SEPARATED INVISCID GAS FLOW PAST A DISK AND A BODY WITH MAXIMUM CRITICAL MACH NUMBERS

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The problem of the separated axisymmetric subsonic flow of an inviscid perfect gas with the specific heat ratio 1.4 past a disk in accordance with the Riabouchinsky scheme is solved using the method developed in [1]. Formulas relating the main parameters with the base pressure coefficient and the Mach number at the free boundary are presented. Formulas which make it possible to determine the shape of the body of revolution giving the maximum critical Mach numbers are also derived.

1. Various schemes of steady inviscid fluid jet flow with a constant pressure in the separation zone may be used to analyze separated gas flows at high Reynolds numbers [2]. In what follows, the separated axisymmetric gas flow past a disk is examined using the Riabouchinsky scheme. According to this scheme, the free boundary shed from the disk is matched smoothly with a second, "fictitious", disk, coaxial to the first and of the same magnitude. Thus, the flow is symmetric about a certain plane normal to the freestream velocity.

We will consider the steady subsonic axisymmetric potential isentropic flow of an inviscid perfect gas past a disk in accordance with the Riabouchinsky scheme. The gasdynamic parameters in the flow under consideration are related by the well-known formulas:

$$M^2 = \frac{2}{k+1}\lambda^2\left(1 - \frac{k-1}{k+1}\lambda^2\right)^{-1}, \quad p = p_0\left(1 + \frac{k-1}{2}M^2\right)^{k/(1-k)}$$
$$\rho = \rho_0\left(1 + \frac{k-1}{2}M^2\right)^{1/(1-k)}, \quad a^2 = \frac{kp}{\rho} \tag{1.1}$$

The following notation is used: λ, reduced velocity; M, Mach number; p, pressure; ρ, density; a, speed of sound; k, specific heat ratio; p_0 and ρ_0, the stagnation values of p and ρ. In what follows, the inviscid perfect gas with $k=1.4$ will be called the air-like gas.

Let p_a, ρ_a, M_a, and a_a be the freestream values of p, ρ, M, and a; V_a is the freestream velocity, p_c and M_c are the values of p and M at the free boundary, X is the disk drag, L is the length of the separation zone, R is the radius of the section of the separation zone by the plane of symmetry, and R_0 is the disk radius ($R_0=1$). The drag coefficient C_x and the pressure coefficient in the separation zone (the base pressure coefficient) Q are determined from the formulas

$$C_x = 2X/(\pi\rho_a V_a^2 R_0^2), \qquad Q = 2(p_a - p_c)/(\rho_a V_a^2)$$

Rewriting the latter expression in terms of (1.1) yields

$$M_c^2 = \frac{2}{k-1}\left[\left(1 + \frac{k-1}{2}M_a^2\right)\left(1 - \frac{1}{2}kQM_a^2\right)^{(1-k)/k} - 1\right] \tag{1.2}$$

An efficient numerical-analytic method for the calculation of the compressible flow past a circular cone in accordance with the Riabouchinsky scheme was developed in [1] for any arbitrarily preassigned relationships $M(\lambda)$ and $\nu(\lambda)$, $\nu = \rho/\rho_0$. In the case of a disk in an incompressible flow ($M_a = M_c = 0$), the main flow parameters, which will be given the superscript "°", can be computed on the range $0.15 \le Q \le 1$ from the formulas [1]

$$C_x^\circ = 0.72084 + 0.11178(1+Q)^{-1} + 0.92960Q$$
$$R^\circ = 0.96034 + 0.08936Q^{-1}\ln Q + 0.42422Q^{-1} \tag{1.3}$$
$$L^\circ = 0.20970 - 0.60585Q^{-1}\ln Q + 1.78362Q^{-1}$$

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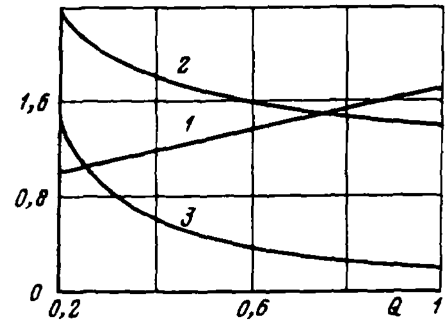

Fig. 1

The Q-dependence of the parameters C_x°, L°, and R° is plotted in Fig. 1 (curves *1–3* correspond to the quantities C_x°, R°, and $0.1L^\circ$).

