];
(3) a wider application of sensitivity analysis results in the construction of more efficient algorithms and accelerating convergence of the existing algorithms for solving geometric programming problems, following the example of [13,17];
(4) integration of sensitivity analysis computations into principal algorithms for solving geometric programs, similar to the work in nonlinear programming [20];
(5) continued efforts to apply the existing and to develop new sensitivity analysis results to analyze special classes of problems originating from applications of geometric programming.

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