The separated flow of air-like gas past a disk was calculated for $Q=0.25$, 0.3, 0.4, 0.5, and 0.6 and $M_c^2=0.2$, 0.4, 0.6, 0.8, and 1 using the method of [1]. The calculations were carried out on a 50×50 grid; the value of the freestream Mach number M_a corresponding to given Q and M_c can be determined from (1.2). From the results thus obtained the following approximate formulas for C_x, L, and R were constructed using the method of least squares:

$$C_x = \alpha C_x^\circ, \quad L = \beta L^\circ, \quad R = \delta R^\circ$$

$$\mu = 1 + f M_c^2 + g M_c^3 + h M_c^4, \quad \mu = \alpha,\ \beta,\ \delta \tag{1.4}$$

$$d = d_1 + d_2 Q^{-1} \ln Q + d_3 Q^{-1}, \quad d = f,\ g,\ h, \quad d_j = f_j,\ g_j,\ h_j$$

Here, C_x°, L°, and R° are functions of Q given by the formulas (1.3). The values of the coefficients f_j, g_j, and h_j in the approximating formulas (1.4) are given in Table 1, together with the values of the maximum relative approximation errors ε. The M_c^2-dependence of the parameters α, β, and δ is presented in Fig. 2, the curves *1–3* corresponding to $Q=0.25$, 0.4, and 0.6.

The formulas (1.3) and (1.4), together with the data 47 Table 1, make it possible to determine the quantities C_x, L, and R on the range $0.25 \le Q \le 0.6$; $0 \le M_c \le 1$. Note that for all the cases considered the radius of the separation zone cross-section increases monotonically from the disk to the plane of symmetry of the flow, while the absolute value of the curvature of the free boundary arc in a meridional plane decreases monotonically.

2. An important characteristic of a body in an inviscid gas flow is the critical Mach number M_*, i.e., the minimum value of the freestream Mach number M_a for which the flow velocity attains the critical value somewhere on the body surface. The bodies achieving the greatest possible M_* within a class of bodies satisfying certain geometrical restrictions can be of practical interest.

Consider axisymmetric inviscid flow past bodies of revolution which satisfy one of the following conditions:

$$R/L \ge r_0, \quad S/L^2 \ge s_0, \quad W/L^3 \ge w_0 \tag{2.1}$$

where L is the body length, R is the mid-section radius, S is the cross-sectional area in a meridional half-plane, W is the body volume, and r_0, s_0, and w_0 are given constants. The bodies of revolution with the maximum value of M_* within the above-mentioned class are those formed by two coaxial disks of equal magnitude transverse to the freestream and the free boundary connecting them, at every point of which the gas velocity is critical [3] (max M_* is attained if the strict equality in (2.1) is fulfilled).

Thus, the shape of a body of revolution with the properties mentioned above (for brevity, we shall call such bodies optimum) can be determined from the solution of the problem of the flow past a disk in accordance with the Riabouchinsky scheme, with the critical velocity on the free boundary. Although this fact has been known for a fairly long time, the shape of the optimum body has not yet been determined because a suitable technique for solving the problem has not been developed.

We calculated axisymmetric potential isentropic flows of perfect gas with the specific heat ratio $k=1.4$ past a disk of radius R_0 in accordance with the Riabouchinsky scheme, subject to the condition $M_c=1$, for $M_a=M_*=0.5$, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.83, 0.85, and 0.87. From the above it appears that the configurations thus obtained are the optimum bodies for an air-like gas.

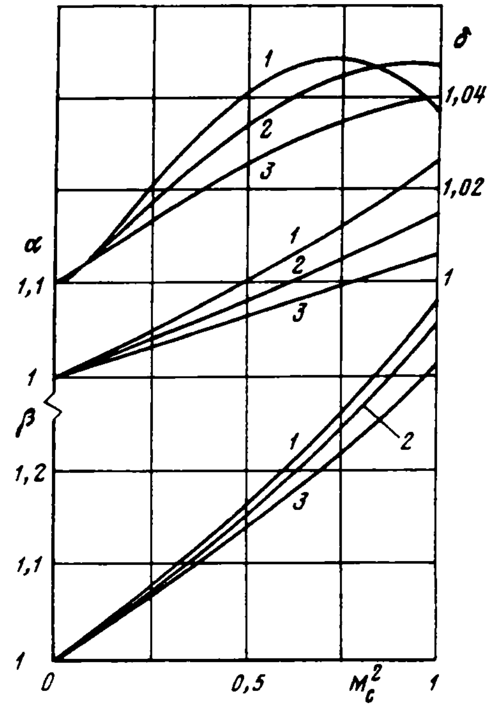

Fig. 2

TABLE 1

μ	d_j	$j=1$	2	3	$\varepsilon \cdot 10^4$
	f_j	-0.10521	0.06738	0.17566	
α	g_j	0.24632	-0.07326	-0.19309	5
	h_j	-0.16489	0.03909	0.12627	
	f_j	0.20163	0.01268	0.04005	
β	g_j	-0.05491	0.00202	0.00625	15
	h_j	-0.08536	0.06838	0.14857	
	f_j	0.03122	0.02190	0.01362	
δ	g_j	-0.11785	-0.02330	0.09410	11
	h_j	0.04666	0.03428	-0.04260	

The contours of the optimum bodies obtained are plotted in Fig. 3 in the meridional half-plane (x, r) to the left of the plane of symmetry $x=0$ for $M_* = 0.87$, 0.85, 0.8, 0.7, and 0.6 (curves *1*–*5*, respectively; $R_0 = 1$). The main geometric characteristics of the optimum bodies together with the values of the pressure coefficient Q on their lateral surfaces are presented in Table 2. In this table $\Omega = (W/(\pi L^3))^{1/2}$, while the coefficient Q is determined, in accordance with (1.2), from the formula

$$Q = \frac{2}{kM_*^2}\left[1 - \left(\frac{2}{k+1} + \frac{k-1}{k+1}M_*^2\right)^{k/(k-1)}\right]$$

Obviously, conditions of the form $M_* \le F_1(R/L)$, $M_* \le F_2(S/L^2)$, $M_* \le F_3(\Omega)$ hold for an arbitrary body of revolution in an air-like gas stream; here, F_1, F_2, and F_3 are monotone decreasing functions of their arguments, the strict equality being fulfilled only in the case of optimum bodies. By approximating the data of Table 2, bearing in mind that $F_1(0) = F_2(0) = F_3(0) = 1$, the following formulas were constructed:

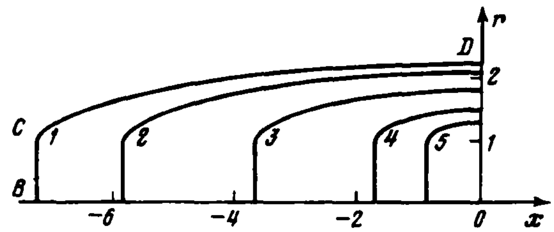

Fig. 3

TABLE 2

M.	Q	$L/(2R_0)$	R/L	S/L^2	Ω
0.5	2.1334	0.4633	1.2999	1.2450	1.2462
0.55	1.6583	0.6390	0.9868	0.9361	0.9375
0.6	1.2943	0.8811	0.7549	0.7086	0.7102
0.65	1.0085	1.2216	0.5797	0.5378	0.5395
0.7	0.7791	1.7223	0.4426	0.4053	0.4071
0.75	0.5912	2.4773	0.3356	0.3031	0.3049
0.8	0.4346	3.6926	0.2499	0.2222	0.2240
0.83	0.3526	4.8283	0.2059	0.1813	0.1830
0.85	0.3020	5.8485	0.1793	0.1569	0.1585
0.87	0.2544	7.2425	0.1541	0.1338	0.1354

$$F_1 = (1 - 0.54152 r_1 + 1.70875 r_1^2 - 0.34186 r_1^3)^{-1}, \quad r_1 = (R/L)^{1/3}$$
$$F_2 = (1 - 0.48238 s_1 + 1.69462 s_1^2 - 0.35413 s_1^3)^{-1}, \quad s_1 = (S/L^2)^{1/3} \tag{2.2}$$
$$F_2 = (1 - 0.49209 w_1 + 1.70871 w_1^2 - 0.35938 w_1^3)^{-1}, \quad w_1 = (\Omega)^{1/3}$$

The error in approximating the calculated data by the formulas (2.2) does not exceed 0.11% for F_1 and 0.07% for F_2 and F_3 for $F_1, F_2, F_3 \geq 0.4$.

Let BC be the disk generator and CD the optimum-body lateral-surface generator in the meridional half-plane (x, r) (see Fig. 3). Let θ be the inclination of the velocity to the axis of symmetry x and $\theta_0 = \pi/2$ be the value of θ on BC. It was shown in [1] that the curvature of the arc CD increases without bound as the point C is approached. A more detailed analysis shows that the following estimates hold in the vicinity of the point C of the arc CD:

$$x/x_c + 1 = O(\sigma^3), \quad r/r_c - 1 = O(\sigma^2) \quad (\sigma \to 0) \tag{2.3}$$

where $\sigma = \theta_0 - \theta$, x_c and r_c are the values of x and r at the point C $(x_c = -L/2, r_c = R_0)$. We introduce a parametric variable t by setting $x/x_c = \cos t$ on CD. Let $t = \pi - \omega$; then, in accordance with (2.3), we have

$$\sigma = O(\omega^{2/3}), \quad r/r_c - 1 = O(\omega^{4/3}) \quad (\omega \to 0)$$

On the basis of the above considerations and the calculated results, approximating formulas allowing the determination of the shape of the optimum body generator, were constructed:

$$\frac{x}{R_0} = b\cos t, \quad b = \frac{L}{2R_0}, \quad t \in [0, \pi], \quad \frac{r}{R_0} = 1 + \left[\sum_{k=1}^{7} a_k \sin(2k-1)t\right]^{4/3} \tag{2.4}$$

TABLE 3

M_*	a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\varepsilon_1 \cdot 10^4$	$\varepsilon_2 \cdot 10^4$
0.5	0.30838	0.00485	0.00088	0.00028	0.00012	0.00006	0.00004	0.3	10
0.55	0.37070	0.00614	0.00112	0.00036	0.00015	0.00008	0.00005	0.3	11
0.6	0.44250	0.00781	0.00145	0.00047	0.00020	0.00010	0.00006	0.5	12
0.65	0.52695	0.01002	0.00190	0.00061	0.00026	0.00013	0.00007	0.6	13
0.7	0.62749	0.01300	0.00256	0.00083	0.00035	0.00018	0.00010	0.9	15
0.75	0.74922	0.01709	0.00351	0.00115	0.00049	0.00024	0.00013	1.2	18
0.8	0.90073	0.02284	0.00495	0.00169	0.00073	0.00035	0.00019	1.6	22
0.83	1.01391	0.02729	0.00604	0.00207	0.00087	0.00041	0.00022	1.9	32
0.85	1.09814	0.03088	0.00694	0.00241	0.00101	0.00047	0.00025	1.8	45
0.87	1.19864	0.03494	0.00797	0.00280	0.00117	0.00055	0.00031	3.8	61

The values of the coefficients a_k obtained by means of a Fourier analysis are given in Table 3 together with the values of ε_1, the maximum relative error in approximating the quantity r, and ε_2, the maximum error in approximating the quantity θ: $\varepsilon_2 = |\theta - \tan^{-1}(r_t/x_t)|$.

Additional calculations carried out for intermediate values of M. showed that using the formulas (2.4) to approximate the coefficients b and a_k from Tables 2 and 3 by splines usually results in an increase in ε_1, though by no more than an order of magnitude, while the value of ε_2 varies only slightly.

Thus, the data presented make it possible to design optimum bodies of revolution in an air-like gas on the range $0.5 \le M. \le 0.87$. The method [1] can also be applied when $M_a < 0.5$; however, when M_a exceeds 0.87, the calculation errors grow quickly (the technique for checking the calculation accuracy is described in [1]); then the iteration procedure, used in solving the problem, no longer converges.

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